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# Pace and motor control optimization for a runner

Amandine Aftalion\*

Emmanuel Trélat<sup>†</sup>

## Abstract

Our aim is to present a new model which encompasses pace optimization and motor control effort for a runner on a fixed distance. We see that for long races, the long term behaviour is well approximated by a turnpike problem. We provide numerical simulations quite consistent with this approximation which leads to a simplified problem. We are also able to estimate the effect of slopes and ramps.

## 1 Introduction

The process of running involves a control phenomenon in the human body. Indeed, the optimal pace to run a fixed distance requires to use the maximal available propulsive force and energy in order to produce the optimal running strategy. This optimal strategy is a combination of cost and benefit: a runner usually wants to finish first or beat the record but minimizing his effort. The issue of finding the optimal pacing is a crucial one in sports sciences [5, 10, 12, 13, 21, 26, 27] and is still not solved. In tactical races, depending on the level of the athlete, and the round on the competition (heating, semi-final or final), the strategy is not always the same: the pacing can either be U-shaped (the start and the finish are quicker), J-shaped (greater finishing pace) or reverse J-shaped (greater starting pace).

In this paper, we want to model this effort minimization as a control problem, solve it and find estimates of the velocity using the turnpike theory of [25]. We will build on a model introduced by Keller [16], improved by [1, 2, 3, 7, 8, 19]. The extension by [1, 2, 3] is sufficiently accurate to model real races. We add a motivation equation inspired from the analysis of motor control in the human body [17]. This is related to the minimal intervention principle [22] so that human effort is minimized through penalty terms. We have developed this model for the 200m in [4] and extend it here for middle distance races.

Let us go back to the various approaches based on Newton's second law and energy conservation. Let  $d > 0$  be the prescribed distance to run. Let  $x(t)$  be the position,  $v(t)$  the velocity,  $e(t)$  the anaerobic energy,  $f(t)$  the propulsive force per unit mass. Newton's second law allows to relate force and acceleration through:

$$\dot{x}(t) = v(t) \qquad x(0) = 0, \quad x(t_f) = d, \qquad (1)$$

$$\dot{v}(t) = -\frac{v(t)}{\tau} + f(t) \qquad v(0) = v^0, \qquad (2)$$

where  $\tau$  is the friction coefficient related to the runner's economy. A first approach by Keller [16] consists in putting the control on the force  $f(t)$ , assuming a bound  $0 \leq f(t) \leq f_M$  and writing an

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energy balance between the variation of aerobic energy, anaerobic energy and the power developed by the propulsive force:

$$\dot{e}(t) = \bar{\sigma} - f(t)v(t) \quad e(0) = e^0, \quad e(t) \geq 0, \quad e(t_f) = 0, \quad (3)$$

where  $e^0$  is the initial energy and  $\bar{\sigma}$  is the energetic equivalent per unit time of  $\dot{V}O_2$ , the volume of oxygen used by a unit of time and is assumed to be constant. The control problem is to minimize the time  $t_f$  to run the prescribed distance  $d = \int_0^{t_f} v(t) dt$ . This model is able to predict times of races but fails to predict the precise velocity profile.

The improvement in [2], with respect to Keller's model, is to better encompass the aerobic effects: indeed  $\dot{V}O_2$ , the volume of oxygen which is transformed into energy, is not constant in time, but increases at the beginning of the race from the rest value to reach a maximal value, then is almost constant and decreases at the end of the race. In [2], the function  $\sigma$  is assumed to depend on the anaerobic energy, and decreases when the anaerobic supplies gets too low. This leads to the following function  $\sigma(e)$ , which is to be thought as  $\sigma(e(t))$ , where  $e(t)$  decreases:

$$\sigma(e) = \begin{cases} \bar{\sigma} \frac{e}{e^0 \gamma_1} + \sigma_f \left(1 - \frac{e}{e^0 \gamma_1}\right) & \text{if } \frac{e}{e^0} < \gamma_1 \\ \bar{\sigma} & \text{if } \frac{e}{e^0} \geq \gamma_1 \quad \text{and} \quad e^0 - e \geq \gamma_2 \\ (\bar{\sigma} - \sigma_r) \frac{e^0 - e}{\gamma_2} + \sigma_r & \text{if } e^0 - e < \gamma_2 \end{cases} \quad (4)$$

where  $\bar{\sigma}$  is the maximal value of  $\sigma$ ,  $\sigma_f$  is the final value at the end of the race,  $\sigma_r$  the rest value,  $e^0$  is the initial value of energy,  $\gamma_1 e^0$  the critical energy at which the flow of aerobic energy into the anaerobic container starts to depend on the residual anaerobic energy and  $\gamma_2$  the energy at which the maximal oxygen uptake  $\bar{\sigma}$  is achieved. The parameters  $e^0$ ,  $\gamma_1$ ,  $\gamma_2$ ,  $\bar{\sigma}$ ,  $\sigma_f$ ,  $\sigma_r$  depend on the runner and on the length of the race. The dependence of  $\sigma$  on time is well known [15]: up to 400 m, the function  $\sigma$  is increasing with time and decreasing in energy, that is  $\gamma_2 = e^0$ ; for larger distance, the maximal value  $\bar{\sigma}$  is reached. The longer the race, the longer is the plateau at  $\sigma = \bar{\sigma}$ .

The model introduced in [1, 2] is an optimal control problem for which the control is the derivative of the propulsive force and the aim is to minimize the final time  $t_f$  to reach a prescribed distance  $d$ . Indeed, the first idea is to put the control on the propulsive force itself as in Keller's paper [16], but the solution yields derivatives of the force which are too big with respect to human ones.

