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# On Generalisations of the AVD Conjecture to Digraphs<sup>☆</sup>

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## Abstract

Given an undirected graph, in the AVD (edge-colouring) Conjecture, the goal is to find a proper edge-colouring with the least number of colours such that every two adjacent vertices are incident to different sets of colours. More precisely, the conjecture says that, a few exceptions apart, every graph  $G$  should admit such an edge-colouring with at most  $\Delta(G) + 2$  colours. Several aspects of interest behind this problem have been investigated over the recent years, including verifications of the conjecture for particular graph classes, general approximations of the conjecture, and multiple generalisations.

In this paper, following a recent work of Sopena and Woźniak, generalisations of the AVD Conjecture to digraphs are investigated. More precisely, four of the several possible ways of generalising the conjecture are focused upon. We completely settle one of our four variants, while, for the three remaining ones, we provide partial results.

*Keywords:* AVD Conjecture; proper edge-colourings; digraphs.

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## 1. Introduction

One of the most central notions of graph theory is that of proper edge-colourings. Given an undirected graph  $G$ , a *proper  $k$ -edge-colouring*  $\phi$  of  $G$  is an assignment  $\phi : E(G) \rightarrow \{1, \dots, k\}$  of colours to the edges such that no two adjacent edges (*i.e.*, incident to a same vertex) get assigned the same colour. We are usually interested in determining the *chromatic index*  $\chi'(G)$  of  $G$ , which refers to the smallest  $k$  such that  $G$  admits a proper  $k$ -edge-colouring. Perhaps the most important result regarding the chromatic index of graphs is Vizing's Theorem [8], which states that, for every graph  $G$ , we have  $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$  (where  $\Delta(G)$  denotes the maximum degree of a vertex in  $G$ ). Even though that result means the chromatic index of any graph is one of only two possible values, it is important to recall that determining the chromatic index of a graph is an NP-complete problem in general [4].

In several contexts, it might be convenient to have edge-colourings of graphs that are not only proper, but also have additional properties. When considering such a stronger form of proper edge-colourings, an interesting question is about the least number of additional colours needed to construct one for any given graph.

In this work, we are mostly interested in proper edge-colourings that allow to distinguish adjacent vertices according to their respective sets of incident colours. More precisely,

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35 given a proper edge-colouring  $\phi$  of a graph  $G$ , for every vertex  $u$ , we can compute  $S(u)$ ,  
36 which is the set of colours assigned by  $\phi$  to the edges incident to  $u$ . Note that, by the  
37 properness of  $\phi$ , we always have  $|S(u)| = d(u)$ . Now, we say that  $\phi$  is *distinguishing* if,  
38 for every edge  $uv$ , we have  $S(u) \neq S(v)$ . We denote by  $\text{ndi}(G)$  (where “ndi” stands for  
39 “neighbour-distinguishing index”) the least  $k$  such that  $G$  admits a distinguishing proper  
40  $k$ -edge-colouring (if any). Actually, it is easy to see that  $\text{ndi}(G)$  is defined if and only if  
41  $G$  has no connected component isomorphic to  $K_2$  (just consider a proper edge-colouring  
42 assigning a distinct colour to every edge). Thus, regarding those notions, we are interested  
43 in *nice graphs*, which are the graphs with no connected component isomorphic to  $K_2$ .

44 We employ the simple terminology and notations above for the sake of the current  
45 work’s legibility. It is worth mentioning, however, that this terminology and these notations  
46 vary from one work to another in the literature. In particular, our notion of distinguishing  
47 edge-colouring is sometimes called *adjacent vertex-distinguishing edge-colouring*, *neighbour-*  
48 *distinguishing edge-colouring*, *adjacent strong edge-colouring* or *1-strong edge-colouring*.  
49 Our parameter  $\text{ndi}$  is sometimes written  $\chi'_{\text{avd}}$  and called the *adjacent vertex-distinguishing*  
50 *chromatic index*. Even more different terms and notations are used in some works.

51 Clearly, we have  $\chi'(G) \leq \text{ndi}(G)$  for every nice graph  $G$ . Regarding the concerns  
52 above, a natural question to ask is how large can  $\text{ndi}(G)$  be in general. It can be noted  
53 that  $\text{ndi}(C_5) = \Delta(C_5) + 3 = 5$ , where  $C_5$  denotes the cycle of length 5. However, it is  
54 believed that  $C_5$  should be the only nice graph  $G$  for which  $\text{ndi}(G)$  is that large. This leads  
55 to the following conjecture raised by Zhang, Liu, and Wang in 2002 [11].

56 **AVD Conjecture.** *For every nice connected graph  $G \neq C_5$ , we have  $\text{ndi}(G) \leq \Delta(G) + 2$ .*

57 Comparing the AVD Conjecture to Vizing’s Theorem, the conjecture indicates that,  
58 in general, at most two additional colours might be necessary to turn a normal proper  
59  $\chi'(G)$ -edge-colouring of a graph  $G$  into a distinguishing one.

60 Several interesting results towards the AVD Conjecture have been obtained since its  
61 introduction. In particular, the conjecture was verified for bipartite graphs and subcubic  
62 graphs [2]. It was also proven in [1], that every nice graph  $G$  verifies  $\text{ndi}(G) \leq 3\Delta(G)$ . We  
63 refer the interested reader to [6] for more details.

64 In graph theory, a common line of research is, given a particular problem defined on  
65 undirected graphs, to wonder about its counterparts on digraphs. Regarding the AVD  
66 Conjecture, this is a promising prospect due to the numerous generalisations that can be  
67 considered. Indeed, by a proper arc-colouring of a digraph  $D$ , note that every vertex  $u$  can  
68 now be associated two sets  $S^+(u)$  and  $S^-(u)$  of incident colours, where  $S^+(u)$  is the set  
69 of colours assigned to the arcs out-going from  $u$ , and  $S^-(u)$  is the set of colours assigned  
70 to the arcs in-coming to  $u$ . Recall that, in the directed context, a *proper arc-colouring*  
71  $\phi$  of  $D$  verifies that two arcs out-going from a same vertex are assigned distinct colours,  
72 and similarly for two arcs in-coming to a same vertex. Under that definition, note that an  
73 arc out-going from  $u$  and an arc in-coming to  $u$  can be assigned the same colour. Note  
74 also that, because  $\phi$  is proper, we have  $|S^+(u)| = d^+(u)$  and  $|S^-(u)| = d^-(u)$ , which is  
75 reminiscent of the similar property of proper edge-colourings of undirected graphs.

