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Path decompositions of perturbed reflected Brownian motions

by

Elie Aïdékon¹, Yueyun Hu², and Zhan Shi³

*In honour of Professor Ron Doney
on the occasion of his 80th birthday*

Summary. We are interested in path decompositions of a perturbed reflecting Brownian motion (PRBM) at the hitting times and at the minimum. Our study relies on the loop soups developed by Lawler and Werner [10] and Le Jan [13]-[14], in particular on a result discovered by Lupu [15] identifying the law of the excursions of the PRBM above its past minimum with the loop measure of Brownian bridges.

Keywords. Perturbed reflecting Brownian motion, path decomposition, Brownian loop soup, Poisson–Dirichlet distribution.

2010 Mathematics Subject Classification. 60J65.

1 Introduction

Let $(B_t, t \geq 0)$ be a standard one-dimensional Brownian motion. Let

$$\mathfrak{L}_t := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{\{0 < B_s \leq \varepsilon\}} ds, \quad \text{a.s.},$$

be the local time at time t and position 0. We take a continuous version of $(\mathfrak{L}_t, t \geq 0)$. Let $\mu \in \mathbb{R} \setminus \{0\}$ be a fixed parameter. Consider the perturbed reflecting Brownian motion (PRBM)

$$(1.1) \quad X_t := |B_t| - \mu \mathfrak{L}_t, \quad t \geq 0.$$

The PRBM family contains two important special members: Brownian motion ($\mu = 1$; this is seen using Lévy’s identity), and the three-dimensional Bessel process ($\mu = -1$; seen by means of Lévy’s and Pitman’s identities).

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The PRBM, sometimes also referred to as the μ -process and appearing in the literature as the limiting process in the winding problem for three-dimensional Brownian motion around lines (Le Gall and Yor [12]), turns out to have remarkable properties such as the Ray–Knight theorems (Le Gall and Yor [11], Werner [24], Perman [18], Perman and Werner [19]), and Lévy’s arc sine law (Petit [20], Carmona, Petit and Yor [4]). The process can also be viewed as (non-reflecting) Brownian motion perturbed by its one-sided maximum (Davis [6], Perman and Werner [19], Chaumont and Doney [5]). As explained on page 100 of Yor [25], these simply formulated and beautiful results were proved for the PRBM because of the scaling property and the strong Markov property of $(|B|, \mathfrak{L})$ via excursion theory.

We study in this paper path decompositions of the PRBM. Apart from their own interests, these decompositions will be exploited in forthcoming work to construct the dual of general Jacobi stochastic flows, extending the work of Bertoin and Le Gall [2] who proved that the Jacobi flows of parameters $(0, 0)$ and $(2, 2)$ are dual with each other. These stochastic flows are connected to other important probabilistic objects in the study of population genetics such as flows of Fleming–Viot processes. Technically, our study of the PRBM often relies on the powerful tool of loop soups (Lawler and Werner [10], Le Jan [13]–[14]), and in particular, on a result discovered by Lupu [15] identifying the law of the excursions of the PRBM above its past minimum with the loop measure of Brownian bridges. The point of view via loop soups has two main advantages: (i) it allows us to rediscover easily some previously known results, and (ii) thanks to the nice independence structure in the Poisson point process representation of the loop measure, it helps to make arguments of conditioning rigorous.

Our path decompositions focus on two families of random times of a recurrent PRBM: first hitting times (Section 3), and times at which the PRBM reaches its past minimum (Section 4). To illustrate the kind of results we have obtained, let us state two examples. The first one, Theorem 3.2, yields in the special case $\mu = 1$ the classical Williams’ Brownian path decomposition theorem (Revuz and Yor [21], Theorem VII.4.9). The second, Theorem 4.3, in the special case $\mu = 1$, states as follows:

Theorem 4.3 (special case $\mu = 1$). *Considering B up to the first time that \mathfrak{L} reaches 1, we decompose the path at the minimum into the post- and time reversed pre- minimum processes. The two processes, if we glue their excursions below the absolute value of the minimum, are independent and distributed as X^1 and X^2 if we glue their excursions below the level $H^{1,2}$, where X^1 and X^2 are independent three-dimensional Bessel processes and $H^{1,2}$ is the last level at which the sum of the total local times of X^1 and X^2 equals 1.*

The rest of this paper is organized as follows.

- Section 2: we recall Lupu [15]’s connection between the PRBM and the Brownian loop soup, and some known results on the PRBM. We also study the minimums of the PRBM considered up to its inverse local times (the process J defined in (2.4)). The main (new) result in this section is Proposition 2.6, a description of the jumping times of J ;
- Section 3: we study the path decomposition at the hitting time of the PRBM. We prove that conditioned on its minimum, the PRBM can be split into four independent

processes (see Figure 1) and describe their laws in Theorems 3.2 and 3.3;

- Section 4: we study the path decomposition at the minimum of the PRBM considered up to its inverse local time. Theorem 4.3 deals with the recurrent case and describes the laws of the post- and time reversed pre- minimum processes. A similar decomposition is obtained in Proposition 4.4 for the transient case (see Figure 2);

- Section 5: we extend the perturbed Bessel process studied in Doney, Warren and Yor [7] to the perturbed Bessel process with a positive local time at 0. The main result in this section (Theorem 5.3) gives an extension of Theorem 2.2 of Doney, Warren and Yor [7].

2 Preliminaries

This section is divided into three subsections. We recall in Section 2.1 Lupu [15]’s description (Proposition 2.2) on the excursions of the PRBM above its past minimum in terms of Brownian loop soup, and we collect the intensities of various Poisson point processes in Lemma 2.3. In Section 2.2, we study the minimum process $J(x)$ defined in (2.4) and describe the jump times of $J(\cdot)$ by means of Poisson–Dirichlet distributions in Proposition 2.6. Finally in Section 2.3 we recall some known results on the PRBM. We also introduce some notations (in particular Notations 2.1 and 2.11) which are used throughout the whole paper.

2.1 The Brownian loop soup

Lupu [15] showed a connection between perturbed reflected Brownian motions and the Brownian loop soup. We rely on [15] and review this connection in this subsection. A rooted loop is a continuous finite path $(\gamma(t), 0 \leq t \leq T(\gamma))$, such that $\gamma(0) = \gamma(T(\gamma))$ (Section 3.1, p. 29). On the space of rooted loops, one defines the measure (Definition 3.8, p. 38)

$$\mu_{\text{loop}}(d\gamma) := \int_{t>0} \int_{x \in \mathbb{R}} P_{x,x}^t(d\gamma) p_t(x,x) dx \frac{dt}{t},$$

where $P_{x,x}^t$ is the distribution of the Brownian bridge of length t from x to x , and $p_t(x,x)$ is the heat kernel $p_t(x,x) = \frac{1}{\sqrt{2\pi t}}$. An unrooted loop is the equivalence class of all loops obtained from one another by time-shift, and μ_{loop}^* denotes the projection of μ_{loop} on the space of unrooted loops. For any fixed $\beta > 0$, the Brownian loop soup of intensity measure β is the Poisson point process on the space of unrooted loops with intensity measure given by $\beta \mu_{\text{loop}}^*$ (Definition 4.2, p. 60). We denote it by \mathcal{L}_β .

For any real q , we let $\gamma - q$ denote the loop $(\gamma(t) - q, 0 \leq t \leq T(\gamma))$. We write $\min \gamma$, resp. $\max \gamma$ for the minimum, resp. maximum of γ . If γ denotes a loop, the loop γ rooted at its minimum is the rooted loop obtained by shifting the starting time of the loop to the hitting time of $\min \gamma$. Similarly for the loop γ rooted at its maximum. By an abuse of notation, we will often write γ for its range. For example $0 \in \gamma$ means that γ visits the point 0.

Similarly to Lupu [15], Section 5.2, define

$$\mathcal{Q}_\beta^\uparrow := \{\min \gamma, \gamma \in \mathcal{L}_\beta\}, \quad \mathcal{Q}_\beta^\downarrow := \{\max \gamma, \gamma \in \mathcal{L}_\beta\}.$$

For any $q \in \mathcal{Q}_\beta^\uparrow$ and $\gamma \in \mathcal{L}_\beta$ such that $\min \gamma = q$, define \mathbf{e}_q^\uparrow as the loop $\gamma - q$ rooted at its minimum. It is an excursion above 0. Define similarly, for any $q \in \mathcal{Q}_\beta^\downarrow$, \mathbf{e}_q^\downarrow as the excursion below 0 given by $q - \gamma$ rooted at its maximum. The point measure $\{(q, \mathbf{e}_q^\downarrow), q \in \mathcal{Q}_\beta^\downarrow\}$ has the same distribution as $\{-(q, \mathbf{e}_q^\uparrow), q \in \mathcal{Q}_\beta^\uparrow\}$.

On the other hand, considering a process X , denote by $\{(q, \mathbf{e}_{X,q}^\uparrow), q \in \mathcal{Q}_X^\uparrow\}$ the excursions of the process $(I_t, X_t - I_t)$ away from $\mathbb{R} \times \{0\}$ (q is seen as a real number), where $I_t := \inf_{0 \leq s \leq t} X_s$ for any $t \geq 0$. Similarly, we denote by $\{(q, \mathbf{e}_{X,q}^\downarrow), q \in \mathcal{Q}_X^\downarrow\}$ the excursions of the process $(S_t, S_t - X_t)$ away from $\mathbb{R} \times \{0\}$ with $S_t := \sup_{0 \leq s \leq t} X_s$.

Notation 2.1. For $\delta > 0$, let \mathbb{P}^δ (resp. $\mathbb{P}^{(-\delta)}$) be the probability measure under which $(X_t)_{t \geq 0}$ is distributed as the PRBM $(|B_t| - \mu \mathfrak{L}_t)_{t \geq 0}$ defined in (1.1) with $\mu = \frac{2}{\delta}$ (resp. $\mu = -\frac{2}{\delta}$). Furthermore, we let $\tilde{\mathbb{P}}^\delta$ (resp. $\tilde{\mathbb{P}}^{(-\delta)}$) be the probability measure under which $(X_t)_{t \geq 0}$ is distributed as $(-X_t)_{t \geq 0}$ under \mathbb{P}^δ (resp. under $\mathbb{P}^{(-\delta)}$).

Note that under \mathbb{P}^δ , $(X_t)_{t \geq 0}$ is recurrent whereas under $\mathbb{P}^{(-\delta)}$, $\lim_{t \rightarrow \infty} X_t = +\infty$ a.s.

The following proposition is for example Proposition 5.2 of [15]. One can also see it from [11] or [1] (together with Proposition 3.18 of [15]).

Proposition 2.2 (Lupu [15]). *Let $\delta > 0$. The point measure $\{(q, \mathbf{e}_{X,q}^\uparrow), q \in \mathcal{Q}_X^\uparrow\}$ is distributed under \mathbb{P}^δ , respectively $\mathbb{P}^{(-\delta)}$, as $\{(q, \mathbf{e}_q^\uparrow), q \in \mathcal{Q}_{\frac{\delta}{2}}^\uparrow \cap (-\infty, 0)\}$, resp. $\{(q, \mathbf{e}_q^\uparrow), q \in \mathcal{Q}_{\frac{\delta}{2}}^\uparrow \cap (0, \infty)\}$. By symmetry, the point measure $\{(q, \mathbf{e}_{X,q}^\downarrow), q \in \mathcal{Q}_X^\downarrow\}$ is distributed under $\tilde{\mathbb{P}}^\delta$, respectively $\tilde{\mathbb{P}}^{(-\delta)}$, as $\{(q, \mathbf{e}_q^\downarrow), q \in \mathcal{Q}_{\frac{\delta}{2}}^\downarrow \cap (0, \infty)\}$, resp. $\{(q, \mathbf{e}_q^\downarrow), q \in \mathcal{Q}_{\frac{\delta}{2}}^\downarrow \cap (-\infty, 0)\}$.*

Actually, Proposition 5.2 [15] states the previous proposition in a slightly different way. In the same way that standard Brownian motion can be constructed from its excursions away from 0, Lupu shows that one can construct the perturbed Brownian motions from the Brownian loop soup by “gluing” the loops of the Brownian loop soup rooted at their minimum and ordered by decreasing minima.

We close this section by collecting the intensities of various Poisson point processes. It comes from computations of [15].

Denote by \mathbf{n} the Itô measure on Brownian excursions and \mathbf{n}^+ (resp. \mathbf{n}^-) the restriction of \mathbf{n} on positive excursions (resp. negative excursions). For any loop γ , let $\ell_\gamma^0 := \lim_{\varepsilon \rightarrow 0} \int_0^{T(\gamma)} 1_{\{0 < \gamma(t) < \varepsilon\}} dt$ be its (total) local time at 0.

In the following lemma, we identify a Poisson point process with its atoms.

Lemma 2.3. *Let $\delta > 0$.*

- (i) *The collection $\{(q, \mathbf{e}_q^\uparrow), q \in \mathcal{Q}_{\frac{\delta}{2}}^\uparrow\}$ is a Poisson point process of intensity measure $\delta da \otimes \mathbf{n}^+(d\mathbf{e})$.*

- (ii) The collection $\{(q, \mathbf{e}_q^\downarrow), q \in \mathcal{Q}_{\frac{\delta}{2}}^\downarrow \text{ such that } \mathbf{e}_q^\downarrow \subset (0, \infty)\}$ is a Poisson point process of intensity measure $\delta 1_{\{a>0\}} da \otimes 1_{\{\min \mathbf{e} > -a\}} \mathbf{n}^-(d\mathbf{e})$.
- (iii) The collection $\{\min \gamma, \gamma \in \mathcal{L}_{\frac{\delta}{2}} \text{ such that } 0 \in \gamma\}$ is a Poisson point process of intensity measure $\frac{\delta}{2|a|} 1_{\{a<0\}} da$.
- (iv) Let $m > 0$. Then the collection $\{\ell_\gamma^0, \gamma \in \mathcal{L}_{\frac{\delta}{2}} \text{ such that } \min \gamma \in [-m, 0], 0 \in \gamma\}$ is a Poisson point process of intensity measure $1_{\{\ell > 0\}} \frac{\delta}{2\ell} e^{-\ell/2m} d\ell$.
- (v) The collection $\{(\ell_\gamma^0, \gamma), \gamma \in \mathcal{L}_{\frac{\delta}{2}} \text{ such that } 0 \in \gamma\}$ is a Poisson point process of intensity measure $\frac{\delta}{2} 1_{\{\ell > 0\}} \frac{d\ell}{\ell} \mathbb{P}((B_t, 0 \leq t \leq \tau_\ell^B) \in d\gamma)$, where $\tau_\ell^B := \inf\{s > 0 : \mathfrak{L}_s > \ell\}$ denotes the inverse of the Brownian local time.

