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# Mathematical physics vs Philosophy: Hegel, Pythagorean triples, Spinors and Clifford Algebras

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**Abstract** The german philosopher G.W.F. Hegel (1770-1831) in his *Phenomenology of Spirit* developed a negative conception of mathematics (for him, the pursuit of equality transforms objects into corpses and leaves them in an inert or dismembered state). This conception is however essentially based on a study of the Euclidean demonstration of the Pythagorean theorem which remains superficial. Not only are there many other proofs, but what is at stake in Pythagoras' theorem refers to complex structures unnoticed by Hegel and which he could not know: relationship between quadratic form and square of a linear form, geometric algebra, spinors and rotations in the space, all concepts of great importance in modern physics. But these are also very close, in fact, to what Hegel privileged: dialectical synthesis and movement. Thus, it is mathematics, and especially mathematical physics, which has now something hegelian, and maybe more than (contemporary) philosophy.

**Key words.** Hegel, Pythagorean triples, spinors, twistors, Clifford algebras.

## 1 Introduction

It is well known that Hegel had a bad opinion of mathematics. Even if, over time, and under the pressure of facts (notably the expansion of differential calculus and Gauss's arithmetic research) this opinion has changed<sup>1</sup>, it remains that his initial view was negative. This has never been more clearly expressed than in the famous

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<sup>1</sup>I recalled this evolution and the texts concerned in [Parrochia 93].

text of the preface to the *Phenomenology of Spirit* (1807) (see [Hegel 77], § 42-45, 24-26) which takes for example the famous demonstration by Euclid of the Pythagorean theorem.

Let us recall briefly that the famous proposition I, 47 of Euclid's Elements says that "in right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle" (see[Euclid 56], I, 349).

The well-know proof, that we will not detail, remains to saying that the sum of the areas of the two squares on the legs ( $a$  and  $b$ ) of the triangle ( $a, b, c$ ) equals the area of the square on the hypotenuse ( $c$ ) (see Fig.1).

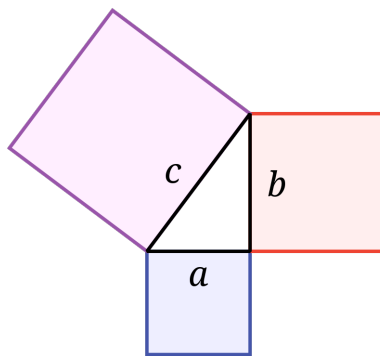


Figure 1: Scheme of Euclid's proof of Pythagorean theorem

Considering this demonstration, Hegel states the following thesis :

1. The process of mathematical proof does not belong to the object<sup>2</sup>; it is a function that takes place outside the matter in hand.
2. In mathematics, construction and proof contain, no doubt, true propositions, but the content, for Hegel, is "false". For example, in the Pythagorean theorem proof, the triangle is taken to pieces, and its parts made into other figures to which the construction in the triangle gives rise. "It is only at the end that we find again reinstated the triangle we are really concerned with; it was lost sight of in the course of the construction, and was present merely in fragments, that

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<sup>2</sup>The English word "object" is the translation of the German word "Gegenstand", which literally means "what is posed in front". This is why the French translator of the *Phenomenology of Spirit*, B. Bourgeois, suggests writing "object" with a hyphen ("ob-ject"), in order to recall that this word "designates the content that the spirit, splitting up inside its primary unity (the soul) opposes, objects to itself, to become properly *consciousness*". One then reserves the word "object" to translate "Objekt" (see [Hegel 18], 76).

belonged to other wholes. Thus we find negativity of content coming in here too, a negativity which would have to be called falsity, just as much as in the case of the movement of the notion where thoughts that are taken to be fixed pass away and disappear."