Nevertheless, putting the control on the derivative of the force seems artificial. Therefore, in this paper, we introduce a new equation which limits the variation of the force and is inspired from the motor control theory of [17]. This new equation relies on the neural drive  $u(t)$ . This leads to the following system:

$$\dot{x}(t) = v(t) \quad x(0) = 0, \quad x(t_f) = d, \quad (5)$$

$$\dot{v}(t) = -\frac{v(t)}{\tau} + f(t) - g\beta(x(t)) \quad v(0) = v^0, \quad (6)$$

$$\dot{f}(t) = \gamma \left( u(t)(F_{\max} - f(t)) - f(t) \right) \quad f(t) \geq 0, \quad (7)$$

$$\dot{e}(t) = \sigma(e(t)) - f(t)v(t) \quad e(0) = e^0, \quad e(t) \geq 0, \quad e(t_f) = 0, \quad (8)$$

where  $e^0$  is the initial energy,  $\tau$  the friction coefficient related to the runner's economy,  $F_{\max}$  is a threshold upper bound for the force,  $\gamma$  the time constant of motor activation and  $u(t)$  the neural drive which will be our control. We have also added a dependence on the slope  $\beta(x)$  at distance  $x$  from the start, where we denote by  $g$  the gravity. Let us point out that for this problem on a

straight line, there is no need for a bound on the maximal force. This comes from (7), the equation on the force.

The optimization problem consists in minimizing the difference between the cost and the benefit. In [17], the expected cost is proportional to the motor control which is the  $L^2$  norm of the neural drive  $u(t)$ . One could think of adding other costs, for instance in walking modeling, the cost is proportional to the jerk, which is the  $L^2$  norm of the derivative of the centrifugal acceleration [6, 9]. In our case, as a start, we assume the runner is running on a straight line, so that we do not take this into account. On the other hand, the benefit is proportional to the reward, and can be estimated for instance to be proportional to  $-t_f$ . Indeed, one could imagine the reward is a fixed amount to which is subtracted a number proportional to the difference between the world record and the final time. Similarly, one could add other benefits or costs linked to multiple attempts or the presence of a supporting audience. In this paper, we choose to model the simplest case where the benefit is the final time and the cost is the motor control. This leads to the following minimization:

$$\min \left( t_f + \frac{\alpha}{2} \int_0^{t_f} u(t)^2 dt \right) \quad (9)$$

where  $\alpha > 0$  is a weight to be determined. Of course, the solution to this optimal control problem is very sensitive on  $\alpha$ , as we will see below.

The turnpike theory in optimal control stipulates that, under general assumptions, the optimal solution of an optimal control problem in fixed final time remains essentially constant, except at the beginning and at the end of the time-frame. We refer the reader to [25] for a complete state-of-the-art and bibliography on the turnpike theory. Actually, according to [25], due to the particular symplectic structure of the first-order optimality system derived from the Pontryagin maximum principle, the optimal solution (which is a triple consisting of the optimal state, co-state or adjoint vector, and optimal control) is, except around the terminal points, exponentially close to a steady-state, which is itself the optimal solution of an associated *static* optimal control problem. This is the so-called *turnpike phenomenon*. Of course, such an optimal steady-state is an equilibrium of the control system.

Therefore, to apply the turnpike theory, we need to work with a fixed final time so we choose to parameterize our problem by the distance  $s$  rather than the time  $t$  so that  $t'(s) = \frac{1}{v(s)}$ . The new equations in terms of  $s$  have a fixed final distance  $d$  and become

$$(OCP) \quad \left\{ \begin{array}{l} \min \int_0^d \left( \frac{1}{v(s)} + \frac{\alpha}{2} \frac{u(s)^2}{v(s)} \right) ds \\ v'(s) = -\frac{1}{\tau} + \frac{f(s)}{v(s)} - g \frac{\beta(s)}{v(s)} \quad v(0) = v^0, \\ e'(s) = \frac{\sigma(e(s))}{v(s)} - f(s) \quad e(0) = e^0, \quad e(d) = 0, \\ f'(s) = \frac{\gamma}{v(s)} \left( u(s)(F_{\max} - f(s)) - f(s) \right) \end{array} \right.$$

As soon as the race is sufficiently long (above 1500 m), one notices the existence of a limiting problem where  $v$  and  $f$  are constant and  $e$  is linearly decreasing. We will see that the turnpike theory of [25] (see also [23]) provides very accurate estimates for these mean velocity, force and the energy decrease. We want to use this to simplify the runner's model for potential software applications.

## 2 Numerical simulations

Optimization and numerical implementation of the optimal control problem (OCP) are done by combining automatic differentiation softwares with the modeling language AMPL [11] and expert optimization routines with the open-source package IpOpt [28]. This allows to solve for the velocity  $v$ , force  $f$ , energy  $e$  in terms of the distance providing the optimal strategy and the final time. As advised in [23, 25], we initialize the optimization algorithm at the turnpike solution that we find below.

We have chosen numerical parameters to match the real race of 1500 m described in [14] so that  $d = 1500$ . The final time is 245 seconds. They are middle distance runners successful in French regional races. The  $\dot{V}O_{2\max}$  is 66 ml/mn/kg which leads to a maximum value  $\bar{\sigma} = 22$  of  $\sigma$ : since one liter of oxygen produces an energy about 21.1 kJ via aerobic cellular mechanisms, this provides an estimate of the available energy per  $kg$  per second which is  $66/60 \times 21.1 \simeq 22$ . The decrease in  $\dot{V}O_2$  at the end of the race is of about 10% when the anaerobic energy left is 15%. Therefore, we choose the final value of  $\sigma$  to be 10% less than the maximal value, that is  $\sigma_f = 20$ , and we choose  $\gamma_1 = 0.15$ . The other parameters are  $\sigma_r = 6$ ,  $\gamma_2 = 566$ ,  $\alpha = 10^{-5}$ ,  $F_{\max} = 8$ ,  $\tau = 0.932$ ,  $e^0 = 4651$ ,  $\gamma = 0.0025$ ,  $v^0 = 3$ .