76 For a digraph  $D$ , its *chromatic index*  $\chi'(D)$  is the least  $k$  such that  $D$  admits proper  
77  $k$ -arc-colourings. By the definition of proper arc-colourings, note that we always have  
78  $\chi'(D) \geq \Delta^*(D) = \max\{\Delta^+(D), \Delta^-(D)\}$ , where  $\Delta^+(D)$  and  $\Delta^-(D)$  denote the maximum  
79 out-degree and maximum in-degree, respectively, over all vertices of  $D$ . In contrast with  
80 the undirected context of Vizing’s Theorem, it is known that the chromatic index of a  
81 digraph  $D$  is always precisely the natural lower bound  $\Delta^*(D)$  (see, *e.g.* [10]).

82 Since every vertex gets assigned two sets of colours in any proper arc-colouring  $\phi$  of  
83 a digraph, there are plenty of ways of considering that two adjacent vertices are distin-  
84 guished by  $\phi$ , and thus, many possibilities for defining a directed counterpart to the AVD  
85 Conjecture. To the best of our knowledge, this line of research was not considered until  
86 quite recently, by Sopena and Woźniak in [7]. In their variant of the problem, they consider  
87 that two adjacent vertices  $u$  and  $v$  are distinguished by a proper arc-colouring when either  
88  $S^+(u) \neq S^+(v)$  or  $S^-(u) \neq S^-(v)$ . They conjectured that every digraph  $D$  admits such a  
89 distinguishing proper arc-colouring using at most  $\Delta^*(D) + 1$  colours, and they proved that  
90  $2\Delta^*(D)$  colours are enough to construct one.

91 In the line of the investigations initiated by Sopena and Woźniak, we consider several  
92 directed variants of the AVD Conjecture. Specifically, we consider four variants, in which,  
93 for every two adjacent vertices, we ask that a single of the two set parameters differs.  
94 Precisely, our general terminology is as follows. Let  $D$  be a digraph, and  $\phi$  a proper arc-  
95 colouring of  $D$ . To each of the set parameters  $S^+$  and  $S^-$ , we associate a sign, namely  
96  $+$  and  $-$ , respectively. Now, for any two signs  $\alpha, \beta \in \{+, -\}$ , we say that  $\phi$  is  $(\alpha, \beta)$ -  
97 *distinguishing* if, for every arc  $\vec{uv}$  of  $D$ , the set parameter of  $u$  associated to  $\alpha$  is different  
98 from the set parameter of  $v$  associated to  $\beta$ . We denote by  $\text{ndi}_{\alpha, \beta}(D)$  the least  $k$  such that  
99  $D$  admits an  $(\alpha, \beta)$ -distinguishing proper  $k$ -arc-colouring (if any).

100 Note that this general terminology encapsulates a series of four colouring problems, each  
101 of which has behaviours that are more or less reminiscent of the original AVD Conjecture.  
102 In terms of colouring behaviours, note that the  $(+, -)$  version is the closest to the original  
103 problem, as, by a proper arc-colouring of a digraph, assigning a colour to an arc  $\vec{uv}$  affects  
104  $S^+(u)$  and  $S^-(v)$  which are precisely the two set parameters that are required to be different.  
105 From this perspective, the  $(+, +)$  and  $(-, -)$  versions are a bit farther from the original  
106 conjecture, while the  $(-, +)$  version is the most distant. Still, for all four of the versions,  
107 recall that the number of colours needed in a distinguishing proper arc-colouring is strongly  
108 dependent on the chromatic index.

109 **Observation 1.1.** *Let  $\alpha, \beta \in \{+, -\}$ . For every digraph  $D$  such that  $\text{ndi}_{\alpha, \beta}(D)$  is defined,*  
110 *we have  $\chi'(D) \leq \text{ndi}_{\alpha, \beta}(D)$ .*

111 Although our four variants of the AVD Conjecture have their own peculiar behaviours,  
112 we feel that, in general, for any of the variants,  $\Delta^*(D) + 2$  colours should always be enough  
113 to construct a desired distinguishing proper arc-colouring. Our goal in this paper is to  
114 provide evidence towards this intuition. Section 2 is dedicated to investigating the  $(+, +)$   
115 and  $(-, -)$  versions of the AVD Conjecture, while Section 3 is dedicated to the  $(+, -)$   
116 version, and Section 4 is dedicated to the  $(-, +)$  variant. We provide a tight result on the  
117  $(+, -)$  variant, while we provide partial results for the other variants.

## 118 2. The $(+, +)$ and $(-, -)$ versions

119 Note that a  $(+, +)$ -distinguishing proper arc-colouring of a given digraph  $D$  directly  
120 yields a  $(-, -)$ -distinguishing proper arc-colouring of  $\tilde{D}$ , the digraph obtained from  $D$  by  
121 reversing the direction of each arc. We can thus focus on the  $(+, +)$  version of the AVD  
122 Conjecture in this section, as these results apply to the  $(-, -)$  version as well, through this  
123 arc reversing operation.

124 First off, we note that all digraphs admit a  $(+, +)$ -distinguishing proper arc-colouring.  
125 In other words, we do not need a notion of nice digraphs in this context.

126 **Proposition 2.1.** *Every digraph admits a  $(+, +)$ -distinguishing proper arc-colouring.*

127 *Proof.* Let  $D$  be any digraph with arcs  $a_1, \dots, a_m$ . Consider the arc-colouring  $\phi$  that sets  
 128  $\phi(a_i) = i$  for every  $i \in \{1, \dots, m\}$ . Clearly,  $\phi$  is proper. It is also  $(+, +)$ -distinguishing,  
 129 since, for every vertex  $u$  with an out-going arc  $a_i$ , only the set  $S^+(u)$  contains  $i$ .  $\square$

130 Note that, for any directed cycle  $D$  of odd length, we have  $\Delta^*(D) = 1$ , but we need  
 131 to use  $\Delta^*(D) + 2 = 3$  colours (two consecutive arcs must be assigned distinct colours in  
 132 any  $(+, +)$ -distinguishing proper arc-colouring of a directed cycle). We believe this is the  
 133 maximum value of  $\text{ndi}_{+,+}(D)$  for a digraph  $D$ .

134 **Conjecture 2.2.** *For every digraph  $D$ , we have  $\text{ndi}_{+,+}(D) \leq \Delta^*(D) + 2$ .*

135 Towards Conjecture 2.2, for every digraph  $D$ , we can easily establish an upper bound on  
 136  $\text{ndi}_{+,+}(D)$  that is linear in  $\Delta^*(D)$  by exploiting its relationship with proper edge-colourings  
 137 of  $\text{und}(D)$ , the undirected graph underlying  $D$ .