Proof. Items (i) and (ii) come from Proposition 3.18, p. 45 of [15]. Item (iii) comes from (i) and the fact that $\mathbf{n}^+(r \in \mathbf{e}) = \frac{1}{2r}$ for any $r > 0$ where, here and in the sequel, \mathbf{e} denotes a Brownian excursion. We prove now (iv). The intensity measure is given by

$$\delta \int_{-m}^0 \mathbf{n}^+(\ell_\mathbf{e}^{|a|} \in d\ell, |a| \in \mathbf{e}) da$$

where $\ell_\mathbf{e}^r$ denotes the local time at r of the excursion \mathbf{e} . Under \mathbf{n}^+ , conditionally on $|a| \in \mathbf{e}$, the excursion after hitting $|a|$ is a Brownian motion killed at 0. Therefore

$$\mathbf{n}^+(\ell_\mathbf{e}^{|a|} \in d\ell \mid |a| \in \mathbf{e}) = \mathbb{P}_{|a|}(\mathfrak{L}_{T_0^B}^{|a|} \in d\ell) = \frac{1}{2|a|} e^{-\frac{\ell}{2|a|}} d\ell,$$

where under $\mathbb{P}_{|a|}$, the Brownian motion B starts at $|a|$ and $\mathfrak{L}_{T_0^B}^{|a|}$ denotes its local time at position $|a|$ up to $T_0^B := \inf\{t > 0 : B_t = 0\}$, and the last equality follows from the standard Brownian excursion theory. Hence the intensity measure is given by

$$\delta \int_{-m}^0 \frac{1}{4a^2} e^{-\frac{\ell}{2|a|}} d\ell da = \frac{\delta}{2\ell} e^{-\ell/2m} d\ell.$$

Finally, (v) comes from Corollary 3.12, equation (3.3.5) p. 39 of [15]. \square

2.2 The Poisson–Dirichlet distribution

For a vector $\mathcal{D} = (D_1, D_2, \dots)$ and a real r , we denote by $r\mathcal{D}$ the vector (rD_1, rD_2, \dots) . We recall that for $a, b > 0$, the density of the gamma(a, b) distribution is given by

$$\frac{1}{\Gamma(a)b^a} x^{a-1} e^{-\frac{x}{b}} 1_{\{x>0\}},$$

and the density of the beta(a, b) distribution is

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1} 1_{\{x \in (0,1)\}}.$$

We introduce the Poisson–Dirichlet distribution, relying on Perman, Pitman and Yor [17]. Let $\beta > 0$. Consider a Poisson point process of intensity measure $\frac{\beta}{x}e^{-x}1_{\{x>0\}}dx$ and denote by $\Delta_{(1)} \geq \Delta_{(2)} \geq \dots$ its atoms. We can see them also as the jump sizes, ordered decreasingly, of a gamma subordinator of parameters $(\beta, 1)$ up to time 1. The sum $T := \sum_{i \geq 1} \Delta_{(i)}$ has gamma($\beta, 1$) distribution. The random variable on the infinite simplex defined by

$$(P_{(1)}, P_{(2)}, \dots) := \left(\frac{\Delta_{(1)}}{T}, \frac{\Delta_{(2)}}{T}, \dots \right)$$

has the Poisson–Dirichlet distribution with parameter β ([9]), and is independent of T ([16], also Corollary 2.3 of [17]).

Consider a decreasingly ordered positive vector $(\xi_{(1)}, \xi_{(2)}, \dots)$ of finite sum $\sum_{i \geq 1} \xi_{(i)} < \infty$. A size-biased random permutation, denoted by (ξ_1, ξ_2, \dots) , is a permutation of $(\xi_{(1)}, \xi_{(2)}, \dots)$ such that, conditionally on $\xi_1 = \xi_{(i_1)}, \dots, \xi_j = \xi_{(i_j)}$, the term ξ_{j+1} is chosen to be $\xi_{(k)}$ for $k \notin \{i_1, \dots, i_j\}$ with probability $\frac{\xi_{(k)}}{\sum_{i \geq 1} \xi_{(i)} - (\xi_{i_1} + \dots + \xi_{i_j})}$. The indices $(i_j, j \geq 1)$ can be constructed by taking i.i.d. exponential random variables of parameter 1, denoted by $(\varepsilon_i, i \geq 1)$, and by ordering \mathbb{N} decreasingly with respect to the total order $k_1 \leq k_2$ if and only if $\xi_{(k_1)}/\varepsilon_{k_1} \leq \xi_{(k_2)}/\varepsilon_{k_2}$ (Lemma 4.4 of [17]). A result due to McCloskey [16] says that the $(P_i, i \geq 1)$ obtained from the $(P_{(1)}, P_{(2)}, \dots)$ by size-biased ordering can also be obtained via the stick-breaking construction:

$$P_i = (1 - U_i) \prod_{j=1}^{i-1} U_j$$

where $U_i, i \geq 1$ are i.i.d. with law beta($\beta, 1$). Let

$$(2.1) \quad \mathcal{D}_\beta := (D_1, D_2, \dots)$$

be the point measure in $[0, 1]$ defined by $D_i := \prod_{j=1}^i U_j$, for $i \geq 1$.

Lemma 2.4. *Let $m > 0, \beta > 0$ and Ξ_β be a Poisson point process of intensity measure $\frac{\beta}{a}1_{\{a>0\}}da$. We denote by $a_1^{(m)} > a_2^{(m)} > \dots$ the points of Ξ_β belonging to $[0, m]$. Then $(a_i^{(m)}, i \geq 1)$ is distributed as $m\mathcal{D}_\beta$.*

Proof. For $0 \leq a \leq 1$,

$$\mathbb{P}\left(\frac{1}{m}a_1^{(m)} \leq a\right) = \exp\left(-\int_{am}^m \frac{\beta}{x}dx\right) = a^\beta.$$

Therefore it is a beta($\beta, 1$) distribution. Conditionally on $\{a_1^{(m)} = a\}$, the law of Ξ_β restricted to $[0, a_1^{(m)})$ is the one of Ξ_β restricted to $[0, a)$. By iteration we get the Lemma. \square

Denote by $L(t, r), r \in \mathbb{R}$ and $t \geq 0$, the local time of X at time t and position r . Let

$$(2.2) \quad \tau_r(t) := \inf\{s \geq 0 : L(s, r) > t\},$$

be the inverse local time of X . Denote by

$$(2.3) \quad T_r := \inf\{t \geq 0 : X_t = r\}$$

the hitting time of r . We are interested in the process $(J(x))_{x \geq 0}$ defined as follows:

$$(2.4) \quad J(x) := \inf\{X_s, s \leq \tau_0(x)\}, \quad x \geq 0.$$

Observe that under \mathbb{P}^δ , J is a Markov process. It has been studied in Section 4 of [3]. The perturbed reflected Brownian motions are related to the Poisson–Dirichlet via the following proposition.

Proposition 2.5. *Let $\delta > 0$. Under \mathbb{P}^δ , the range $\{J(x), x > 0\}$ is distributed as $-\Xi_\beta$ with $\beta = \frac{\delta}{2}$. Consequently, for any $m > 0$, the range of J in $[-m, 0]$, ordered increasingly, is distributed as $-m\mathcal{D}_\beta$.*

Proof. Recall Proposition 2.2. Note that the range $\{J(x), x > 0\}$ is equal to $\{\min \gamma : 0 \in \gamma, \gamma \in \mathcal{L}_{\frac{\delta}{2}}\}$, the first statement is (iii) of Lemma 2.3. The second statement is Lemma 2.4. \square

Under \mathbb{P}^δ , for $m > 0$, let

$$(2.5) \quad T_{-m}^J := \inf\{x > 0 : J(x) \leq -m\}$$

be the first passage time of $-m$ by J . The main result in this subsection is the following description of the jump times of J before its first passage time of $-m$:

Proposition 2.6. *For $m > 0$, let $x_1^{(m)} > x_2^{(m)} > \dots$ denote the jumping times of J before time T_{-m}^J . Under \mathbb{P}^δ :*

- (i) ([25]) T_{-m}^J follows a gamma($\frac{\delta}{2}, 2m$) distribution. Consequently, for any $x > 0$, $\frac{-1}{J(x)}$ follows a gamma($\frac{\delta}{2}, \frac{2}{x}$) distribution.
- (ii) T_{-m}^J is independent of $\frac{1}{T_{-m}^J}(x_1^{(m)}, x_2^{(m)}, \dots)$.
- (iii) $\frac{1}{T_{-m}^J}(x_1^{(m)}, x_2^{(m)}, \dots)$ is distributed as \mathcal{D}_β with $\beta = \frac{\delta}{2}$.

Statement (i) of Proposition 2.6 is not new. It is contained in Proposition 9.1, Chapter 9.2, p. 123, of Yor [25]. For the sake of completeness we give here another proof of (i) based on Lemma 2.3.

Proof. (i) Let for $i \geq 1$, $d_i^{(m)} := (x_{i-1}^{(m)} - x_i^{(m)})/T_{-m}^J$ where $x_0^{(m)} := T_{-m}^J$. Proposition 2.2 says that $\{\ell_\gamma^0, \gamma \in \mathcal{L}_{\frac{\delta}{2}} \text{ such that } \min \gamma \in (-m, 0], 0 \in \gamma\}$ forms a Poisson point process of intensity measure $1_{\{\ell > 0\}} \frac{\delta}{2\ell} e^{-\ell/2m} d\ell$, whose atoms are exactly the (non-ordered) sequence $\{x_{i-1}^{(m)} - x_i^{(m)}, i \geq 1\}$. Note that $T_{-m}^J = L(T_{-m}, 0) = \sum_{\min \gamma \in (-m, 0], 0 \in \gamma} \ell_\gamma^0$.

Let $\{d_{(1)}^{(m)} > d_{(2)}^{(m)} > \dots\}$ be the ordered sequence of $\{d_i^{(m)}\}_{i \geq 1}$. Then, (iv) of Lemma 2.3 and the properties of the Poisson–Dirichlet distribution recalled at the beginning of the section imply that T_{-m}^J is independent of the point measure $\{d_{(1)}^{(m)}, d_{(2)}^{(m)}, \dots\}$ (not knowing a priori the order) and that $T_{-m}^J/2m$ follows a $\text{gamma}(\frac{\delta}{2}, 1)$ distribution. Also, the second statement of (i) comes from the observation that $\{J(x) > -m\} = \{T_{-m}^J > x\}$. This proves (i).

(ii) and (iii): It remains to show that the vector $(d_1^{(m)}, d_2^{(m)}, \dots)$ is a size-biased ordering of $\{d_{(1)}^{(m)}, d_{(2)}^{(m)}, \dots\}$, and that this size-biased ordering is still independent of T_{-m}^J .

To this end, denote by $\{-m_i, i \in \mathcal{I}\}$ the range of J . By Proposition 2.5, the point measure $\{\ln(m_i), i \in \mathcal{I}\}$ is a Poisson point process on \mathbb{R} of intensity measure $\frac{\delta}{2}$.

When J jumps at some $-m_i$, the time to jump at $-m_{i+1}$ is exponentially distributed with parameter $\mathbf{n}^-(\min \mathbf{e} \leq m_i) = \frac{1}{2m_i}$ (it is the local time at 0 of a Brownian motion when it hits level $-m_i$, by Markov property of the process (X, I) under \mathbb{P}^δ).

Denote by ε_i the exponential of parameter 1 obtained as the waiting time between jumps to $-m_i$ and to $-m_{i+1}$, divided by $2m_i$. Conditionally on $\{(m_i, i \in \mathcal{I})\}$, the random variables $(\varepsilon_i, i \in \mathcal{I})$ are i.i.d. and exponentially distributed with parameter 1. Then $\{(\ln(m_i), \varepsilon_i), i \in \mathcal{I}\}$ is a Poisson point process on $\mathbb{R} \times \mathbb{R}_+$ of intensity measure $\frac{\delta}{2} dt e^{-x} dx$. It is straightforward to check that $\{(\ln(2m_i \varepsilon_i), \varepsilon_i), i \in \mathcal{I}\}$ is still a Poisson point process with the same intensity measure.

