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Some unexpected properties of Littlewood-Richardson coefficients

Maxime Pelletier and Nicolas Ressayre

July 5, 2021

Abstract

We are interested in identities between Littlewood-Richardson coefficients, and hence in comparing different tensor product decompositions of the irreducible modules of the linear group $\mathrm{GL}_n(\mathbb{C})$. A family of partitions – called near-rectangular – is defined, and we prove a stability result which basically asserts that the decomposition of the tensor product of two representations associated to near-rectangular partitions does not depend on n . Given a partition λ , of length at most n , denote by $V_n(\lambda)$ the associated simple $\mathrm{GL}_n(\mathbb{C})$ -module. We conjecture that, if λ is near-rectangular and μ any partition, the decompositions of $V_n(\lambda) \otimes V_n(\mu)$ and $V_n(\lambda)^* \otimes V_n(\mu)$ coincide modulo a mysterious bijection. We prove this conjecture if μ is also near-rectangular and report several computer-assisted computations which reinforce our conjecture.

1 Introduction

In this paper we study some properties of Littlewood-Richardson coefficients when some partitions are near-rectangular (see below for a precise definition). Let $n \geq 2$ be an integer. The irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ are parametrized by all non-increasing sequences of n integers. As is often the case, we focus on the polynomial representations among those, which correspond to the sequences containing only non-negative integers, also called partitions with at most n parts. Denote by $V_n(\lambda)$ the representation of $\mathrm{GL}_n(\mathbb{C})$ associated to such a partition λ . The Littlewood-Richardson coefficient $c_{\lambda\mu}^\nu$ appears in the tensor product decomposition

$$V_n(\lambda) \otimes V_n(\mu) = \bigoplus_{\nu \in \Lambda_n} c_{\lambda\mu}^\nu V_n(\nu),$$

where Λ_n denotes the set of partitions of length at most n . Denote moreover by $V_n(\lambda)^*$ the $\mathrm{GL}_n(\mathbb{C})$ -module which is dual to $V_n(\lambda)$.

Obviously, given λ and μ two partitions with at most n parts, $V_n(\lambda) \otimes V_n(\mu)$ and $V_n(\lambda)^* \otimes V_n(\mu)$ are not isomorphic as $\mathrm{GL}_n(\mathbb{C})$ -modules. Nevertheless, we may want to compare their decompositions in irreducible modules.

Problem 1. Compare the two $\mathrm{GL}_n(\mathbb{C})$ -modules, $V_n(\lambda) \otimes V_n(\mu)$ and $V_n(\lambda)^* \otimes V_n(\mu)$.

For example, Coquereaux-Zuber [CZ11] showed that the sums of the multiplicities of these two representations coincide. For $\lambda \in \Lambda_n$, set $\lambda^* = \lambda_1 - \lambda_n \geq \lambda_1 - \lambda_{n-1} \geq \dots \geq \lambda_1 - \lambda_2 \geq 0$. Then $V_n(\lambda)^*$ and $V_n(\lambda^*)$ only differ by a tensor power of the determinant. More precisely, $V_n(\lambda)^*$ is the irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ corresponding to the sequence $-\lambda_n \geq -\lambda_{n-1} \geq \dots \geq -\lambda_1$. As a consequence $V_n(\lambda^*)$ is isomorphic to $V_n(\lambda)^*$ tensored by the λ_1 -th power of the determinant representation of $\mathrm{GL}_n(\mathbb{C})$.

Following these observations, the Coquereaux-Zuber's result can be written as

$$\sum_{\nu \in \Lambda_n} c_{\lambda\mu}^\nu = \sum_{\nu \in \Lambda_n} c_{\lambda^*\mu}^\nu. \quad (1)$$

More generally, we may want to compare $\{c_{\lambda\mu}^\nu : \nu \in \Lambda_n\}$ and $\{c_{\lambda^*\mu}^\nu : \nu \in \Lambda_n\}$ as multisets. For instance, for $n = 3$, $\lambda = (5, 3)$ and $\mu = (6, 3)$, $V_3(\lambda) \otimes V_3(\mu)$ decomposes as

$$\begin{aligned} & V_3(7, 5, 5) + V_3(7, 7, 3) + V_3(8, 8, 1) + V_3(9, 4, 4) + V_3(9, 8) + V_3(10, 7) \\ & + V_3(11, 6) + V_3(11, 3, 3) + V_3(11, 4, 2) + V_3(11, 5, 1) + V_3(6, 6, 5) \\ & + 2V_3(7, 6, 4) + 2V_3(8, 5, 4) + 2V_3(8, 7, 2) + 2V_3(9, 7, 1) + 2V_3(10, 4, 3) \\ & + 2V_3(10, 5, 2) + 2V_3(10, 6, 1) + 3V_3(8, 6, 3) + 3V_3(9, 5, 3) + 3V_3(9, 6, 2), \end{aligned}$$

while $V_3(\lambda^*) \otimes V_3(\mu)$ expands as

$$\begin{aligned} & V_3(7, 7, 2) + V_3(8, 4, 4) + V_3(10, 3, 3) + V_3(8, 8) + V_3(9, 7) + V_3(10, 6) \\ & + V_3(11, 3, 2) + V_3(11, 4, 1) + V_3(6, 5, 5) + V_3(6, 6, 4) + 2V_3(7, 5, 4) \\ & + 2V_3(7, 6, 3) + 2V_3(8, 7, 1) + 2V_3(9, 4, 3) + 2V_3(9, 6, 1) + 2V_3(10, 4, 2) \\ & + 2V_3(10, 5, 1) + 3V_3(8, 5, 3) + 3V_3(8, 6, 2) + 3V_3(9, 5, 2) + V_3(11, 5). \end{aligned}$$

One can then notice that the multiplicities in the two expansions are the same: 11 occurrences of “1”, 7 occurrences of “2” and 3 occurrences of “3” in both cases. A natural question is then: is this always true?

We check in this article that it is the case for $\mathrm{GL}_3(\mathbb{C})$ in general (note that this was already proven by Coquereaux and Zuber in [CZ14]; they even give several proofs of this fact in this paper from 2014): using a computer, we are able to compute explicitly, for $n = 3$, the function

$$\begin{aligned} (\Lambda_n^2) \times \mathbb{N} & \longrightarrow \mathbb{N} \\ (\lambda, \mu, c) & \longmapsto \mathrm{Nb}_n(c_{\lambda\mu}^\bullet > c) := \#\{\nu \in \Lambda_n : c_{\lambda\mu}^\nu > c\}. \end{aligned}$$

See Section 4 for details. In this introduction we report on this computation as follows:

Proposition 1. *The function*

$$\begin{aligned} \mathrm{Nb}_3(c_{\lambda\mu}^\bullet > c) : \Lambda_3 \times \Lambda_3 \times \mathbb{N} & \longrightarrow \mathbb{N} \\ (\lambda, \mu, c) & \longmapsto \#\{\nu \in \Lambda_n : c_{\lambda\mu}^\nu > c\} \end{aligned}$$

is piecewise polynomial of degree 2 with 7 cones¹. Moreover,

$$\mathrm{Nb}_3(c_{\lambda\mu}^\bullet > c) = \mathrm{Nb}_3(c_{\lambda^*\mu}^\bullet > c). \quad (2)$$

¹Namely, the cone $\Lambda_3 \times \Lambda_3 \times \mathbb{N}$ is the support of a fan with 7 maximal cones. And, on each one of these 7 cones, the function $\mathrm{Nb}_3(c_{\lambda\mu}^\bullet > c)$ is given by a polynomial in (λ, μ, c) of degree 2.

