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On Generalisations of the AVD Conjecture to Digraphs[☆]

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Abstract

Given an undirected graph, in the AVD (edge-colouring) Conjecture, the goal is to find a proper edge-colouring with the least number of colours such that every two adjacent vertices are incident to different sets of colours. More precisely, the conjecture says that, a few exceptions apart, every graph G should admit such an edge-colouring with at most $\Delta(G) + 2$ colours. Several aspects of interest behind this problem have been investigated over the recent years, including verifications of the conjecture for particular graph classes, general approximations of the conjecture, and multiple generalisations.

In this paper, following a recent work of Sopena and Woźniak, generalisations of the AVD Conjecture to digraphs are investigated. More precisely, four of the several possible ways of generalising the conjecture are focused upon. We completely settle one of our four variants, while, for the three remaining ones, we provide partial results.

Keywords: AVD Conjecture; proper edge-colourings; digraphs.

1. Introduction

One of the most central notions of graph theory is that of proper edge-colourings. Given an undirected graph G , a *proper k -edge-colouring* ϕ of G is an assignment $\phi : E(G) \rightarrow \{1, \dots, k\}$ of colours to the edges such that no two adjacent edges (*i.e.*, incident to a same vertex) get assigned the same colour. We are usually interested in determining the *chromatic index* $\chi'(G)$ of G , which refers to the smallest k such that G admits a proper k -edge-colouring. Perhaps the most important result regarding the chromatic index of graphs is Vizing's Theorem [8], which states that, for every graph G , we have $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$ (where $\Delta(G)$ denotes the maximum degree of a vertex in G). Even though that result means the chromatic index of any graph is one of only two possible values, it is important to recall that determining the chromatic index of a graph is an NP-complete problem in general [4].

In several contexts, it might be convenient to have edge-colourings of graphs that are not only proper, but also have additional properties. When considering such a stronger form of proper edge-colourings, an interesting question is about the least number of additional colours needed to construct one for any given graph.

In this work, we are mostly interested in proper edge-colourings that allow to distinguish adjacent vertices according to their respective sets of incident colours. More precisely,

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35 given a proper edge-colouring ϕ of a graph G , for every vertex u , we can compute $S(u)$,
36 which is the set of colours assigned by ϕ to the edges incident to u . Note that, by the
37 properness of ϕ , we always have $|S(u)| = d(u)$. Now, we say that ϕ is *distinguishing* if,
38 for every edge uv , we have $S(u) \neq S(v)$. We denote by $\text{ndi}(G)$ (where “ndi” stands for
39 “neighbour-distinguishing index”) the least k such that G admits a distinguishing proper
40 k -edge-colouring (if any). Actually, it is easy to see that $\text{ndi}(G)$ is defined if and only if
41 G has no connected component isomorphic to K_2 (just consider a proper edge-colouring
42 assigning a distinct colour to every edge). Thus, regarding those notions, we are interested
43 in *nice graphs*, which are the graphs with no connected component isomorphic to K_2 .

44 We employ the simple terminology and notations above for the sake of the current
45 work’s legibility. It is worth mentioning, however, that this terminology and these notations
46 vary from one work to another in the literature. In particular, our notion of distinguishing
47 edge-colouring is sometimes called *adjacent vertex-distinguishing edge-colouring*, *neighbour-*
48 *distinguishing edge-colouring*, *adjacent strong edge-colouring* or *1-strong edge-colouring*.
49 Our parameter ndi is sometimes written χ'_{avd} and called the *adjacent vertex-distinguishing*
50 *chromatic index*. Even more different terms and notations are used in some works.

51 Clearly, we have $\chi'(G) \leq \text{ndi}(G)$ for every nice graph G . Regarding the concerns
52 above, a natural question to ask is how large can $\text{ndi}(G)$ be in general. It can be noted
53 that $\text{ndi}(C_5) = \Delta(C_5) + 3 = 5$, where C_5 denotes the cycle of length 5. However, it is
54 believed that C_5 should be the only nice graph G for which $\text{ndi}(G)$ is that large. This leads
55 to the following conjecture raised by Zhang, Liu, and Wang in 2002 [11].

56 **AVD Conjecture.** *For every nice connected graph $G \neq C_5$, we have $\text{ndi}(G) \leq \Delta(G) + 2$.*

57 Comparing the AVD Conjecture to Vizing’s Theorem, the conjecture indicates that,
58 in general, at most two additional colours might be necessary to turn a normal proper
59 $\chi'(G)$ -edge-colouring of a graph G into a distinguishing one.

60 Several interesting results towards the AVD Conjecture have been obtained since its
61 introduction. In particular, the conjecture was verified for bipartite graphs and subcubic
62 graphs [2]. It was also proven in [1], that every nice graph G verifies $\text{ndi}(G) \leq 3\Delta(G)$. We
63 refer the interested reader to [6] for more details.

64 In graph theory, a common line of research is, given a particular problem defined on
65 undirected graphs, to wonder about its counterparts on digraphs. Regarding the AVD
66 Conjecture, this is a promising prospect due to the numerous generalisations that can be
67 considered. Indeed, by a proper arc-colouring of a digraph D , note that every vertex u can
68 now be associated two sets $S^+(u)$ and $S^-(u)$ of incident colours, where $S^+(u)$ is the set
69 of colours assigned to the arcs out-going from u , and $S^-(u)$ is the set of colours assigned
70 to the arcs in-coming to u . Recall that, in the directed context, a *proper arc-colouring*
71 ϕ of D verifies that two arcs out-going from a same vertex are assigned distinct colours,
72 and similarly for two arcs in-coming to a same vertex. Under that definition, note that an
73 arc out-going from u and an arc in-coming to u can be assigned the same colour. Note
74 also that, because ϕ is proper, we have $|S^+(u)| = d^+(u)$ and $|S^-(u)| = d^-(u)$, which is
75 reminiscent of the similar property of proper edge-colourings of undirected graphs.