3. The real defect of this kind of knowledge affects its process of knowing as much as its material. As to that process, one does not see any necessity in the construction. An external purpose controls it. Concerning the material, it only consists of space and numerical units (das Eins).
4. For all that, philosophy has nothing to do with mathematics. In mathematics, knowledge advances along the lines of bare equality, of abstract identity, which is "lifeless, not being self-moved, does not bring about distinction within its essential nature; does not attain to essential opposition or unlikeness; and hence involves no transition of one opposite element into its other, no qualitative, immanent movement, no self-movement."
5. The main concern of mathematics is quantity, "a form of difference that does not touch the essential nature". In geometry, "it abstracts from the fact that it is the notion which separates space into its dimensions, and determines the connections between them and in them. It does not consider, for example, the relation of line to surface, and when it compares the diameter of a circle with its circumference, it runs up against their incommensurability, i.e. a relation in terms of the notion, an infinite element, that escapes mathematical determination."
6. Even applied mathematics does not take in account true concrete realities. Of course this one "treats of time, as also of motion, and other concrete things as well; but it picks up from experience synthetic propositions – i.e. statements of their relations, which are determined by their conceptual nature – and merely applies its formulae to those propositions assumed to start with."
7. Philosophy, on the contrary, does not deal with a determination that is non-essential, but with a determination so far as it is an essential factor. The abstract or unreal is not its element and content, but the real, what is self-establishing, has life within itself, existence in its very notion. It is the process that creates its own moments in its course, and goes through them all; and the whole of this movement constitutes its positive content and its truth. This movement includes, therefore, within it the negative factor as well, the element which would be named falsity if it could be considered one from which we had to abstract. The element that disappears has rather to be looked at as itself

essential, not in the sense of being something fixed, that has to be cut off from truth and allowed to lie outside it, heaven knows where; just as similarly the truth is not to be held to stand on the other side as an immovable lifeless positive element. Appearance is the process of arising into being and passing away again, a process that itself does not arise and does not pass away, but is per se, and constitutes reality and the life-movement of truth. "The truth is thus the bacchanalian revel, where not a member is sober".

8. In consequence, mathematics cannot be a useful model for philosophy. "It is not difficult to see that the method of propounding a proposition, producing reasons for it and then refuting its opposite by reasons too, is not the form in which truth can appear. Truth moves itself by its very nature; but the method just mentioned is a form of knowledge external to its material. Hence it is peculiar to mathematics and must be left to mathematics, which, as already indicated, takes for its principle the relation of quantity, a relation alien to the notion, and gets its material from lifeless space, and the equally lifeless numerical unit."
9. Generally speaking, Hegel protests against any schematizing formalism. However, it will allow himself to make use of triplicity "now that the triplicity, adopted in the system of Kant... has been raised to its significance as an absolute method" so that true form is thereby set up in its true content, and the conception of science has come to light.

## 2 The multiple proofs of Pythagorean theorem

Hegel's criticism against mathematical thought, which was already beginning to meet limits in its time, is no longer in season today (see [Larvor 99], 24). But it has, in fact, never really been admissible. We can make it a particularly striking demonstration by taking precisely the example of the right triangle which is the one that was mentioned above.

Some of the Euclid's proofs of the famous proposition I. 47 in the *Elements* (see [Euclid 56], 349) are classified as:

1. Proofs by rearrangement (see, for example, Heath's proof as reported in [Euclid 56], 354-355 or in [Benson 99], 172-173)).
2. Proofs by dissection without rearrangement (like Einstein's proof (see [Schroeder 12]),

3-4)).

3. Proofs using similar triangles (already known in the Antiquity).

In all these proofs, the initial triangle is either divided into other triangles, or inserted into a more complex figure in which it disappears or, let's say, only occupies an inessential place.

Hegel's criticism can at best strike this kind of proofs. But those are far from being the only ones.

We know that there are in fact hundreds of proofs of Pythagoras' theorem, not to say thousands. In his famous book *The Pythagorean proposition*, Elisha Scoot Loomis presents a collection of 370 proofs, grouped into the four following categories: Algebraic (109 proofs), Geometric (255), Quaternionic (4) ; and those based on mass and velocity, Dynamic (2). This author even asserts that the number of algebraic proofs is limitless – as is also the number of geometric proofs (see [Loomis 68], viii).

For many of these proofs, Hegel's reasoning does not hold water.

## 2.1 Algebraic proofs

For example, in a certain number of "algebraic" proofs, the triangle is not dismembered, but multiplied. This is the case, for example, in the following proof, which assumes the geometric arrangement reported below:

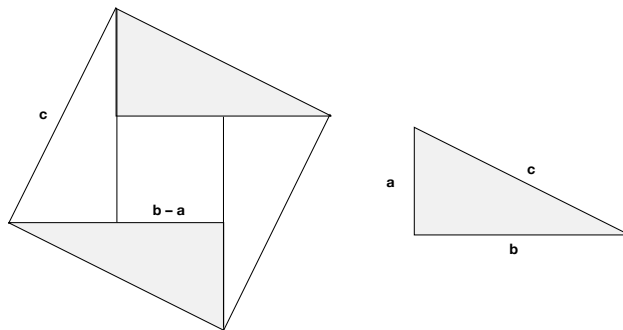


Figure 2: An algebraic proof of Pythagorean theorem

As we can see, the theorem can be proved algebraically using four copies of a right triangle with sides  $a$ ,  $b$  and  $c$ , arranged inside a square with side  $c$  as in Fig. 2. The

triangles are similar with area  $\frac{1}{2}ab$ , while the small square has side  $b - a$  and area  $(b - a)^2$ . The area of the large square is therefore:

$$(b - a)^2 + 4\frac{ab}{2} = (b - a)^2 + 2ab = b^2 - 2ab + a^2 + 2ab = a^2 + b^2.$$

But this is a square with side  $c$  and area  $c^2$ , so:

$$c^2 = a^2 + b^2.$$

This proof (or similar proof) would have already been known from the Hindu mathematician Bhaskara (12th century) and would not be much different from much older proof, which can be found in the Chinese classic *Zhoubi Suanjing* (The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven), which gives a reasoning for the (3, 4, 5) triangle (see next section). In China, it is called the "Gougu theorem" (see [Crease 08],25; [Cullen 07], 139).

There is a lot of other proofs of this kind.

## 2.2 A proof from Calculus

But we can also make use of certain advances in mathematics, which have occurred since antiquity, for example, differential calculus, moreover known from Hegel. In Fig.3, we just study how changes in a side of the right triangle produce a change in the hypotenuse.

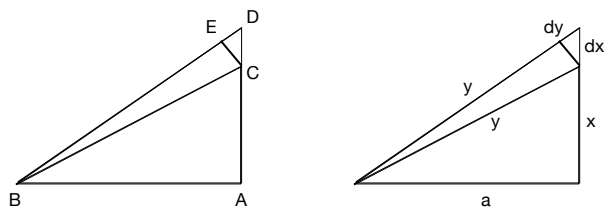


Figure 3: A proof of Pythagorean theorem using calculus

The triangle  $ABC$  is a right triangle, as shown in the upper part of the diagram, with  $BC$  the hypotenuse. At the same time the triangle lengths are measured as shown, with the hypotenuse of length  $y$ , the side  $AC$  of length  $x$  and the side  $AB$  of length  $a$ , as seen above.

If  $x$  is increased by a small amount  $dx$  by extending the side  $AC$  slightly to  $D$ , then  $y$  also increases by  $dy$ . These form two sides of a triangle,  $CDE$ , which (with  $E$  chosen so  $CE$  is perpendicular to the hypotenuse) is a right triangle approximately similar to  $ABC$ . Therefore, the ratios of their sides must be the same, that is:

$$\frac{dy}{dx} = \frac{x}{y}.$$

This can be rewritten as  $y dy = x dx$ , which is a differential equation that can be solved by direct integration:

$$\int y dy = \int x dx,$$

giving:

$$y^2 = x^2 + C.$$

The constant can be deduced from  $x = 0$ , then we can pose  $y = a$  and obtain the equation:

$$y^2 = x^2 + a^2.$$

Maybe one would say that this is more of an intuitive proof than a formal one. But it can be made more rigorous if proper limits are used in place of  $dx$  and  $dy$ .

## 2.3 A vectorial proof

Let us now consider the following Fig. 4.

Assume that the triangle  $ABC$ , formed of the vectors  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$ , is located in a Euclidean vector space  $E$  in which the scalar product (or dot product) of two Euclidean vectors  $\mathbf{a}$  and  $\mathbf{b}$  is defined as usual by:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta.$$

where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ . In this case, as we know, if the vectors  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal (their angle is  $\pi/2$ ), then  $\cos(\frac{\pi}{2}) = 0$  implies  $\mathbf{a} \cdot \mathbf{b} = 0$ .



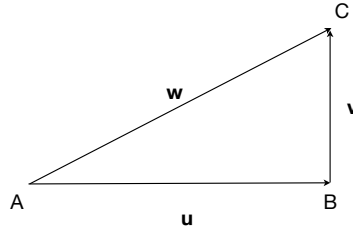


Figure 4: A vectorial proof of Pythagorean theorem

At the other extreme, if they are codirectional, then we also know the angle between them is zero and we get:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\|$$

This implies that the dot product of a vector  $\mathbf{a}$  with itself is:

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2,$$

which obviously gives:

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}},$$

the formula for the Euclidean length of the vector.