We then aim to suggest a generalization to $\mathrm{GL}_n(\mathbb{C})$ for any n : we will say that a partition $\lambda \in \Lambda_n$ is *near-rectangular* if $\lambda = \lambda_1 \lambda_2^{n-2} \lambda_n$ for some integers $\lambda_1 \geq \lambda_2 \geq \lambda_n$; that is, if $\lambda_2 = \dots = \lambda_{n-1}$. We formulate the following

Conjecture 1. *Let λ and μ in Λ_n . If λ is near-rectangular then*

$$\forall c \in \mathbb{N} \quad \#\{\nu \in \Lambda_n : c_{\lambda\mu}^\nu = c\} = \#\{\nu \in \Lambda_n : c_{\lambda^*\mu}^\nu = c\}.$$

Equivalently, we conjecture that there exists a bijection $\varphi : \Lambda_n \rightarrow \Lambda_n$, depending on λ and μ such that

$$\forall \nu \in \Lambda_n \quad c_{\lambda\mu}^\nu = c_{\lambda^*\mu}^{\varphi(\nu)},$$

if λ is near-rectangular. In other words, the multisets $\{c_{\lambda\mu}^\nu : \nu \in \Lambda_n\}$ and $\{c_{\lambda^*\mu}^\nu : \nu \in \Lambda_n\}$ are expected to be equal, if λ is near-rectangular.

Note that any partition of size ≤ 3 is near-rectangular. Then, both the last assertion of Proposition 1, and the result of [CZ14] are equivalent to

Corollary 2. *Conjecture 1 holds for $n = 3$.*

In literature, there are a lot of equalities between Littlewood-Richardson coefficients. There are symmetries (see [BR20] and references therein), stabilities (see [BOR15]), reductions (see [CM11]). None of these numerous results really seems to explain whether or why Conjecture 1 should hold. Moreover, there are various combinatorial models for the Littlewood-Richardson coefficients (see [Ful97, Lit95, Zel81, KT99, Vak06, Cos09]). None of these models seems to prove Conjecture 1 easily.

What made this Conjecture 1 even more unexpected for us is that it seems that, should the aforementioned bijection $\varphi : \Lambda_n \rightarrow \Lambda_n$ exist, it does not seem to be expressible simply in terms of λ and μ . Indeed, even if $n = 3$, we remark in Section 4 that $(\lambda, \mu, \nu) \rightarrow (\lambda^*, \mu, \varphi(\nu))$ cannot be linear.

We checked, using Sagemath, Conjecture 1 on a few million examples for $\mathrm{GL}_4(\mathbb{C})$, $\mathrm{GL}_5(\mathbb{C})$, $\mathrm{GL}_6(\mathbb{C})$, and $\mathrm{GL}_{10}(\mathbb{C})$: see Section 7.2 for details. Moreover, we also give in Section 7.2 an example showing that the assumption on λ is truly necessary.

We now consider the tensor products of representations associated to two near-rectangular partitions. Observe that a partition λ of length l parametrizes representations $V_n(\lambda)$ of $\mathrm{GL}_n(\mathbb{C})$ for any $n \geq l$. A priori, $c_{\lambda\mu}^\nu$ could depend on n . It is a classical result (see e.g. [Ful97]) of stability that it actually does not. Our second result is a similar stability result but for partitions of arbitrarily large length. Indeed, fix two near-rectangular partitions $\lambda = \lambda_1 \lambda_2^{n-2} \lambda_n$ and $\mu = \mu_1 \mu_2^{n-2} \mu_n$. We prove that the decomposition of $V_n(\lambda) \otimes V_n(\mu)$ does not depend on $n \geq 4$ but only on the six integers $\lambda_1, \lambda_2, \lambda_n, \mu_1, \mu_2$ and μ_n . More precisely, we get a simple expression of the Littlewood-Richardson coefficients that appear in this tensor product, expression independent of n .

Proposition 3. *Let $n \geq 4$. Let $\lambda = \lambda_1 \lambda_2^{n-2}$ and $\mu = \mu_1 \mu_2^{n-2}$ be two near-rectangular partitions. Let ν be a partition with at most n parts. Then $c_{\lambda\mu}^\nu = 0$ unless $\nu = \nu_1 \nu_2 (\lambda_2 + \mu_2)^{n-4} \nu_{n-1} \nu_n$ for four integers ν_1, ν_2, ν_{n-1} and ν_n such that $\nu_1 \geq \nu_2 \geq \lambda_2 + \mu_2 \geq \nu_{n-1} \geq \nu_n$. In this case*

$$c_{\lambda\mu}^\nu = \#(\llbracket M, m \rrbracket),$$

where

$$M = \max(0, \lambda_2 + \mu_1 - \nu_1, -\mu_2 + \nu_n)$$

and

$$m = \min(\lambda_1 + \mu_1 - \nu_1, \lambda_2 + \mu_1 - \nu_2, -\lambda_2 - \mu_2 + \nu_{n-1} + \nu_n, -\mu_2 + \nu_{n-1}).$$

In particular, this value does not depend on $n \geq 4$.

In Proposition 3, for simplicity we assumed that $\lambda_n = \mu_n = 0$. Yet nothing is lost with such an assumption, since $V_n(\lambda_1 \lambda_2^{n-2} \lambda_n) \simeq \det^{\lambda_n} \otimes V_n((\lambda_1 - \lambda_n)(\lambda_2 - \lambda_n)^{n-2})$.

Proposition 3 positively answers [PW20, Question 2] by giving a much stronger result. It also allows to check a particular case of Conjecture 1. Indeed, with the help of a computer we computed $\text{Nb}_4(c_{\lambda\mu}^\bullet > c)$ for λ and μ near-rectangular. Let $\Lambda_n^{\text{nr}} = \{\lambda_1 \lambda_2^{n-2} \lambda_n : \lambda_1 \geq \lambda_2 \geq \lambda_n\}$ be the set of near-rectangular partitions of length at most n .

Proposition 4. *The function*

$$\begin{aligned} \text{Nb}_4(c_{\lambda\mu}^\bullet > c) : \Lambda_4^{\text{nr}} \times \Lambda_4^{\text{nr}} \times \mathbb{N} &\longrightarrow \mathbb{N} \\ (\lambda, \mu, c) &\longmapsto \#\{\nu \in \Lambda_n : c_{\lambda\mu}^\nu > c\} \end{aligned}$$

is piecewise polynomial of degree 3 with 36 cones. Moreover,

$$\text{Nb}_4(c_{\lambda\mu}^\bullet > c) = \text{Nb}_4(c_{\lambda^*\mu}^\bullet > c). \quad (3)$$

The 36 polynomial functions and cones are given in Section 6.3. As a consequence of Propositions 3 and 4, we get

Corollary 5. *Let $n \geq 4$. Conjecture 1 holds for $\text{GL}_n(\mathbb{C})$ assuming in addition that μ is near-rectangular.*

A much weaker version of Conjecture 1 is

Conjecture 2. *If $\lambda \in \Lambda_n$ is near-rectangular then*

$$\#\{\nu \in \Lambda_n : c_{\lambda\mu}^\nu \neq 0\} = \#\{\nu \in \Lambda_n : c_{\lambda^*\mu}^\nu \neq 0\}.$$

Equivalently, we ask whether, for $\lambda \in \Lambda_n^{\text{nr}}$,

$$\forall \mu \in \Lambda_n \quad \text{Nb}_n(c_{\lambda\mu}^\bullet > 0) = \text{Nb}_n(c_{\lambda^*\mu}^\bullet > 0).$$

For $n = 4$ and λ near-rectangular, we computed $\text{Nb}_4(c_{\lambda\mu}^\bullet > 0)$ and checked Conjecture 2. Here we report on this computation as follows (see Section 6.4 for details).