76 For a digraph D , its *chromatic index* $\chi'(D)$ is the least k such that D admits proper
77 k -arc-colourings. By the definition of proper arc-colourings, note that we always have
78 $\chi'(D) \geq \Delta^*(D) = \max\{\Delta^+(D), \Delta^-(D)\}$, where $\Delta^+(D)$ and $\Delta^-(D)$ denote the maximum
79 out-degree and maximum in-degree, respectively, over all vertices of D . In contrast with
80 the undirected context of Vizing’s Theorem, it is known that the chromatic index of a
81 digraph D is always precisely the natural lower bound $\Delta^*(D)$ (see, *e.g.* [10]).

82 Since every vertex gets assigned two sets of colours in any proper arc-colouring ϕ of
83 a digraph, there are plenty of ways of considering that two adjacent vertices are distin-
84 guished by ϕ , and thus, many possibilities for defining a directed counterpart to the AVD
85 Conjecture. To the best of our knowledge, this line of research was not considered until
86 quite recently, by Sopena and Woźniak in [7]. In their variant of the problem, they consider
87 that two adjacent vertices u and v are distinguished by a proper arc-colouring when either
88 $S^+(u) \neq S^+(v)$ or $S^-(u) \neq S^-(v)$. They conjectured that every digraph D admits such a
89 distinguishing proper arc-colouring using at most $\Delta^*(D) + 1$ colours, and they proved that
90 $2\Delta^*(D)$ colours are enough to construct one.

91 In the line of the investigations initiated by Sopena and Woźniak, we consider several
92 directed variants of the AVD Conjecture. Specifically, we consider four variants, in which,
93 for every two adjacent vertices, we ask that a single of the two set parameters differs.
94 Precisely, our general terminology is as follows. Let D be a digraph, and ϕ a proper arc-
95 colouring of D . To each of the set parameters S^+ and S^- , we associate a sign, namely
96 $+$ and $-$, respectively. Now, for any two signs $\alpha, \beta \in \{+, -\}$, we say that ϕ is (α, β) -
97 *distinguishing* if, for every arc \vec{uv} of D , the set parameter of u associated to α is different
98 from the set parameter of v associated to β . We denote by $\text{ndi}_{\alpha, \beta}(D)$ the least k such that
99 D admits an (α, β) -distinguishing proper k -arc-colouring (if any).

100 Note that this general terminology encapsulates a series of four colouring problems, each
101 of which has behaviours that are more or less reminiscent of the original AVD Conjecture.
102 In terms of colouring behaviours, note that the $(+, -)$ version is the closest to the original
103 problem, as, by a proper arc-colouring of a digraph, assigning a colour to an arc \vec{uv} affects
104 $S^+(u)$ and $S^-(v)$ which are precisely the two set parameters that are required to be different.
105 From this perspective, the $(+, +)$ and $(-, -)$ versions are a bit farther from the original
106 conjecture, while the $(-, +)$ version is the most distant. Still, for all four of the versions,
107 recall that the number of colours needed in a distinguishing proper arc-colouring is strongly
108 dependent on the chromatic index.

109 **Observation 1.1.** *Let $\alpha, \beta \in \{+, -\}$. For every digraph D such that $\text{ndi}_{\alpha, \beta}(D)$ is defined,*
110 *we have $\chi'(D) \leq \text{ndi}_{\alpha, \beta}(D)$.*

111 Although our four variants of the AVD Conjecture have their own peculiar behaviours,
112 we feel that, in general, for any of the variants, $\Delta^*(D) + 2$ colours should always be enough
113 to construct a desired distinguishing proper arc-colouring. Our goal in this paper is to
114 provide evidence towards this intuition. Section 2 is dedicated to investigating the $(+, +)$
115 and $(-, -)$ versions of the AVD Conjecture, while Section 3 is dedicated to the $(+, -)$
116 version, and Section 4 is dedicated to the $(-, +)$ variant. We provide a tight result on the
117 $(+, -)$ variant, while we provide partial results for the other variants.

118 2. The $(+, +)$ and $(-, -)$ versions

119 Note that a $(+, +)$ -distinguishing proper arc-colouring of a given digraph D directly
120 yields a $(-, -)$ -distinguishing proper arc-colouring of \tilde{D} , the digraph obtained from D by
121 reversing the direction of each arc. We can thus focus on the $(+, +)$ version of the AVD
122 Conjecture in this section, as these results apply to the $(-, -)$ version as well, through this
123 arc reversing operation.

124 First off, we note that all digraphs admit a $(+, +)$ -distinguishing proper arc-colouring.
125 In other words, we do not need a notion of nice digraphs in this context.

126 **Proposition 2.1.** *Every digraph admits a $(+, +)$ -distinguishing proper arc-colouring.*

127 *Proof.* Let D be any digraph with arcs a_1, \dots, a_m . Consider the arc-colouring ϕ that sets
 128 $\phi(a_i) = i$ for every $i \in \{1, \dots, m\}$. Clearly, ϕ is proper. It is also $(+, +)$ -distinguishing,
 129 since, for every vertex u with an out-going arc a_i , only the set $S^+(u)$ contains i . \square

130 Note that, for any directed cycle D of odd length, we have $\Delta^*(D) = 1$, but we need
 131 to use $\Delta^*(D) + 2 = 3$ colours (two consecutive arcs must be assigned distinct colours in
 132 any $(+, +)$ -distinguishing proper arc-colouring of a directed cycle). We believe this is the
 133 maximum value of $\text{ndi}_{+,+}(D)$ for a digraph D .

134 **Conjecture 2.2.** *For every digraph D , we have $\text{ndi}_{+,+}(D) \leq \Delta^*(D) + 2$.*

135 Towards Conjecture 2.2, for every digraph D , we can easily establish an upper bound on
 136 $\text{ndi}_{+,+}(D)$ that is linear in $\Delta^*(D)$ by exploiting its relationship with proper edge-colourings
 137 of $\text{und}(D)$, the undirected graph underlying D .