Now consider Fig. 4. We can say that :

$$\mathbf{w} = \mathbf{u} + \mathbf{v},$$

and by bilinearity of the scalar product, we get:

$$\|\mathbf{w}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}).$$

But as:

$$\mathbf{u} \cdot \mathbf{v} = 0,$$

Then, we get,:

$$\|\mathbf{w}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2,$$

which is the vectorial version of Pythagorean theorem.

In all these demonstrations, the triangle is by no means dismembered and Hegelian criticism does not apply. But we can go even further by showing that in reality, what is in question behind Pythagoras' statement refers to synthetic physico-mathematical structures much deeper than the simple figure of the triangle, which is only an appearance.

### 3 The unity of Pythagorean triples

In fact, what Hegel has not seen – maybe he could not – is that the most important in the Pythagorean formula is not the triangle in itself but the relation between the three quantities  $a, b$  and  $c$ , which constitute what we call now a «triple», and in our case a «Pythagorean triple».

As we have seen before, Pythagorean triples – despite their name – are not a Greek invention. In China, during the Han Dynasty (202 BC to 220 AD), something like Pythagorean triples appear in *The Nine Chapters on the Mathematical Art*, together with a mention of right triangles (see [Kangshen 99], 488). Some even believe the theorem arose before in China, where it is alternatively known as the "Shang Gao theorem", named after the Duke of Zhou's astronomer and mathematician, whose reasoning composed most of what was in the *Zhoubi Suanjing* (see [Wen-tsün 08], 158).

It is also possible that an intuitive knowledge of Pythagoras' relationship would be much older than Chinese mathematics, since it could have its roots in ancient Mesopotamia and, beyond, in the Egypt of the pyramids. As Thom (see [Thom 67]) has shown, the cuneiform tablet known as Plimpton 322 (see [Neugebauer 45] and, for the multiple interpretations see [Proust 15]) from Mesopotamia enlists 15 Pythagorean triples and is dated for almost 2000 BCE. The second pyramid of Giza is based on the 3-4-5 triangle quite perfectly and was build before 2500 BCE. "It has also been argued that many megalithic constructions include Pythagorean triples" (see [Kocik 07]).

In any case, as the Greek only knew integers and rational numbers, we do not loose in generality by restricting our view to these cases, i.e triples of integers.

In a previous work (see [Parrochia 12], 58-59), I quoted a lecture pronounced by Trautman (see [Trautman 90]) in Belgium in 1987, whose summary was published in 1990. This text explained that the Pythagorean equation, in the interpretation

of Diophante, enveloped in itself an extraordinarily modern synthetic notion, the notion of "spinor".

I would like to show today that something of this notion – not the concept, of course, nor the word – is already in Euclid and has extensions far beyond him.

But let us remain, for the moment, inside the Euclid's Elements, even if the existence of Pythagorean triples, that is triples of natural numbers  $(a, b, c)$  satisfying:

$$a^2 + b^2 = c^2, \tag{1}$$

has been known, in fact, for thousands of years.

Apparently, the problem of finding rational numbers which could be made the size of right-angled triangles or of finding square numbers which are the sum of two squares – as Heath has shown: see [Euclid 56], I, 356-357 – had been, for sure, investigated by Pythagoras and Plato<sup>3</sup>. Indeed Euclid himself, in the Book III of *the Elements*, Lemma 1 ([Euclid 56], III, 63-64), has well explained his own method of forming such triangles.

1. Firstable, he sets out the segments  $AB$  bisected at  $C$  and  $AC$  bisected at  $D$ , such that  $AD = DC = AC/2$  (see Fig. 5).
2. An additional condition is assumed :  $AB$  and  $BC$  must be either both even or both odd, which means that  $AC$  is even (because whether an even number is subtracted from an even number or an odd number from an odd number, the remainder is even).
3. Finally, the quantities  $AB$  and  $BC$  must be either similar plane numbers, or square numbers which are themselves also similar plane numbers.



Figure 5: Euclid's schema of Book II, Lemma I

From Proposition 6, Book II, Euclid establishes the following:

$$AB \cdot BC + CD^2 = BD^2. \tag{2}$$

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<sup>3</sup>As Heath (see [Euclid 56], I., 357) has shown, H. G. Zeuthen and M. Cantor have both attempted to reconstruct their respective methods, without really achieving anything other than conjectures which are always questionable.