138 **Proposition 2.3.** *For every digraph  $D$ , we have  $\text{ndi}_{+,+}(D) \leq 2\Delta^*(D) + 2$ .*

139 *Proof.* Let  $G$  be the undirected multigraph obtained from  $D$  by replacing each arc by an  
 140 edge. Note that  $G$  might indeed have parallel edges, but the maximum multiplicity  $\mu(G)$   
 141 of its edges is 2. Also,  $\Delta(G) \leq \Delta^-(D) + \Delta^+(D) \leq 2\Delta^*(D)$ . Now, by Vizing's Theorem [9],  
 142 there is a proper edge-colouring  $\phi_G$  of  $G$  using  $\Delta(G) + \mu(G) = \Delta(G) + 2$  colours. We infer  
 143  $\phi_G$  to an arc-colouring  $\phi_D$  of  $D$  by simply transferring the colour of an edge in  $G$  to the  
 144 corresponding arc in  $D$ . By the properness of  $\phi_G$ , note that  $\phi_D$  is also proper. Also, for  
 145 every arc  $uv$  of  $D$ , we get that no arc out-going from  $v$  is assigned colour  $\phi_D(uv)$ , and thus,  
 146  $S^+(u) \neq S^+(v)$  since  $\phi_D(uv) \notin S^+(v)$ . Thus,  $\phi_D$  is also  $(+, +)$ -distinguishing. Since  $\phi_D$   
 147 uses at most  $2\Delta^*(D) + 2$  colours, the result follows.  $\square$

148 Using a different approach, we can slightly improve the upper bound in Proposition 2.3.

149 **Theorem 2.4.** *For every digraph  $D$ , we have  $\text{ndi}_{+,+}(D) \leq 2\Delta^*(D) + 1$ .*

150 *Proof.* The proof is by induction on  $k$ , the number of vertices with out-degree at least 1. If  
 151  $k = 1$ , then  $D$  has only one vertex with out-going arcs, in which case  $D$  is an out-star. In  
 152 this case, a  $(+, +)$ -distinguishing proper  $\Delta^*(D)$ -arc-colouring  $\phi$  is obtained by assigning a  
 153 distinct colour to each arc of  $D$ . Indeed,  $\phi$  is clearly proper, and we have  $S^+(v) = \emptyset$  for  
 154 every leaf  $v$  of  $D$  and  $S^+(u) \neq \emptyset$  for the center  $u$ .

155 Assume the claim holds for  $k$  up to some value  $x$ , and consider that  $D$  has  $x + 1$  vertices  
 156 with out-degree at least 1. Let us consider  $u$  to be any vertex with out-degree at least 1, and  
 157 out-going arcs  $u\vec{v}_1, \dots, u\vec{v}_d$  (where  $d \geq 1$ ). Consider  $D' = D - \{u\vec{v}_1, \dots, u\vec{v}_d\}$  the digraph  
 158 obtained from  $D$  by removing all arcs out-going from  $u$ . By the induction hypothesis,  $D'$   
 159 has a  $(+, +)$ -distinguishing proper  $(2\Delta^*(D) + 1)$ -arc-colouring  $\phi_{D'}$  (since  $\Delta^*(D) \geq \Delta^*(D')$ ),  
 160 which we would like to extend to a  $(+, +)$ -distinguishing proper  $(2\Delta^*(D) + 1)$ -arc-colouring  
 161  $\phi_D$  of  $D$ , i.e., to the arcs  $u\vec{v}_1, \dots, u\vec{v}_d$ . Note that assigning a colour to any arc  $u\vec{v}_i$  only  
 162 affects  $S^+(u)$ . Thus, our goal is to assign colours to the  $u\vec{v}_i$ 's in a proper way, so that the  
 163 resulting  $S^+(u)$  is different from  $S^+(w)$  for every neighbour  $w$  of  $u$  in  $D$ . Note that there  
 164 are at most  $2\Delta^*(D)$  such neighbours  $w$  around  $u$ .

165 For every arc  $u\vec{v}_i$ , in terms of properness, the colour assigned to  $u\vec{v}_i$  must differ from  
 166 the colours assigned to the at most  $\Delta^*(D) - 1$  other arcs in-coming to  $v_i$ . Since we are  
 167 using a set of  $2\Delta^*(D) + 1$  colours, there are at least  $\Delta^*(D) + 2$  colours that can, from that  
 168 point of view, freely be assigned to  $u\vec{v}_i$ . For every  $i \in \{1, \dots, d\}$ , let us denote by  $L_i$  the  
 169 set of these colours. Our goal now, is to choose distinct elements (to ensure properness)  
 170 from  $L_1, \dots, L_d$ , one from each of the  $L_i$ 's, in such a way that the union of these elements  
 171 avoids the sets of colours of the at most  $2\Delta^*(D)$  neighbours of  $u$ . This is something that  
 172 can always be done, according to the following claim:

173 **Claim 2.5.** Let  $L_1, \dots, L_d$  be  $d \geq 1$  sets each containing at least  $d + 2$  elements. Then,  
 174 there are at least  $2d + 1$  different combinations  $e_1, \dots, e_d$  of elements, such that  $e_i \in L_i$  for  
 175 every  $i \in \{1, \dots, d\}$ , and all the  $e_i$ 's are distinct.

176 *Proof of the claim.* The proof is by induction on  $d$ . For  $d = 1$ , we have  $|L_1| = 3$ . Every  
 177 single element of  $L_1$  is a correct choice as  $e_1$ , and thus, there are three correct combinations.  
 178 Assume now that the claim holds for every  $d$  up to some  $x$ , and assume  $d = x + 1$ . Without  
 179 loss of generality, we may assume  $1 \in L_1$ . Set  $L'_2 = L_2 \setminus \{1\}, \dots, L'_d = L_d \setminus \{1\}$ . By the  
 180 induction hypothesis, we can produce  $2(d - 1) + 1 = 2d - 1$  combinations  $e'_2, \dots, e'_d$  of  
 181 distinct elements from the  $L'_i$ 's (one from each set). Each such combination  $e'_2, \dots, e'_d$ ,  
 182 together with 1, yields a combination  $1, e'_2, \dots, e'_d$  that is valid for  $L_1, \dots, L_d$ . Thus, we  
 183 already know how to generate  $2d - 1$  different combinations of distinct elements from  
 184  $L_1, \dots, L_d$ , all of which contain the element 1.