138 **Proposition 2.3.** *For every digraph D , we have $\text{ndi}_{+,+}(D) \leq 2\Delta^*(D) + 2$.*

139 *Proof.* Let G be the undirected multigraph obtained from D by replacing each arc by an
 140 edge. Note that G might indeed have parallel edges, but the maximum multiplicity $\mu(G)$
 141 of its edges is 2. Also, $\Delta(G) \leq \Delta^-(D) + \Delta^+(D) \leq 2\Delta^*(D)$. Now, by Vizing's Theorem [9],
 142 there is a proper edge-colouring ϕ_G of G using $\Delta(G) + \mu(G) = \Delta(G) + 2$ colours. We infer
 143 ϕ_G to an arc-colouring ϕ_D of D by simply transferring the colour of an edge in G to the
 144 corresponding arc in D . By the properness of ϕ_G , note that ϕ_D is also proper. Also, for
 145 every arc uv of D , we get that no arc out-going from v is assigned colour $\phi_D(uv)$, and thus,
 146 $S^+(u) \neq S^+(v)$ since $\phi_D(uv) \notin S^+(v)$. Thus, ϕ_D is also $(+, +)$ -distinguishing. Since ϕ_D
 147 uses at most $2\Delta^*(D) + 2$ colours, the result follows. \square

148 Using a different approach, we can slightly improve the upper bound in Proposition 2.3.

149 **Theorem 2.4.** *For every digraph D , we have $\text{ndi}_{+,+}(D) \leq 2\Delta^*(D) + 1$.*

150 *Proof.* The proof is by induction on k , the number of vertices with out-degree at least 1. If
 151 $k = 1$, then D has only one vertex with out-going arcs, in which case D is an out-star. In
 152 this case, a $(+, +)$ -distinguishing proper $\Delta^*(D)$ -arc-colouring ϕ is obtained by assigning a
 153 distinct colour to each arc of D . Indeed, ϕ is clearly proper, and we have $S^+(v) = \emptyset$ for
 154 every leaf v of D and $S^+(u) \neq \emptyset$ for the center u .

155 Assume the claim holds for k up to some value x , and consider that D has $x + 1$ vertices
 156 with out-degree at least 1. Let us consider u to be any vertex with out-degree at least 1, and
 157 out-going arcs $u\vec{v}_1, \dots, u\vec{v}_d$ (where $d \geq 1$). Consider $D' = D - \{u\vec{v}_1, \dots, u\vec{v}_d\}$ the digraph
 158 obtained from D by removing all arcs out-going from u . By the induction hypothesis, D'
 159 has a $(+, +)$ -distinguishing proper $(2\Delta^*(D) + 1)$ -arc-colouring $\phi_{D'}$ (since $\Delta^*(D) \geq \Delta^*(D')$),
 160 which we would like to extend to a $(+, +)$ -distinguishing proper $(2\Delta^*(D) + 1)$ -arc-colouring
 161 ϕ_D of D , i.e., to the arcs $u\vec{v}_1, \dots, u\vec{v}_d$. Note that assigning a colour to any arc $u\vec{v}_i$ only
 162 affects $S^+(u)$. Thus, our goal is to assign colours to the $u\vec{v}_i$'s in a proper way, so that the
 163 resulting $S^+(u)$ is different from $S^+(w)$ for every neighbour w of u in D . Note that there
 164 are at most $2\Delta^*(D)$ such neighbours w around u .

165 For every arc $u\vec{v}_i$, in terms of properness, the colour assigned to $u\vec{v}_i$ must differ from
 166 the colours assigned to the at most $\Delta^*(D) - 1$ other arcs in-coming to v_i . Since we are
 167 using a set of $2\Delta^*(D) + 1$ colours, there are at least $\Delta^*(D) + 2$ colours that can, from that
 168 point of view, freely be assigned to $u\vec{v}_i$. For every $i \in \{1, \dots, d\}$, let us denote by L_i the
 169 set of these colours. Our goal now, is to choose distinct elements (to ensure properness)
 170 from L_1, \dots, L_d , one from each of the L_i 's, in such a way that the union of these elements
 171 avoids the sets of colours of the at most $2\Delta^*(D)$ neighbours of u . This is something that
 172 can always be done, according to the following claim:

173 **Claim 2.5.** *Let L_1, \dots, L_d be $d \geq 1$ sets each containing at least $d + 2$ elements. Then,*
 174 *there are at least $2d + 1$ different combinations e_1, \dots, e_d of elements, such that $e_i \in L_i$ for*
 175 *every $i \in \{1, \dots, d\}$, and all the e_i 's are distinct.*

176 *Proof of the claim.* The proof is by induction on d . For $d = 1$, we have $|L_1| = 3$. Every
 177 single element of L_1 is a correct choice as e_1 , and thus, there are three correct combinations.
 178 Assume now that the claim holds for every d up to some x , and assume $d = x + 1$. Without
 179 loss of generality, we may assume $1 \in L_1$. Set $L'_2 = L_2 \setminus \{1\}, \dots, L'_d = L_d \setminus \{1\}$. By the
 180 induction hypothesis, we can produce $2(d - 1) + 1 = 2d - 1$ combinations e'_2, \dots, e'_d of
 181 distinct elements from the L'_i 's (one from each set). Each such combination e'_2, \dots, e'_d ,
 182 together with 1, yields a combination $1, e'_2, \dots, e'_d$ that is valid for L_1, \dots, L_d . Thus, we
 183 already know how to generate $2d - 1$ different combinations of distinct elements from
 184 L_1, \dots, L_d , all of which contain the element 1.