For the clarity, let us use a modern mathematical language. We will pose:

$$AB = p^2; \quad BC = q^2; \quad CD = \frac{p^2 - q^2}{2}; \quad BD = \frac{p^2 + q^2}{2} \quad (3)$$

By putting (3) in (2), we get:

$$p^2q^2 + \left(\frac{p^2 - q^2}{2}\right)^2 = \left(\frac{p^2 + q^2}{2}\right)^2,$$

which we may write:

$$4p^2q^2 + (p^2 - q^2)^2 = (p^2 + q^2)^2,$$

a formula which satisfies (1) iff:

$$a = p^2 - q^2, \quad b = \pm 2pq, \quad c = p^2 + q^2, \quad (4)$$

a result found again by Trautman (see [Trautman 98]).

We can add today that, if  $(a, b, c)$  is a Pythagorean triple, then at least one of the number  $a$  and  $b$  is even. Moreover, if  $t \in \mathbb{Z}$ , then  $(b, a, c)$  and  $(ta, tb, tc)$  are also Pythagorean. A Pythagorean triple  $(a, b, c)$  is *standard* if  $c > 0$  and either, the triple  $(a, b, c)$  is said to be "relatively prime" (rp) and  $b$  is even or  $(a/2, b/2, c/2)$  is a triple of rp-integers and  $b/2$  is odd. Every Pythagorean triple can be written as  $(ta, tb, tc)$ , where  $t \in \mathbb{Z}$ , the integers  $(a, b, c)$  are rp and  $c > 0$ . If  $b$  is even, then  $(a, b, c)$  is standard. If  $b$  is odd, then  $(2a, 2b, 2c)$  is standard.

So, as some mathematicians have long noted, the standard triples are in one-to-one correspondence with relatively prime pairs  $(p, q)$ ,  $\gcd(p, q) = 1$ ,  $p > q$ , such that exactly one of  $(p, q)$  is even (see [Sierpinski 62]; [Tausky-Tood 82]). In other words, as Trautman said, "there is a bijection between the set of directions in  $\mathbb{Z}^2$  and the set of 'null directions' in  $\mathbb{Z}^3$ " (see [Trautman 98], 412)<sup>4</sup>.

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<sup>4</sup>The space of directions is a space where the directions must be well defined. Here they are indicated by right triangle hypotenuses, insofar as the data of rectangular coordinates defines each time a straight line with a certain slope (the hypotenuse of the right triangle). The principal null directions of a spacetime are a fundamental set of invariant directions which play an important

## 4 Spinors and the modular group

This may be explained as follow: in fact, Pythagorean triples can likewise be encoded into a square matrix of the form:

$$X = \begin{bmatrix} c + b & a \\ a & c - b \end{bmatrix},$$

with:

$$\det X = c^2 - a^2 - b^2,$$

and we get  $\det X = 0$  precisely when  $(a, b, c)$  is a Pythagorean triple. If  $X$  corresponds to a Pythagorean triple then, as a matrix, it must have rank 1.

Since  $X$  is symmetric, it follows from a result in linear algebra that there is a column vector  $\xi = [p \ q]^T$  such that the outer product:

$$X = 2 \begin{bmatrix} p \\ q \end{bmatrix} [p \ q] = 2\xi\xi^T, \tag{5}$$

holds, where  $T$  denotes the matrix transpose. The vector  $\xi$  is called a *spinor* (for the Lorentz group  $\text{SO}(1, 2)$ )<sup>5</sup>.

In more abstract terms, the real meaning of the Euclid formula is that each primitive Pythagorean triple can be written as the outer product with itself of a spinor with integer entries, as in (5).

So, as it appears, we are very far from the simple geometric characterization of the right triangle, to which Hegel's understanding was limited.

But we can go further.

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role in studying the geometry of the spacetime. Let  $g$  be the metric of the spacetime and let  $W$  be the Weyl tensor for  $g$ . A principal null direction is a null vector  $k$  which satisfies the system of non-linear algebraic:

$$k_{[e} W_{a] \ bc} [d \ k_{f]} k^b k^c = 0$$

The principal null directions are calculated from the factorization of the Weyl spinor as a product of rank 1 spinors – each distinct spinor in this factorization determines a principal null direction.