Proposition 6. *The function*

$$\begin{aligned} \text{Nb}_4(c_{\lambda\mu}^\bullet > 0) : \Lambda_4^{\text{nr}} \times \Lambda_4 &\longrightarrow \mathbb{N} \\ (\lambda, \mu) &\longmapsto \#\{\nu \in \Lambda_n : c_{\lambda\mu}^\nu > 0\} \end{aligned}$$

is piecewise quasi-polynomial² of degree 3 with 205 cones. The only congruence occurring is the parity of $\lambda_1 + |\mu|$. Moreover,

$$\text{Nb}_4(c_{\lambda\mu}^\bullet > 0) = \text{Nb}_4(c_{\lambda^*\mu^*}^\bullet > 0). \quad (4)$$

This symmetry with the complete duality $(\lambda, \mu) \mapsto (\lambda^, \mu^*)$ gives an action of $(\mathbb{Z}/2\mathbb{Z})^2$. This group acts on the 205 pairs (cone, quasi-polynomial) with 83 orbits.*

This work is based on numerous computer aided computations with Barvinok [VSB⁺07], Normaliz [BIS] and SageMath [The20]. Details on these computations can be found on the webpage of the second author [Res20, Supplementary material].

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Remark. After a version of this work was posted on ArXiv, Darij Grinberg offered a solution of our main conjecture in [Gri20]. Therein he defines a piecewise linear involution φ from \mathbb{Z}^n to \mathbb{Z}^n satisfying

$$\forall \nu \in \Lambda_n \quad c_{\lambda\mu}^\nu = c_{\lambda^*\mu^*}^{\varphi(\nu)},$$

if λ is near-rectangular, thus solving our conjecture. An amazing fact is that this bijection does not necessarily map a partition to a partition: if $\varphi(\nu)$ is not a partition then $c_{\lambda\mu}^\nu$ simply vanishes, allowing φ to work.

2 Generalities on the function $\text{Nb}_n(c_{\lambda\mu}^\bullet > c)$

Recall that for $\lambda, \mu \in \Lambda_n$ and $c \in \mathbb{N}$ we set

$$\text{Nb}_n(c_{\lambda\mu}^\bullet > c) = \#\{\nu \in \Lambda_n : c_{\lambda\mu}^\nu > c\}.$$

Since $V_n(\lambda) \otimes V_n(\mu) \simeq V_n(\mu) \otimes V_n(\lambda) \simeq (V_n(\lambda^*) \otimes V_n(\mu^*))^*$ as $\text{SL}_n(\mathbb{C})$ -modules, the function $\text{Nb}_n(c_{\lambda\mu}^\bullet > c)$ satisfies

$$\text{Nb}_n(c_{\lambda\mu}^\bullet > c) = \text{Nb}_n(c_{\mu\lambda}^\bullet > c) = \text{Nb}_n(c_{\lambda^*\mu^*}^\bullet > c) = \text{Nb}_n(c_{\mu^*\lambda^*}^\bullet > c). \quad (5)$$

²See Section 2 right after Proposition 7 for a definition.

Let $1^n \in \Lambda_n$ denote the partition with n parts equal to 1. Then $V_n(1^n)$ is the one dimensional representation of $\mathrm{GL}_n(\mathbb{C})$ given by the determinant. In particular, as an $\mathrm{SL}_n(\mathbb{C})$ -module $V_n(\lambda) \simeq V_n(\lambda - k^n)$ for any k . Set $\Lambda_n^0 = \{\lambda \in \Lambda_n : \lambda_n = 0\}$. It can be seen as the set of dominant weights for the group $\mathrm{SL}_n(\mathbb{C})$. For $\lambda \in \Lambda_n$, set $\bar{\lambda} = \lambda - \lambda_n^n$, the partition obtained by subtracting λ_n to each part of λ . Then

$$\mathrm{Nb}_n(c_{\lambda\mu}^\bullet > c) = \mathrm{Nb}_n(c_{\bar{\lambda}\bar{\mu}}^\bullet > c), \quad \text{and even more} \quad c_{\lambda\mu}^\nu = c_{\bar{\lambda}\bar{\mu}}^{\nu - (\lambda_n + \mu_n)^n}. \quad (6)$$

Set

$$\mathrm{Horn}_n = \{(\lambda, \mu, \nu) \in (\Lambda_n)^3 : c_{\lambda\mu}^\nu \neq 0\}.$$

By a Brion-Knop's result (see [É92]), Horn_n is a finitely generated semigroup. The Knutson-Tao saturation theorem [KT99] shows that Horn_n is the set of integer points in a convex cone, the Horn cone. The Horn cone is polyhedral and the minimal list of inequalities defining it is known (see *e.g.* [Ful00, Bel01, KTW04]). These inequalities contain the Weyl inequalities

$$\nu_{i+j-1} \leq \lambda_i + \mu_j \quad \text{whenever } i + j - 1 \leq n; \quad (7)$$

and are all of the form

$$\sum_{k \in K} \nu_k \leq \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j, \quad (8)$$

for some triples (I, J, K) of three subsets of $\{1, \dots, n\}$ of the same cardinality.

Proposition 7. *Fix $n \geq 0$. The function*

$$\begin{aligned} \mathrm{Nb}_n(c_{\lambda\mu}^\bullet > 0) : \Lambda_n \times \Lambda_n &\longrightarrow \mathbb{N} \\ (\lambda, \mu) &\longmapsto \#\{\nu \in \Lambda_n : c_{\lambda\mu}^\nu > 0\}. \end{aligned}$$

is piecewise quasi-polynomial.

This means that there exist a sub-lattice Λ of $\mathbb{Z}^{2n} \supset \Lambda_n \times \Lambda_n$ of finite index and a collection of piecewise polynomial functions (see Footnote 1) parametrized by the classes in \mathbb{Z}^{2n}/Λ . Applying to (λ, μ) the piecewise polynomial function corresponding to its class, one then gets $\mathrm{Nb}_n(c_{\lambda\mu}^\bullet > 0)$.

Proof. We have

$$\mathrm{Nb}_n(c_{\lambda\mu}^\bullet > 0) = \#\left(\mathrm{Horn}_n \cap (\{(\lambda, \mu)\} \times \Lambda_n)\right). \quad (9)$$

Consider the Horn cone $\mathrm{Horn}_n^{\mathbb{Q}}$ generated by Horn_n . By the discussion above this proposition, it is defined as a subset of \mathbb{Q}^{3n} by an explicit list of linear inequalities, namely the Horn inequalities (8). Knutson-Tao's saturation Theorem [KT99] asserts that Horn_n is precisely the set of integer points (that is, belonging to $(\Lambda_n)^3$) in $\mathrm{Horn}_n^{\mathbb{Q}}$. Now, equality (9) describes $\mathrm{Nb}_n(c_{\lambda\mu}^\bullet > 0)$ as the number of integer points in the affine section of the Horn cone obtained by fixing (λ, μ) .

Moreover each inequality 8 of the Horn cone depends linearly on (λ, μ) . Now the conclusion is a consequence of the general theory of multivariate Ehrhart polynomials (see *e.g.* [BBDL⁺19, Theorem 1.1] or [Stu95]). \square

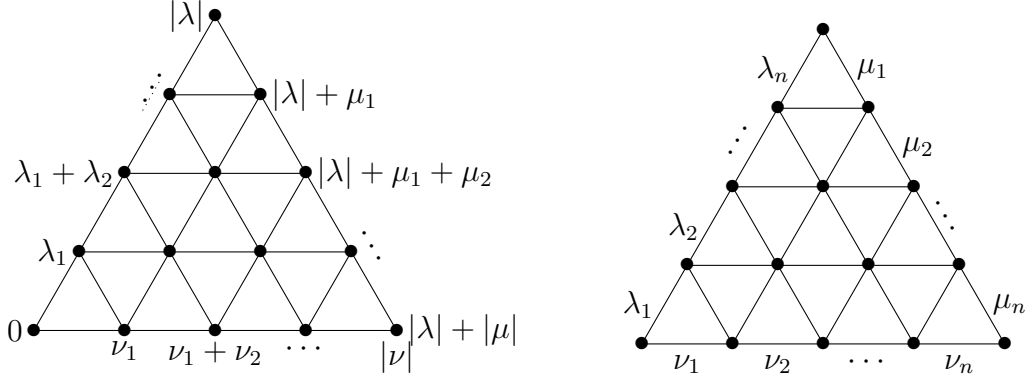


Figure 1: Hives with boundary conditions

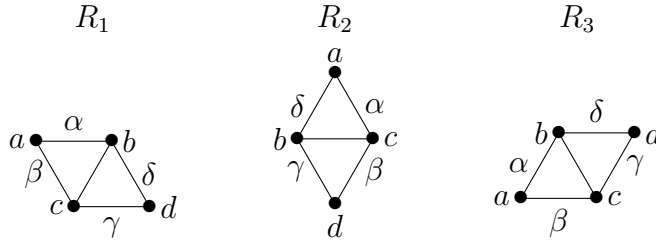


Figure 2: Rhombi

3 The hive model

For later use, we shortly review the hive model that expresses the Littlewood-Richardson coefficients as the number of integer points in polyhedra. Fix an integer $n \geq 2$.