185 All that remains to do is to generate two more combinations. To ensure this, we exhibit
 186 two such different combinations not containing the element 1. To that aim, let us choose
 187 arbitrary distinct elements e_2, \dots, e_d different from 1 from L_2, \dots, L_d . This is possible
 188 since each L_i has size at least $d + 2$. Now, since L_1 also has size at least $d + 2$, there are
 189 at least three elements e_1, e'_1, e''_1 that are different from e_2, \dots, e_d . At least two of e_1, e'_1, e''_1
 190 must be different from 1. These two elements together with e_2, \dots, e_d yield our remaining
 191 two different combinations. \diamond

192 Now, by Claim 2.5, we can choose a combination e_1, \dots, e_d of distinct elements from
 193 L_1, \dots, L_d (where $e_i \in L_i$ for every $i \in \{1, \dots, d\}$) such that no w of the at most $2\Delta^*(D)$
 194 neighbours of u in D verifies $S^+(w) = \{e_1, \dots, e_d\}$. To finish the extension of $\phi_{D'}$ to ϕ_D ,
 195 it now suffices to set $\phi_D(u\vec{v}_i) = e_i$ for every $i \in \{1, \dots, d\}$. \square

196 3. The $(+, -)$ version

197 In this section, we focus on the $(+, -)$ version of the AVD Conjecture, in which, by a
 198 proper arc-colouring of a digraph, it is required that $S^+(u) \neq S^-(v)$ holds for every arc
 199 $u\vec{v}$. Recall that this distinction condition is, out of the four ones we are considering, the
 200 closest to the original distinction condition behind the original AVD Conjecture. Indeed,
 201 by a proper arc-colouring ϕ of some digraph, for every arc $u\vec{v}$ the colour $\phi(u\vec{v})$ contributes
 202 to both $S^+(u)$ and $S^-(v)$, which are precisely the two set parameters that are asked to
 203 differ for u and v in the $(+, -)$ version.

204 Compared to the $(+, +)$ version, there are digraphs admitting no $(+, -)$ -distinguishing
 205 proper arc-colourings. The smallest such digraph is a single arc $u\vec{v}$, as $S^+(u) = S^-(v)$ in
 206 any proper arc-colouring. Actually, the case of such a pathological arc can be generalised
 207 in the following way. We say that an arc $u\vec{v}$ of a digraph is *lonely* if $d^+(u) = d^-(v) = 1$.
 208 Note that, indeed, $\text{ndi}_{+,-}(D)$ is not defined for every digraph D containing a lonely arc. It
 209 turns out that lonely arcs are the only source of non-colourability in the $(+, -)$ version.
 210 That is, if we define a $(+, -)$ -*nice digraph* as a digraph D with $\text{ndi}_{+,-}(D)$ being defined,
 211 then being $(+, -)$ -nice is equivalent to having no lonely arcs.

212 **Proposition 3.1.** *A digraph is $(+, -)$ -nice if and only if it has no lonely arc.*

213 *Proof.* Consider any digraph D , and let us denote by a_1, \dots, a_m its arcs in an arbitrary
 214 fashion. Let ϕ be the arc-colouring of D where $\phi(a_i) = i$ for every $i \in \{1, \dots, m\}$. Clearly
 215 ϕ is proper. We claim ϕ is $(+, -)$ -distinguishing if and only if D has no lonely arc. Indeed,
 216 consider an arc $u\vec{v}$. If $d^+(u) \neq d^-(v)$, then for sure $S^+(u) \neq S^-(v)$ since $|S^+(u)| \neq |S^-(v)|$.
 217 Thus, assume $d^+(u) = d^-(v)$. If $d^+(u) = d^-(v) \geq 2$, then, if we denote by v' another

218 out-neighbour of u , we have $\phi(\vec{uv'}) \in S^+(u)$ and $\phi(\vec{uv'}) \notin S^-(v)$, and thus $S^+(u) \neq S^-(v)$.
 219 The only remaining case is when $d^+(u) = d^-(v) = 1$, which is precisely the case where \vec{uv}
 220 is lonely, and we necessarily have $S^+(u) = S^-(v)$ by any proper arc-colouring of D . \square

221 We are actually able to prove a tight general upper bound on $\text{ndi}_{+,-}(D)$ for $(+, -)$ -nice
 222 digraphs D . We prove that bound right away, because most of the remarks we can raise
 223 on the $(+, -)$ version of the AVD Conjecture actually follow from our proof scheme.

224 Our proof relies on an equivalence between the $(+, -)$ version and particular cases of
 225 the original AVD Conjecture. This equivalence is with respect to the following notions
 226 and definitions. For a digraph D , by the *bipartite graph associated to D* , we refer to the
 227 undirected bipartite graph $B(D)$ constructed as follows:

- 228 • The two partite classes of $B(D)$ are V^+ and V^- .
- 229 • For every vertex u of D , we add a vertex u^+ to V^+ and a vertex u^- to V^- .
- 230 • For every arc \vec{uv} of D , we add the edge u^+v^- to $B(D)$.

231 In some sense, $B(D)$ is obtained from D by splitting the out-going part and the in-
 232 coming part of every vertex. Note that $B(D)$ is always balanced, in the sense that $|V^+| =$
 233 $|V^-|$. Also, we can infer some additional properties of $B(D)$ from properties of D .

234 **Observation 3.2.** *For every digraph D :*

- 235 • $\Delta(B(D)) = \Delta^*(D)$.
- 236 • $B(D)$ is nice if and only if D is $(+, -)$ -nice.

237 *Proof.* The first item is because, for every vertex u of D , the value of $d^+(u)$ (in D) equals
 238 the value of $d(u^+)$ (in $B(D)$), and similarly $d^-(u)$ equals $d(u^-)$. The second item is because
 239 an isolated edge u^+v^- in $B(D)$ corresponds to a lonely arc \vec{uv} in D , and *vice versa*. \square

240 We are now ready to prove our main result in this section.

241 **Theorem 3.3.** *The $(+, -)$ version of the AVD Conjecture is equivalent to the (original)
 242 AVD Conjecture in bipartite graphs.*

243 *Proof.* The notion of the associated bipartite graph is the key behind this equivalence.
 244 Indeed, finding a $(+, -)$ -distinguishing proper k -arc-colouring of some $(+, -)$ -nice digraph
 245 is equivalent to finding a distinguishing proper k -edge-colouring of some nice undirected
 246 bipartite graph.