Suppose that we enumerated the range of J with $\mathcal{I} = \mathbb{Z}$ so that $(-m_i, i \geq 1)$ are the atoms of the range in $(-m, 0)$ ranked increasingly. Then, $2m_i \varepsilon_i = T_{-m}^J d_i^{(m)} =: \xi_i$ for any $i \geq 1$. We deduce that, conditionally on $\{T_{-m}^J d_i^{(m)}, i \geq 1\}$, the vector $(\varepsilon_1, \varepsilon_2, \dots)$ consists of i.i.d. random variables exponentially distributed with parameter 1, and $i \leq j$ if and only if $\xi_i/\varepsilon_i \geq \xi_j/\varepsilon_j$. From the description of size-biased ordering at the beginning of this section, we conclude that the $(\xi_i, i \geq 1)$ are indeed size-biased ordered, hence also $(d_1^{(m)}, d_2^{(m)}, \dots)$. \square

Since $T_{-m}^J = L(T_{-m}, 0)$, the statement (i) of Proposition 2.6 says

$$(2.6) \quad L(T_{-m}, 0) \stackrel{(\text{law})}{=} \text{gamma}\left(\frac{\delta}{2}, 2m\right).$$

Corollary 2.7. *Let $\delta > 0$. Under \mathbb{P}^δ , the collection of jumping times of J is distributed as Ξ_β with $\beta = \frac{\delta}{2}$.*

Proof. From Proposition 2.6 and Lemma 2.4, we can couple the jumping times of J which are strictly smaller than the passage time of $-m$ with Ξ_β restricted to $[0, Z_m]$ where Z_m is $\text{gamma}(\frac{\delta}{2}, 2m)$ distributed, independent of Ξ_β . Letting $m \rightarrow +\infty$ gives the Corollary. \square

2.3 Some known results

At first we recall two Ray–Knight theorems:

Proposition 2.8 (Le Gall and Yor [11]). *Let $\delta > 0$. Under $\mathbb{P}^{(-\delta)}$, the process $(L(\infty, t), t \geq 0)$ is the square of a Bessel process of dimension δ starting from 0 reflected at 0.*

Proposition 2.9 (Carmona, Petit and Yor [4], Werner [24]). *Let $\delta > 0$ and $a \geq 0$. Under \mathbb{P}^δ , the process $(L(\tau_0(a), -t), t \geq 0)$ is the square of a Bessel process of dimension $(2 - \delta)$ starting from a absorbed at 0.*

We have the following independence result.

Proposition 2.10 (Yor [25], Proposition 9.1). *Let $\delta > 0$. Under \mathbb{P}^δ , for any fixed $x > 0$, $L(T_{J(x)}, 0)/x$ is independent of $J(x)$ and follows a beta($\frac{\delta}{2}, 1$) distribution.*

We introduce some notations which will be used in Section 4.

Notation 2.11. *Let $h \in \mathbb{R}$. We define the process $X^{-,h}$ obtained by gluing the excursions of X below h as follows. Let for $t \geq 0$,*

$$A_t^{-,h} := \int_0^t 1_{\{X_s \leq h\}} ds, \quad \alpha_t^{-,h} := \inf\{u > 0, A_u^{-,h} > t\},$$

with the usual convention $\inf \emptyset := \infty$. Define

$$X_t^{-,h} := X_{\alpha_t^{-,h}}, \quad t < A_\infty^{-,h} := \int_0^\infty 1_{\{X_s \leq h\}} ds.$$

Similarly, we define $A_t^{+,h}$, $\alpha_t^{+,h}$ and $X^{+,h}$ by replacing $X_s \leq h$ by $X_s > h$. When the process is denoted by X with some superscript, the analogous quantities will hold the same superscript. For example for $r \in \mathbb{R}$, $\ell > 0$, $\tau_r^{+,h}(\ell) = \inf\{t > 0 : L^{+,h}(t, r) > \ell\}$, where $L^{+,h}(t, r)$ denotes the local time of $X^{+,h}$ at position r and time t .

Proposition 2.12 (Perman and Werner [19]). *Let $\delta > 0$. Under \mathbb{P}^δ , the two processes $X^{+,0}$ and $X^{-,0}$ are independent. Moreover, $X^{+,0}$ is a reflecting Brownian motion, and the process $(X_t^{-,0}, \inf_{s \leq t} X_s^{-,0})_{t \geq 0}$ is strongly Markovian.*

Let $m > 0$. We look at these processes up to the first time the process X hits level $-m$. In this case, there is a dependence between $X^{-,0}$ and $X^{+,0}$ due to their duration. This dependence is taken care of by conditioning on the (common) local time at 0 of $X^{-,0}$ and $X^{+,0}$. It is the content of the following corollary.

Corollary 2.13. *Let $\delta > 0$. Fix $m > 0$. Under \mathbb{P}^δ , conditionally on $(X_t^{-,0}, t \leq A_{T_{-m}}^{-,0})$, the process $(X_t^{+,0}, t \leq A_{T_{-m}}^{+,0})$ is a reflected Brownian motion stopped at time $\tau_0^{+,0}(\ell)$ where $\ell = L^{-,0}(0, A_{T_{-m}}^{-,0}) = L(T_{-m}, 0)$.*

Proof. By Proposition 2.12, conditionally on $X^{-,0}$, the process $(X_{\tau_0^{+,0}(t)}^{+,0}, t \geq 0)$ is a reflected Brownian motion indexed by its inverse local time. Observe that $A_{T_{-m}}^{+,0} = \tau_0^{+,0}(\ell)$ with $\ell = L^{-,0}(0, T_{-m}^{-,0})$, this proves the Corollary. \square

As mentioned in Section 3 of Werner [24], we have the following duality between $\mathbb{P}^{(-\delta)}$ and \mathbb{P}^δ .

Proposition 2.14 (Werner [24]). *Let $\delta > 0$. For any $m > 0$, the process $(X_{T_{-m}-t}+m, t \leq T_{-m})$ under \mathbb{P}^δ has the distribution of $(X_t, t \leq \mathcal{D}_m)$ under $\mathbb{P}^{(-\delta)}$, where $\mathcal{D}_m := \sup\{t > 0 : X_t = m\}$ denotes the last passage time at m .*

3 Decomposition at a hitting time

The following lemma is Lemma 2.3 in Perman [18], together with the duality stated in Proposition 2.14. Recall that under $\mathbb{P}^{(-2)}$, X is a Bessel process of dimension 3. We refer to (2.3) for the definition of the first hitting time T_r and to (2.4) for the process $J(x)$.

Lemma 3.1 (Perman [18]). *Let $\delta > 0$. Let $m, x > 0$ and $y \in (0, x)$. Define the processes*

$$Z^1 := (X_{T_{J(x)}-t} - J(x))_{t \in [0, T_{J(x)}]} \quad Z^2 := (X_{T_{J(x)}+t} - J(x))_{t \in [0, \tau_0(x) - T_{J(x)}]}.$$

Under $\mathbb{P}^\delta(\cdot | J(x) = -m, L(T_{J(x)}, 0) = y)$:

- (i) Z^1 and Z^2 are independent,
- (ii) Z^1 is distributed as $(X_t)_{t \in [0, \mathcal{D}_m]}$ under $\mathbb{P}^{(-\delta)}(\cdot | L(\infty, m) = y)$,
- (iii) Z^2 is distributed as $(X_t)_{t \in [0, \mathcal{D}_m]}$ under $\mathbb{P}^{(-2)}(\cdot | L(\infty, m) = x - y)$,

with $\mathcal{D}_m := \sup\{t > 0 : X_t = m\}$.

Proof. For the sake of completeness, we give here a proof which is different from Perman [18]'s.

By Proposition 2.2, we can identify under \mathbb{P}^δ the point measure $\{(q, \mathbf{e}_{X,q}^\uparrow), q \in \mathcal{Q}_X^\uparrow\}$ with $\{(q, \mathbf{e}_q^\uparrow), q \in \mathcal{Q}_{\frac{\delta}{2}}^\uparrow \cap (-\infty, 0)\}$. Using the notations in Lemma 2.3, we have

$$L(T_{J(x)}, 0) = \sum_{q \in \mathcal{Q}_{\frac{\delta}{2}}^\uparrow \cap (J(x), 0)} \ell_\gamma^0 < x \leq \sum_{q \in \mathcal{Q}_{\frac{\delta}{2}}^\uparrow \cap [J(x), 0]} \ell_\gamma^0,$$

where in the above sum γ is the (unique) loop in $\mathcal{L}_{\frac{\delta}{2}}$ such that $\min \gamma = q$. Let $\ell^r(\mathbf{e})$ be the local time of the excursion \mathbf{e} at level r . We claim that conditioning on $\{J(x) = -m, L(T_{J(x)}, 0) = y\}$, $\mathbf{e}_{J(x)}^\uparrow$ and $\{(q, \mathbf{e}_q^\uparrow), q \in \mathcal{Q}_{\frac{\delta}{2}}^\uparrow \cap (J(x), 0)\}$ are independent and distributed as a Brownian excursion \mathbf{e} under $\mathbf{n}^+(\cdot | \ell^m(\mathbf{e}) > x - y)$, and $\{(q, \mathbf{e}_q^\uparrow), q \in \mathcal{Q}_{\frac{\delta}{2}}^\uparrow \cap (-m, 0)\}$ conditioned on $\{\xi_m = y\}$ respectively, where $\xi_m := \sum_{q \in \mathcal{Q}_{\frac{\delta}{2}}^\uparrow \cap (-m, 0)} \ell_\gamma^0$.

In fact, let F be a bounded measurable functional on $\mathbb{R} \times \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, and G be a bounded measurable functional on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, where, here and in the sequel, we endow $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ with the (usual) topology of uniform convergence on every compact set. Let f

be a bounded Borel function on \mathbb{R}^2 . Note that $\xi_m = \sum_{q \in \mathcal{Q}_{\frac{\delta}{2}}^{\uparrow} \cap (-m, 0)} \ell^{|q|}(\mathbf{e}_q^{\uparrow})$. By Proposition 2.2, we deduce from the master formula that

$$\begin{aligned} & \mathbb{E}^{\delta} \left[F((q, \mathbf{e}_q^{\uparrow}), q \in \mathcal{Q}_{\frac{\delta}{2}}^{\uparrow} \cap (J(x), 0)) G(\mathbf{e}_{J(x)}^{\uparrow}) f(J(x), L(T_{J(x)}, 0)) \right] \\ &= \mathbb{E}^{\delta} \left[\sum_{m > 0} F((q, \mathbf{e}_q^{\uparrow}), q \in \mathcal{Q}_{\frac{\delta}{2}}^{\uparrow} \cap (-m, 0)) G(\mathbf{e}_{-m}^{\uparrow}) f(-m, \xi_m) 1_{\{\xi_m < x, \ell^m(\mathbf{e}_{-m}^{\uparrow}) > x - \xi_m\}} \right] \\ &= \delta \int_0^{\infty} dm \mathbb{E} \left[F((q, \mathbf{e}_q^{\uparrow}), q \in \mathcal{Q}_{\frac{\delta}{2}}^{\uparrow} \cap (-m, 0)) f(-m, \xi_m) 1_{\{\xi_m < x\}} \right. \\ & \quad \left. \times \int \mathbf{n}^+(d\mathbf{e}) G(\mathbf{e}) 1_{\{\ell^m(\mathbf{e}) > x - \xi_m\}} \right], \end{aligned}$$

by using Lemma 2.3 (i). The claim follows.

Now we observe that conditioning on $\{J(x) = -m, L(T_{J(x)}, 0) = y\}$, Z^2 is measurable with respect to $\mathbf{e}_{J(x)}^{\uparrow}$ whereas Z^1 is to $\{(q, \mathbf{e}_q^{\uparrow}), q \in \mathcal{Q}_{\frac{\delta}{2}}^{\uparrow} \cap (J(x), 0)\}$, we get the (i); moreover, Z^2 is distributed as $(\mathbf{e}_t)_{t \in [0, \sigma_{x-y}^m]}$, under $\mathbf{n}^+(\cdot | \ell^m(\mathbf{e}) > x - y)$, where $\sigma_{x-y}^m := \inf\{t > 0 : \ell_t^m(\mathbf{e}) = x - y\}$ with $\ell_t^m(\mathbf{e})$ being the local time at level m at time t . The latter process has the same law as $(X_t)_{t \in [0, \mathcal{D}_m]}$ under $\mathbb{P}^{(-2)}(\cdot | L(\infty, m) = x - y)$.⁴ We get (iii).

To prove (ii), we denote by \widehat{e} the time-reversal of a loop e . By Proposition 2.14, $\{(m + q, \widehat{\mathbf{e}}_{q, X}^{\uparrow}), q \in \mathcal{Q}_X^{\uparrow} \cap (-m, 0)\}$ under \mathbb{P}^{δ} is distributed as $\{(q, \mathbf{e}_{q, X}^{\uparrow}), q \in \mathcal{Q}_X^{\uparrow} \cap (0, m)\}$ under $\mathbb{P}^{-\delta}$. Note that under $\mathbb{P}^{\delta}(\cdot | J(x) = -m, L(T_{J(x)}, 0) = y)$, Z^1 can be constructed from $\{(m + q, \widehat{\mathbf{e}}_{q, X}^{\uparrow}), q \in \mathcal{Q}_X^{\uparrow} \cap (-m, 0)\}$. Then Z^1 is distributed as $(X_t)_{0 \leq t \leq \mathcal{D}_m}$ under $\mathbb{P}^{(-\delta)}(\cdot | \sum_{q \in \mathcal{Q}_X^{\uparrow} \cap (0, m)} \ell^{m-q}(\mathbf{e}_{q, X}^{\uparrow}) = y)$. Finally remark that $\sum_{q \in \mathcal{Q}_X^{\uparrow} \cap (0, m)} \ell^{m-q}(\mathbf{e}_{q, X}^{\uparrow}) = L(\mathcal{D}_m, m) = L(\infty, m)$. We get (ii). This completes the proof of Lemma 3.1. \square

Fix $m > 0$. The following Theorems 3.2 and 3.3 describe the path decomposition of (X_t) at T_{-m} . Let $\mathbf{g}_m := \sup\{t \in [0, T_{-m}] : X_t = 0\}$. Recall that $I_{\mathbf{g}_m} = \inf_{0 \leq s \leq \mathbf{g}_m} X_s$. Define

$$\mathbf{d}_m := \inf\{t > \mathbf{g}_m : X_t = I_{\mathbf{g}_m}\}.$$

Theorem 3.2. *Let $\delta > 0$. Fix $m > 0$. Under \mathbb{P}^{δ} , the random variable $\frac{1}{m} I_{\mathbf{g}_m}$ is beta($\frac{\delta}{2}, 1$) distributed. Moreover, conditionally on $\{I_{\mathbf{g}_m} = -a\}$, the four processes*

$$\begin{aligned} & (X_t, t \in [0, T_{-a}]), \\ & (X_{\mathbf{g}_m - t}, t \in [0, \mathbf{g}_m - T_{-a}]), \\ & (-X_{\mathbf{g}_m + t}, t \in [0, \mathbf{d}_m - \mathbf{g}_m]), \\ & (X_{\mathbf{d}_m + t} + a, t \in [0, T_{-m} - \mathbf{d}_m]), \end{aligned}$$

⁴Under $\mathbf{n}^+(\cdot | \ell^m(\mathbf{e}) > x - y)$, an excursion up to the inverse local time $x - y$ at position m is a three-dimensional Bessel process, up to the hitting time of m , followed by a Brownian motion starting at m stopped at local time at level m given by $x - y$, this Brownian motion being conditioned on not touching 0 during that time. By excursion theory, the time-reversed process is distributed as a Brownian motion starting at level m stopped at the hitting time of 0 conditioned on the local time at m being equal to $x - y$. We conclude by William's time reversal theorem (Corollary VII.4.6 of [21]).

are independent, with law respectively the one of:

- (i) X under \mathbb{P}^δ up to the hitting time of $-a$;
- (ii) a Brownian motion up to the hitting time of $-a$;
- (iii) a Bessel process of dimension 3 from 0 stopped when hitting a ;
- (iv) X under \mathbb{P}^δ conditionally on $\{T_{-(m-a)} < T_a\}$.