185 All that remains to do is to generate two more combinations. To ensure this, we exhibit  
 186 two such different combinations not containing the element 1. To that aim, let us choose  
 187 arbitrary distinct elements  $e_2, \dots, e_d$  different from 1 from  $L_2, \dots, L_d$ . This is possible  
 188 since each  $L_i$  has size at least  $d + 2$ . Now, since  $L_1$  also has size at least  $d + 2$ , there are  
 189 at least three elements  $e_1, e'_1, e''_1$  that are different from  $e_2, \dots, e_d$ . At least two of  $e_1, e'_1, e''_1$   
 190 must be different from 1. These two elements together with  $e_2, \dots, e_d$  yield our remaining  
 191 two different combinations.  $\diamond$

192 Now, by Claim 2.5, we can choose a combination  $e_1, \dots, e_d$  of distinct elements from  
 193  $L_1, \dots, L_d$  (where  $e_i \in L_i$  for every  $i \in \{1, \dots, d\}$ ) such that no  $w$  of the at most  $2\Delta^*(D)$   
 194 neighbours of  $u$  in  $D$  verifies  $S^+(w) = \{e_1, \dots, e_d\}$ . To finish the extension of  $\phi_{D'}$  to  $\phi_D$ ,  
 195 it now suffices to set  $\phi_D(u\vec{v}_i) = e_i$  for every  $i \in \{1, \dots, d\}$ .  $\square$

### 196 3. The $(+, -)$ version

197 In this section, we focus on the  $(+, -)$  version of the AVD Conjecture, in which, by a  
 198 proper arc-colouring of a digraph, it is required that  $S^+(u) \neq S^-(v)$  holds for every arc  
 199  $u\vec{v}$ . Recall that this distinction condition is, out of the four ones we are considering, the  
 200 closest to the original distinction condition behind the original AVD Conjecture. Indeed,  
 201 by a proper arc-colouring  $\phi$  of some digraph, for every arc  $u\vec{v}$  the colour  $\phi(u\vec{v})$  contributes  
 202 to both  $S^+(u)$  and  $S^-(v)$ , which are precisely the two set parameters that are asked to  
 203 differ for  $u$  and  $v$  in the  $(+, -)$  version.

204 Compared to the  $(+, +)$  version, there are digraphs admitting no  $(+, -)$ -distinguishing  
 205 proper arc-colourings. The smallest such digraph is a single arc  $u\vec{v}$ , as  $S^+(u) = S^-(v)$  in  
 206 any proper arc-colouring. Actually, the case of such a pathological arc can be generalised  
 207 in the following way. We say that an arc  $u\vec{v}$  of a digraph is *lonely* if  $d^+(u) = d^-(v) = 1$ .  
 208 Note that, indeed,  $\text{ndi}_{+,-}(D)$  is not defined for every digraph  $D$  containing a lonely arc. It  
 209 turns out that lonely arcs are the only source of non-colourability in the  $(+, -)$  version.  
 210 That is, if we define a  $(+, -)$ -nice digraph as a digraph  $D$  with  $\text{ndi}_{+,-}(D)$  being defined,  
 211 then being  $(+, -)$ -nice is equivalent to having no lonely arcs.

212 **Proposition 3.1.** A digraph is  $(+, -)$ -nice if and only if it has no lonely arc.

213 *Proof.* Consider any digraph  $D$ , and let us denote by  $a_1, \dots, a_m$  its arcs in an arbitrary  
 214 fashion. Let  $\phi$  be the arc-colouring of  $D$  where  $\phi(a_i) = i$  for every  $i \in \{1, \dots, m\}$ . Clearly  
 215  $\phi$  is proper. We claim  $\phi$  is  $(+, -)$ -distinguishing if and only if  $D$  has no lonely arc. Indeed,  
 216 consider an arc  $u\vec{v}$ . If  $d^+(u) \neq d^-(v)$ , then for sure  $S^+(u) \neq S^-(v)$  since  $|S^+(u)| \neq |S^-(v)|$ .  
 217 Thus, assume  $d^+(u) = d^-(v)$ . If  $d^+(u) = d^-(v) \geq 2$ , then, if we denote by  $v'$  another

218 out-neighbour of  $u$ , we have  $\phi(\vec{uv'}) \in S^+(u)$  and  $\phi(\vec{uv'}) \notin S^-(v)$ , and thus  $S^+(u) \neq S^-(v)$ .  
 219 The only remaining case is when  $d^+(u) = d^-(v) = 1$ , which is precisely the case where  $\vec{uv}$   
 220 is lonely, and we necessarily have  $S^+(u) = S^-(v)$  by any proper arc-colouring of  $D$ .  $\square$

221 We are actually able to prove a tight general upper bound on  $\text{ndi}_{+,-}(D)$  for  $(+, -)$ -nice  
 222 digraphs  $D$ . We prove that bound right away, because most of the remarks we can raise  
 223 on the  $(+, -)$  version of the AVD Conjecture actually follow from our proof scheme.

224 Our proof relies on an equivalence between the  $(+, -)$  version and particular cases of  
 225 the original AVD Conjecture. This equivalence is with respect to the following notions  
 226 and definitions. For a digraph  $D$ , by the *bipartite graph associated to  $D$* , we refer to the  
 227 undirected bipartite graph  $B(D)$  constructed as follows:

- 228 • The two partite classes of  $B(D)$  are  $V^+$  and  $V^-$ .
- 229 • For every vertex  $u$  of  $D$ , we add a vertex  $u^+$  to  $V^+$  and a vertex  $u^-$  to  $V^-$ .
- 230 • For every arc  $\vec{uv}$  of  $D$ , we add the edge  $u^+v^-$  to  $B(D)$ .

231 In some sense,  $B(D)$  is obtained from  $D$  by splitting the out-going part and the in-  
 232 coming part of every vertex. Note that  $B(D)$  is always balanced, in the sense that  $|V^+| =$   
 233  $|V^-|$ . Also, we can infer some additional properties of  $B(D)$  from properties of  $D$ .

234 **Observation 3.2.** *For every digraph  $D$ :*

- 235 •  $\Delta(B(D)) = \Delta^*(D)$ .
- 236 •  $B(D)$  is nice if and only if  $D$  is  $(+, -)$ -nice.

237 *Proof.* The first item is because, for every vertex  $u$  of  $D$ , the value of  $d^+(u)$  (in  $D$ ) equals  
 238 the value of  $d(u^+)$  (in  $B(D)$ ), and similarly  $d^-(u)$  equals  $d(u^-)$ . The second item is because  
 239 an isolated edge  $u^+v^-$  in  $B(D)$  corresponds to a lonely arc  $\vec{uv}$  in  $D$ , and *vice versa*.  $\square$

240 We are now ready to prove our main result in this section.

241 **Theorem 3.3.** *The  $(+, -)$  version of the AVD Conjecture is equivalent to the (original)  
 242 AVD Conjecture in bipartite graphs.*

243 *Proof.* The notion of the associated bipartite graph is the key behind this equivalence.  
 244 Indeed, finding a  $(+, -)$ -distinguishing proper  $k$ -arc-colouring of some  $(+, -)$ -nice digraph  
 245 is equivalent to finding a distinguishing proper  $k$ -edge-colouring of some nice undirected  
 246 bipartite graph.