In [17], the equivalent of  $\alpha$  is determined by experimental data. In our case, we have noticed that, according to  $\alpha$ , either  $u$  is negative with a minimum or changes sign with a minimum and a maximum. Also, when  $\alpha$  gets too small,  $\dot{f}$  is almost constant. The choice of  $\alpha$  is made such that the second term of the objective is a small perturbation of the first one, that is can act at most on the tenth of second for the final time.

With these parameters, we simulate the optimal control problem (OCP) and plot the velocity  $v$ , the propulsive force  $f$ , the motor control  $u$  and  $\sigma$  vs the distance in Figure 1.

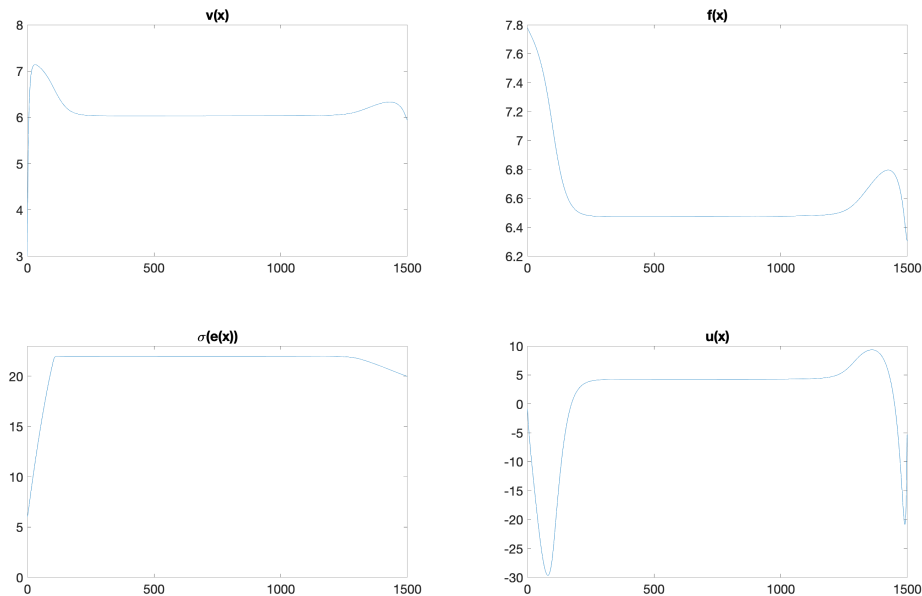


Figure 1: Velocity  $v$ , force  $f$ , aerobic energy  $\sigma(e)$  and motor control  $u$  vs distance on a 1500 m.

The velocity increases until reaching a peak value, then decreases to a mean value, before the final sprint at the end of the race. This is consistent with usual tactics which consist in an even pace until the last lap where the final sprint starts. This final sprint takes place when the function  $\sigma$  starts decreasing. The velocity and force follow the same profile.

We point out that we obtain a turnpike limit with a mean motor control, mean force and mean velocity in the middle of the race (with  $\bar{v} = 6.06$ ) that we want to analyze analytically. We will also try to construct local models for the beginning and end of the race.

### 3 Main results using turnpike estimates

In this section, we will assume a flat race, that is the slope  $\beta(x)$  is identically zero.

The optimal control problem (OCP) involves a state variable, namely, the energy  $e(t)$ , that is decreasing and thus has no equilibrium. The turnpike theory has been extended in [23] to this situation when the function  $\sigma$  is constant, and in this case, the above steady-state is replaced by a partial steady-state (namely,  $v$  and  $f$  are steady), and  $e(t)$  is approximated by an affine function satisfying the imposed constraints  $e(0) = 0$  and  $e(t_f) = 0$ . In what follows, we denote the approximating turnpike trajectory with an upper bar. More precisely, in what follows, we denote by  $t \mapsto (\bar{v}_c, \bar{e}_c(t), \bar{f}_c)$  the turnpike trajectory (parameterized in time here; but it can equivalently be considered as parameterized in distance), defined on the interval  $[0, \bar{t}_c]$  so that  $\bar{v}_c$  and  $\bar{f}_c$  are steady-states (equilibrium of the control dynamics) and  $\bar{e}_c(t)$  is affine and satisfies the terminal constraints  $\bar{e}_c(0) = 0$  and  $\bar{e}_c(\bar{t}_c) = 0$ , while  $d = \bar{v}_c \bar{t}_c$ .

We next elaborate to show how the turnpike theory can be applied to (OCP) and ultimately provides very accurate approximate solutions.

Assume as a first step that  $\sigma(e)$  is constant equal to  $\bar{\sigma}$ , then, the turnpike theory [23, 25] yields that, for long races, the velocity is almost constant and  $\bar{v}_c = \bar{f}_c \tau$  is the solution of the equation

$$\frac{d\bar{v}_c^2}{\tau} - \bar{\sigma}d = e^0 \bar{v}_c. \quad (10)$$

This comes from (8) assuming that  $\sigma, f, v$  are constant and that  $e$  is affine. The mean velocity  $\bar{v}_c$  can be solved from (10) to get

$$\bar{v}_c = \frac{e^0 \tau}{2d} + \sqrt{\bar{\sigma} \tau + \left(\frac{e^0 \tau}{2d}\right)^2}. \quad (11)$$