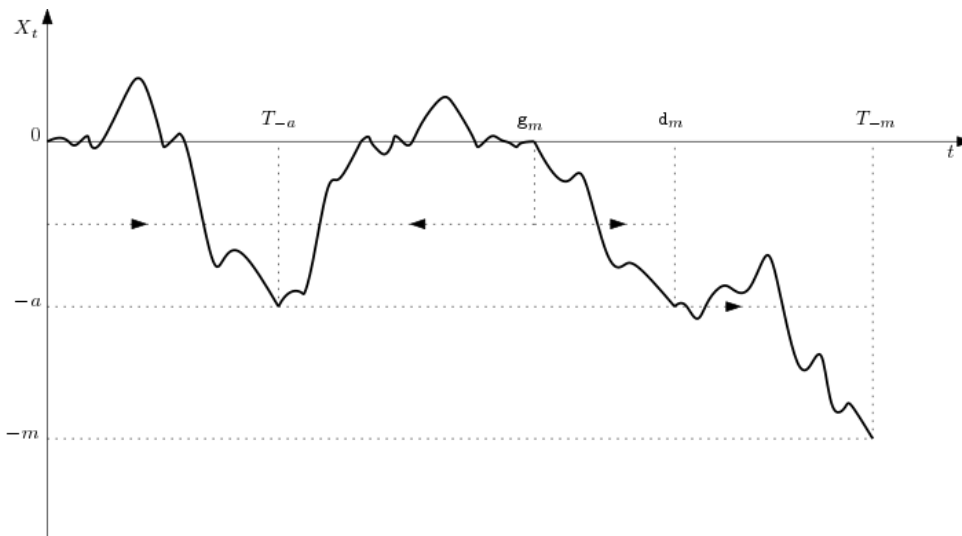


Figure 1: The four processes in Theorem 3.2.

Proof. To get the distribution of $\frac{1}{m} I_{\mathbf{g}_m}$ we proceed as follows: under \mathbb{P}^δ , X is measurable with respect to its excursions above the infimum, that we denoted by $(\mathbf{e}_{q,X}^\uparrow, q \in \mathcal{Q}_X^\uparrow)$, that we identify with $(\mathbf{e}_q^\uparrow, q \in \mathcal{Q}_{\frac{\delta}{2}}^\uparrow)$ by Proposition 2.2. The variable $I_{\mathbf{g}_m}$ is the global minimum of the loops γ such that $\min \gamma > -m$ and $0 \in \gamma$. By Lemma 2.3 (iii), we get the law of $I_{\mathbf{g}_m}$ (see also the computation in the proof of Lemma 2.4 together with Proposition 2.5).

Let $\tilde{\gamma}$ be the loop such that $\min \tilde{\gamma} = I_{\mathbf{g}_m}$ and call $-a = I_{\mathbf{g}_m}$ its minimum. Conditioning on $\tilde{\gamma}$ and loops hitting $(-\infty, -a)$, the loops γ such that $\min \gamma > -a$ are distributed as the usual Brownian loop soup $\mathcal{L}_{\frac{\delta}{2}}$ in $(-a, \infty)$. It gives (i) by Proposition 2.2. Conditioning on $\min \tilde{\gamma} = I_{\mathbf{g}_m} = -a$ and on loops hitting $(-\infty, -a)$, the loop $\tilde{\gamma} - \min \tilde{\gamma}$ has the measure $\mathbf{n}^+(\mathrm{d}\mathbf{e} | \max \mathbf{e} > a)$. Therefore (ii) and (iii) comes from the usual decomposition of the Itô measure. Finally conditioning on $\min \tilde{\gamma} = I_{\mathbf{g}_m} = -a$, the collection of loops γ with $\min \gamma \in (-m, -a)$ is distributed as the Brownian loop soup $\mathcal{L}_{\frac{\delta}{2}}$ restricted to loops γ such that $\min \gamma \in (-m, -a)$ conditioned on the event that none of these loops hit 0. We deduce (iv). \square

The following theorem gives the path decomposition when conditioning on $(L(T_{-m}, 0), I_{\mathbf{g}_m})$. Recall (2.6) for the law of $L(T_{-m}, 0)$.

Theorem 3.3. *We keep the notations of Theorem 3.2. Under \mathbb{P}^δ ,*

(i) *the density of $(L(T_{-m}, 0), |I_{\mathbf{g}_m}|)$ is given by $\frac{a^{-2}}{2\Gamma(\frac{\delta}{2})}(\frac{x}{2m})^{\frac{\delta}{2}}e^{-\frac{x}{2a}}$ for $x > 0$ and $0 < a < m$.*

(ii) *conditionally on $\{L(T_{-m}, 0) = x, I_{\mathbf{g}_m} = -a\}$, the three processes*

$$\begin{aligned} &(X_t, t \in [0, \mathbf{g}_m]), \\ &(-X_{\mathbf{g}_m+t}, t \in [0, \mathbf{d}_m - \mathbf{g}_m]), \\ &(X_{\mathbf{d}_m+t} + a, t \in [0, T_{-m} - \mathbf{d}_m]), \end{aligned}$$

are independent and distributed respectively as

$$\begin{aligned} &(X_t, t \in [0, \tau_0(x)]) \text{ under } \mathbb{P}^\delta(\cdot | J(x) = -a), \\ &a \text{ Bessel process of dimension 3 starting from 0 stopped when hitting } a, \\ &X \text{ under } \mathbb{P}^\delta \text{ conditionally on } \{T_{-(m-a)} < T_a\}; \end{aligned}$$

Proof. (i) In view of (2.6), it is enough to show

$$(3.1) \quad \mathbb{P}^\delta(|I_{\mathbf{g}_m}| \in da | L(T_{-m}, 0) = x) = \frac{x}{2a^2} e^{-\frac{x}{2a} + \frac{x}{2m}} 1_{\{0 < a < m\}} da.$$

To this end, we shall prove that conditionally on $\{L(T_{-m}, 0) = x\}$, $I_{\mathbf{g}_m}$ is distributed as $\inf_{0 \leq t \leq \tau_0^B(x)} B(t)$ conditioned on $\{\inf_{0 \leq t \leq \tau_0^B(x)} B(t) > -m\}$, where $\tau_0^B(x) := \inf\{t > 0 : \mathfrak{L}_t > x\}$ denotes the first time when the local time at 0 of B attains x .

Consider the Brownian loop soup $\mathcal{L}_{\frac{\delta}{2}}$. In this setting (recalling Proposition 2.2),

$$I_{\mathbf{g}_m} = \inf_{\gamma \in \mathcal{L}_{\frac{\delta}{2}}} \left\{ q : q = \min \gamma > -m, 0 \in \gamma \right\},$$

and

$$L(T_{-m}, 0) = \sum_{\gamma \in \mathcal{L}_{\frac{\delta}{2}}} \ell_\gamma^0 1_{\{\min \gamma \in (-m, 0), 0 \in \gamma\}}.$$

From (v) of Lemma 2.3, conditionally on $\{\ell_\gamma^0 : \gamma \in \mathcal{L}_{\frac{\delta}{2}}, 0 \in \gamma\}$, the loops γ such that $0 \in \gamma$ are independent Brownian motions stopped at τ_0^ℓ with $\ell = \ell_\gamma^0$. Then, the loops γ such that $0 \in \gamma$ and $\min \gamma > -m$ are merely independent Brownian motions stopped at local time given by ℓ_γ^0 , conditioned on not hitting $-m$. The conditional density (3.1) of $I_{\mathbf{g}_m}$ follows from standard Brownian excursion theory.

(ii) By Theorem 3.2, conditionally on $\{I_{\mathbf{g}_m} = -a\}$, the three processes

$$\begin{aligned} &(X_t, t \in [0, \mathbf{g}_m]), \\ &(-X_{\mathbf{g}_m+t}, t \in [0, \mathbf{d}_m - \mathbf{g}_m]), \\ &(X_{\mathbf{d}_m+t} + a, t \in [0, T_{-m} - \mathbf{d}_m]), \end{aligned}$$

are independent. Since $L(T_{-m}, 0)$ is measurable with respect to $\sigma(X_t, t \in [0, \mathbf{g}_m])$, we obtain the independence of the three processes in (ii) and the claimed laws of the latter two processes in (ii).

To complete the proof of (ii), we are going to show that for any bounded continuous functional Φ on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, for all $x > 0$ and $0 < a < m$,

$$(3.2) \quad \begin{aligned} & \mathbb{E}^\delta[\Phi(X_t, t \in [0, \mathbf{g}_m]) \mid L(T_{-m}, 0) = x, I_{\mathbf{g}_m} = -a] \\ &= \mathbb{E}^\delta[\Phi(X_t, t \in [0, \tau_0(x)]) \mid J(x) = -a]. \end{aligned}$$

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a bounded continuous function. By Theorem 3.2,

$$(3.3) \quad \begin{aligned} & \mathbb{E}^\delta[\Phi(X_t, t \in [0, \mathbf{g}_m])f(L(T_{-m}, 0), I_{\mathbf{g}_m})] \\ &= \int \mathbb{P}^\delta(-I_{\mathbf{g}_m} \in da) \mathbb{E}^\delta[\Phi(X^{1,a}, X^{2,a})f(L^0(X^{1,a}) + L^0(X^{2,a}), -a)], \end{aligned}$$

where $X_s^{1,a} := X_s, s \leq T_{-a}$ and $X_s^{2,a} := X_{T_{-a}+s}, s \leq \mathbf{g}_m - T_{-a}$ are independent, and we used the fact that $L(T_{-m}, 0) = L^0(X^{1,a}) + L^0(X^{2,a})$ the sum of the total local times at position 0 of $X^{1,a}$ and $X^{2,a}$. Remark that $X^{1,a}$ is distributed as X up to T_{-a} and $X^{2,a} + a$ is a three-dimensional Bessel process up to its last passage time at a . We claim that for a.e. $0 < a < m$ and $x > 0$,

$$(3.4) \quad \mathbb{E}^\delta[\Phi(X^{1,a}, X^{2,a}) \mid L^0(X^{1,a}) + L^0(X^{2,a}) = x] = \mathbb{E}^\delta[\Phi(X_t, t \in [0, \tau_0(x)]) \mid J(x) = -a].$$

In fact, to get (3.4), we first note that $L^0(X^{1,a})$ is distributed as $L(T_{-a}, 0)$ (under \mathbb{P}^δ) and $L^0(X^{2,a})$ has the same law as the total local time at position a of a three-dimensional Bessel process. Then

$$\mathbb{P}^\delta(L^0(X^{2,a}) \in dz) = \frac{1}{2a} e^{-\frac{z}{2a}}, \quad z > 0,$$

and $L^0(X^{1,a}) \stackrel{(\text{law})}{=} \text{gamma}(\frac{\delta}{2}, 2a)$:

$$\mathbb{P}^\delta(L^0(X^{1,a}) \in dy) = \frac{1}{\Gamma(\frac{\delta}{2})} (2a)^{-\frac{\delta}{2}} y^{\frac{\delta}{2}-1} e^{-\frac{y}{2a}}, \quad y > 0.$$

By convolution the density of $L^0(X^{1,a}) + L^0(X^{2,a})$ is $\frac{1}{\Gamma(1+\frac{\delta}{2})} (2a)^{-1-\frac{\delta}{2}} x^{\frac{\delta}{2}} e^{-\frac{x}{2a}}, x > 0$.

Now let h be a bounded Borel function. Then

$$\begin{aligned} & \mathbb{E}^\delta[\Phi(X^{1,a}, X^{2,a}) h(L^0(X^{1,a}) + L^0(X^{2,a}))] \\ &= \int_0^\infty \int_0^\infty h(y+z) \mathbb{E}^\delta[\Phi(X^{1,a}, X^{2,a}) \mid L^0(X^{1,a}) = y, L^0(X^{2,a}) = z] \frac{(2a)^{-1-\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})} y^{\frac{\delta}{2}-1} e^{-\frac{y+z}{2a}} dy dz \\ &= \int_0^\infty h(x) \int_0^x \mathbb{E}^\delta[\Phi(X^{1,a}, X^{2,a}) \mid L^0(X^{1,a}) = y, L^0(X^{2,a}) = x-y] \frac{(2a)^{-1-\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})} y^{\frac{\delta}{2}-1} e^{-\frac{x}{2a}} dy dx \\ &= \int_0^\infty h(x) \int_0^x \mathbb{E}^\delta[\Phi(X_t, t \leq \tau_0(x)) \mid J(x) = -a, L(T_{J(x)}, 0) = y] \frac{(2a)^{-1-\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})} y^{\frac{\delta}{2}-1} e^{-\frac{x}{2a}} dy dx, \end{aligned}$$

where the last equality is due to Lemma 3.1. Consequently, for all $x > 0$,

$$\begin{aligned} & \mathbb{E}^\delta[\Phi(X^{1,a}, X^{2,a}) | L^0(X^{1,a}) + L^0(X^{2,a}) = x] \\ &= \int_0^x \mathbb{E}^\delta[\Phi(X_t, t \in [0, \tau_0(x)]) | J(x) = -a, L(T_{J(x)}, 0) = y] \frac{\delta}{2} x^{\frac{\delta}{2}} y^{\frac{\delta}{2}-1} dy, \end{aligned}$$

which gives (3.4) as $\mathbb{P}^\delta(L(T_{J(x)}, 0) \in dy) = \frac{\delta}{2} x^{\frac{\delta}{2}} y^{\frac{\delta}{2}-1} 1_{\{0 < y < x\}} dy$.