<sup>5</sup>Though Cartan (see [Cartan 38]) described this kind of structure long before, the explicit notion of "spinor" appears in 1931 in *Physical Review*. The citation refers to spinor analysis developed by B. Van der Waerden. But this mathematician thought the notion was due to Paul Ehrenfest. For justification of this term here, see section 5.

There is a modular group  $\Gamma$ , which is the set of  $2 \times 2$  matrices with integer entries:

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

with determinant equal to one. We have:  $\alpha\delta - \beta\gamma = 1$ . This set forms a group, since the inverse of a matrix in  $\Gamma$  is again in  $\Gamma$ , as is the product of two matrices in  $\Gamma$ . The modular group acts on the collection of all integer spinors. Furthermore, the group is transitive on the collection of integer spinors with relatively prime entries. For if  $[p \ q]^T$  has relatively prime entries, then:

$$\begin{bmatrix} p & -v \\ q & u \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where  $u$  and  $v$  are selected (by the Euclidean algorithm) so that  $pu + qv = 1$ .

By acting on the spinor  $\xi$  in (5), the action of  $\Gamma$  goes over to an action on Pythagorean triples, provided one allows for triples with possibly negative components. Thus if  $A$  is a matrix in  $\Gamma$ , then:

$$\rho(A)X = 2(A\xi)(A\xi)^T = AXA^T. \quad (6)$$

gives rise to an action on the matrix  $X$  in (5). However, this does not give a well-defined action on primitive triples, since it may take a primitive triple to an imprimitive one. As we have seen before with Trautman, it is convenient to call a triple  $(a, b, c)$  *standard* if  $c > 0$  and either  $(a, b, c)$  are relatively prime or  $(a/2, b/2, c/2)$  are relatively prime with  $a/2$  odd. If the spinor  $[p \ q]^T$  has relatively prime entries, then the associated triple  $(a, b, c)$  determined by (5) is a standard triple. It follows that the action of the modular group is transitive on the set of standard triples and we have the exact sequence:

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \mathrm{SL}(2, \mathbb{Z}) \xrightarrow{\rho} \Gamma \rightarrow 1$$

Alternatively, restrict attention to those values of  $p$  and  $q$  for which  $p$  is odd and  $q$  is even. Let the subgroup  $\Gamma(2)$  of  $\Gamma$  be the kernel of the group homomorphism:

$$\Gamma = \mathrm{SL}(2, \mathbb{Z}) \rightarrow \mathrm{SL}(2, \mathbb{Z}_2),$$

where  $SL(2, \mathbb{Z}_2)$  is the special linear group over the finite field  $\mathbb{Z}_2$  of integers modulo 2. Then  $\Gamma(2)$  is the group of unimodular transformations which preserve the parity of each entry. Thus if the first entry of  $\xi$  is odd and the second entry is even, then the same is true of  $A\xi$  for all  $A \in \Gamma(2)$ . In fact, under the action (6), the group  $\Gamma(2)$  acts transitively on the collection of primitive Pythagorean triples.

The group  $\Gamma(2)$  is the free group whose generators are the matrices:

$$U^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad L^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Consequently, every primitive Pythagorean triple can be obtained in a unique way as a product of copies of the matrices  $U^2$  and  $L^2$  (for other possibilities, see [?] and also [Trautman 98]) and the set of positive primitive Pythagorean triples has the structure of a complete, infinite, rooted ternary-tree.(see [Alperin 05], 807-808).

## 5 Spinors in physics

At the end of the previous analysis, we can therefore say that the solution of (1), moreover found by Euclid, then by Diophante, is in fact deduced from (5). As Trautman points out (see [Trautman 90], 187), we can interpret (5) by saying that the "spinor"  $(p, q)$  is a square root of the isotropic vector of component  $a, b, c$ . This observation, which admits a generalization in dimension  $> 3$  is the base of Elie Cartan's approach of spinors.

As Trautman has shown ([Trautman 90], 187-188), the matrix  $X$ , multiplied on the left by:

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

becomes a matrix whose square is a multiple of the unit matrix. If, in addition, we consider the numbers  $a, b, c$  as real numbers and if we replace  $b$  with  $-ib$  (where  $i = \sqrt{-1}$ ), we obtain the formula:

$$\begin{bmatrix} c & a - ib \\ a + ib & -c \end{bmatrix}^2 = a^2 + b^2 + c^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (7)$$

which gives a linear form in  $(a, b, c) \in R^3$  whose square is (proportional to) the fundamental quadratic form of Euclidean space.