- 247 • Let  $D$  be a  $(+, -)$ -nice digraph, and consider the (nice, by Observation 3.2) bipartite  
 248 graph  $B = B(D)$  associated to  $D$ . Let  $\phi_B$  be a distinguishing proper  $k$ -edge-colouring  
 249 of  $B$ . Consider the  $k$ -arc-colouring  $\phi_D$  of  $D$  obtained by setting  $\phi_D(\vec{uv}) = \phi_B(u^+v^-)$   
 250 for every arc  $\vec{uv}$ . Note that  $\phi_D$  is proper because  $\phi_D(\vec{uv}) \neq \phi_D(\vec{uv'})$  when  $v \neq v'$  since  
 251  $\phi_B(u^+v^-) \neq \phi_B(u^+v'^-)$  (by the properness of  $\phi_B$ ), and, similarly,  $\phi_D(\vec{uv}) \neq \phi_D(\vec{u'v})$   
 252 when  $u \neq u'$  since  $\phi_B(u^+v^-) \neq \phi_B(u'^+v^-)$ . Also,  $\phi_D$  is  $(+, -)$ -distinguishing because  
 253  $S^+(u) \neq S^-(v)$  for every arc  $\vec{uv}$  of  $D$  since, in  $B$ ,  $S(u^+) \neq S(v^-)$  (by  $\phi_B$  being  
 254 distinguishing). Thus,  $\phi_D$  is a  $(+, -)$ -distinguishing proper  $k$ -arc-colouring of  $D$ .

255 • Let  $B$  be a nice bipartite graph. Denote by  $(U, V)$  the bipartition of  $B$ . If necessary,  
256 add isolated vertices to  $B$  so that 1)  $B$  is balanced, and 2) there is an ordering of  
257 the vertices in  $U$  and  $V$  such that  $U = \{u_1, \dots, u_n\}$ ,  $V = \{v_1, \dots, v_n\}$ , and  $u_i v_i$  is  
258 not an edge for every  $i \in \{1, \dots, n\}$ . Under those conditions, let  $D$  be the digraph  
259 constructed from  $B$  by adding a vertex  $w_i$  for every  $i \in \{1, \dots, n\}$ , and an arc  $w_i \vec{w}_j$   
260 for every edge  $u_i v_j$  of  $B$ . Clearly,  $B$  is the bipartite graph  $B(D)$  associated to  $D$   
261 (where  $U$  plays the role of  $V^+$  and  $V$  plays the role of  $V^-$ ). We now have the desired  
262 equivalence by the arguments used to deal with the previous case.  $\square$

263 The equivalence established in the proof of Theorem 3.3 has a series of consequences  
264 on the  $(+, -)$  version of the AVD Conjecture. In particular, the fact that  $\text{ndi}_{+,-}(D) =$   
265  $\text{ndi}(B(D))$  and  $\Delta^*(D) = \Delta(B(D))$  hold for every  $(+, -)$ -nice digraph  $D$  yields some side  
266 results. For instance, it is known that there exist nice bipartite graphs  $G$  with  $\text{ndi}(G) =$   
267  $\Delta(G) + 2$ , see [2]. In the same paper, the authors proved that  $\Delta(G) + 2$  is actually the  
268 maximum value of  $\text{ndi}(G)$  for a nice bipartite graph  $G$ . In other words, the AVD Conjecture  
269 holds for nice bipartite graphs. For our problem, these remarks directly imply:

270 **Corollary 3.4.** *For every  $(+, -)$ -nice digraph  $D$ , we have  $\text{ndi}_{+,-}(D) \leq \Delta^*(D) + 2$ . Fur-*  
271 *thermore, this upper bound cannot be decreased in general.*

#### 272 4. The $(-, +)$ version

273 We now consider the  $(-, +)$  version of the AVD Conjecture, in which two vertices  $u$   
274 and  $v$  that are adjacent through an arc  $u\vec{v}$  are required, by a proper arc-colouring, to verify  
275  $S^-(u) \neq S^+(v)$ . Recall that this version is, in some sense, the variant we are considering  
276 that is the farthest from the original conjecture. This is mainly because the colour  $\phi(u\vec{v})$   
277 of an arc  $u\vec{v}$  by an arc-colouring  $\phi$  of a digraph contributes nothing to  $S^-(u)$  and  $S^+(v)$ ,  
278 which are the parameters of  $u$  and  $v$  that must differ.

279 In the present context, again, not all digraphs admit  $(-, +)$ -distinguishing proper arc-  
280 colourings. To see this is true, consider the case of a digraph  $D$  containing an arc  $s\vec{t}$   
281 such that  $d^-(s) = 0$  (source) and  $d^+(t) = 0$  (sink). Clearly,  $D$  does not admit a  $(-, +)$ -  
282 distinguishing proper arc-colouring, since we always have  $S^-(s) = \emptyset = S^+(t)$ . Note that  
283 the situation remains unchanged if we add the arc  $t\vec{s}$  to  $D$ , since, here, we would always get  
284  $S^-(s) = \alpha = S^+(t)$ , when assigning a colour  $\alpha$  to  $t\vec{s}$ . If  $D$  has two such adjacent vertices,  
285 we say that  $D$  has a *bad configuration*. Again, in this variant, a  $(-, +)$ -nice digraph is a  
286 digraph  $D$  for which  $\text{ndi}_{-,+}(D)$  is defined. Actually, bad configurations are the only reason  
287 why some digraphs are not  $(-, +)$ -nice:

288 **Proposition 4.1.** *A digraph is  $(-, +)$ -nice if and only if it has no bad configuration.*

289 *Proof.* Let  $D$  be any digraph with arcs  $a_1, \dots, a_m$ , and let  $\phi$  be the arc-colouring of  $D$   
290 where  $\phi(a_i) = i$  for every  $i \in \{1, \dots, m\}$ . We claim that  $\phi$ , which is clearly proper, is  
291  $(-, +)$ -distinguishing if and only if  $D$  has no bad configuration. Indeed, let us focus on an  
292 arc  $u\vec{v}$ . If  $d^-(u) \neq d^+(v)$ , then  $S^-(u) \neq S^+(v)$  because  $|S^-(u)| \neq |S^+(v)|$ . So, let us assume  
293 that  $d^-(u) = d^+(v)$ . If  $d^-(u) = d^+(v) \geq 2$ , then  $S^-(u) \neq S^+(v)$  due to there being at least  
294 one arc out-going from  $v$  that is not in-coming to  $u$ . The same holds if  $d^-(u) = d^+(v) = 1$   
295 and the arc in-coming to  $u$  is different from the arc out-going from  $v$ . So, there are only  
296 two cases remaining: 1)  $d^-(u) = d^+(v) = 1$  and  $v\vec{u}$  is an arc, and 2)  $d^-(u) = d^+(v) = 0$ .  
297 In both cases,  $D$  has a bad configuration, and  $\phi$  cannot be  $(-, +)$ -distinguishing.  $\square$



298 We note that there are  $(-, +)$ -nice digraphs  $D$  with  $\text{ndi}_{-,+}(D) = \Delta^*(D) + 2$ . Every  
 299 odd-length directed cycle is an example of such a digraph. We think this might be the  
 300 maximum value of  $\text{ndi}_{-,+}(D)$  for a  $(-, +)$ -nice digraph  $D$ , which would be reminiscent of  
 301 the AVD Conjecture.