Recalling from Theorem 3.2 that $\mathbb{P}^\delta(-I_{\mathbf{g}_m} \in da) = \frac{\delta}{2} m^{-\frac{\delta}{2}} a^{\frac{\delta}{2}-1} da$ for $0 < a < m$. Plugging (3.4) into (3.3) gives that

$$\begin{aligned} (3.5) \quad & \mathbb{E}^\delta[\Phi(X_t, t \in [0, \mathbf{g}_m]) f(L(T_{-m}, 0), I_{\mathbf{g}_m})] \\ &= \int_0^\infty \int_0^m \mathbb{E}^\delta[\Phi(X_t, t \in [0, \tau_0(x)]) | J(x) = -a] f(x, -a) \frac{a^{-2}}{2\Gamma(\frac{\delta}{2})} \left(\frac{x}{2m}\right)^{\frac{\delta}{2}} e^{-\frac{x}{2a}} da dx, \end{aligned}$$

which readily yields (3.2) and completes the proof of the Proposition. \square

Corollary 3.4. *Let us keep the notations of Theorem 3.2. Let $x > 0$ and $m > a > 0$. Under \mathbb{P}^δ , the conditional law of the process $(X_t, t \in [0, \mathbf{g}_m])$ given $\{L(T_{-m}, 0) = x\}$ is equal to the (unconditional) law of $(X_t, t \in [0, \tau_0(x)])$ biased by $c_{m,x,\delta} |J(x)|^{\frac{\delta}{2}-1} 1_{\{J(x) > -m\}}$, with*

$$c_{m,x,\delta} := \Gamma\left(\frac{\delta}{2}\right) \left(\frac{x}{2}\right)^{1-\frac{\delta}{2}} e^{\frac{x}{2m}}.$$

Proof. Let Φ be a bounded continuous functional on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$. Recall from (2.6) that the density function of $L(T_{-m}, 0)$ is $\frac{1}{\Gamma(\frac{\delta}{2})} (2m)^{-\frac{\delta}{2}} x^{\frac{\delta}{2}-1} e^{-\frac{x}{2m}}$, $x > 0$. Considering some f in (3.5) which only depends on the first coordinate, we see that for all $x > 0$,

$$\begin{aligned} & \mathbb{E}^\delta[\Phi(X_t, t \in [0, \mathbf{g}_m]) | L(T_{-m}, 0) = x] \\ &= \int_0^m \mathbb{E}^\delta[\Phi(X_t, t \in [0, \tau_0(x)]) | J(x) = -a] \frac{x}{2} a^{-2} e^{-\frac{x}{2a} + \frac{x}{2m}} da \\ (3.6) \quad &= c_{m,x,\delta} \mathbb{E}^\delta[\Phi(X_t, t \in [0, \tau_0(x)]) | J(x)|^{\frac{\delta}{2}-1} 1_{\{J(x) > -m\}}], \end{aligned}$$

by using the fact that the density of $|J(x)|$ is $a \rightarrow \frac{1}{\Gamma(\frac{\delta}{2})} \left(\frac{x}{2}\right)^{\frac{\delta}{2}} a^{-\frac{\delta}{2}-1} e^{-\frac{x}{2a}}$. This proves Corollary 3.4. \square

Remark 3.5. Note that the conditional expectation term in the left-hand-side of (3.6) is a continuous function of (m, x) , this fact will be used later on.

As an application of the above decomposition results, we give another proof of Proposition 2.6.

Another proof of Proposition 2.6. Notice that $T_{-m}^J = L(T_{-m}, 0)$. Conditioning on $T_{-m}^J = x$: by Corollary 3.4, $x_1^{(m)}$ is distributed as $L(T_{J(x)}, 0)$ under \mathbb{P}^δ biased by $|J(x)|^{\frac{\delta}{2}-1} 1_{\{J(x) > -m\}}$. By the independence of $L(T_{J(x)}, 0)$ and $J(x)$ of Proposition 2.10, the biased law of $L(T_{J(x)}, 0)$ is the same as under \mathbb{P}^δ , hence is x times a beta($\frac{\delta}{2}, 1$) random variable. Moreover, conditionally on $x_1^{(m)} = y$ and $J(x) = -m_1$, the process before $T_{J(x)}$ is simply the process X under \mathbb{P}^δ before hitting T_{-m_1} conditioned on $L(T_{-m_1}, 0) = y$ (by Corollary 3.4 and Lemma 3.1). Therefore we can iterate and get Proposition 2.6. \square

4 Decomposition at the minimum

Let $\delta > 0$. Let X_1 be a copy of the process X under $\mathbb{P}^{(-\delta)}$ and X_2 be an independent Bessel process of dimension 3, both starting at 0. Recall Notation 2.11. From our notations, $L^1(\infty, r)$, resp. $L^2(\infty, r)$, denotes the total local time at height r of X_1 , resp. X_2 , while $X_1^{-,h}, X_2^{-,h}$ are obtained by gluing the excursions below h of X_1 and X_2 respectively. We set

$$(4.1) \quad H^{1,2} := \sup\{r \geq 0 : L^1(\infty, r) + L^2(\infty, r) = 1\}.$$

Proposition 2.8 yields that the process $L^1(\infty, r) + L^2(\infty, r), r \geq 0$ is distributed as the square of a Bessel process of dimension $\delta + 2$, starting from 0. Then $H^{1,2} < \infty$ a.s.

Lemma 4.1. *Let $\delta > 0$. Let $m > 0$ and $x \in (0, 1)$. Conditionally on $\{H^{1,2} = m, L^1(\infty, H^{1,2}) = x\}$:*

- (i) $X_1^{-,H^{1,2}}$ and $X_2^{-,H^{1,2}}$ are independent;
 - (ii) $X_1^{-,H^{1,2}}$ is distributed as $(X_t^{-,m}, t < A_\infty^{-,m})$ under $\mathbb{P}^{(-\delta)}(\cdot | L^1(\infty, m) = x)$;
 - (iii) $X_2^{-,H^{1,2}}$ is distributed as $(X_t^{-,m}, t < A_\infty^{-,m})$ under $\mathbb{P}^{(-2)}(\cdot | L^2(\infty, m) = 1 - x)$,
- where $A_\infty^{-,m} = \int_0^\infty 1_{\{X_t \leq m\}} dt$ is the total lifetime of the process $X^{-,m}$.

Proof. First we describe the law of $(X_t^{-,m}, t < A_\infty^{-,m})$ under $\mathbb{P}^{(-\delta)}(\cdot | L^1(\infty, m) = x)$. Let $\mathcal{D}_m := \sup\{t > 0 : X_t \leq m\}$ be the last passage time of X at m [note that under $\mathbb{P}^{(-\delta)}$, $X_t \rightarrow \infty$ as $t \rightarrow \infty$]. By the duality of Proposition 2.14, $\{X_{\mathcal{D}_m - t} - m, 0 \leq t \leq \mathcal{D}_m\}$, under $\mathbb{P}^{(-\delta)}$, has the same law as $\{X_t, 0 \leq t \leq T_{-m}\}$ under \mathbb{P}^δ . Corollary 3.4 gives then the law of $(X_t, t \leq \mathcal{D}_m)$ under $\mathbb{P}^{(-\delta)}(\cdot | L(\infty, m) = x)$. The process $(X_t^{-,m}, t < A_\infty^{-,m})$ is a measurable function of $(X_t, t \leq \mathcal{D}_m)$. Note that $m \rightarrow (X_1^{-,m}, X_2^{-,m})$ is continuous.⁵ From Corollary 3.4, we may find a regular version of the law of $(X_t^{-,m}, t < A_\infty^{-,m})$ under $\mathbb{P}^{(-\delta)}(\cdot | L(\infty, m) = x)$ such that for any bounded continuous functional F on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, the application

$$(m, x) \mapsto \mathbb{E}^{(-\delta)}[F(X_t^{-,m}, t < A_\infty^{-,m}) | L(\infty, m) = x]$$

⁵For instance, we may show that for any $T > 0$, almost surely $\sup_{0 \leq t \leq T} |X_t^{-,m'} - X_t^{-,m}| \rightarrow 0$ as $m' \rightarrow m$. Let us give a proof by contradiction. Suppose there exists some $\varepsilon_0 > 0$, a sequence (t_k) in $[0, T]$ and $m_k \rightarrow m$ such that $|X_{t_k}^{-,m_k} - X_{t_k}^{-,m}| > \varepsilon_0$. Write for simplification $s_k := \alpha_{t_k}^{-,m}$ and $s'_k := \alpha_{t_k}^{-,m_k}$. Consider the case $m_k > m$ (the other direction can be treated in a similar way). Then $s_k \geq s'_k$ and $|X_{s'_k} - X_{s_k}| > \varepsilon_0$ for all k . Since $X_{s_k} \leq m$ and $X_{s'_k} \leq m_k$, either $X_{s_k} \leq m - \frac{\varepsilon_0}{2}$ or $X_{s'_k} \leq m - \frac{\varepsilon_0}{2}$ for all large k . Consider for example the case $X_{s'_k} \leq m - \frac{\varepsilon_0}{2}$. By the uniform continuity of X_t on every compact, there exists some $\delta_0 > 0$ such that $X_u \leq m$ for all $|u - s'_k| \leq \delta_0$ and $k \geq 1$. Then for any $s \geq s'_k$, $\int_{s'_k}^s 1_{\{X_u \leq m\}} du \geq \min(\delta_0, s - s'_k)$. Note that by definition, $\int_{s'_k}^{s_k} 1_{\{X_u \leq m\}} du = t_k - \int_0^{s'_k} 1_{\{X_u \leq m\}} du = \int_m^{m_k} L(s'_k, x) dx \leq \zeta(m_k - m)$, with $\zeta := \sup_{x \in \mathbb{R}} L(\alpha_T^{-,m}, x)$. It follows that for all sufficiently large k , $0 \leq s_k - s'_k \leq \zeta(m_k - m)$. Consequently $X_{s_k} - X_{s'_k} \rightarrow 0$ as $m_k \rightarrow m$, in contradiction with the assumption that $|X_{s'_k} - X_{s_k}| > \varepsilon_0$ for all k . This proves the continuity of $m \rightarrow X^{-,m}$.

is continuous.

Now let us write $H := H^{1,2}$ for concision. Let F_1 and F_2 be two bounded continuous functionals on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ and $g : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be a bounded continuous function. Let $H_n := 2^{-n} \lfloor 2^n H \rfloor$ for any $n \geq 1$. By the continuity of $m \rightarrow (X_1^{-,m}, X_2^{-,m})$ and that of $(L^1(\infty, m), L^2(\infty, m))$, we have

$$\mathbb{E} \left[F_1(X_1^{-,H}) F_2(X_2^{-,H}) g(H, L^1(\infty, H)) \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[F_1(X_1^{-,H_n}) F_2(X_2^{-,H_n}) g(H_n, L^1(\infty, H_n)) \right].$$

Note that

$$\begin{aligned} & \mathbb{E} \left[F_1(X_1^{-,H_n}) F_2(X_2^{-,H_n}) g(H_n, L^1(\infty, H_n)) \right] \\ &= \sum_{j=0}^{\infty} \mathbb{E} \left[F_1(X_1^{-, \frac{j}{2^n}}) F_2(X_2^{-, \frac{j}{2^n}}) g\left(\frac{j}{2^n}, L^1(\infty, \frac{j}{2^n})\right) 1_{\{\frac{j}{2^n} \leq H < \frac{j+1}{2^n}\}} \right]. \end{aligned}$$

By the independence property of Corollary 2.13 and the duality of Proposition 2.14, conditioning on $\{L^1(\infty, \frac{j}{2^n}), L^2(\infty, \frac{j}{2^n})\}$, the processes $(X_1^{-, \frac{j}{2^n}}, X_2^{-, \frac{j}{2^n}})$ are independent, and independent of $(X_1^{+, \frac{j}{2^n}}, X_2^{+, \frac{j}{2^n}})$. Since $\{\frac{j}{2^n} \leq H < \frac{j+1}{2^n}\}$ is measurable with respect to $\sigma(X_1^{+, \frac{j}{2^n}}, X_2^{+, \frac{j}{2^n}})$, we get that for each $j \geq 0$,

$$\begin{aligned} & \mathbb{E} \left[F_1(X_1^{-, \frac{j}{2^n}}) F_2(X_2^{-, \frac{j}{2^n}}) g\left(\frac{j}{2^n}, L^1(\infty, \frac{j}{2^n})\right) 1_{\{\frac{j}{2^n} \leq H < \frac{j+1}{2^n}\}} \right] \\ &= \mathbb{E} \left[\Phi_1\left(\frac{j}{2^n}, L^1(\infty, \frac{j}{2^n})\right) \Phi_2\left(\frac{j}{2^n}, L^2(\infty, \frac{j}{2^n})\right) g\left(\frac{j}{2^n}, L^1(\infty, \frac{j}{2^n})\right) 1_{\{\frac{j}{2^n} \leq H < \frac{j+1}{2^n}\}} \right], \end{aligned}$$

where

$$\Phi_1(m, x) := \mathbb{E}[F_1(X_1^{-,m}) | L^1(\infty, m) = x], \quad \Phi_2(m, x) := \mathbb{E}[F_2(X_2^{-,m}) | L^2(\infty, m) = x].$$