We observe that this increases with  $e^0, \tau$  (which is the inverse of friction) and  $\bar{\sigma}$ , but is not related to the maximal force. In the case of our simulations,  $\bar{v}_c = 6.19$  which is slightly overestimated. If one takes into account the full shape of  $\sigma(e)$ , made up of three parts, then the turnpike theory allows us to derive a limiting problem: the velocity curve is made up of three parts. We will find

$$v(s) = \begin{cases} v_0 e^{-t/\tau} + \left(v_{\max} + \frac{t}{t_1}(\bar{v} - v_{\max})\right)(1 - e^{-t/\tau}) & \text{if } 0 \leq t \leq t_1, \\ \bar{v} & \text{if } t_1 \leq t \leq t_f - t_2, \\ \bar{v} e^{\frac{\bar{\sigma} - \sigma_f}{e^0 \gamma_1}(t - (t_f - t_2))} & \text{if } t_f - t_2 \leq t \leq t_f. \end{cases} \quad (12)$$

Therefore, the curve goes from the initial velocity  $v_0$  to a maximum velocity  $v_{\max}$ , then down to  $\bar{v}$ , which is the turnpike value. At the end of the race, the velocity increases to the final velocity  $v_f = \bar{v} e^{\frac{\bar{\sigma} - \sigma_f}{e^0 \gamma_1}}$ . This type of curve is quite consistent with velocity curves in the sports literature, see for instance [10, 13], and with our simulations illustrated in Figure 1. We notice that the sprint

velocity at the end of the race is higher for large values of  $\bar{\sigma}$  that is  $\dot{V}O_{2\max}$ , because  $\sigma_f$  is always around 10 to 20% smaller than  $\bar{\sigma}$ . It is also higher when  $\gamma_1$  is small, that is when the runner can maintain his  $\dot{V}O_{2\max}$  for small oxygen deficit or small anaeroby energy left.

We will also find that

$$t_1 \simeq \frac{\gamma_2}{\frac{\bar{v}^2}{\tau} - \sigma_r}, \quad t_f - t_2 \simeq \frac{e^0 \gamma_1}{\frac{\bar{v}^2}{\tau} - \sigma_f}$$

so that  $t_1$  increases with  $\gamma_2$ , while  $t_f - t_2$  increases with  $\gamma_1$ . We note that this estimate for  $t_f - t_2$  yields 35.96 seconds, and 16.95 for  $t_1$ , which is very good.

If the track goes uphill or downhill with a constant rate  $\lambda$ , then the velocity equation is modified to take into account the slope [1, 2] and in the turnpike estimate, this becomes

$$v = \tau f - g\tau\lambda, \quad (13)$$

where  $\lambda$  is positive when the track goes up. If the slope is constant for the whole race, the turnpike estimate can be computed. Considering the energy  $e^0$  is the same, the decrease in velocity due to a slope is

$$\Delta v = -g\lambda\tau \left( \frac{1}{2} + \frac{1}{\sqrt{\frac{\bar{v}}{4} + \bar{\sigma} \left(\frac{d}{e^0}\right)^2}} \right). \quad (14)$$

But if the slope is constant for a small part of the race, then the variation of velocity cannot be computed locally because the whole mean velocity of the race is influenced by a local change of slope as we will see in the last part of the paper.

### 3.1 Turnpike estimate

Assume the turnpike holds for the whole race, then we have  $v = \bar{v}$ ,  $f = \bar{f}$ ,  $u = \bar{u}$  and

$$\bar{f} = \frac{\bar{v}}{\tau}, \quad \bar{u} = \frac{\bar{f}}{F_{\max} - \bar{f}}. \quad (15)$$

But we have to integrate

$$\bar{e}'(t) = \sigma(\bar{e}(t)) - \frac{\bar{v}^2}{\tau}, \quad \bar{e}(0) = e^0$$

using the three parts for  $\sigma$ . In the central part of the race, which is of duration  $\Delta t_{\text{turn}}$ ,  $\sigma$  is constant and we find

$$e^0(1 - \gamma_1) - \gamma_2 = \Delta t_{\text{turn}} \left( \frac{\bar{v}^2}{\tau} - \bar{\sigma} \right).$$

The initial and final parts of the race produce exponential terms namely, if we define the times  $t_1$  and  $t_2$  such that  $e(t_2) = \gamma_1 e^0$  and  $e(t_1) = e^0 - \gamma_2$ , then

$$\frac{\bar{\sigma} - \sigma_r}{\frac{\bar{v}^2}{\tau} - \sigma_r} = 1 - e^{-\frac{(\bar{\sigma} - \sigma_r)t_1}{\gamma_2}} \quad \text{and} \quad \frac{\bar{\sigma} - \sigma_f}{\frac{\bar{v}^2}{\tau} - \sigma_f} = 1 - e^{-\frac{(\bar{\sigma} - \sigma_f)(\bar{t} - t_2)}{e^0 \gamma_1}} \quad (16)$$

where  $\bar{t}$  is the final time of the turnpike trajectory defined by  $\bar{e}(\bar{t}) = 0$ . If the initial and final parts are not too long, then (16) can be approximated by

$$t_1 \simeq \frac{\gamma_2}{\frac{\bar{v}^2}{\tau} - \sigma_r} \quad \text{and} \quad \bar{t} - t_2 \simeq \frac{e^0 \gamma_1}{\frac{\bar{v}^2}{\tau} - \sigma_f}. \quad (17)$$

Therefore, for the total distance  $d$ , we find

$$\bar{t} = \frac{d}{\bar{v}} = \frac{e^0(1 - \gamma_1) - \gamma_2}{\frac{\bar{v}^2}{\tau} - \bar{\sigma}} - \frac{\gamma_2}{\bar{\sigma} - \sigma_r} \ln \left( 1 - \frac{\bar{\sigma} - \sigma_r}{\frac{\bar{v}^2}{\tau} - \sigma_r} \right) - \frac{e^0 \gamma_1}{\bar{\sigma} - \sigma_f} \ln \left( 1 - \frac{\bar{\sigma} - \sigma_f}{\frac{\bar{v}^2}{\tau} - \sigma_f} \right). \quad (18)$$