302 **Conjecture 4.2.** *For every  $(-, +)$ -nice digraph  $D$ , we have  $\text{ndi}_{-,+}(D) \leq \Delta^*(D) + 2$ .*

303 Towards Conjecture 4.2, contrarily to what was done in Section 2, note that using  
 304 proper edge-colourings does not yield a linear upper bound (in  $\Delta^*(D)$ ) on  $\text{ndi}_{-,+}(D)$  for  
 305 every  $(-, +)$ -nice digraph  $D$ , as the distinction condition in the  $(-, +)$  version, in some  
 306 sense, asks arcs at distance 2 to be different. Instead, an upper bound on  $\text{ndi}_{-,+}(D)$  can be  
 307 expressed, for instance, as a function of the *strong chromatic index*  $\chi'_s(\text{und}(D))$  of  $\text{und}(D)$ ,  
 308 which is the smallest number of colours in an edge-colouring of  $\text{und}(D)$  where no two edges  
 309 at distance at most 2 get assigned a same colour. Such an upper bound would be quadratic  
 310 in  $\Delta^*(D)$  (see, for instance, [3]).

311 Using different arguments, for every  $(-, +)$ -nice digraph  $D$ , we provide an upper bound  
 312 on  $\text{ndi}_{-,+}(D)$  that is linear in  $\Delta^*(D)$ . Just as in Section 3, this is by exploiting some  
 313 relationship between an arc-colouring of  $D$  and an edge-colouring of  $B(D)$ , the bipartite  
 314 graph associated to  $D$ . However, note that using the relationship is not so natural in the  
 315 present context. Indeed, by a proper edge-colouring of  $B(D)$ , having  $S(u^+) \neq S(v^-)$  is not  
 316 so relevant regarding  $D$ , as, when transposing the edge-colouring to an arc-colouring of  $D$ ,  
 317 this would yield  $S^+(u) \neq S^-(v)$ , which is not required in the  $(-, +)$  version. Also, it might  
 318 be that we want  $S^+(u)$  and  $S^-(v)$  to differ in  $D$ , while  $S(u^+)$  and  $S(v^-)$  are not required  
 319 to differ in  $B(D)$  because  $u^+v^-$  is not an edge. To deal with such issues, we will consider  
 320 distinguishing proper edge-colourings of  $B(D)$  verifying strong distinction conditions.

321 Before proceeding with the proof, it is important to point out that lonely arcs, though  
 322 they do not prevent  $\text{ndi}_{-,+}(D)$  to be defined for a  $(-, +)$ -nice digraph  $D$ , have a peculiar  
 323 behaviour (they yield isolated edges in  $B(D)$ ) that will force us to handle them separately.  
 324 In particular, we will make use of the following property of lonely arcs:

325 **Observation 4.3.** *Removing a lonely arc from a digraph cannot create new lonely arcs.*

326 *Proof.* Let  $D$  be a digraph, and let  $D' = D - \vec{uv}$  be the digraph obtained from  $D$  by  
 327 removing a lonely arc  $\vec{uv}$ . Assume  $D'$  has a lonely arc  $\vec{xy}$  which is not lonely in  $D$ . Then,  
 328 either  $u = x$  or  $v = y$ . In the first case, we deduce that  $u$  has out-degree 2 in  $D$ , while, in the  
 329 second case, we deduce that  $v$  has in-degree 2 in  $D$ . In both cases, we get a contradiction  
 330 to the loneliness of  $\vec{uv}$ .  $\square$

331 We are now ready to prove our main result in this section.

332 **Theorem 4.4.** *For every  $(-, +)$ -nice digraph  $D$ , we have  $\text{ndi}_{-,+}(D) \leq 3\Delta^*(D)$ .*

333 *Proof.* Let  $L$  be the set of all lonely arcs of  $D$ , and set  $D' = D - L$ . By Observation 4.3,  
 334  $D'$  has no lonely arcs. Let  $B = B(D')$  be the bipartite graph associated to  $D'$ . By  
 335 Observation 3.2,  $D'$  is nice, and  $\Delta = \Delta(B) \leq \Delta^*(D') \leq \Delta^*(D)$ . Recall that the bipartition  
 336 of  $B$  is denoted by  $(V^+, V^-)$ . Note that  $B$  may have several connected components. In  
 337 what follows, we need to dedicate a special care to some of them. Specifically, a connected  
 338 component of  $B$  is said to be *bad* if it is a star with center in  $V^+$  (and at least two leaves,  
 339 in  $V^-$ , since  $B$  is nice). In what follows, we obtain a  $(-, +)$ -distinguishing proper  $3\Delta$ -arc-  
 340 colouring  $\phi_D$  of  $D$  by first colouring the edges of the non-bad connected components of  
 341  $B$  (and transferring the assigned colours to corresponding arcs in  $D$ ), then colouring, in

342  $D$ , the arcs corresponding to edges of the bad connected components of  $B$ , and eventually  
 343 colouring the lonely arcs of  $D$ .

344 We start by constructing a proper  $3\Delta$ -edge-colouring  $\phi_B$  of the non-bad connected  
 345 components of  $B$ , such that all vertices  $v \in V^+$  verify  $S(v) \in \mathcal{X}$  while all vertices  $v \in V^-$   
 346 verify  $S(v) \in \mathcal{Y}$ , for some disjoint sets  $\mathcal{X}, \mathcal{Y}$  of subsets of  $\{1, \dots, 3\Delta\}$ . To achieve this, we  
 347 split  $\{1, \dots, 3\Delta\}$  into the three smaller sets  $\mathcal{R} = \{1, \dots, \Delta\}$ ,  $\mathcal{B} = \{\Delta + 1, \dots, 2\Delta\}$ , and  
 348  $\mathcal{G} = \{2\Delta + 1, \dots, 3\Delta\}$  of colours, which we assign as follows.

349 Let us focus on a non-bad connected component of  $B$ . Abusing the notation, let us  
 350 denote by  $B$  this connected component. Recall that there are at least two edges in  $B$ , since  
 351  $B$  is nice. Pick an arbitrary vertex  $u^* \in V^+$ . This defines a partition of  $V(B)$  into *layers*  
 352  $V_0, \dots, V_d$ , where every  $V_i$  contains the vertices of  $B$  that are at distance exactly  $i$  from  
 353  $u^*$ . Since  $B$  is not bad, we have  $d \geq 2$ . Note that  $V_0 = \{u^*\}$ , that every edge of  $B$  joins  
 354 vertices in two consecutive  $V_i$ 's, that every vertex in some  $V_i$  with  $i \neq 0$  has a neighbour  
 355 in  $V_{i-1}$ , and that the union of all  $V_i$ 's with even index is exactly  $V^+$  while the union of all  
 356  $V_i$ 's with odd index is exactly  $V^-$ . Given an edge  $uv$  with  $u \in V_i$  and  $v \in V_{i+1}$ , we say  
 357 that  $uv$  is *downward* from the point of view of  $u$ , while it is *upward* from that of  $v$ .