By Remark 3.5, Φ_1 and Φ_2 are continuous functions in (m, x) . Taking the sum over j we get that

$$\begin{aligned} & \mathbb{E} \left[F_1(X_1^{-,H_n}) F_2(X_2^{-,H_n}) g(H_n, L^1(\infty, H_n)) \right] \\ &= \mathbb{E} \left[\Phi_1(H_n, L^1(\infty, H_n)) \Phi_2(H_n, L^2(\infty, H_n)) g(H_n, L^1(\infty, H_n)) \right]. \end{aligned}$$

Since Φ_1 and Φ_2 are bounded and continuous, the dominated convergence theorem yields that

$$\begin{aligned} & \mathbb{E} \left[F_1(X_1^{-,H}) F_2(X_2^{-,H}) g(H, L^1(\infty, H)) \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[F_1(X_1^{-,H_n}) F_2(X_2^{-,H_n}) g(H_n, L^1(\infty, H_n)) \right] \\ (4.2) \quad &= \mathbb{E} \left[\Phi_1(H, L^1(\infty, H)) \Phi_2(H, L^2(\infty, H)) g(H, L^1(\infty, H)) \right], \end{aligned}$$

proving Lemma 4.1 as $L^2(\infty, H) = 1 - L^1(\infty, H)$. \square

Remark 4.2. Let $\delta > 2$. Consider the process X under $\mathbb{P}^{(-\delta)}$ and the total local time $L(\infty, r)$ of X at position $r \geq 0$. For $x > 0$, let

$$H_x := \sup\{r \geq 0 : L(\infty, r) = x\}.$$

By Proposition 2.8, $H_x < \infty$, $\mathbb{P}^{(-\delta)}$ -a.s. Note that the same arguments leading to (4.2) shows that

$$\mathbb{E}\left[F_1(X^{-,H_x})g(H_x, L(\infty, H_x))\right] = \mathbb{E}\left[\Phi_1(H_x, L(\infty, H_x))g(H_x, L(\infty, H_x))\right],$$

where $\Phi_1(m, x) := \mathbb{E}[F_1(X^{-,m}) | L(\infty, m) = x]$ for $m > 0, x > 0$. Since $L(\infty, H_x) = x$, we obtain that conditionally on $\{H_x = m\}$, the process X^{-,H_x} is distributed as $(X_t^{-,m}, t < A_\infty^{-,m})$ under $\mathbb{P}^{(-\delta)}(\cdot | L(\infty, m) = x)$.

Recall (2.3), (2.4) and (4.1). The main result in this section is the following path decomposition of (X_t) at $T_{J(1)} = \inf\{t \in [0, \tau_0(1)] : X_t = J(1)\}$, the unique time before $\tau_0(1)$ at which X reaches its minimum $J(1)$.

Theorem 4.3. Let $\delta > 0$. Define $Z_1 := (X_{T_{J(1)}-t} - J(1))_{t \in [0, T_{J(1)}]}$ and $Z_2 := (X_{T_{J(1)}+t} - J(1))_{t \in [0, \tau_0(1) - T_{J(1)}]}$. Under \mathbb{P}^δ , the couple of processes

$$\left(Z_1^{-,|J(1)|}, Z_2^{-,|J(1)|}\right)$$

is distributed as $(X_1^{-,H^{1,2}}, X_2^{-,H^{1,2}})$.

Proof. From Lemmas 3.1 and 4.1, it remains to prove that the joint law of $(|J(1)|, L(T_{J(1)}, 0))$ is the same as $(H^{1,2}, L^1(\infty, H^{1,2}))$. Recall the law of $(J(1), L(T_{J(1)}, 0))$ from Propositions 2.6 (i) and 2.10. Define a process $(Y_t, -\infty < t < \infty)$ with values in $[0, 1]$ defined by time-change as

$$Y_{A_m} = \frac{L^1(\infty, m)}{L^1(\infty, m) + L^2(\infty, m)},$$

where $A_m := \int_1^m \frac{dh}{L^1(\infty, h) + L^2(\infty, h)}$ for any $m > 0$ (as such $\lim_{m \rightarrow 0} A_m = -\infty$ a.s.). Following Warren and Yor [23], equation (3.1), we call Jacobi process of parameters $d, d' \geq 0$ the diffusion with generator $2y(1-y)\frac{d^2}{dy^2} + (d - (d+d')y)\frac{d}{dy}$. We claim that Y is a stationary Jacobi process of parameter $(\delta, 2)$, independent of $(L^1(\infty, m) + L^2(\infty, m), m \geq 0)$. It is a consequence of Proposition 8 of Warren and Yor [23]. Let us see why.

First, notice that $\frac{L^1(\infty, m)}{L^1(\infty, m) + L^2(\infty, m)}$ is a beta($\frac{\delta}{2}, 1$)-random variable for any $m > 0$, because $L^1(\infty, m)$ and $L^2(\infty, m)$ are independent and distributed as gamma($\frac{\delta}{2}, 2m$) and gamma($1, 2m$) respectively, by Proposition 2.8 and the duality in Proposition 2.14. It is independent of $L^1(\infty, m) + L^2(\infty, m)$, hence by the Markov property, also of $(L^1(\infty, h) + L^2(\infty, h), h \geq m)$.

Let $t_0 \in \mathbb{R}$. By Proposition 8 of [23], for any $m \in (0, 1)$, conditioning on $(L^1(\infty, h) + L^2(\infty, h), h \geq m)$, the process $(Y_{h+A_m}, h \geq 0)$ is distributed as a Jacobi process starting

from a $\text{beta}(\frac{\delta}{2}, 1)$ random variable, hence stationary. Notice that A_m is measurable with respect to $\sigma(L^1(\infty, h) + L^2(\infty, h), h \geq m)$. We deduce that, conditioned on $(L^1(\infty, h) + L^2(\infty, h), h \geq m)$ and $A_m \leq t_0$, the process $(Y_h, h \geq t_0)$ is a Jacobi process starting from a $\text{beta}(\frac{\delta}{2}, 1)$ random variable. Letting $m \rightarrow 0$ we see that Y is a stationary Jacobi process of parameter $(\delta, 2)$, independent of $(L^1(\infty, m) + L^2(\infty, m), m \geq 0)$.

Since $L^1(\infty, H^{1,2}) = Y_{A_{H^{1,2}}}$ and $A_{H^{1,2}}, H^{1,2}$ are measurable with respect to $\sigma\{L^1(\infty, m) + L^2(\infty, m), m \geq 0\}$, we deduce that the random variable $L^1(\infty, H^{1,2})$ follows the $\text{beta}(\frac{\delta}{2}, 1)$ distribution and that $H^{1,2}$ and $L^1(\infty, H^{1,2})$ are independent. Finally, the random variable $H^{1,2}$ is the exit time of a square Bessel process of dimension $2 + \delta$ by Proposition 2.8, hence is distributed as $|J(1)|$ (Exercise (1.18), Chapter XI of Revuz and Yor [21]). \square

The rest of this section is devoted to a path decomposition of X under $\mathbb{P}^{(-\delta)}$ for $\delta > 2$. For $x > 0$, let as in Remark 4.2,

$$H_x := \sup\{r \geq 0 : L(\infty, r) = x\}.$$

Define

$$S_x := \sup\{t \geq 0 : X_t = H_x\}, \quad \widehat{J}_x := \inf\{X_t, t \geq T_{H_x}\} - H_x,$$

where as before $T_{H_x} := \inf\{t \geq 0 : X_t = H_x\}$ is the hitting time of H_x by X .

Write $C_x := H_x + \widehat{J}_x$. We consider the following three processes:

$$\begin{aligned} X^{(1)} &:= (X_{S_x-t} - H_x, t \in [0, S_x - T_{H_x}]), \\ X^{(2)} &:= -(X_{T_{H_x}-t} - H_x, t \in [0, T_{H_x} - D_x]), \\ X^{(3)} &:= (X_{D_x-t} - C_x, t \in [0, D_x]), \end{aligned}$$

where $D_x := \sup\{t < T_{H_x} : X_t = C_x\}$.

Furthermore, let $X^{(1),-}$ be the process $X^{(1)}$ obtained by removing all its positive excursions:

$$X_t^{(1),-} := X_{\alpha_t^{(1),-}}^{(1)},$$

with $\alpha_t^{(1),-} := \inf\{s > 0 : \int_0^s 1_{\{X_u^{(1)} \leq 0\}} du\}$ and $t \leq \int_0^{S_x - T_{H_x}} 1_{\{X_u^{(1)} \leq 0\}} du$.

Proposition 4.4. *Let $\delta > 2$ and $x, a > 0$.*

(i) *Under $\mathbb{P}^{(-\delta)}$, $\frac{1}{|\widehat{J}_x|}$ is distributed as $\text{gamma}(\frac{\delta}{2}, \frac{2}{x})$.*

(ii) *Under $\mathbb{P}^{(-\delta)}(\cdot | \widehat{J}_x = -a)$, the three processes $X^{(1),-}$, $X^{(2)}$, $X^{(3)}$ are independent and distributed respectively as*

- $(X_t, 0 \leq t \leq \tau_0(x))$, under $\mathbb{P}^\delta(\cdot | J(x) = -a)$, after removing all excursions above 0;
- a Bessel process $(R_t)_{0 \leq t \leq T_a}$ of dimension 3 starting from 0 killed at $T_a := \inf\{t > 0 : R_t = a\}$;
- $(X_t, 0 \leq t \leq T_{-a(1-u)/u})$ under $\mathbb{P}^\delta(\cdot | T_{-a(1-u)/u} < T_a)$, where $u \in [0, 1]$ is independently chosen according to the law $\text{beta}(\frac{\delta}{2} - 1, 1)$.

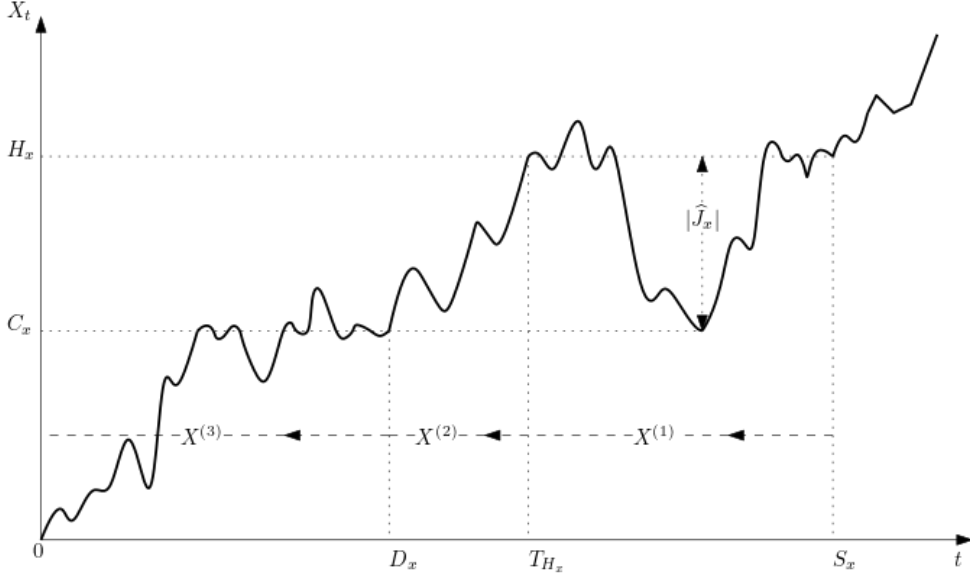


Figure 2: Under $\mathbb{P}^{(-\delta)}$, $X_t \rightarrow \infty$ a.s.

Since \hat{J}_x under $\mathbb{P}^{(-\delta)}$ is distributed as $J(x)$ under \mathbb{P}^δ , we observe that the (unconditional) law of $X^{(1),-}$ under $\mathbb{P}^{(-\delta)}$ is equal to that $(X_t, 0 \leq t \leq \tau_0(x))$, under \mathbb{P}^δ , after removing all excursions above 0.

Proof. Let $m > 0$. By Remark 4.2, conditionally on $\{H_x = m\}$, the process X^{-,H_x} is distributed as $(X_t^{-,m}, t < A_\infty^{-,m})$ under $\mathbb{P}^{(-\delta)}(\cdot | L(\infty, m) = x)$.

Note that conditionally on $\{H_x = m\}$, $S_x = \sup\{t > 0 : X_t \leq m\}$ is the last passage time of X at m . By the duality of Proposition 2.14, under $\mathbb{P}^{(-\delta)}(\cdot | L(\infty, m) = x)$, the process $(X_{A_\infty^{-,m-t}}^{-,m} - m, t < A_\infty^{-,m})$ is distributed as $\{X_t^{-,0}, 0 \leq t \leq A_{T-m}^{-,0}\}$ under $\mathbb{P}^\delta(\cdot | L(T-m, 0) = x)$, the process $(X_t, 0 \leq t \leq T-m)$ obtained by removing all positive excursions. Furthermore, remark that $\hat{J}(x)$ corresponds to $I_{\mathbf{g}_m}$ which is defined for the process $(X_t, 0 \leq t \leq T-m)$ under $\mathbb{P}^\delta(\cdot | L(T-m, 0) = x)$. Then $\mathbb{P}^{(-\delta)}(|\hat{J}_x| \in \cdot | H_x = m) = \mathbb{P}^\delta(|I_{\mathbf{g}_m}| \in \cdot | L(T-m, 0) = x)$. We deduce that for any $0 < a < m$, the conditional law of the process $(X_{A_\infty^{-,m-t}}^{-,m} - m, t < A_\infty^{-,m})$ under $\mathbb{P}^{(-\delta)}(\cdot | H_x = m, \hat{J}_x = -a)$ is the same as the conditional law of the process $\{X_t^{-,0}, 0 \leq t \leq A_{T-m}^{-,0}\}$ under $\mathbb{P}^\delta(\cdot | L(T-m, 0) = x, I_{\mathbf{g}_m} = -a)$.