This is complicated to solve in  $\bar{v}$  but can be safely approximated by

$$d \simeq \frac{\bar{v} e^0 \gamma_1}{\frac{\bar{v}^2}{\tau} - \sigma_f} + \bar{v} \frac{e^0(1 - \gamma_1) - \gamma_2}{\frac{\bar{v}^2}{\tau} - \bar{\sigma}} + \frac{\bar{v} \gamma_2}{\frac{\bar{v}^2}{\tau} - \sigma_r}. \quad (19)$$

The advantage of this equation is that it allows to compute  $\bar{v}$ , and therefore  $\bar{f}$ ,  $\bar{t}$ ,  $t_1$ ,  $t_2$ . Numerically, this yields  $\bar{v} = 6.06$ ,  $\bar{t} = 247.55$ . The intermediate times can be computed from (17). These approximations are very good with respect to our simulations. The final stage is 35.96 s and initial 16.95 s from (17). This also yields the distances of each part by multiplying by  $\bar{v}$ . Note that they do not depend on  $\alpha$  nor on  $F_{\max}$ .

Note that this turnpike calculation can be used the other way round: if one knows the mean velocity,  $d$ ,  $\tau$  and  $\bar{\sigma}$ , it yields an estimate of the energy  $e^0$  used while running, as well as the aerobic part which is  $\bar{\sigma}d/\bar{v}$ . This energy is expressed in J/kg because everything is by unit of mass.

The next step is to identify reduced problems for the beginning (interval  $(0, t_1)$ ) and end of the race (interval  $(0, \bar{t} - t_2)$ ). The two are not totally equivalent since at the beginning we have an initial constraint for  $v$  and therefore the motor control does not seem to play such a role to reach the turnpike, whereas on the final part, the final velocity is free and therefore, it will help to use the adjoint equations given by the Pontryagin maximum principle.

### 3.2 Estimates for the beginning of the race

The full problem is to consider the equations for  $v$ ,  $f$ ,  $e$  with boundary conditions

$$v(0) = v^0, \quad v(t_1) = \bar{v}, \quad f(t_1) = \bar{f}, \quad e(0) = e^0, \quad e(t_1) = e^0 - \gamma_2. \quad (20)$$

Here  $f(0)$  is free and we minimize (9). An issue is to determine how this problem is sensitive to  $F_{\max}$  and  $\alpha$ .

Our aim is to reduce even further this problem.

The first idea is to estimate  $t_1$ . It can be estimated from the turnpike estimate of the previous section, which provides a reasonable estimate. It can also be estimated from the fact that we estimate the final stage of the race and then use that the total distance is prescribed and the middle distance is estimated from the turnpike. Now let us assume  $t_1$  is prescribed. If we fix the interval  $(0, t_1)$ , we have the equations for  $v$  and  $f$  with

$$v(0) = v^0, \quad v(t_1) = \bar{v}, \quad f(0) = f^0, \quad f(t_1) = \bar{f}. \quad (21)$$

Here  $f^0$  is unknown and we want to minimize the motor control only.

Given the behaviour, it is not too wrong to assume that  $f$  is linear decreasing (or exponentially decreasing), forget the equation for  $f$  and just solve the  $v$  equation with this assumption and then compute

$$\int_0^{t_1} f(t)v(t) dt = \int_0^{t_1} \left( \frac{de}{dt} - \sigma(e) \right) dt.$$



In this regime,  $\sigma$  is linear, and this equation can be integrated explicitly. Indeed let  $A = \frac{\bar{\sigma} - \sigma_r}{\gamma_2}$ , then

$$-\gamma_2 = \frac{\sigma_r \gamma_2}{\bar{\sigma} - \sigma_r} (1 - e^{-At_1}) - e^{-At_1} \int_0^{t_1} f v e^{At} dt. \quad (22)$$

A first estimate can assume  $v = \tau f$  in this integral and that  $f$  is linear, that is  $f(t) = f^0 + t(\bar{f} - f^0)/t_1$ , then we find a first approximation of  $f^0$ :

$$-\gamma_2 = \frac{\sigma_r \gamma_2}{\bar{\sigma} - \sigma_r} (1 - e^{-At_1}) - e^{-At_1} \int_0^{t_1} \tau^2 \left( f^0 + t \frac{\bar{f} - f^0}{t_1} \right)^2 e^{At} dt. \quad (23)$$

This can be integrated analytically or numerically to determine  $f^0$ . In our case,  $f^0 = 7.95$  and the maximal velocity  $v_{\max}$  is roughly  $\tau f^0$ . This yields (12). We recall that this problem is with a fixed prescribed time  $t_1$  on which we will come back later.

### 3.3 End of the race

One option is to keep the equations for  $v$ ,  $f$ ,  $e$  with initial values

$$v(t_2) = \bar{v}, \quad f(t_2) = \bar{f}, \quad e(t_2) = e^0 - \gamma_2, \quad (24)$$

and minimize (9). This yields a nice simulation in Figure 2. We observe that  $f$  and  $v\tau$  are very close, as expected.

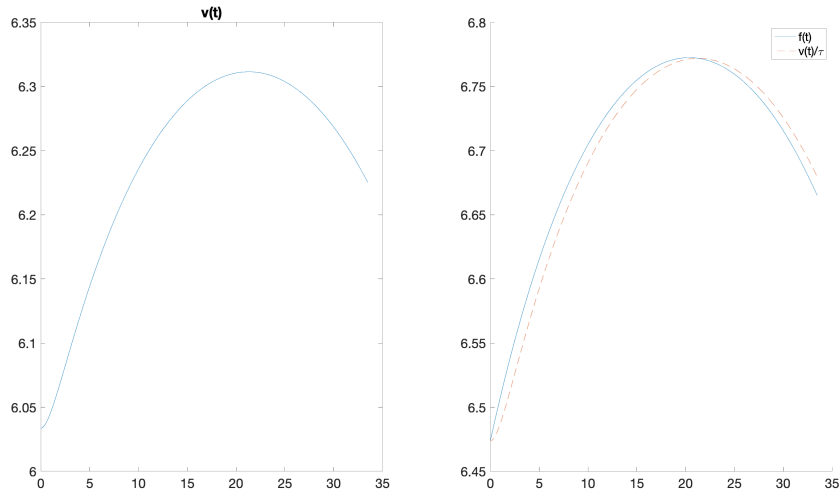


Figure 2: Velocity and force for the reduced problem modelling the end of the race. The force is compare to the value  $v/\tau$ .