Let λ, μ and ν in Λ_n such that $|\nu| = |\lambda| + |\mu|$. Otherwise $c_{\lambda\mu}^\nu = 0$. Label the $\frac{(n+1)(n+2)}{2}$ vertices of the triangles on Figure 1 with integers, with the labels on the boundaries determined by λ, μ and ν as drawn on the left.

Then, neighbouring vertices define three distinct types of rhombus (see Figure 2), each coming with its own constraint condition on the integers labelling the vertices. For each extracted rhombus we impose the following constraint (using the notation of Figure 2):

$$b + c \geq a + d \tag{10}$$

Alternatively, one can label the edges of the drawing rather than the vertices: each edge is labelled by the difference between the values on its vertices, with an orientation as shown on the right of Figure 1. Then each rhombus gives also a constraint on the labels of the edges, equivalent to the previous one on the labels of the vertices (still using the notation of Figure 2):

$$\beta \geq \delta \quad \text{or equivalently} \quad \alpha \geq \gamma. \tag{11}$$

The equivalence between these two inequalities comes from the fact that $\alpha + \delta = \beta + \gamma$.

By definition a hive is a labelling in $\mathbb{Z}^{\frac{(n+1)(n+2)}{2}}$ satisfying Inequalities (10) for each one of the $3\frac{n(n-1)}{2}$ rhombi. The Knutson-Tao's result is

Theorem 8. (see [KT99, Appendix]) *Let λ, μ and ν in Λ_n . Then $c_{\lambda\mu}^\nu$ is the number of hives with boundary conditions determined by λ, μ and ν as on the left of Figure 1.*

4 The case of GL_3

It is known that the function $(\Lambda_n)^3 \rightarrow \mathbb{N}$, $(\lambda, \mu, \nu) \mapsto c_{\lambda\mu}^\nu$ is piecewise polynomial (see [Ras04]) of degree $\frac{n^2-3n+2}{2}$. For $n = 3$, we get precisely the following.

Proposition 9. *Let $\lambda = (\lambda_1, \lambda_2, 0)$, $\mu = (\mu_1, \mu_2, 0)$, and $\nu = (\nu_1, \nu_2, \nu_3)$ in Λ_3 be such that $|\nu| = |\lambda| + |\mu|$. Then $c_{\lambda\mu}^\nu$ is the number of integer points in the interval*

$$\llbracket \max(\mu_1 - \lambda_2, \mu_2, \nu_1 - \lambda_1, \mu_1 - \nu_3, \nu_2 - \lambda_2, \mu_1 + \mu_2 - \nu_2), \min(\mu_1, \nu_1 - \lambda_2, \mu_1 + \mu_2 - \nu_3) \rrbracket.$$

This statement is well known and can easily be checked using the hive model. Indeed, once λ, μ and ν are fixed, there is only one interior entry x to choose in order to determine the hive. The solution set to the system of the 9 rhombus inequalities is an interval from which the interval of the statement is obtained by translation by $\lambda_1 + \lambda_2$.

Proposition 9 implies that, for any nonnegative integer c , $c_{\lambda\mu}^\nu > c$ if and only if, for any linear form φ and ψ appearing in the min and max respectively, we have $\varphi - \psi \geq c$. Namely $c_{\lambda\mu}^\nu > c$ if and only if

$$\begin{array}{ll} \lambda_1 - \lambda_2 - c \geq 0 & \lambda_2 - c \geq 0 \\ \mu_1 - \mu_2 - c \geq 0 & \mu_2 - c \geq 0 \\ \nu_1 - \nu_2 - c \geq 0 & \nu_2 - \nu_3 - c \geq 0 \\ \lambda_1 + \mu_1 - \nu_1 - c \geq 0 & \lambda_1 + \mu_1 - \nu_2 - \nu_3 - c \geq 0 \\ \lambda_1 + \mu_2 - \nu_2 - c \geq 0 & \lambda_1 + \lambda_2 + \mu_1 - \nu_1 - \nu_3 - c \geq 0 \\ \lambda_1 - \nu_3 - c \geq 0 & \lambda_1 + \lambda_2 + \mu_2 - \nu_2 - \nu_3 - c \geq 0 \\ \lambda_2 + \mu_1 - \nu_2 - c \geq 0 & \lambda_1 + \mu_1 + \mu_2 - \nu_1 - \nu_3 - c \geq 0 \\ \mu_1 - \nu_3 - c \geq 0 & \lambda_2 + \mu_1 + \mu_2 - \nu_2 - \nu_3 - c \geq 0 \\ \lambda_2 + \mu_2 - \nu_3 - c \geq 0 & \lambda_1 + \lambda_2 + \mu_1 + \mu_2 - \nu_1 - \nu_2 - c \geq 0 \end{array} \quad (12)$$

and

$$|\nu| = |\lambda| + |\mu|. \quad (13)$$

Note that, for $c = 0$, we recover the 6 inequalities saying that λ, μ and ν are dominant, the 6 Weyl inequalities and the 6 others inequalities of the Horn cone (see *e.g.* [Ful00]).

We now want to compute the function mapping (λ, μ, c) to the number of solutions of this system of inequalities in ν . The method consists in restating this problem in the language of vector partition functions as in [Stu95].

Start with the 18×8 matrix H whose rows give the coefficients of the 18 inequalities (12).
Set

$$\Lambda = \{(\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3, c) \in \mathbb{Z}^8 : |\nu| = |\lambda| + |\mu|\}$$

and

$$\Lambda^+ = \{(\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3, c) \in \Lambda : \lambda, \mu, \nu \text{ dominant and } c \geq 0\}.$$

Let $\widetilde{\text{Horn}}_3$ denote the set of points in Λ^+ that satisfy the inequalities (12).

To get nonnegative variables, we make the following change of coordinates

$$\begin{aligned} a_1 &= \lambda_1 - \lambda_2 - c & b_1 &= \mu_1 - \mu_2 - c & c_1 &= \nu_1 - \nu_2 - c \\ a_2 &= \lambda_2 - c & b_2 &= \mu_2 - c & c_2 &= \nu_2 - c \end{aligned}$$

Then $\widetilde{\text{Horn}}_3$ identifies with $\widetilde{\text{Horn}}'_3 = \{X \in \mathbb{N}^7 \mid NX \geq 0\}$, where

$$N = \begin{pmatrix} -1 & -2 & -1 & -2 & 1 & 3 & -3 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & -1 & -2 & 1 & 2 & -2 \\ 0 & 1 & 1 & 1 & 0 & -1 & 1 \\ -1 & -2 & 0 & -1 & 1 & 2 & -2 \\ -1 & -1 & -1 & -1 & 1 & 2 & -2 \\ 0 & -1 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 & 1 & 1 & -1 \\ 1 & 2 & 1 & 2 & -1 & -2 & 2 \end{pmatrix}.$$

Set $\tilde{N} = (N \mid -I_{13})$ in such a way that

$$\begin{aligned} \widetilde{\text{Horn}}'_3 &\simeq \{(X, Y) \in \mathbb{N}^7 \times \mathbb{N}^{13} \mid NX = Y\} \\ &\simeq \{X \in \mathbb{N}^{20} \mid \tilde{N}X = 0\}. \end{aligned}$$

We now proceed to the affine section mentioned in the proof of Proposition 7. Thus, up to our changes of variables, the function $(\lambda, \mu, c) \mapsto \text{Nb}_3(c_{\lambda\mu}^\bullet > c)$ is the map

$$\begin{aligned} \mathbb{N}^5 &\longrightarrow \mathbb{N} \\ Y &\longmapsto \#\{X \in \mathbb{N}^{15} : \tilde{M}X = -BY\}, \end{aligned}$$

where $\tilde{M} = (M \mid -I_{13})$, M is the matrix formed by columns 5 and 6 of the matrix N and B is the matrix formed by the other columns of N .