- 247 • Let D be a $(+, -)$ -nice digraph, and consider the (nice, by Observation 3.2) bipartite
 248 graph $B = B(D)$ associated to D . Let ϕ_B be a distinguishing proper k -edge-colouring
 249 of B . Consider the k -arc-colouring ϕ_D of D obtained by setting $\phi_D(\vec{uv}) = \phi_B(u^+v^-)$
 250 for every arc \vec{uv} . Note that ϕ_D is proper because $\phi_D(\vec{uv}) \neq \phi_D(\vec{uv'})$ when $v \neq v'$ since
 251 $\phi_B(u^+v^-) \neq \phi_B(u^+v'^-)$ (by the properness of ϕ_B), and, similarly, $\phi_D(\vec{uv}) \neq \phi_D(\vec{u'v})$
 252 when $u \neq u'$ since $\phi_B(u^+v^-) \neq \phi_B(u'^+v^-)$. Also, ϕ_D is $(+, -)$ -distinguishing because
 253 $S^+(u) \neq S^-(v)$ for every arc \vec{uv} of D since, in B , $S(u^+) \neq S(v^-)$ (by ϕ_B being
 254 distinguishing). Thus, ϕ_D is a $(+, -)$ -distinguishing proper k -arc-colouring of D .

255 • Let B be a nice bipartite graph. Denote by (U, V) the bipartition of B . If necessary,
 256 add isolated vertices to B so that 1) B is balanced, and 2) there is an ordering of
 257 the vertices in U and V such that $U = \{u_1, \dots, u_n\}$, $V = \{v_1, \dots, v_n\}$, and $u_i v_i$ is
 258 not an edge for every $i \in \{1, \dots, n\}$. Under those conditions, let D be the digraph
 259 constructed from B by adding a vertex w_i for every $i \in \{1, \dots, n\}$, and an arc $w_i \vec{w}_j$
 260 for every edge $u_i v_j$ of B . Clearly, B is the bipartite graph $B(D)$ associated to D
 261 (where U plays the role of V^+ and V plays the role of V^-). We now have the desired
 262 equivalence by the arguments used to deal with the previous case. \square

263 The equivalence established in the proof of Theorem 3.3 has a series of consequences
 264 on the $(+, -)$ version of the AVD Conjecture. In particular, the fact that $\text{ndi}_{+,-}(D) =$
 265 $\text{ndi}(B(D))$ and $\Delta^*(D) = \Delta(B(D))$ hold for every $(+, -)$ -nice digraph D yields some side
 266 results. For instance, it is known that there exist nice bipartite graphs G with $\text{ndi}(G) =$
 267 $\Delta(G) + 2$, see [2]. In the same paper, the authors proved that $\Delta(G) + 2$ is actually the
 268 maximum value of $\text{ndi}(G)$ for a nice bipartite graph G . In other words, the AVD Conjecture
 269 holds for nice bipartite graphs. For our problem, these remarks directly imply:

270 **Corollary 3.4.** *For every $(+, -)$ -nice digraph D , we have $\text{ndi}_{+,-}(D) \leq \Delta^*(D) + 2$. Fur-*
 271 *thermore, this upper bound cannot be decreased in general.*

272 4. The $(-, +)$ version

273 We now consider the $(-, +)$ version of the AVD Conjecture, in which two vertices u
 274 and v that are adjacent through an arc $u\vec{v}$ are required, by a proper arc-colouring, to verify
 275 $S^-(u) \neq S^+(v)$. Recall that this version is, in some sense, the variant we are considering
 276 that is the farthest from the original conjecture. This is mainly because the colour $\phi(u\vec{v})$
 277 of an arc $u\vec{v}$ by an arc-colouring ϕ of a digraph contributes nothing to $S^-(u)$ and $S^+(v)$,
 278 which are the parameters of u and v that must differ.

279 In the present context, again, not all digraphs admit $(-, +)$ -distinguishing proper arc-
 280 colourings. To see this is true, consider the case of a digraph D containing an arc $s\vec{t}$
 281 such that $d^-(s) = 0$ (source) and $d^+(t) = 0$ (sink). Clearly, D does not admit a $(-, +)$ -
 282 distinguishing proper arc-colouring, since we always have $S^-(s) = \emptyset = S^+(t)$. Note that
 283 the situation remains unchanged if we add the arc $t\vec{s}$ to D , since, here, we would always get
 284 $S^-(s) = \alpha = S^+(t)$, when assigning a colour α to $t\vec{s}$. If D has two such adjacent vertices,
 285 we say that D has a *bad configuration*. Again, in this variant, a $(-, +)$ -nice digraph is a
 286 digraph D for which $\text{ndi}_{-,+}(D)$ is defined. Actually, bad configurations are the only reason
 287 why some digraphs are not $(-, +)$ -nice:

288 **Proposition 4.1.** *A digraph is $(-, +)$ -nice if and only if it has no bad configuration.*

289 *Proof.* Let D be any digraph with arcs a_1, \dots, a_m , and let ϕ be the arc-colouring of D
 290 where $\phi(a_i) = i$ for every $i \in \{1, \dots, m\}$. We claim that ϕ , which is clearly proper, is
 291 $(-, +)$ -distinguishing if and only if D has no bad configuration. Indeed, let us focus on an
 292 arc $u\vec{v}$. If $d^-(u) \neq d^+(v)$, then $S^-(u) \neq S^+(v)$ because $|S^-(u)| \neq |S^+(v)|$. So, let us assume
 293 that $d^-(u) = d^+(v)$. If $d^-(u) = d^+(v) \geq 2$, then $S^-(u) \neq S^+(v)$ due to there being at least
 294 one arc out-going from v that is not in-coming to u . The same holds if $d^-(u) = d^+(v) = 1$
 295 and the arc in-coming to u is different from the arc out-going from v . So, there are only
 296 two cases remaining: 1) $d^-(u) = d^+(v) = 1$ and $v\vec{u}$ is an arc, and 2) $d^-(u) = d^+(v) = 0$.
 297 In both cases, D has a bad configuration, and ϕ cannot be $(-, +)$ -distinguishing. \square

298 We note that there are $(-, +)$ -nice digraphs D with $\text{ndi}_{-,+}(D) = \Delta^*(D) + 2$. Every
 299 odd-length directed cycle is an example of such a digraph. We think this might be the
 300 maximum value of $\text{ndi}_{-,+}(D)$ for a $(-, +)$ -nice digraph D , which would be reminiscent of
 301 the AVD Conjecture.