In the following, we will assume  $v \simeq f\tau$ , which removes an equation. Nevertheless, we have to keep both the minimization on time and motor control. In this case, we have a free interval that we will need to optimize. The value of the size of the interval can either come from the turnpike estimate or, from a new estimate that we explain below. Note that  $t_2 - t_1$  is totally determined

by the turnpike equation in the sense

$$t_2 - t_1 = \frac{e^0(1 - \gamma_1) - \gamma_2}{\frac{v^2}{\tau} - \bar{\sigma}}.$$

In our case, the value is 194.64. Equivalently, the distance or duration of turnpike is prescribed.

Let us apply the Pontryagin maximum principle (see [18, 20, 24]), which states that any optimal solution  $(x(\cdot), v(\cdot), f(\cdot), e(\cdot), u(\cdot))$  of the optimal control under consideration problem must be the projection onto the state space of an extremal, consisting of the optimal solution and of an adjoint vector, or co-state,  $(p_x(\cdot), p_v(\cdot), p_f(\cdot), p_e(\cdot))$  satisfying Hamiltonian equations. In our case, the Hamiltonian of the problem is

$$H = p_x v + p_v \left( -\frac{v}{\tau} + f \right) + p_e (\sigma(e) - f v) + p_f \gamma (u(F_{\max} - f) - f) + p^0 + p^0 \frac{\alpha}{2} u^2$$

and the adjoint equations are

$$\begin{aligned} \dot{p}_x &= 0 \\ \dot{p}_v &= -p_x + \frac{1}{\tau} p_v + p_e f \\ \dot{p}_e &= -p_e \sigma'(e) \\ \dot{p}_f &= -p_v + p_e v + p_f \gamma (u + 1) \end{aligned}$$

Assuming that there is no abnormal extremal (this is verified on our numerical simulations, we take  $p^0 = -1$  and the condition  $\frac{\partial H}{\partial u}$  of the Pontryagin maximum principle yields

$$u = p_f \frac{\gamma}{\alpha} (F_{\max} - f).$$

The transversality conditions are  $p_v(t_f) = 0$ ,  $p_f(t_2) = 0$ ,  $p_f(t_f) = 0$ ,  $H(t_f) = 0$  (in particular,  $u(t_f) = 0$ ).

Like for the state variables, we denote by  $(\bar{p}_v, \bar{p}_f, \bar{p}_e)$ , with an upper bar, the adjoint variable associated with the turnpike values. Indeed, we recall that, with respect to the large literature on the turnpike theory, one of the main novelties of [25] (see also [23]) is to have shown that not only the optimal state and control remain essentially close to a steady-state, but also the adjoint vector coming from the application of the Pontryagin maximum principle. This fact is used next. We recall the turnpike values from (15) and for  $(\bar{p}_v, \bar{p}_f, \bar{p}_e)$ , the following relations hold:

$$\begin{aligned} -p_x + \frac{1}{\tau} \bar{p}_v + \bar{p}_e \bar{f} &= 0 \\ -\bar{p}_v + \bar{p}_e \bar{v} + \bar{p}_f \gamma (\bar{u} + 1) &= 0. \end{aligned}$$

In the following, in order to estimate the duration of this last step, we are going to make two assumptions: firstly, that  $\dot{v}$  is negligible in front of  $v/\tau$  so that  $f(t) \simeq v(t)/\tau$ , and secondly that on the singular arc,  $p_f$  and  $\dot{p}_f$  remain small so that  $p_v = p_e v$ . Moreover, since  $p_x$  is constant, we find

$$p_x = 2 \frac{\bar{p}_e \bar{v}}{\tau} = 2 \frac{p_e v}{\tau}. \quad (25)$$

On the final phase, since  $\dot{p}_e = -p_e \sigma'(e)$ , and we know the initial value  $\bar{p}_e$ , we have an explicit expression for

$$p_e(t) = \bar{p}_e e^{-\frac{\bar{\sigma} - \sigma_f}{e^0 \gamma_1} (t - t_2)}.$$

Therefore, from (25), we find

$$v(t) = \bar{v} e^{\frac{\bar{\sigma} - \sigma_f}{e^0 \gamma_1} (t - t_2)}. \quad (26)$$

This yields  $f(t)$  since  $f(t) \simeq v(t)/\tau$ . Then we need to integrate the energy equation using the explicit expressions for  $v$  and  $f$  to find with  $t_{\text{end}} = t_f - t_2$  and  $A = \frac{\bar{\sigma} - \sigma_f}{e^0 \gamma_1} \simeq 0.0028$ ,

$$\sigma_f e^{-A t_{\text{end}}} + \frac{1}{\tau} \bar{v}^2 e^{A t_{\text{end}}} = A \gamma_1 e^0 + \sigma_f + \frac{1}{\tau}.$$

Numerically, we find  $t_{\text{end}} = 31.48$ . We notice that this estimate added to the turnpike estimate of 194.64 and the beginning of the race of 16.95 yields a final time of 243, which is very close to our simulation of 244 seconds.