Dirac's approach in the constitution of the equation of the relativistic electron wave function amounts to representing the whole coordinates by differential operators and to expressing a quadratic form of 4-dimensional space-time like the square of a linear form, which is a simple generalization of the initial Pythagorean formula. by restricting itself to dimension 3, formula (7) becomes:

$$\left[ \begin{array}{cc} \frac{\partial}{\partial c} & \frac{\partial}{\partial a} - i \frac{\partial}{\partial b} \\ \frac{\partial}{\partial a} + i \frac{\partial}{\partial b} & -\frac{\partial}{\partial c} \end{array} \right]^2 = \Delta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (8)$$

where  $\Delta$  is the Laplacian. We even have a Pauli operator which takes the form:

$$\vec{\sigma}\text{grad} = \begin{bmatrix} \frac{\partial}{\partial c} & \frac{\partial}{\partial a} - i \frac{\partial}{\partial b} \\ \frac{\partial}{\partial a} + i \frac{\partial}{\partial b} & -\frac{\partial}{\partial c} \end{bmatrix} \quad (9)$$

To explain the abnormal nature of the radiation from certain atoms – for example, an even (and not odd) number of spectral lines in the hydrogen atom –, physicists put forward the hypothesis that these doublets were due to the fact that the electron possessed an internal angular momentum (or spin) equal to  $\hbar/2$ . Wolfgang Pauli transformed this hypothesis into a coherent theory. In this theory, the wave function of the electron is a field of spinors on  $\mathbb{R}^3$  with two components. The unimodular unitary group  $SU(2)$  which appears here acts on the spinors and is a double and universal covering of  $SO(3)$ . For mathematicians, this group identifies with the group  $Spin(3)$ . And, for all  $n \geq 3$ , there is a simply connected group  $Spin(n)$  and an exact sequence of group homomorphisms:

$$1 \rightarrow \mathbb{Z}_2 \rightarrow Spin(n) \rightarrow SO(n) \rightarrow 1.$$

But Spin groups are defined from algebras called "Clifford algebras".



## 6 Clifford algebras and orthogonal transformations

Clifford considered a natural generalization of formula (7) to quadratic forms with  $n$  variables and signature  $k, l$ . Given a quadratic form of the type:

$$\sum_{\mu, \nu=1}^n g_{\mu\nu} x_\mu x_\nu = x_1^2 + \dots + x_k^2 - (x_{k+1}^2 + \dots + x_{k+l}^2),$$

it is a question of finding a family of  $n$  complex matrices  $\gamma_\mu$  with  $n$  rows and  $n$  columns,  $\gamma_\mu \in \mathbb{C}^n$  so that:

$$\left( \sum \gamma_\mu x_\mu \right)^2 = I \sum_{\mu, \nu=1}^n g_{\mu\nu} x_\mu x_\nu,$$

where  $I = \mathbb{C}^n$ . One shows that this problem always has solutions.

To stay as close as possible to our starting point (Pythagoras' formula), let's restrict ourselves to the case where the  $x_\mu$  are integers and the square of the linear form must be expressed by a defined quadratic form positive, that is:

$$\left( \sum_{i=1}^n x_i \right)^2 = x_1^2 + \dots + x_n^2. \quad (10)$$

In this case, we know that (10) is satisfied if we introduce in the previous forms coefficients  $e_i, e_j$  which belong to  $\mathbb{Z}^n$  and are such that:

$$\begin{aligned} \mathbf{e}_i^2 &= 1, & i &= 1, 2, \dots, n, \\ \mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i &= 0, & i &\neq j. \end{aligned}$$

where  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  is the canonical basis of  $\mathbb{Z}^n$ . The associative and distributive algebra generated by the set  $\mathbb{Z}^n$  with the above product rules may be called universal Clifford algebra of the set  $\mathbb{Z}^n$  and denoted  $Cl(\mathbb{Z}^n)$ .

It therefore finally appears that Pythagoras' theorem is only superficially linked to the geometric idea of a triangle. It is actually a special case of the general equivalence of a quadratic form to the square of a linear form.

Much more, the elements  $p$  and  $q$ , which make it possible to satisfy equality (1), constitute a new particular mathematical entity (different from scalars, vectors or even tensors) connected to the idea of orthogonal rotation in space.