Note that \tilde{M} is not unimodular: the lowest common multiple of the maximal minors is not 1, but 6. There are 83 such nonzero minors. Then [Stu95] implies that $(\lambda, \mu, c) \mapsto \text{Nb}(c_{\lambda\mu}^\bullet > c)$ is piecewise quasi-polynomial with chambers obtained by intersecting some

83 explicit simplicial cones. We used [VSB⁺07], an implementation of Barvinok algorithm [Bar94] to compute this function. The surprise was that we got only polynomial functions and only 7 cones. Actually, the software gave 36 cones that can be glued to give those 7 described in Proposition 10.

Proposition 10. *We use the basis of fundamental weights to set $\lambda = k_1\varpi_1 + k_2\varpi_2 = (k_1 + k_2, k_2)$ and $\mu = l_1\varpi_1 + l_2\varpi_2 = (l_1 + l_2, l_2)$. Then $\text{Nb}(c_{\lambda\mu}^\bullet > c) = 0$ unless*

$$c \leq \min(k_1, k_2, l_1, l_2).$$

Moreover, this cone decomposes into 7 cones C_1, \dots, C_7 on which $\text{Nb}(c_{\lambda\mu}^\bullet > c)$ is given by polynomial functions P_1, \dots, P_7 . Five of these seven pairs (C_i, P_i) are kept unchanged by switching k_1 and k_2 . The two others are swapped by this operation.

In particular, Conjecture 1 holds for GL_3 .

In the basis of fundamental weights, we are interested in the function

$$\psi : \begin{array}{ccc} \mathbb{N}^5 & \longrightarrow & \mathbb{N} \\ (k_1, k_2, l_1, l_2, c) & \longmapsto & \#\{\nu \in \Lambda_3 \mid c_{k_1\varpi_1+k_2\varpi_2, l_1\varpi_1+l_2\varpi_2}^\nu > c\} \end{array} .$$

Notice moreover that switching k_1 and k_2 corresponds then to taking λ^* . Define now the following seven polynomials in k_1, k_2, l_1, l_2, c :

$$P_1 = 2c^2 - c(k_1 + k_2 + l_1 + l_2 + 2) - \frac{1}{2}(k_1 + k_2 - l_1 - l_2)^2 + k_1k_2 + l_1l_2 + \frac{1}{2}(k_1 + k_2 + l_1 + l_2) + 1$$

$$P_2 = 3c^2 - 3c(k_1 + k_2 + 1) + \frac{1}{2}(k_1 + k_2)^2 + k_1k_2 + \frac{3}{2}(k_1 + k_2) + 1$$

$$P_3 = 3c^2 - 3c(l_1 + l_2 + 1) + \frac{1}{2}(l_1 + l_2)^2 + l_1l_2 + \frac{3}{2}(l_1 + l_2) + 1$$

$$P_4 = \frac{5}{2}c^2 - c\left(2k_1 + 2k_2 + l_1 + \frac{5}{2}\right) + k_1k_2 + (k_1 + k_2)(l_1 + 1) - \frac{l_1}{2}(l_1 - 1) + 1$$

$$P_5 = \frac{5}{2}c^2 - c\left(2k_1 + 2k_2 + l_2 + \frac{5}{2}\right) + k_1k_2 + (k_1 + k_2)(l_2 + 1) - \frac{l_2}{2}(l_2 - 1) + 1$$

$$P_6 = \frac{5}{2}c^2 - c\left(k_1 + 2l_1 + 2l_2 + \frac{5}{2}\right) + l_1l_2 + (l_1 + l_2)(k_1 + 1) - \frac{k_1}{2}(k_1 - 1) + 1$$

$$P_7 = \frac{5}{2}c^2 - c\left(k_2 + 2l_1 + 2l_2 + \frac{5}{2}\right) + l_1l_2 + (l_1 + l_2)(k_2 + 1) - \frac{k_2}{2}(k_2 - 1) + 1$$

Notice that P_1, \dots, P_5 are symmetric in k_1, k_2 , whereas P_6 and P_7 are swapped when one swaps k_1 and k_2 . One might also add that P_3 is the image of P_2 under the involution corresponding to swapping λ and μ – i.e. swapping (k_1, k_2) and (l_1, l_2) –, as P_6 is the image of P_4 and P_7 the one of P_5 under this same involution.

Then, for $c \geq 0$, $k_1 \geq c$, $k_2 \geq c$, $l_1 \geq c$, and $l_2 \geq c$, the function ψ is given by the following piecewise polynomial:

Cones of polynomiality	Polynomial giving ψ
$C_1 : k_1 + k_2 \geq \max(l_1, l_2) + c, \quad l_1 + l_2 \geq \max(k_1, k_2) + c$	P_1
$C_2 : k_1 + k_2 \leq \min(l_1, l_2) + c$	P_2
$C_3 : l_1 + l_2 \leq \min(k_1, k_2) + c$	P_3
$C_4 : l_1 + c \leq k_1 + k_2 \leq l_2 + c$	P_4
$C_5 : l_2 + c \leq k_1 + k_2 \leq l_1 + c$	P_5
$C_6 : k_1 + c \leq l_1 + l_2 \leq k_2 + c$	P_6
$C_7 : k_2 + c \leq l_1 + l_2 \leq k_1 + c$	P_7

One can then see that the cones C_1 to C_5 are stable under permutation of k_1 and k_2 whereas the cones C_6 and C_7 are swapped when k_1 and k_2 are. Thus, for all $k_1, k_2, l_1, l_2, c \geq 0$,

$$\psi(k_1, k_2, l_1, l_2, c) = \psi(k_2, k_1, l_1, l_2, c),$$

which proves Proposition 10.

Remark. The last part of Proposition 10 asserts that there exists a bijection $(\Lambda_3^0)^2 \times \Lambda_3 \rightarrow (\Lambda_3^0)^2 \times \Lambda_3$, $(\lambda, \mu, \nu) \mapsto (\lambda^*, \mu, \tilde{\nu})$ such that

$$c_{\lambda\mu}^\nu = c_{\lambda^*\mu}^{\tilde{\nu}}.$$

One could hope for such a bijection to be linear. Unfortunately, it CANNOT.

One can check this claim as follows. The matrix of such a linear bijection $\varphi : (\Lambda_3^0)^2 \times \Lambda_3 \rightarrow (\Lambda_3^0)^2 \times \Lambda_3$, $(\lambda, \mu, \nu) \mapsto (\lambda^*, \mu, \tilde{\nu})$ would only depend on 7 vectors ν^1, \dots, ν^7 in \mathbb{Z}^3 . Assume that this matrix does exist.

For example, the image of $(\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3) = (1, 0, 0, 0, 1, 0, 0)$ is then $(1, 1, 0, 0, \nu^1 + \nu^5)$. Moreover this image has to correspond to some nonzero Littlewood-Richardson coefficient. Hence it is necessary that $\nu_1 + \nu_5 = (1, 1, 0)$.

Similarly, considering the image of $(0, 0, 1, 1, 1, 1, 0)$, one gets that it must be $(0, 0, 1, 1, \nu^3 + \nu^4 + \nu^5 + \nu^6) = (0, 0, 1, 1, 1, 1, 0)$, and then $\nu^3 + \nu^4 + \nu^5 + \nu^6 = (1, 1, 0)$.