In the real life,  $v$  has a short decrease at the very end of the race where the assumption  $p_v = p_e v$  is no longer true. But this changes very slightly the estimate on  $t_f - t_2$  and is not meaningful for a runner, so we can safely ignore it for our approximations.

Our distance is made up of 3 parts: the turnpike distance which is totally determined by  $\gamma_1$  and  $\gamma_2$  and the distance run in the initial and final parts. Of course, since the sum is prescribed, only one of the two is free. So for instance, in the final phase if we determine the duration of this final phase by some estimate like above, the initial phase has to match the total distance, but nevertheless is safely estimated from the turnpike.

## 4 Comparison with a real 1500 m

The runners' oxygen uptake was recorded in [14] by means of a telemetric gas exchange system. This allowed to observe that the  $\dot{V}O_2$  reached a peak in around 450m from start, with a significant decrease between 450 and 550 meters. Then the  $\dot{V}O_2$  remained constant for 800 meters, before a decrease of 10% at the end of the race. To match more precisely the  $\dot{V}O_2$  curve of [14], we add an extra piece to the curve of  $\sigma$ , before the long mean value  $\bar{\sigma}$ : after the initial increase, there is a local maximum before decreasing to the constant turnpike value:

$$\sigma(e) = \begin{cases} \bar{\sigma} \frac{e}{e^0 \gamma_1} + \sigma_f \left(1 - \frac{e}{e^0 \gamma_1}\right) & \text{if } \frac{e}{e^0} < \gamma_1 \\ \bar{\sigma} & \text{if } \gamma_1 \leq \frac{e}{e^0} \leq \gamma_+ \\ \bar{\sigma} + 0.8 \frac{e - \gamma_+ e^0}{e^0 - \gamma_2 - \gamma_+ e^0} & \text{if } \frac{e}{e^0} \geq \gamma_+ \text{ and } e^0 - e > \gamma_2 \\ (\bar{\sigma} + 0.8 - \sigma_r) \frac{e^0 - e}{\gamma_2} + \sigma_r & \text{if } e^0 - e < \gamma_2 \end{cases} \quad (27)$$

We take roughly the same parameters as before except for  $\gamma_2 = 2000$  and  $\gamma_+ = 1 - \gamma_2/e^0 - 400/e^0$ . The others are  $\sigma_r = 6$ ,  $\sigma_f = 20$ ,  $\bar{\sigma} = 22$ ,  $\gamma_1 = 0.15$ ,  $F_{\text{max}} = 8$ ,  $\tau = 1.032$ ,  $e^0 = 4651$ ,  $\gamma = 0.0025$ ,  $v_0 = 1$ . Then we see in Figure 3 that the velocity has a local minimum in the region where  $\sigma$  has a local maximum, which matches exactly the velocity profile in [14]. Small variations in  $\sigma$  always provide variations in the velocity profile with the opposite sense.

It is well known that successful athletes in a race are not so much those who speed up a lot at the end but those who avoid slowing down too much. We have noticed that if the maximal force at the beginning of the race is too high, then the velocity tends to fall down at the end of the race, leading to a bad performance. For a final in a world competition, it is observed in [13]

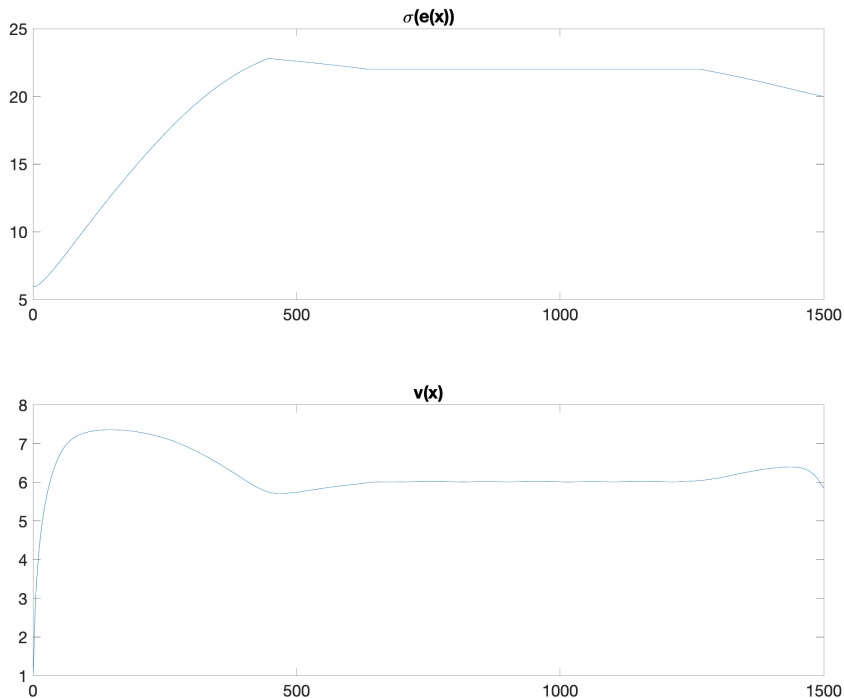


Figure 3: Modified  $\sigma$  in four pieces and optimal velocity vs distance for a 1500 m

that the best strategy is J-shaped, which means reaching maximal speed at the end of the race. But this is not available to all athletes. The runners profile of these simulations are not world champions but only successful in French regional races. Therefore, their pacing strategy is either U-shaped (the start and the finish are quicker) or reverse J-shaped (greater starting pace). This is very dependent on the relative values of running economy  $\tau$ , anaerobic energy  $e^0$  and profile of  $\dot{V}O_2$ . Moreover, top runners use pace variation according to bends as their winning tactics [5], but this is not active on the level of runners we have analyzed in this paper.