Then we may apply Theorem 3.3 (ii) to see that conditionally on $\{H_x = m, \hat{J}_x = -a\}$, $X^{(1),-}$, $X^{(2)}$, and $X^{(3)}$ are independent, and

- $X^{(1),-}$ is distributed as $(X_t^{-,0}, t \leq A_{\tau_0(x)}^{-,0})$ under $\mathbb{P}^\delta(\cdot | J(x) = -a)$, where $(X_t^{-,0}, t \leq A_{\tau_0(x)}^{-,0})$ is the process obtained from $(X_t, 0 \leq t \leq \tau_0(x))$ by removing all positive excursions;
- $X^{(2)}$ is distributed as a three-dimensional Bessel process $(R_t)_{0 \leq t \leq T_a}$ killed at $T_a := \inf\{t > 0 : R_t = a\}$;
- $X^{(3)}$ is distributed as $(X_t, 0 \leq t \leq T_{-(m-a)})$ under $\mathbb{P}^\delta(\cdot | T_{-(m-a)} < T_a)$.

Moreover

$$\begin{aligned}\mathbb{P}^{(-\delta)}(|\widehat{J}_x| \in da \mid H_x = m) &= \mathbb{P}^\delta(I_{\mathbf{g}_m} \in da \mid L(T_{-m}, 0) = x) \\ &= \frac{x}{2a^2} e^{-\frac{x}{2a} + \frac{x}{2m}} 1_{\{0 < a < m\}} da,\end{aligned}$$

where the last equality follows from (3.1).

Recall ([8]) the law of H_x under $\mathbb{P}^{(-\delta)}$: For $\delta > 2$,

$$\mathbb{P}(H_x \in dm)/dm = \frac{\left(\frac{x}{2}\right)^{\frac{\delta}{2}-1}}{\Gamma\left(\frac{\delta}{2}-1\right)} m^{-\frac{\delta}{2}} e^{-\frac{x}{2m}}, \quad m > 0.$$

We get

$$\mathbb{P}^{(-\delta)}(|\widehat{J}_x| \in da) = \frac{1}{\Gamma\left(\frac{\delta}{2}\right)} \left(\frac{x}{2}\right)^{\frac{\delta}{2}} a^{-(\frac{\delta}{2}+1)} e^{-\frac{x}{2a}} da, \quad a > 0,$$

which implies (i).

For any bounded continuous functionals F_1, F_2, F_3 on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, we have

$$\begin{aligned}& \mathbb{E}^{(-\delta)}[F_1(X^{(1, \leq 0)}) F_2(X^{(2)}) F_3(X^{(3)}) \mid \widehat{J}_x = -a] \\ &= \int_a^\infty \frac{\mathbb{P}^{(-\delta)}(|\widehat{J}_x| \in da, H_x \in dm)}{\mathbb{P}(|\widehat{J}_x| \in da)} \mathbb{E}^{(-\delta)}[F_1(X^{(1, -)}) F_2(X^{(2)}) F_3(X^{(3)}) \mid \widehat{J}_x = -a, H_x = m] \\ &= \left(\frac{\delta}{2} - 1\right) \int_a^\infty dma^{\frac{\delta}{2}-1} m^{-\frac{\delta}{2}} \mathbb{E}^\delta[F_1(X_t^{-,0}, t \leq A_{\tau_0(x)}^{-,0}) \mid J(x) = -a] \mathbb{E}[F_2(R_t, t \leq T_a)] \\ & \quad \times \mathbb{E}^\delta[F_3(X_t, t \leq T_{-(m-a)}) \mid T_{-(m-a)} < T_a] \\ &= \mathbb{E}^\delta[F_1(X_t^{-,0}, t \leq A_{\tau_0(x)}^{-,0}) \mid J(x) = -a] \mathbb{E}[F_2(R_t, t \leq T_a)] \times \\ & \quad \left(\frac{\delta}{2} - 1\right) \int_0^1 duu^{\frac{\delta}{2}-2} \mathbb{E}^\delta[F_3(X_t, t \leq T_{-a(1-u)/u}) \mid T_{-a(1-u)/u} < T_a],\end{aligned}$$

which gives (ii) and completes the proof of the Proposition. \square

5 The perturbed Bessel process and its rescaling at a stopping time

We rely on the paper of Doney, Warren and Yor [7], restricting our attention to the case of dimension $d = 3$. For $\kappa < 1$, the κ -perturbed Bessel process of dimension $d = 3$ starting from $a \geq 0$ is the process $(R_{3,\kappa}, t \geq 0)$ solution of

$$(5.1) \quad R_{3,\kappa}(t) = a + W_t + \int_0^t \frac{ds}{R_{3,\kappa}(s)} + \kappa(S_t^{R_{3,\kappa}} - a),$$

where $S_t^{R_{3,\kappa}} = \sup_{0 \leq s \leq t} R_{3,\kappa}(s)$ and W is a standard Brownian motion. For $a > 0$, it can be constructed as the law of X under the measure $\mathbb{P}^{3,\kappa}$ defined by

$$(5.2) \quad \mathbb{P}^{3,\kappa}|_{\mathcal{F}_t} = \frac{1}{a^{1-\kappa}} \frac{X_{t \wedge T_0}}{(S_{t \wedge T_0})^\kappa} \widetilde{\mathbb{P}}^\delta|_{\mathcal{F}_t}$$

where $\delta := 2(1 - \kappa)$ and $\tilde{\mathbb{P}}^\delta$ is defined in Notation 2.1. The next proposition is very related to Lemma 5.1 of [7].

Proposition 5.1. *Let $\kappa < 1$ and $\delta := 2(1 - \kappa)$. The point process $\{(q, \mathbf{e}_{X,q}^\downarrow), q \in \mathcal{Q}_X^\downarrow\}$ under $\mathbb{P}^{3,\kappa}$ is distributed as the Poisson point process*

$$\{(q, \mathbf{e}_q^\downarrow), q \in \mathcal{Q}_{\frac{\delta}{2}}^\downarrow \text{ such that } \mathbf{e}_q^\downarrow \subset (0, q)\}.$$

Consequently, the excursions of $R_{3,\kappa}$ below its supremum, seen as unrooted loops, are distributed as the loops of the Brownian loop soup $\mathcal{L}_{\frac{\delta}{2}}$ which entirely lie in the positive half-line.

Remark 5.2. (i) The intensity measure of this Poisson point process has been computed in (ii) of Lemma 2.3.

(ii) Similarly to Section 5.1 of Lupu [15], one can construct the process $R_{3,\kappa}$ from the loops of the Brownian loop soup $\mathcal{L}_{\frac{\delta}{2}}$ which entirely lie in $(0, \infty)$, by rooting them at their maxima and gluing them in the increasing order of their maxima.

Proof of Proposition 5.1. For any Borel nonnegative function F , and any $0 < s < s'$, we compute

$$\mathbb{E}^{3,\kappa} \left[F \left((q, \mathbf{e}_{X,q}^\downarrow), q \in \mathcal{Q}_X^\downarrow \cap [s, s'] \right) \right].$$

Notice that the integrand is measurable with respect to the σ -algebra $\sigma(X_t, t \in [T_s^{R_{3,\kappa}}, T_{s'}^{R_{3,\kappa}}])$, where $T_s^{R_{3,\kappa}} := \inf\{t > 0 : R_{3,\kappa}(t) = s\}$. Since $(R_{3,\kappa}, S^{R_{3,\kappa}})$ is a Markov process, by the strong Markov property at time $T_s^{R_{3,\kappa}}$ and the absolute continuity (5.2) with $a = s$ there, the previous expectation is equal to

$$s^{\kappa-1} \frac{s'}{(s')^\kappa} \tilde{\mathbb{E}}^\delta \left[F \left((q, \mathbf{e}_{X,q}^\downarrow), q \in \mathcal{Q}_X^\downarrow \cap [s, s'] \right) 1_{\{T_0 \circ \theta_{T_s} > T_{s'}\}} \right]$$

where θ is the shift operator. Write

$$F \left((q, \mathbf{e}_{X,q}^\downarrow), q \in \mathcal{Q}_X^\downarrow \cap [s, s'] \right) 1_{\{T_0 \circ \theta_{T_s} > T_{s'}\}} = F \left((q, \mathbf{e}_{X,q}^\downarrow), q \in \mathcal{Q}_X^\downarrow \cap [s, s'], -q \notin \mathbf{e}_{X,q}^\downarrow \right) 1_{\mathcal{E}}$$

where \mathcal{E} is the event that the set of $(q, \mathbf{e}_{X,q}^\downarrow)$ such that $-q \in \mathbf{e}_{X,q}^\downarrow$ is empty.

Recall that the collection of $(q, \mathbf{e}_{q,X}^\downarrow)$ for $q \in \mathcal{Q}_X^\downarrow$ is a Poisson point process by Proposition 2.2. By the independence property of Poisson point processes, we deduce that

$$\mathbb{E}^{3,\kappa} \left[F \left((q, \mathbf{e}_{X,q}^\downarrow), q \in \mathcal{Q}_X^\downarrow \cap [s, s'] \right) \right] = c \tilde{\mathbb{E}}^\delta \left[F \left((q, \mathbf{e}_{X,q}^\downarrow), q \in \mathcal{Q}_X^\downarrow \cap [s, s'], -q \notin \mathbf{e}_{X,q}^\downarrow \right) \right]$$

for some constant c which is necessarily 1. Use again Proposition 2.2 to complete the proof. \square

In the rest of this section, we will extend the definition of perturbed Bessel processes to allow some positive local time at 0.

Let $x \geq 0$. We define a kind of *perturbed Bessel process* $R_{3,\kappa}^x$ with local time x at position 0. More precisely, for $\kappa < 1$ and $\delta = 2(1 - \kappa)$, we denote by $R_{3,\kappa}^x$ the process obtained by concatenating $-X^\delta$ up to time $\tau_0(x)$, biased by $|J(x)|^{\frac{\delta}{2}-1}$, followed by a Bessel of dimension 3 killed when hitting $|J(x)|$, followed by the κ -perturbed process $R_{3,\kappa}$ starting from $|J(x)|$. Corollary 3.4⁶ shows that, when $x > 0$, $R_{3,\kappa}^x$ is the limit in distribution of the process $(-X_t, t \leq T_{-m})$ under $\mathbb{P}^\delta(\cdot | L(T_{-m}, 0) = x)$ as $m \rightarrow \infty$. Clearly when $x = 0$, $R_{3,\kappa}^0$ coincides with $R_{3,\kappa}$ defined previously in (5.1).

Theorem 5.3. *Suppose $\kappa < 1$ and let $\delta := 2(1 - \kappa)$. Fix $m > 0$ and $x \geq 0$. Let the space-change*

$$\theta(z) := \begin{cases} -\frac{mz}{m+z} & \text{if } z \geq 0, \\ -z & \text{if } z < 0, \end{cases}$$

and the time-change

$$A_t := \int_0^t (\theta'(R_{3,\kappa}^x(s)))^2 ds, \quad t \geq 0.$$

If \tilde{X} is defined via $\theta(R_{3,\kappa}^x(t)) := \tilde{X}_{A_t}$, then \tilde{X} is distributed as $(X_t, 0 \leq t \leq T_{-m})$ under $\mathbb{P}^\delta(\cdot | L(T_{-m}, 0) = x)$.

Theorem 5.3 is an extension of Theorem 2.2 equation (2.5) of Doney, Warren, Yor [7] [which corresponds to the case $x = 0$ and $m = 1$].

Proof. First we describe the excursions above infimum of X under $\mathbb{P}^\delta(\cdot | L(T_{-m}, 0) = x)$ in terms of the Brownian loop soup $\mathcal{L}_{\frac{\delta}{2}}$. For the loops which hit 0, we use again the same observation as in the proof of (3.1): conditionally on $\{\ell_\gamma^0 : \gamma \in \mathcal{L}_{\frac{\delta}{2}}, 0 \in \gamma\}$, the loops γ such that $\min \gamma > -m$ are independent Brownian motions stopped at local time given by ℓ_γ^0 , conditioned on not hitting $-m$.

Remark that the set $\{\ell_\gamma^0 : \gamma \in \mathcal{L}_{\frac{\delta}{2}}, 0 \in \gamma, \min \gamma > -m\}$ is equal to the (non-ordered) set $\{x_{i-1}^{(m)} - x_i^{(m)}\}_{i \geq 1}$ in the notation of Proposition 2.6, and $L(T_{-m}, 0) = T_{-m}^J$. By Proposition 2.6, conditionally on $\{L(T_{-m}, 0) = x\}$, the ordered sequence of $(\ell_\gamma^0 : \gamma \in \mathcal{L}_{\frac{\delta}{2}}, 0 \in \gamma, \min \gamma > -m)$ is distributed as $x(P_{(1)}, P_{(2)}, \dots)$, where $(P_{(1)}, P_{(2)}, \dots)$ has the Poisson–Dirichlet distribution of parameter $\frac{\delta}{2}$.

Note that the loops γ of $\mathcal{L}_{\frac{\delta}{2}}$ such that $\gamma \subset (-m, 0)$ are independent of $L(T_{-m}, 0)$. Then the excursions above infimum of X under $\mathbb{P}^\delta(\cdot | L(T_{-m}, 0) = x)$ consists in the superposition of:

- the loops γ of the Brownian loop soup $\mathcal{L}_{\frac{\delta}{2}}$ such that $\gamma \subset (-m, 0)$;

⁶Note that $R_{3,\kappa}$, starting from $a := |J(x)|$, is distributed as the opposite of a PRBM X starting from $-a$ and “conditioned to stay negative”.

- an independent collection of independent Brownian motions stopped at local time $x(P_{(1)}, P_{(2)}, \dots)$, where $(P_{(1)}, P_{(2)}, \dots)$ has the Poisson–Dirichlet distribution of parameter $\frac{\delta}{2}$.