302 **Conjecture 4.2.** *For every $(-, +)$ -nice digraph D , we have $\text{ndi}_{-,+}(D) \leq \Delta^*(D) + 2$.*

303 Towards Conjecture 4.2, contrarily to what was done in Section 2, note that using
 304 proper edge-colourings does not yield a linear upper bound (in $\Delta^*(D)$) on $\text{ndi}_{-,+}(D)$ for
 305 every $(-, +)$ -nice digraph D , as the distinction condition in the $(-, +)$ version, in some
 306 sense, asks arcs at distance 2 to be different. Instead, an upper bound on $\text{ndi}_{-,+}(D)$ can be
 307 expressed, for instance, as a function of the *strong chromatic index* $\chi'_s(\text{und}(D))$ of $\text{und}(D)$,
 308 which is the smallest number of colours in an edge-colouring of $\text{und}(D)$ where no two edges
 309 at distance at most 2 get assigned a same colour. Such an upper bound would be quadratic
 310 in $\Delta^*(D)$ (see, for instance, [3]).

311 Using different arguments, for every $(-, +)$ -nice digraph D , we provide an upper bound
 312 on $\text{ndi}_{-,+}(D)$ that is linear in $\Delta^*(D)$. Just as in Section 3, this is by exploiting some
 313 relationship between an arc-colouring of D and an edge-colouring of $B(D)$, the bipartite
 314 graph associated to D . However, note that using the relationship is not so natural in the
 315 present context. Indeed, by a proper edge-colouring of $B(D)$, having $S(u^+) \neq S(v^-)$ is not
 316 so relevant regarding D , as, when transposing the edge-colouring to an arc-colouring of D ,
 317 this would yield $S^+(u) \neq S^-(v)$, which is not required in the $(-, +)$ version. Also, it might
 318 be that we want $S^+(u)$ and $S^-(v)$ to differ in D , while $S(u^+)$ and $S(v^-)$ are not required
 319 to differ in $B(D)$ because u^+v^- is not an edge. To deal with such issues, we will consider
 320 distinguishing proper edge-colourings of $B(D)$ verifying strong distinction conditions.

321 Before proceeding with the proof, it is important to point out that lonely arcs, though
 322 they do not prevent $\text{ndi}_{-,+}(D)$ to be defined for a $(-, +)$ -nice digraph D , have a peculiar
 323 behaviour (they yield isolated edges in $B(D)$) that will force us to handle them separately.
 324 In particular, we will make use of the following property of lonely arcs:

325 **Observation 4.3.** *Removing a lonely arc from a digraph cannot create new lonely arcs.*

326 *Proof.* Let D be a digraph, and let $D' = D - \vec{uv}$ be the digraph obtained from D by
 327 removing a lonely arc \vec{uv} . Assume D' has a lonely arc \vec{xy} which is not lonely in D . Then,
 328 either $u = x$ or $v = y$. In the first case, we deduce that u has out-degree 2 in D , while, in the
 329 second case, we deduce that v has in-degree 2 in D . In both cases, we get a contradiction
 330 to the loneliness of \vec{uv} . \square

331 We are now ready to prove our main result in this section.

332 **Theorem 4.4.** *For every $(-, +)$ -nice digraph D , we have $\text{ndi}_{-,+}(D) \leq 3\Delta^*(D)$.*

333 *Proof.* Let L be the set of all lonely arcs of D , and set $D' = D - L$. By Observation 4.3,
 334 D' has no lonely arcs. Let $B = B(D')$ be the bipartite graph associated to D' . By
 335 Observation 3.2, D' is nice, and $\Delta = \Delta(B) \leq \Delta^*(D') \leq \Delta^*(D)$. Recall that the bipartition
 336 of B is denoted by (V^+, V^-) . Note that B may have several connected components. In
 337 what follows, we need to dedicate a special care to some of them. Specifically, a connected
 338 component of B is said to be *bad* if it is a star with center in V^+ (and at least two leaves,
 339 in V^- , since B is nice). In what follows, we obtain a $(-, +)$ -distinguishing proper 3Δ -arc-
 340 colouring ϕ_D of D by first colouring the edges of the non-bad connected components of
 341 B (and transferring the assigned colours to corresponding arcs in D), then colouring, in

342 D , the arcs corresponding to edges of the bad connected components of B , and eventually
 343 colouring the lonely arcs of D .

344 We start by constructing a proper 3Δ -edge-colouring ϕ_B of the non-bad connected
 345 components of B , such that all vertices $v \in V^+$ verify $S(v) \in \mathcal{X}$ while all vertices $v \in V^-$
 346 verify $S(v) \in \mathcal{Y}$, for some disjoint sets \mathcal{X}, \mathcal{Y} of subsets of $\{1, \dots, 3\Delta\}$. To achieve this, we
 347 split $\{1, \dots, 3\Delta\}$ into the three smaller sets $\mathcal{R} = \{1, \dots, \Delta\}$, $\mathcal{B} = \{\Delta + 1, \dots, 2\Delta\}$, and
 348 $\mathcal{G} = \{2\Delta + 1, \dots, 3\Delta\}$ of colours, which we assign as follows.

349 Let us focus on a non-bad connected component of B . Abusing the notation, let us
 350 denote by B this connected component. Recall that there are at least two edges in B , since
 351 B is nice. Pick an arbitrary vertex $u^* \in V^+$. This defines a partition of $V(B)$ into *layers*
 352 V_0, \dots, V_d , where every V_i contains the vertices of B that are at distance exactly i from
 353 u^* . Since B is not bad, we have $d \geq 2$. Note that $V_0 = \{u^*\}$, that every edge of B joins
 354 vertices in two consecutive V_i 's, that every vertex in some V_i with $i \neq 0$ has a neighbour
 355 in V_{i-1} , and that the union of all V_i 's with even index is exactly V^+ while the union of all
 356 V_i 's with odd index is exactly V^- . Given an edge uv with $u \in V_i$ and $v \in V_{i+1}$, we say
 357 that uv is *downward* from the point of view of u , while it is *upward* from that of v .