Now the image of $(1, 0, 1, 1, 1, 1, 1)$ has to be a ray of the Horn cone. We deduce that this image is $(1, 1, 1, 1, 2, 2, 0)$, and thus $\nu^1 + \nu^3 + \nu^4 + \nu^5 + \nu^6 + \nu^7 = (2, 2, 0)$.

Combining these three constraints one gets $\nu^5 = \nu^7$, which contradicts the invertibility of φ .

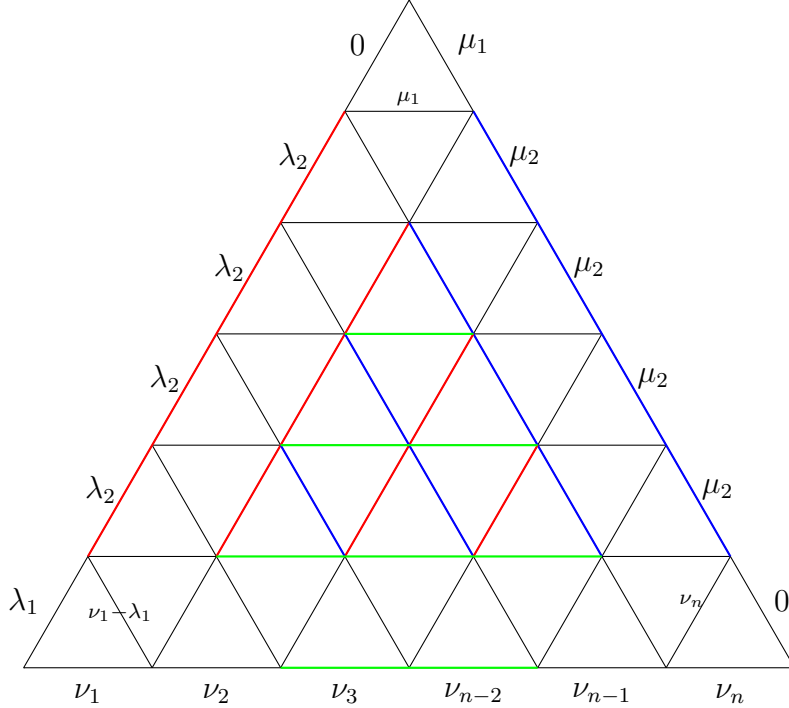
Note also that the linear automorphisms of $(\Lambda_3)^3$ preserving the Littlewood-Richardson coefficients are proved to form a group of cardinality 288 (so big !) in [BR20].

5 A stability result

In this section, we are interested in the Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$ with λ and μ near-rectangular. Using the hive model, we give a proof of Proposition 3 stated in the

introduction.

Proof of Proposition 3. We know that the Littlewood-Richardson coefficient $c_{\lambda\mu}^\nu$ is equal to the number of hives with exterior edges labelled by λ , μ , and ν . Let us assume that $c_{\lambda\mu}^\nu > 0$, meaning that such a hive exists, and consider any hive like this. In this hive, a certain number of edges have a label that is already fixed by these “boundary conditions”. These edges and values are shown in the following picture (made for the sake of example with a hive of size 6, but the picture is strictly the same for all sizes at least 4):

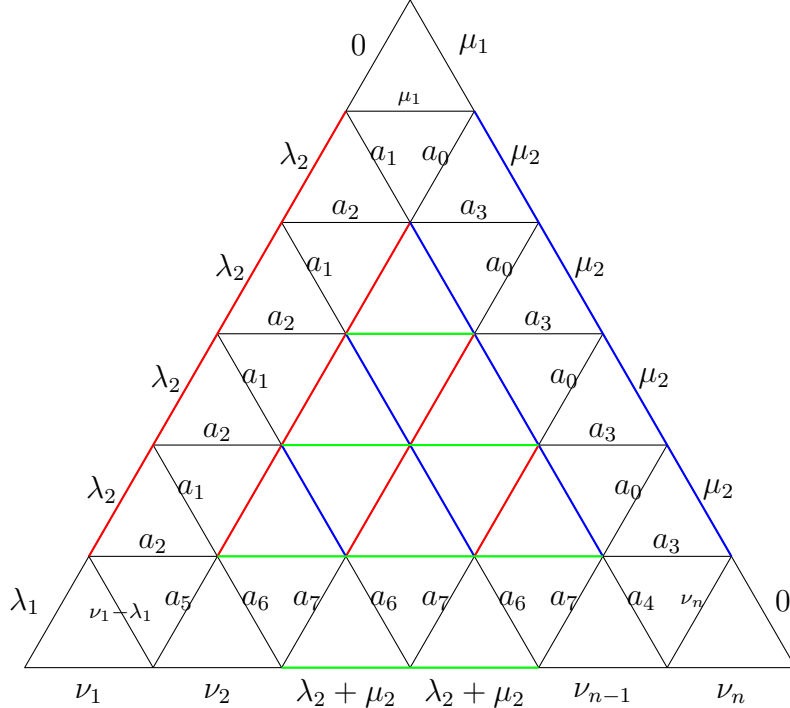


Then, using a number of hive conditions (including the fact that, in any triangle, the values on two of the edges determine the value on the third one), we obtain the following values for some interior edges: all the edges parallel to the north-west side of the triangle, with no endpoint on the other two sides, correspond to the value λ_2 ; the ones parallel to the north-east side, with no endpoint on the other two sides, have the value μ_2 ; finally the edges parallel to the south side, strictly inside the triangle and with no endpoint on the other two sides, must have the value $\lambda_2 + \mu_2$ ³.

Thus the edges on the south side corresponding to the parts ν_3 to ν_{n-2} must in fact have the value $\lambda_2 + \mu_2$. That is, ν must be of the aforementioned form: $\nu = \nu_1 \nu_2 (\lambda_2 + \mu_2)^{n-4} \nu_{n-1} \nu_n$. Notice moreover that, even if $n = 4$, two rhombi inequalities show immediately that one must still have $\nu_2 \geq \lambda_2 + \mu_2 \geq \nu_{n-1}$.

³These three kinds of edges are coloured on both pictures in this proof: the first kind (parallel to the north-west) in red, the second kind (north-east) in blue, and the third one (south) in green.

From now on we assume that ν has this particular form. Consider once again a hive with exterior edges labelled by λ , μ , and ν . The same conditions as before apply, and the values of all the remaining edges have then to be chosen in order to determine completely the hive. Let us name these values a_0, a_1, \dots, a_7 as shown on the following picture:



The fact that many of these edges must be given the same value comes everytime from the fact that, in any rhombus inside a hive, if two opposite edges have the same value, then it must also be the case for the other pair of opposite edges.

Using now once again the fact that, in any triangle, the values corresponding to two edges determine the value on the third one, these 8 integers a_0, \dots, a_7 are related by the following equations:

$$\left\{ \begin{array}{l} a_0 + a_1 = \mu_1 \\ a_0 + \mu_2 = a_3 \\ \lambda_2 + a_1 = a_2 \\ \nu_1 - \lambda_1 + a_5 = a_2 \\ a_4 + \nu_n = a_3 \\ a_4 + a_7 = \nu_{n-1} \\ a_5 + a_6 = \nu_2 \\ a_6 + a_7 = \lambda_2 + \mu_2 \end{array} \right. \iff \left\{ \begin{array}{l} a_1 = \mu_1 - a_0 \\ a_2 = \lambda_2 + \mu_1 - a_0 \\ a_3 = \mu_2 + a_0 \\ a_4 = \mu_2 - \nu_n + a_0 \\ a_5 = \lambda_1 + \lambda_2 + \mu_1 - \nu_1 - a_0 \\ a_6 = \lambda_2 + 2\mu_2 - \nu_{n-1} - \nu_n + a_0 \\ a_7 = -\mu_2 + \nu_{n-1} + \nu_n - a_0 \end{array} \right.$$

(let us recall that the equality $|\lambda| + |\mu| = |\nu|$ means that $\lambda_1 + 2\lambda_2 + \mu_1 + 2\mu_2 = \nu_1 + \nu_2 + \nu_{n-1} + \nu_n$). In particular, this means that the hive is for instance entirely determined by the value of a_0 .