## 5 Running uphill or downhill

If we assume a slope  $\beta(x)$  which is constant equal to  $\lambda$ , the new turnpike estimate is

$$\bar{v} = \frac{(e^0 - dg\lambda)\tau}{2d} + \sqrt{\bar{\sigma}\tau + \left(\frac{(e^0 - dg\lambda)\tau}{2d}\right)^2}.$$

If the slope is small, one can make an asymptotic expansion in terms of  $\lambda$  to find the difference in velocity (14).

Nevertheless, because the energy is involved, a change of slope, even locally implies a change of the turnpike velocity on the whole race. A slope is never a local effect. We have chosen to put slopes

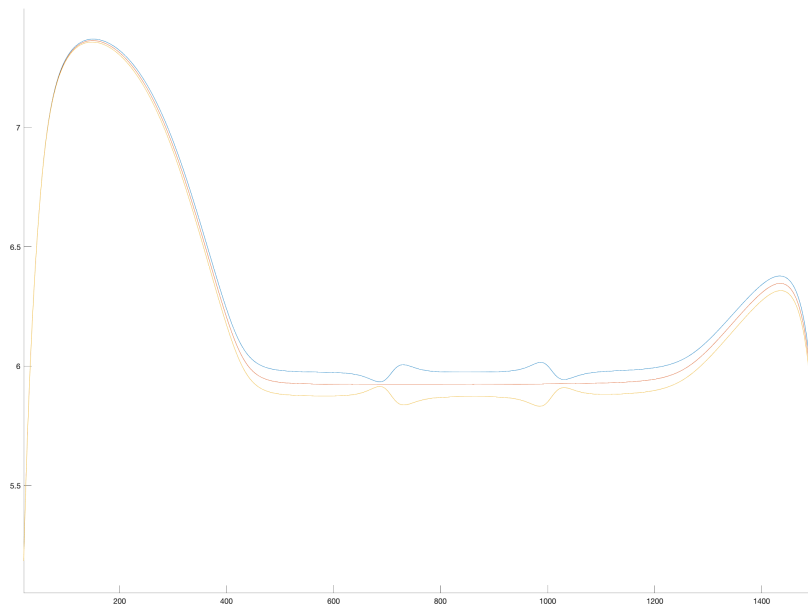


Figure 4: Velocity vs distance for a 1500 m, on a flat track (red) and with a track with a 2% slope for 300 m (orange) or 2% ramp for 300 m (blue) between 700 and 1000 m.

and ramps of 2% for 300 m. We see in Figure 4 that without slope we have an intermediate turnpike value, but with a slope or ramp even only for 300 m, the whole turnpike velocity is modified.

To illustrate further the slope effect, we have put a periodic slope and ramp of 200 m between 300 m and 1200 m. We use the same parameters as in the previous section. We see in Figure 5 that the turnpike velocity is affected. When going down, a runner speeds at the end of the ramp, but his velocity has a local maximum at the middle of the ramp. Similarly, it has a local minimum at the middle of the slope. The variations in velocity are very small since they are of order of a few percents. But this allows to understand that slopes and ramps are not local perturbations on the pacing profile.

## 6 Conclusion

We have provided a model for pace optimization. This involves a control problem in order to use the maximal available propulsive force and energy to produce the optimal running strategy and minimize the time to run and the motor control. For sufficiently long races (above 1500 m), the optimal strategy is well approximated by a turnpike problem that we describe. Simplified estimates for the peak velocity and velocity profiles related to aerobic, anaerobic energy and effect of the motor control are obtained and fit the simulations. The effect of the parameters and slope and ramps are analyzed. The potential applications of this turnpike theory would be to derive a simpler model for pacing strategy that could be encompassed in a running app.

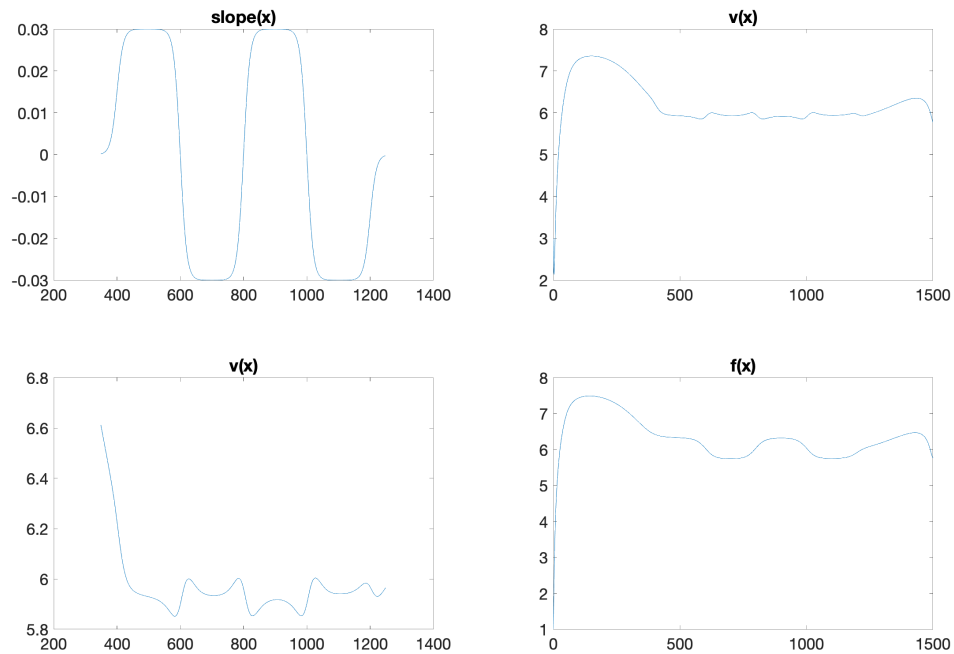


Figure 5: Slope, velocity, zoom on the velocity and force for a 1500 m with slopes and ramps of 3% for 200 m.

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