358 For every  $i \in \{1, \dots, d\}$ , we split  $V_i$  into  $V_i'$  and  $V_i''$ , where  $V_i'$  contains the vertices  
 359 of  $V_i$  with no downward edges, while  $V_i''$  contains the vertices in  $V_i$  with downward edges.  
 360 Note that  $V_d'' = \emptyset$ , and recall that  $V_1'' \neq \emptyset$  since  $B$  is not bad. We now construct  $\phi_B$  by  
 361 assigning colours in  $\mathcal{R}$ ,  $\mathcal{B}$ , and  $\mathcal{G}$  to some sets  $F_R$ ,  $F_B$ , and  $F_G$  of edges of  $B$ , that are  
 362 defined as follows:

- 363 • For all  $i \in \{0, \dots, d-1\}$  with  $i \equiv 0 \pmod{4}$ , add all edges in  $E(B[V_i \cup V_{i+1}'])$  to  $F_R$ ,  
 364 and all edges in  $E(B[V_i \cup V_{i+1}''])$  to  $F_G$ .
- 365 • For all  $i \in \{1, \dots, d-1\}$  with  $i \equiv 1 \pmod{4}$ , add all edges in  $E(B[V_i \cup V_{i+1}])$  to  $F_B$ .
- 366 • For all  $i \in \{1, \dots, d-1\}$  with  $i \equiv 2 \pmod{4}$ , add all edges in  $E(B[V_i \cup V_{i+1}'])$  to  $F_R$ ,  
 367 and all edges in  $E(B[V_i \cup V_{i+1}''])$  to  $F_B$ .
- 368 • For all  $i \in \{1, \dots, d-1\}$  with  $i \equiv 3 \pmod{4}$ , add all edges in  $E(B[V_i \cup V_{i+1}])$  to  $F_G$ .

369 Note that  $F_R \cup F_B \cup F_G = E(B)$ . Furthermore, each of  $B[F_R]$ ,  $B[F_B]$ , and  $B[F_G]$   
 370 induces a subgraph of  $B$  with maximum degree at most  $\Delta$ . By Vizing's Theorem,  $B[F_R]$   
 371 admits a proper  $\Delta$ -edge-colouring with colours from  $\mathcal{R}$ ,  $B[F_B]$  admits a proper  $\Delta$ -edge-  
 372 colouring with colours from  $\mathcal{B}$ , and  $B[F_G]$  admits a proper  $\Delta$ -edge-colouring with colours  
 373 from  $\mathcal{G}$ . Altogether, this yields a proper  $3\Delta$ -edge-colouring  $\phi_B$  of  $B$  with colours from  
 374  $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G}$ .

375 In terms of vertex colours, by  $\phi_B$ , we get:

- 376 •  $S(u^*)$  has either elements from both  $\mathcal{R}$  and  $\mathcal{G}$  only (case where  $V_1' \neq \emptyset$  and  $V_1'' \neq \emptyset$ )  
 377 or elements from  $\mathcal{G}$  only (case where  $V_1' = \emptyset$ ).
- 378 • Consider a vertex  $u \in V_1$ . On the one hand, if  $u \in V_1'$ , then  $S(u)$  contains only one  
 379 element, from  $\mathcal{R}$ . On the other hand, if  $u \in V_1''$ , then  $S(u)$  contains exactly one  
 380 element from  $\mathcal{G}$  and at least one element from  $\mathcal{B}$  (and no element from  $\mathcal{R}$ ).
- 381 • More generally, for every  $u \in V_i$  with  $i > 1$ :
  - 382 – Assume  $i \equiv 0 \pmod{4}$ . On the one hand, if  $u \in V_i'$ , then  $S(u) \subseteq \mathcal{G}$ . On the other  
 383 hand, if  $u \in V_i''$ , then  $S(u)$  contains at least one element from  $\mathcal{G}$ , and perhaps  
 384 elements from  $\mathcal{R}$  (and no elements from  $\mathcal{B}$ ).

- 385 – Assume  $i \equiv 1 \pmod{4}$ . On the one hand, if  $u \in V_i'$ , then  $S(u) \subseteq \mathcal{R}$ . On the other  
386 hand, if  $u \in V_i''$ , then  $S(u)$  contains at least one element from  $\mathcal{G}$ , and at least  
387 one element from  $\mathcal{B}$  (and no elements from  $\mathcal{R}$ ).
- 388 – Assume  $i \equiv 2 \pmod{4}$ . On the one hand, if  $u \in V_i'$ , then  $S(u) \subseteq \mathcal{B}$ . On the other  
389 hand, if  $u \in V_i''$ , then  $S(u)$  contains at least one element from  $\mathcal{B}$ , and perhaps  
390 elements from  $\mathcal{R}$  (and no elements from  $\mathcal{G}$ ).
- 391 – Assume  $i \equiv 3 \pmod{4}$ . On the one hand, if  $u \in V_i'$ , then  $S(u) \subseteq \mathcal{R}$ . On the other  
392 hand, if  $u \in V_i''$ , then  $S(u)$  contains at least one element from  $\mathcal{B}$ , and at least  
393 one element from  $\mathcal{G}$  (and no elements from  $\mathcal{R}$ ).

394 It is then easy to check that  $\phi_B$  is distinguishing, since vertices in  $V^+$  and in  $V^-$  have  
395 sets of colours in disjoint sets  $\mathcal{X}$  and  $\mathcal{Y}$ . In particular, the fact that  $V_1'' \neq \emptyset$  (because  $B$  is  
396 not bad) implies that  $S(u^*)$  contains at least one element from  $\mathcal{G}$ .

397 By applying the arguments above to all the non-bad connected components of the  
398 whole of  $B$ , we get  $\phi_B$ , a partial distinguishing proper  $3\Delta$ -edge-colouring of the non-bad  
399 connected components of  $B$ . By arguments used in the proof of Theorem 3.3, the properties  
400 of  $\phi_B$  remain when transforming  $\phi_B$  to a partial arc-colouring  $\phi_{D'}$  of  $D'$ . In particular,  
401  $\phi_{D'}$  is proper and uses at most  $3\Delta$  colours, and, for every arc  $uv$  such that at least one of  
402  $u$  and  $v$  does not belong to a bad connected component of  $B$ , we have  $S^-(u) \neq S^+(v)$ .

403 Our goal now is to extend  $\phi_{D'}$  to a  $(-, +)$ -distinguishing proper  $(3\Delta^*(D))$ -arc-colouring  
404  $\phi_D$  of  $D$ . To that aim, note that two types of arc configurations remain to be coloured  
405 in  $D$ : 1) the configurations corresponding to the bad connected components of  $B$ , and 2)  
406 the lonely arcs. Extending the colouring to such configurations can be proved to always  
407 be possible, via, essentially, counting arguments.