We can now look at all the hive inequalities that must be satisfied by these a_i 's:

$$\begin{array}{lll}
 a_0 \geq 0 & \nu_1 \geq a_2 & a_3 \geq \nu_n \\
 \lambda_2 \geq a_0 & a_1 \geq \nu_1 - \lambda_1 & \nu_n \geq a_0 \\
 a_1 \geq \mu_2 & a_2 \geq \nu_2 & \nu_{n-1} \geq a_3 \\
 & \mu_2 \geq a_6 &
 \end{array}$$

The inequalities of the first column can be obtained from the hive inequalities in the north corner of the hive, those of the second from the south-west corner, and those of the third from the south-east corner (keep in mind that some of them can of course be obtained in several ways). Thanks to the previous relations between the a_i 's, all these inequalities can be expressed in terms of a_0 only, giving in the end exactly the following necessary and sufficient conditions on a_0 to obtain a hive:

$$\begin{cases}
 a_0 \geq \max(0, \lambda_2 + \mu_1 - \nu_1, \nu_n - \mu_2) \\
 a_0 \leq \min(\lambda_2, \mu_1 - \mu_2, \lambda_1 + \mu_1 - \nu_1, \lambda_2 + \mu_1 - \nu_2, -\lambda_2 - \mu_2 + \nu_{n-1} + \nu_n, \nu_n, \nu_{n-1} - \mu_2)
 \end{cases}$$

As a consequence, $c'_{\lambda\mu}$ is the number of integer points in the interval $[M; m]$ where

$$\begin{aligned}
 M &= \max(0, \lambda_2 + \mu_1 - \nu_1, \nu_n - \mu_2), \quad \text{and} \\
 m &= \min(\lambda_2, \mu_1 - \mu_2, \lambda_1 + \mu_1 - \nu_1, \lambda_2 + \mu_1 - \nu_2, -\lambda_2 - \mu_2 + \nu_{n-1} + \nu_n, \nu_n, \nu_{n-1} - \mu_2).
 \end{aligned}$$

Observe finally that $\nu_{n-1} \leq \lambda_2 + \mu_2$ and $\nu_2 \geq \lambda_2 + \mu_2$ give

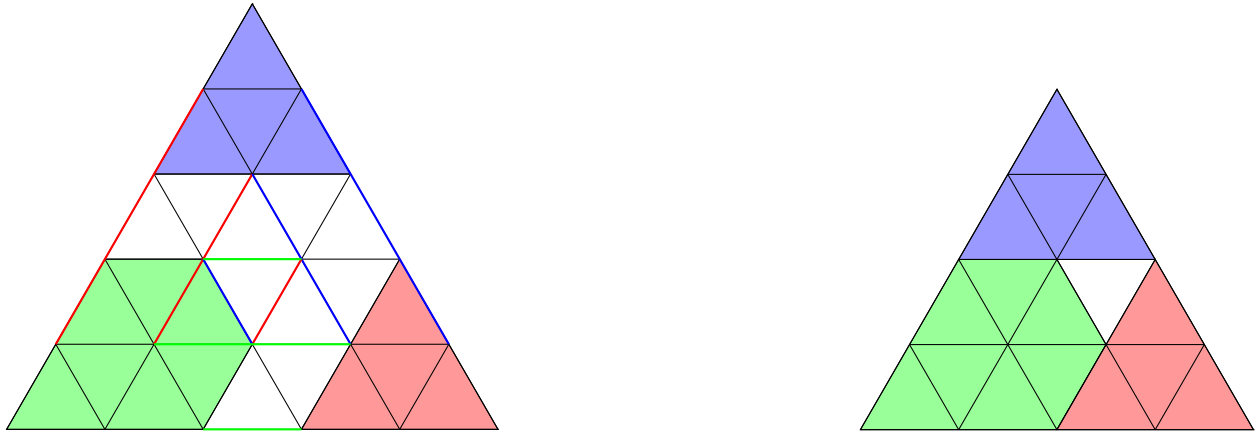
$$\nu_{n-1} - \mu_2 \leq \lambda_2, \quad \lambda_2 + \mu_1 - \nu_2 \leq \mu_1 - \mu_2, \quad -\lambda_2 - \mu_2 + \nu_{n-1} + \nu_n \leq \nu_n.$$

Hence $c'_{\lambda\mu}$ is the cardinality of

$$\llbracket \max(0, \lambda_2 + \mu_1 - \nu_1, -\mu_2 + \nu_n), \min(\lambda_1 + \mu_1 - \nu_1, \lambda_2 + \mu_1 - \nu_2, -\lambda_2 - \mu_2 + \nu_{n-1} + \nu_n, -\mu_2 + \nu_{n-1}) \rrbracket.$$

□

This stability can be interpreted as a proof of the existence of a bijection between sets of hives. Such a bijection can for instance be obtained as follows.



Starting from a hive of size n ($n \geq 4$), consider the three areas coloured in the picture above (on the left): the four triangles in the north corner, the four in the south-east one, and the seven in the south-west one. Then send this hive to the one of size 4 obtained by keeping these three coloured-areas (picture on the right). The rhombus inequalities in the hive show that the values on the edges in these three particular areas determine indeed completely the hive. This means that this map is well defined and that it is truly a bijection.

Remark. Let α, β, γ be three partitions such that $c_{\alpha, \beta}^{\gamma} = 1$. By the Fulton's conjecture (see [KT99] or [Bel07, BKR12, Res11]), we have $c_{k\alpha, k\beta}^{k\gamma} = 1$, for any $k \geq 0$. Let $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ be a second triple of partitions. The stability result of [SS16] (see also [Par19, Pel19]) asserts that $c_{\tilde{\alpha}+k\alpha, \tilde{\beta}+k\beta}^{\tilde{\gamma}+k\gamma}$ does not depend on the integer k big enough.

Returning to the setting of Proposition 3, consider $\alpha = 1^{\lambda_2}$, $\beta = 1^{\mu_2}$ and $\gamma = 1^{\lambda_2 + \mu_2}$. Set also $\tilde{\alpha} = (\lambda_1 \lambda_2 \lambda_2)'$, $\tilde{\beta} = (\mu_1 \mu_2)'$ and $\tilde{\gamma} = (\nu_1 \nu_2 \nu_3 \nu_4)'$, where \square' denote the conjugated partition. Using the invariance of the Littlewood-Richardson coefficient by simultaneous conjugation we get $c_{\lambda\mu}^{\nu} = c_{\tilde{\alpha}+k\alpha, \tilde{\beta}+k\beta}^{\tilde{\gamma}+k\gamma}$ with $k = n - 2$. Thus, the mentioned stability result asserts that $c_{\lambda\mu}^{\nu}$ does not depend on n big enough. Proposition 3 asserts that this sequence is in fact constant for $n \geq 2$.

6 The case of $GL_4(\mathbb{C})$

This section is about $GL_4(\mathbb{C})$. But Proposition 3 allows to extend several results to any $GL_n(\mathbb{C})$ for $n \geq 4$.