358 For every $i \in \{1, \dots, d\}$, we split V_i into V_i' and V_i'' , where V_i' contains the vertices
 359 of V_i with no downward edges, while V_i'' contains the vertices in V_i with downward edges.
 360 Note that $V_d'' = \emptyset$, and recall that $V_1'' \neq \emptyset$ since B is not bad. We now construct ϕ_B by
 361 assigning colours in \mathcal{R} , \mathcal{B} , and \mathcal{G} to some sets F_R , F_B , and F_G of edges of B , that are
 362 defined as follows:

- 363 • For all $i \in \{0, \dots, d-1\}$ with $i \equiv 0 \pmod{4}$, add all edges in $E(B[V_i \cup V_{i+1}'])$ to F_R ,
 364 and all edges in $E(B[V_i \cup V_{i+1}''])$ to F_G .
- 365 • For all $i \in \{1, \dots, d-1\}$ with $i \equiv 1 \pmod{4}$, add all edges in $E(B[V_i \cup V_{i+1}])$ to F_B .
- 366 • For all $i \in \{1, \dots, d-1\}$ with $i \equiv 2 \pmod{4}$, add all edges in $E(B[V_i \cup V_{i+1}'])$ to F_R ,
 367 and all edges in $E(B[V_i \cup V_{i+1}''])$ to F_B .
- 368 • For all $i \in \{1, \dots, d-1\}$ with $i \equiv 3 \pmod{4}$, add all edges in $E(B[V_i \cup V_{i+1}])$ to F_G .

369 Note that $F_R \cup F_B \cup F_G = E(B)$. Furthermore, each of $B[F_R]$, $B[F_B]$, and $B[F_G]$
 370 induces a subgraph of B with maximum degree at most Δ . By Vizing's Theorem, $B[F_R]$
 371 admits a proper Δ -edge-colouring with colours from \mathcal{R} , $B[F_B]$ admits a proper Δ -edge-
 372 colouring with colours from \mathcal{B} , and $B[F_G]$ admits a proper Δ -edge-colouring with colours
 373 from \mathcal{G} . Altogether, this yields a proper 3Δ -edge-colouring ϕ_B of B with colours from
 374 $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G}$.

375 In terms of vertex colours, by ϕ_B , we get:

- 376 • $S(u^*)$ has either elements from both \mathcal{R} and \mathcal{G} only (case where $V_1' \neq \emptyset$ and $V_1'' \neq \emptyset$)
 377 or elements from \mathcal{G} only (case where $V_1' = \emptyset$).
- 378 • Consider a vertex $u \in V_1$. On the one hand, if $u \in V_1'$, then $S(u)$ contains only one
 379 element, from \mathcal{R} . On the other hand, if $u \in V_1''$, then $S(u)$ contains exactly one
 380 element from \mathcal{G} and at least one element from \mathcal{B} (and no element from \mathcal{R}).
- 381 • More generally, for every $u \in V_i$ with $i > 1$:
 - 382 – Assume $i \equiv 0 \pmod{4}$. On the one hand, if $u \in V_i'$, then $S(u) \subseteq \mathcal{G}$. On the other
 383 hand, if $u \in V_i''$, then $S(u)$ contains at least one element from \mathcal{G} , and perhaps
 384 elements from \mathcal{R} (and no elements from \mathcal{B}).

- 385 – Assume $i \equiv 1 \pmod{4}$. On the one hand, if $u \in V_i'$, then $S(u) \subseteq \mathcal{R}$. On the other
386 hand, if $u \in V_i''$, then $S(u)$ contains at least one element from \mathcal{G} , and at least
387 one element from \mathcal{B} (and no elements from \mathcal{R}).
- 388 – Assume $i \equiv 2 \pmod{4}$. On the one hand, if $u \in V_i'$, then $S(u) \subseteq \mathcal{B}$. On the other
389 hand, if $u \in V_i''$, then $S(u)$ contains at least one element from \mathcal{B} , and perhaps
390 elements from \mathcal{R} (and no elements from \mathcal{G}).
- 391 – Assume $i \equiv 3 \pmod{4}$. On the one hand, if $u \in V_i'$, then $S(u) \subseteq \mathcal{R}$. On the other
392 hand, if $u \in V_i''$, then $S(u)$ contains at least one element from \mathcal{B} , and at least
393 one element from \mathcal{G} (and no elements from \mathcal{R}).

394 It is then easy to check that ϕ_B is distinguishing, since vertices in V^+ and in V^- have
395 sets of colours in disjoint sets \mathcal{X} and \mathcal{Y} . In particular, the fact that $V_1'' \neq \emptyset$ (because B is
396 not bad) implies that $S(u^*)$ contains at least one element from \mathcal{G} .

397 By applying the arguments above to all the non-bad connected components of the
398 whole of B , we get ϕ_B , a partial distinguishing proper 3Δ -edge-colouring of the non-bad
399 connected components of B . By arguments used in the proof of Theorem 3.3, the properties
400 of ϕ_B remain when transforming ϕ_B to a partial arc-colouring $\phi_{D'}$ of D' . In particular,
401 $\phi_{D'}$ is proper and uses at most 3Δ colours, and, for every arc $u\vec{v}$ such that at least one of
402 u and v does not belong to a bad connected component of B , we have $S^-(u) \neq S^+(v)$.

403 Our goal now is to extend $\phi_{D'}$ to a $(-, +)$ -distinguishing proper $(3\Delta^*(D))$ -arc-colouring
404 ϕ_D of D . To that aim, note that two types of arc configurations remain to be coloured
405 in D : 1) the configurations corresponding to the bad connected components of B , and 2)
406 the lonely arcs. Extending the colouring to such configurations can be proved to always
407 be possible, via, essentially, counting arguments.