We can see it now very clearly.

A Pythagorean triple can be conceived as a vector  $\mathbf{x} = (a/c, b/c) \in \mathbb{Z}_2$ , such that:

$$a^2 + b^2 = c^2, \quad a, b, c \in \mathbb{Z}, \quad \|\mathbf{x}\| = 1.$$

Let  $\{\mathbf{e}_1, \mathbf{e}_2\}$  be an orthonormal basis of  $\mathbb{Z}_2$ . Aragón-González (et alii) (see [Aragón 09], 9-10) proved that there exists a simple rotation  $R$ , such that:

$$R(\mathbf{y}) = \frac{(1 + \mathbf{x}\mathbf{e}_1)\mathbf{y}(1 + \mathbf{e}_1\mathbf{x})}{2(1 + (\mathbf{x}, \mathbf{e}_1))}.$$

which satisfies  $R(\mathbf{e}_1) = \mathbf{x}$ . Therefore, the orthogonal matrix associated to Pythagorean triple is:

$$[R(\mathbf{e}_1), R(\mathbf{e}_2)] = \begin{bmatrix} \frac{a}{c} & \frac{-2(a+c)c^2y}{2c^3(a+c)} \\ \frac{b}{c} & \frac{2c^2a(a+b)}{2c^3(a+b)} \end{bmatrix} = \begin{bmatrix} \frac{a}{c} & \frac{-b}{c} \\ \frac{b}{c} & \frac{a}{c} \end{bmatrix}, \quad (11)$$

where one uses the fact that  $\|\mathbf{x}\| = 1$ .

As the above orthogonal matrix has rational entries (with respect to the canonical basis), one can apply some properties of coincidence isometries. In particular, Aragón-González (et alii) prove the following:

Let  $R : R^2 \rightarrow R^2$  be a non trivial orthogonal transformation. If the matrix representation of  $R$  with respect to the canonical basis  $\{\mathbf{e}_1, \mathbf{e}_2\}$  has rational entries, then there exists a vector  $\mathbf{a} = (p, q) \in \mathbb{Z}_2$ , with  $\gcd(p, q) = 1$ , such that:

$$R(\mathbf{x}) = \mathbf{a}\mathbf{e}_2\mathbf{x}\mathbf{e}_2\mathbf{a}^{-1}.$$

The matrix associated this orthogonal transformation is:

$$[R(\mathbf{e}_1), R(\mathbf{e}_2)] = \begin{bmatrix} \frac{q^2-p^2}{p^2+q^2} & \frac{2pq}{p^2+q^2} \\ \frac{-2pq}{p^2+q^2} & \frac{p^2-q^2}{p^2+q^2} \end{bmatrix},$$

and a direct comparison with (11) yields

$$\begin{bmatrix} \frac{a}{c} & \frac{-b}{c} \\ \frac{b}{c} & \frac{a}{c} \end{bmatrix} = \begin{bmatrix} \frac{q^2-p^2}{p^2+q^2} & \frac{2pq}{p^2+q^2} \\ \frac{-2pq}{p^2+q^2} & \frac{p^2-q^2}{p^2+q^2} \end{bmatrix},$$

which verifies (4).

So, we can see explicitly that the spinor  $(p, q)$  is associated with non trivial orthogonal transformations which are, in fact, rotations in space.

Thus, the Pythagorean triples are not only associated with trivial right triangles. Pythagoras theorem is a case of equivalence between a quadratic form and the square of a linear form. This equivalence, to be satisfied, supposes the existence of elements taken in an algebra of matrices – a Clifford algebra – whose orthogonal transformations or rotations in space are parametrized by spinors.

In conclusion, behind Pythagoras theorem and the demonstration of the right triangle, exists a very deep rational organization with complex synthetic structures like Clifford algebras, rotations in space and spinors. Who could contest that there are here, with those structures, dialectical syntheses (geometric algebra), self-movement and "life" which place mathematics far beyond the (fairly negative) view that Hegel had of it? Thus, a little mathematics and physics often overcomes the most ingrained philosophical claims.

What Hegel believed to be characteristics peculiar to philosophy, mathematics, little by little, came to make them its own, to the point that certain mathematicians - like Lakatos, for example - can today be rightly taxed with "Hegelians" (see [Larvor 99] if we leave aside the errors of Hegel and if we now recognize in mathematics the qualities which, for Hegel, were those of philosophy.

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