Let $m \rightarrow \infty$, we deduce that the excursions below supremum of $R_{3,\kappa}^x$ (seen as unrooted loops) consists in the superposition of:

- the loops γ of the Brownian loop soup $\mathcal{L}_{\frac{\delta}{2}}$ such that $\min \gamma > 0$;
- an independent collection of independent Brownian motions stopped at local time $x(P_{(1)}, P_{(2)}, \dots)$, where $(P_{(1)}, P_{(2)}, \dots)$ has the Poisson–Dirichlet distribution of parameter $\frac{\delta}{2}$.

Note that via the transformation $\theta(R_{3,\kappa}^x(t)) = \tilde{X}_{A_t}$, the excursions of $R_{3,\kappa}^x$ below their current supremum are transformed into excursions of \tilde{X} above their current infimum. Since the law of $R_{3,\kappa}^x$ is characterized by the law of its excursions below their current supremum (exactly as Remark 5.2 (ii)), and the law of \tilde{X} is characterized by the law of its excursions above their current infimum in a similar way, we only have to focus on the law of those excursions and show that applying the transformation in space θ and in time A_t^{-1} , say Φ ,⁷

(a) the loops γ of the Brownian loop soup $\mathcal{L}_{\frac{\delta}{2}}$ such that $\min \gamma > 0$ are transformed into loops $\tilde{\gamma}$ of $\mathcal{L}_{\delta/2}$ such that $\max \tilde{\gamma} < 0$ and $\min \tilde{\gamma} > -m$;

(b) for any $\ell > 0$, a Brownian motion $(B_t, 0 \leq t \leq \tau_\ell^B)$ stopped at local time ℓ is transformed into $(B_t, 0 \leq t \leq \tau_\ell^B)$ conditioned on $\{\inf_{0 \leq t \leq \tau_\ell^B} B_t > -m\}$.

To prove (a), we remark that for any loop γ with $\min \gamma > 0$, $\min \Phi(\gamma) = \theta(a)$ with $a := \max \gamma$, and $\Phi(\gamma)^\dagger = \Phi(a - \gamma^\dagger) - \theta(a)$. By Lemma 2.3 (i), for any nonnegative measurable function f on $\mathbb{R}_- \times \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, we have

$$\begin{aligned} & \mathbb{E} \left[e^{-\sum_{\gamma \in \mathcal{L}_{\delta/2}, \min \gamma > 0} f(\min \Phi(\gamma), \Phi(\gamma)^\dagger)} \right] \\ &= \exp \left(-\delta \int_0^\infty da \int \mathbf{n}^+(de) (1 - e^{-f(\theta(a), \Phi(a-\epsilon) - \theta(a))}) 1_{\{\max \epsilon < a\}} \right). \end{aligned}$$

Let $h > 0$. Williams' description of the Itô measure says that under $\mathbf{n}^+(\cdot \mid \max \mathbf{e} = h)$, the excursion e can be split into two independent three-dimensional Bessel processes run until they hit h . For a three-dimensional Bessel process R starting from 0, the Itô formula together with the Dubins-Schwarz representation yield that $-\Phi(R)$ is still a three dimensional Bessel process run until it hits m (this can also be seen as a special case of Theorem 2.2 equation (2.5) of Doney, Warren, Yor [7] by taking $\alpha = 0$ there). It follows that under $\mathbf{n}^+(\cdot \mid \max \mathbf{e} = h)$, $\Phi(a - \mathbf{e}) - \theta(a)$ is distributed as \mathbf{e} under $\mathbf{n}^+(\cdot \mid \max \mathbf{e} =$

⁷More precisely for any continuous real-valued process $(\gamma_t, t \geq 0)$, $\Phi(\gamma)$ is the process defined by $\theta(\gamma_t) = \Phi(\gamma) \left(\int_0^t (\theta'(\gamma_s))^2 ds \right)$.

$|\theta(a-h) - \theta(a)|$). Consequently, for any $a > 0$,

$$\begin{aligned}
& \int \mathbf{n}^+(\mathrm{d}\boldsymbol{\epsilon})(1 - e^{-f(\theta(a), \Phi(a-\boldsymbol{\epsilon})-\theta(a))})1_{\{\max \boldsymbol{\epsilon} < a\}} \\
&= \int_0^a \frac{\mathrm{d}h}{2h^2} \int (1 - e^{-f(\theta(a), -\boldsymbol{\epsilon})})\mathbf{n}^+(\mathrm{d}\boldsymbol{\epsilon} \mid \max \boldsymbol{\epsilon} = \theta(a-h) - \theta(a)) \\
&= \frac{m^2}{(m+a)^2} \int_0^{|\theta(a)|} \frac{\mathrm{d}s}{2s^2} \int (1 - e^{-f(\theta(a), -\boldsymbol{\epsilon})})\mathbf{n}^+(\mathrm{d}\boldsymbol{\epsilon} \mid \max \boldsymbol{\epsilon} = s) \\
&= \frac{m^2}{(m+a)^2} \int \mathbf{n}^+(\mathrm{d}\boldsymbol{\epsilon})(1 - e^{-f(\theta(a), -\boldsymbol{\epsilon})})1_{\{\max \boldsymbol{\epsilon} < |\theta(a)|\}},
\end{aligned}$$

where the second equality follows from a change of variables $s = \theta(a-h) - \theta(a)$. It follows that

$$\begin{aligned}
& \mathbb{E} \left[e^{-\sum_{\gamma \in \mathcal{L}_{\frac{\delta}{2}}, \min \gamma > 0} f(\min \Phi(\gamma), \Phi(\gamma)^\dagger)} \right] \\
&= \exp \left(-\delta \int_0^\infty \mathrm{d}a \frac{m^2}{(m+a)^2} \int \mathbf{n}^+(\mathrm{d}\boldsymbol{\epsilon})(1 - e^{-f(\theta(a), -\boldsymbol{\epsilon})})1_{\{\max \boldsymbol{\epsilon} < |\theta(a)|\}} \right) \\
&= \exp \left(-\delta \int_0^m \mathrm{d}y \int \mathbf{n}^+(\mathrm{d}\boldsymbol{\epsilon})(1 - e^{-f(-y, -\boldsymbol{\epsilon})})1_{\{\max \boldsymbol{\epsilon} < y\}} \right),
\end{aligned}$$

after a change of variables $y = |\theta(a)|$. This proves (a).

It remains to show (b). Let $(\boldsymbol{\epsilon}_s, s > 0)$ be the standard Brownian excursion process. It is well known that $(B_s, 0 \leq s \leq \tau_\ell^B)$ can be constructed from $(\boldsymbol{\epsilon}_s, s \leq \ell)$ (see Revuz and Yor [21] Chapter XII, Proposition 2.5). Observe that the process $\Phi(B_s, 0 \leq s \leq \tau_\ell^B)$ can be constructed from $(\Phi(\boldsymbol{\epsilon}_s), s \leq \ell)$ in the same way. To prove (b), it is enough to show that $(\Phi(\boldsymbol{\epsilon}_s), s \leq \ell)$ under the Itô measure \mathbf{n} , is distributed as $(\boldsymbol{\epsilon}_s, s \leq \ell)$ under $\mathbf{n}(\cdot \mid \inf_{s \leq \ell} \min \boldsymbol{\epsilon}_s > -m)$. To this end, we use the same observation as in the proof of (a): for any $h > 0$, under $\mathbf{n}^+(\cdot \mid \max \boldsymbol{\epsilon} = h)$, $\Phi(\boldsymbol{\epsilon})$ is distributed as $-\boldsymbol{\epsilon}$ under $\mathbf{n}^+(\cdot \mid \max \boldsymbol{\epsilon} = |\theta(h)|)$. Consequently, for any nonnegative measurable function f on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$,

$$\begin{aligned}
\int \mathbf{n}^+(\mathrm{d}\boldsymbol{\epsilon})(1 - e^{-f(\Phi(\boldsymbol{\epsilon}))}) &= \int_0^\infty \frac{\mathrm{d}h}{2h^2} \int (1 - e^{-f(-\boldsymbol{\epsilon})})\mathbf{n}^+(\mathrm{d}\boldsymbol{\epsilon} \mid \max \boldsymbol{\epsilon} = |\theta(h)|) \\
&= \int_0^m \frac{\mathrm{d}s}{2s^2} \int (1 - e^{-f(-\boldsymbol{\epsilon})})\mathbf{n}^+(\mathrm{d}\boldsymbol{\epsilon} \mid \max \boldsymbol{\epsilon} = s) \\
&= \int \mathbf{n}^+(\mathrm{d}\boldsymbol{\epsilon})(1 - e^{-f(-\boldsymbol{\epsilon})})1_{\{\max \boldsymbol{\epsilon} < m\}},
\end{aligned}$$

where the second equality follows from a change of variables $s = |\theta(h)|$. It follows that

$$\begin{aligned}
\int \mathbf{n}(\mathrm{d}\boldsymbol{\epsilon})(1 - e^{-f(\Phi(\boldsymbol{\epsilon}))}) &= \int \mathbf{n}^-(\mathrm{d}\boldsymbol{\epsilon})(1 - e^{-f(-\boldsymbol{\epsilon})}) + \int \mathbf{n}^+(\mathrm{d}\boldsymbol{\epsilon})(1 - e^{-f(-\boldsymbol{\epsilon})})1_{\{\max \boldsymbol{\epsilon} < m\}} \\
&= \int \mathbf{n}(\mathrm{d}\boldsymbol{\epsilon})(1 - e^{-f(\boldsymbol{\epsilon})})1_{\{\min \boldsymbol{\epsilon} > -m\}},
\end{aligned}$$

which together with the exponential formula for the excursion process, yield that $(\Phi(\mathbf{e}_s), s \leq \ell)$ under \mathbf{n} is distributed as $(\mathbf{e}_s, s \leq \ell)$ under $\mathbf{n}(\cdot | \inf_{s \leq \ell} \min \mathbf{e}_s > -m)$. This completes the proof of Theorem 5.3. \square

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References

- [1] Abraham, R. and Mazliak, L. (1998). Branching properties of Brownian paths and trees. *Expo. Math.* **16**, No. 1, 59–73.
- [2] Bertoin, J. and Le Gall, J.-F. (2005). Stochastic flows associated to coalescent processes. II. Stochastic differential equations. *Ann. Inst. H. Poincaré Probab. Statist.* **41**, 307–333.
- [3] Carmona, P., Petit, F. and Yor, M. (1994). Sur les fonctionnelles exponentielles de certains processus de Lévy. *Stochastics and Stochastic Reports* **42**, 71–101.
- [4] Carmona, P., Petit, F. and Yor, M. (1994). Some extensions of the arc sine law as partial consequences of the scaling property of Brownian motion. *Probab. Theory Related Fields* **100**, 1–29.
- [5] Chaumont, L. and Doney, R.A. (1999). Pathwise uniqueness for perturbed versions of Brownian motion and reflected Brownian motion. *Probab. Theory Related Fields* **113**, 519–534.
- [6] Davis, B. (1999). Brownian motion and random walk perturbed at extrema. *Probab. Theory Related Fields* **113**, 501–518.
- [7] Doney, R.A., Warren, J. and Yor, M. (1998). Perturbed Bessel processes. *Séminaire de probabilités* **32**, 237–249.
- [8] Gôing-Jaesckhe, A. and Yor, M. (2003). A survey and some generalizations of Bessel processes. *Bernoulli* **9**, 313–349.
- [9] Kingman, J.F.C. (1975). Random discrete distributions. *J. R. Statist. Soc.* **37**, 1–22.
- [10] Lawler, G.F. and Werner, W. (2004). The Brownian loop soup. *Probab. Theory Related Fields* **44**, 197–217.
- [11] Le Gall, J.-F. and Yor, M. (1986). Excursions Browniennes et carrés de processus de Bessel. *C. R. Acad. Sci. Paris*, **302**, 641–643.

- [12] Le Gall, J.-F. and Yor, M. (1987). Étude asymptotique des enlacements du mouvement brownien autour des droites de l'espace. *Probab. Theory Related Fields*, **74**, 617–635.
- [13] Le Jan, Y. (2010). Markov loops and renormalization. *Ann. Probab.*, **38**, 1280–1319.
- [14] Le Jan, Y. (2011). *Markov Paths, Loops and Fields*. École d'été Saint-Flour XXXVIII 2008, Lecture Notes in Mathematics **2026**. Springer, Berlin.
- [15] Lupu, T. (2018). Poisson ensembles of loops of one-dimensional diffusions. *Mém. Soc. Math. Fr. (N.S.)*, **158**.
- [16] McCloskey, J.W. (1965). A model for the distribution of individuals by species in an environment. *Unpublished Ph.D. thesis, Michigan State University*.
- [17] Perman, M., Pitman, J. and Yor, M. (1992). Size-biased sampling of Poisson point processes and excursions. *Probab. Theory Related Fields* **92**, 21–39.
- [18] Perman, M. (1996). An excursion approach to Ray–Knight theorems for perturbed Brownian motion. *Stoch. Proc. Appl.* **63**, 67–74.
- [19] Perman, M. and Werner, W. (1997). Perturbed Brownian motions. *Probab. Theory Related Fields* **108**, 357–383.
- [20] Petit, F. (1992). Quelques extensions de la loi de l'arcsinus. *C. R. Acad. Sci. Paris* **315**, 855–858.
- [21] Revuz, D. and Yor, M. (1999). Continuous martingales and Brownian motion. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, Third Edition.
- [22] Warren, J. (2005). A stochastic flow arising in the study of local times. *Probab. Theory Related Fields* **133**, 559–572.
- [23] Warren, J. and Yor, M. (1998). The Brownian burglar: conditioning Brownian motion by its local time process. *Sém. Probab. XXXII*, pp. 328–342, Lecture Notes Math. **1686**, Springer, Berlin.
- [24] Werner, W. (1995). Some remarks on perturbed reflecting brownian motion. *Sém. Probab.*, **29**, 37–43.
- [25] Yor, M. (1992). Some aspects of Brownian motion. Part I. *Lectures in Mathematics ETH Zürich*, Birkhäuser Verlag, Basel.