408 First, consider a bad connected component in B . By definition, this connected compo-
409 nent is a star with unique vertex u^+ in V^+ being its center, and being adjacent to $k \geq 2$
410 leaves v_1^-, \dots, v_k^- in V^- . Back in D , this corresponds to a vertex u with out-neighbours
411 v_1, \dots, v_k , where u has no other out-going arcs while all the v_i 's have no other in-coming
412 arcs. For every v_i , note that if all arcs in-coming to u and all arcs out-going from v_i
413 have already been coloured, then $S^-(u) \neq S^+(v_i)$. This is because either both u^- and v_i^+
414 belong to non-bad connected components of B (in which case the distinction comes from
415 how B was edge-coloured), because v_i^+ is the center of a bad connected component in B
416 whose associated bad configuration in D was treated earlier (in which case the distinction
417 comes from the upcoming counting arguments), or because u^- is a leaf in a bad connected
418 component of B whose associated bad configuration in D was treated earlier (in which
419 case the distinction comes from similar arguments as in the previous case). Thus, when
420 colouring the arcs $u\vec{v}_1, \dots, u\vec{v}_k$, we only need to make sure that 1) all arcs out-going from
421 u are assigned distinct colours, that 2) for every v_i and every out-neighbour w of v_i , the
422 resulting set $S^-(v_i)$ is distinct from the set $S^+(w)$, and that 3) the resulting set $S^+(u)$ is
423 distinct from the set $S^-(w)$ of every in-neighbour w of u . Since $d^+(v_i) \leq \Delta^*(D)$ for every
424 v_i , regarding the second condition, there is a set L_i of at least $2\Delta^*(D)$ colours that can be
425 freely assigned to $u\vec{v}_i$ without violating that condition. We are now in a weaker condition
426 than that of the statement of Claim 2.5: we have sets L_1, \dots, L_d of at least $2\Delta^*(D)$ ele-
427 ments, and we have to find a combination e_1, \dots, e_d of their elements such that all e_i 's are
428 distinct, each e_i lies in L_i , and the set $\{e_1, \dots, e_d\}$ is different from $S^-(w)$ for each w of
429 the at most $\Delta^*(D)$ in-neighbours w of u in D . By Claim 2.5, there is a such combination
430 e_1, \dots, e_d , and we can correctly extend the arc-colouring by setting $\phi_D(u\vec{v}_i) = e_i$ for every
431 $i \in \{1, \dots, d\}$.

432 Once no configuration corresponding to a bad connected component of B remains in
433 D , we are left with assigning a colour to each of the lonely arcs in L . Let us focus on a
434 lonely arc \vec{uv} of L that remains to be coloured. Recall that having $S^-(u) \neq S^+(v)$ does
435 not depend on how $\phi_D(\vec{uv})$ is chosen; that distinction condition is either already met by
436 previous colouring arguments, or will be met by how other lonely arcs will be coloured
437 later on. By definition of a lonely arc, recall that \vec{uv} is the only arc in-coming to v and
438 the only arc out-going from u . Hence, at the moment, $S^+(u) = S^-(v) = \emptyset$, and assigning
439 a colour to \vec{uv} will completely determine $S^+(u) = S^-(v)$. Also, since v has no other arc
440 coming in and u has no other arc going out, assigning any colour to \vec{uv} cannot break the
441 properness of ϕ_D . So, we just need to assign any colour to \vec{uv} so that $S^+(u)$ is different
442 from $S^-(w)$ for every w such that \vec{wu} is an arc, and so that $S^-(v)$ is different from $S^+(w)$
443 for every w such that \vec{vw} is an arc. There are at most $\Delta^*(D)$ such w for u , while there
444 are at most $\Delta^*(D)$ such w for v , hence at most $2\Delta^*(D)$ constraints in total. Since we are
445 using a set of $3\Delta^*(D)$ colours, there is a free one that we can assign to \vec{uv} by ϕ_D , without
446 raising conflicts in terms of properness or distinction.

447 Once all lonely arcs of L have been treated that way, ϕ_D is a $(-, +)$ -distinguishing
448 proper $(3\Delta^*(D))$ -arc-colouring of D . \square

449 5. Conclusion

450 Our goal in this paper was to investigate directed counterparts of the AVD Conjecture
451 where, by a proper arc-colouring of a digraph, adjacent vertices are required to be
452 distinguished by a given one of their two set parameters. For each of the four resulting
453 variants, we believe that, for any nice (specific to the variant) digraph D , there should be
454 a proper $(\Delta^*(D) + 2)$ -arc-colouring which is as desired. We verified this for the $(+, -)$
455 variant (Corollary 3.4), while we only verified weaker statements for the other variants
456 (Theorems 2.4 and 4.4).

457 An interesting aspect of this line of research lies in the differences between the original
458 AVD Conjecture and each of the four variants, and also in the differences between these
459 four variants. For instance, the notion of nice digraphs varies greatly from one variant to
460 another. Also, the effects of colouring an arc in the four variants are more or less distant
461 from the effects of colouring an edge in the original conjecture. In terms of inherent
462 hardness, the $(+, -)$ version of the AVD Conjecture seems to be the easiest one, as we
463 proved it is equivalent to a very restricted case of the original conjecture (Theorem 3.3).
464 We have the feeling that the $(-, +)$ version should be the hardest one, particularly due to
465 the fact that the colouring mechanisms are a bit less local.

466 Regarding further work, of course the most important direction would be to tackle
467 Conjectures 2.2 and 4.2. It could also be interesting to investigate our four variants of the
468 AVD Conjecture in restricted classes of digraphs such as tournaments or acyclic digraphs.
469 Lastly, as mentioned earlier, distinguishing adjacent vertices by a single set parameter is
470 only one possible way for defining a directed counterpart to the AVD Conjecture, but
471 playing with combinations of the two set parameters, just like Sopena and Woźniak did
472 in [7], might lead to other interesting problems as well.

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