408 First, consider a bad connected component in  $B$ . By definition, this connected compo-  
409 nent is a star with unique vertex  $u^+$  in  $V^+$  being its center, and being adjacent to  $k \geq 2$   
410 leaves  $v_1^-, \dots, v_k^-$  in  $V^-$ . Back in  $D$ , this corresponds to a vertex  $u$  with out-neighbours  
411  $v_1, \dots, v_k$ , where  $u$  has no other out-going arcs while all the  $v_i$ 's have no other in-coming  
412 arcs. For every  $v_i$ , note that if all arcs in-coming to  $u$  and all arcs out-going from  $v_i$   
413 have already been coloured, then  $S^-(u) \neq S^+(v_i)$ . This is because either both  $u^-$  and  $v_i^+$   
414 belong to non-bad connected components of  $B$  (in which case the distinction comes from  
415 how  $B$  was edge-coloured), because  $v_i^+$  is the center of a bad connected component in  $B$   
416 whose associated bad configuration in  $D$  was treated earlier (in which case the distinction  
417 comes from the upcoming counting arguments), or because  $u^-$  is a leaf in a bad connected  
418 component of  $B$  whose associated bad configuration in  $D$  was treated earlier (in which  
419 case the distinction comes from similar arguments as in the previous case). Thus, when  
420 colouring the arcs  $uv_1, \dots, uv_k$ , we only need to make sure that 1) all arcs out-going from  
421  $u$  are assigned distinct colours, that 2) for every  $v_i$  and every out-neighbour  $w$  of  $v_i$ , the  
422 resulting set  $S^-(v_i)$  is distinct from the set  $S^+(w)$ , and that 3) the resulting set  $S^+(u)$  is  
423 distinct from the set  $S^-(w)$  of every in-neighbour  $w$  of  $u$ . Since  $d^+(v_i) \leq \Delta^*(D)$  for every  
424  $v_i$ , regarding the second condition, there is a set  $L_i$  of at least  $2\Delta^*(D)$  colours that can be  
425 freely assigned to  $uv_i$  without violating that condition. We are now in a weaker condition  
426 than that of the statement of Claim 2.5: we have sets  $L_1, \dots, L_d$  of at least  $2\Delta^*(D)$  ele-  
427 ments, and we have to find a combination  $e_1, \dots, e_d$  of their elements such that all  $e_i$ 's are  
428 distinct, each  $e_i$  lies in  $L_i$ , and the set  $\{e_1, \dots, e_d\}$  is different from  $S^-(w)$  for each  $w$  of  
429 the at most  $\Delta^*(D)$  in-neighbours  $w$  of  $u$  in  $D$ . By Claim 2.5, there is a such combination  
430  $e_1, \dots, e_d$ , and we can correctly extend the arc-colouring by setting  $\phi_D(uv_i) = e_i$  for every  
431  $i \in \{1, \dots, d\}$ .

432 Once no configuration corresponding to a bad connected component of  $B$  remains in  
433  $D$ , we are left with assigning a colour to each of the lonely arcs in  $L$ . Let us focus on a  
434 lonely arc  $\vec{uv}$  of  $L$  that remains to be coloured. Recall that having  $S^-(u) \neq S^+(v)$  does  
435 not depend on how  $\phi_D(\vec{uv})$  is chosen; that distinction condition is either already met by  
436 previous colouring arguments, or will be met by how other lonely arcs will be coloured  
437 later on. By definition of a lonely arc, recall that  $\vec{uv}$  is the only arc in-coming to  $v$  and  
438 the only arc out-going from  $u$ . Hence, at the moment,  $S^+(u) = S^-(v) = \emptyset$ , and assigning  
439 a colour to  $\vec{uv}$  will completely determine  $S^+(u) = S^-(v)$ . Also, since  $v$  has no other arc  
440 coming in and  $u$  has no other arc going out, assigning any colour to  $\vec{uv}$  cannot break the  
441 properness of  $\phi_D$ . So, we just need to assign any colour to  $\vec{uv}$  so that  $S^+(u)$  is different  
442 from  $S^-(w)$  for every  $w$  such that  $\vec{wu}$  is an arc, and so that  $S^-(v)$  is different from  $S^+(w)$   
443 for every  $w$  such that  $\vec{vw}$  is an arc. There are at most  $\Delta^*(D)$  such  $w$  for  $u$ , while there  
444 are at most  $\Delta^*(D)$  such  $w$  for  $v$ , hence at most  $2\Delta^*(D)$  constraints in total. Since we are  
445 using a set of  $3\Delta^*(D)$  colours, there is a free one that we can assign to  $\vec{uv}$  by  $\phi_D$ , without  
446 raising conflicts in terms of properness or distinction.

447 Once all lonely arcs of  $L$  have been treated that way,  $\phi_D$  is a  $(-, +)$ -distinguishing  
448 proper  $(3\Delta^*(D))$ -arc-colouring of  $D$ .  $\square$

## 449 5. Conclusion

450 Our goal in this paper was to investigate directed counterparts of the AVD Conjec-  
451 ture where, by a proper arc-colouring of a digraph, adjacent vertices are required to be  
452 distinguished by a given one of their two set parameters. For each of the four resulting  
453 variants, we believe that, for any nice (specific to the variant) digraph  $D$ , there should be  
454 a proper  $(\Delta^*(D) + 2)$ -arc-colouring which is as desired. We verified this for the  $(+, -)$   
455 variant (Corollary 3.4), while we only verified weaker statements for the other variants  
456 (Theorems 2.4 and 4.4).

457 An interesting aspect of this line of research lies in the differences between the original  
458 AVD Conjecture and each of the four variants, and also in the differences between these  
459 four variants. For instance, the notion of nice digraphs varies greatly from one variant to  
460 another. Also, the effects of colouring an arc in the four variants are more or less distant  
461 from the effects of colouring an edge in the original conjecture. In terms of inherent  
462 hardness, the  $(+, -)$  version of the AVD Conjecture seems to be the easiest one, as we  
463 proved it is equivalent to a very restricted case of the original conjecture (Theorem 3.3).  
464 We have the feeling that the  $(-, +)$  version should be the hardest one, particularly due to  
465 the fact that the colouring mechanisms are a bit less local.

466 Regarding further work, of course the most important direction would be to tackle  
467 Conjectures 2.2 and 4.2. It could also be interesting to investigate our four variants of the  
468 AVD Conjecture in restricted classes of digraphs such as tournaments or acyclic digraphs.  
469 Lastly, as mentioned earlier, distinguishing adjacent vertices by a single set parameter is  
470 only one possible way for defining a directed counterpart to the AVD Conjecture, but  
471 playing with combinations of the two set parameters, just like Sopena and Woźniak did  
472 in [7], might lead to other interesting problems as well.

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