6.1 The Horn cone

The set of points in Horn_n with λ and/or μ near-rectangular is the set of integer points on a face of this cone. Proposition 3 implies that the geometry of this face and the Littlewood-Richardson coefficients on it do not depend on $n \geq 4$. We denote by $\overline{\text{Horn}}_n$ the set of points in Horn_n with the first two partitions λ and μ in Λ_n^0 . Then $\text{Horn}_n \simeq \mathbb{Z}^2 \times \overline{\text{Horn}}_n$. Set

$$\text{Horn}_4^{\text{nr}^2} = \{(\lambda, \mu, \nu) \in \text{Horn}_4 : \lambda \text{ and } \mu \text{ are near-rectangular}\}$$

and

$$\text{Horn}_4^{\text{nr}} = \{(\lambda, \mu, \nu) \in \text{Horn}_4 : \lambda \text{ is near-rectangular}\}.$$

The inequalities defining the Horn cone $\text{Horn}_n^{\mathbb{Q}}$ are well known (see Section 2). By convex geometry and explicit computation, one can deduce the minimal lists of inequalities for $\text{Horn}_4^{\text{nr}^2}$ and $\text{Horn}_4^{\text{nr}}$. Softwares like Normaliz [BIS] allow to make the computation.

Proposition 11. Let λ, μ in Λ_4^0 and ν in Λ_4 such that λ and μ are near-rectangular. Then $c_{\lambda\mu}^\nu \neq 0$ if and only if

$$\begin{aligned}
|\lambda| + |\mu| &= |\nu| \\
\nu_1 &\geq \nu_2 & \nu_4 &\geq 0 \\
\nu_3 + \nu_4 &\geq \lambda_2 + \mu_2 \\
\nu_1 + \nu_3 &\geq \lambda_1 + \lambda_2 + \mu_2 & \nu_1 + \nu_3 &\geq \lambda_2 + \mu_1 + \mu_2 \\
\nu_2 &\geq \lambda_2 + \mu_2 \geq \nu_3 \\
\nu_3 &\geq \lambda_2 & \nu_3 &\geq \mu_2 \\
\lambda_1 + \mu_2 &\geq \nu_2 & \lambda_2 + \mu_1 &\geq \nu_2
\end{aligned}$$

Remark. Proposition 3 also implies that $\nu_1 + \nu_4 \geq \lambda_2 + \mu_1$, which is a consequence of these 11 inequalities.

Proposition 12. The cone generated by $\text{Horn}_4^{\text{nr}^2} \cap \overline{\text{Horn}}_4$ has 8 extremal rays generated by the triples (λ, μ, ν) associated to the following inclusions

1. $V(1) \subset V(1) \otimes V(0)$ (twice by permuting the factors);
2. $V(1^3) \subset V(1^3) \otimes V(0)$ (twice by permuting the factors);
3. $V(1^2) \subset V(1) \otimes V(1)$;
4. $V(1^4) \subset V(1) \otimes V(1^3)$ (twice by permuting the factors);
5. $V(2^2 1^2) \subset V(1^3) \otimes V(1^3)$.

Each triple (λ, μ, ν) on one of these extremal rays indexes a Littlewood-Richardson coefficient with value 1. The Hilbert basis of $\text{Horn}_4^{\text{nr}^2} \cap \overline{\text{Horn}}_4$ consists in these 8 generators.

We get similar descriptions for $\text{Horn}_4^{\text{nr}}$.

Proposition 13. Let λ, μ in Λ_4^0 and ν in Λ_4 such that λ is near-rectangular. Then $c_{\lambda\mu}^\nu \neq 0$ if and only if all of the following inequalities hold:

$$\begin{aligned}
|\lambda| + |\mu| &= |\nu| \\
\nu_1 &\geq \nu_2 \geq \nu_3 \geq \nu_4 \geq 0 \\
\lambda_1 &\geq \lambda_2 \geq 0 & \mu_1 &\geq \mu_2 \geq \mu_3 \geq 0 \\
\lambda_1 + \mu_1 &\geq \nu_1 & \min(\lambda_2 + \mu_1, \lambda_1 + \mu_2) &\geq \nu_2 \\
\min(\lambda_2 + \mu_2, \lambda_1 + \mu_3) &\geq \nu_3 & \min(\lambda_1, \mu_1, \lambda_2 + \mu_3) &\geq \nu_4 \\
\nu_1 &\geq \max(\lambda_1, \mu_1 \lambda_2 + \mu_2) & \nu_2 &\geq \max(\mu_2, \lambda_2 + \mu_3) & \nu_3 &\geq \max(\lambda_2, \mu_3) \\
\nu_1 + \nu_2 &\geq \max(\lambda_1 + \lambda_2 + \mu_2, \lambda_2 + \mu_1 + \mu_2) & \nu_1 + \nu_3 &\geq \max(\lambda_1 + \lambda_2 + \mu_3, \lambda_2 + \mu_1 + \mu_3) \\
\nu_2 + \nu_3 &\geq \lambda_2 + \mu_2 + \mu_3 & \nu_1 + \nu_4 &\geq \lambda_2 + \mu_1 \\
\nu_2 + \nu_4 &\geq \lambda_2 + \mu_2 & \nu_3 + \nu_4 &\geq \lambda_2 + \mu_3
\end{aligned}$$

We have 32 facets.

Proposition 14. *The cone generated by $\text{Horn}_4^{\text{nr}} \cap \overline{\text{Horn}}_4$ has 12 extremal rays generated by the triples (λ, μ, ν) associated to the following inclusions*

1. $V(1) \subset V(1) \otimes V(0)$, $V(1) \subset V(111) \otimes V(0)$, $V(1) \subset V(0) \otimes V(1)$, $V(11) \subset V(0) \otimes V(11)$
and $V(111) \subset V(0) \otimes V(111)$;
2. $V(1^2) \subset V(1) \otimes V(1)$;
3. $V(1^4)$ is contained in $V(1) \otimes V(1^3)$ and $V(1^3) \otimes V(1)$;
4. $V(2211)$ is contained in $V(1^3) \otimes V(1^3)$ and $V(211) \otimes V(11)$;
5. $V(1^3) \subset V(1) \otimes V(11)$;
6. $V(21^3) \subset V(1^3) \otimes V(1^2)$.

Each triple (λ, μ, ν) on one of these extremal rays indexes a Littlewood-Richardson coefficient with value 1. The Hilbert basis of $\text{Horn}_4^{\text{nr}} \cap \overline{\text{Horn}}_4$ consists in these 12 generators.

6.2 Special case of self-dual representations

Let k and l be two nonnegative integers and $n \geq 4$. The $\text{SL}_n(\mathbb{C})$ -modules $V_n((2k)k^{n-2})$, $V_n((2l)l^{n-2})$ and hence $V_n((2k)k^{n-2}) \otimes V_n((2l)l^{n-2})$ are self-dual.

In [PW20, Section 8], conjectural values (for $n = 6$) are given for the numbers of isotypic components in $V_n((2k)k^{n-2}) \otimes V_n((2l)l^{n-2})$ and for the numbers of self-dual isotypic components. Here, we prove and extend these formulas.

Corollary 15. *Assume up to symmetry that $l \leq k$. The number of distinct isotypic components in $V_n((2k)k^{n-2}) \otimes V_n((2l)l^{n-2})$ is given by*

$$\begin{cases} l^3 + 3l^2 + 3l + 1 & \text{if } 2l \leq k \\ \frac{1}{3}k^3 - 2k^2l + 4kl^2 - \frac{5}{3}l^3 - k^2 + 4kl - l^2 + \frac{2}{3}k + \frac{5}{3}l + 1 & \text{if } 2l \geq k \end{cases}$$

Proof. By Proposition 3, one may assume that $n = 4$. Then, Proposition 11 shows that $\nu \in \mathbb{Z}^4$ is the highest weight of an isotypic component of $V_4((2k)k^2) \otimes V_4((2l)l^2)$ if and only if (recall that $l \leq k$) all of the following conditions hold:

$$\begin{aligned} 4(k+l) &= |\nu| & \nu_1 &\geq \nu_2 \\ \nu_4 &\geq 0 & \nu_3 + \nu_4 &\geq k+l \\ \nu_1 + \nu_3 &\geq 3k+l & & \\ 2n+m &\geq \nu_2 \geq k+l \geq \nu_3 \geq k & & \end{aligned} \tag{14}$$

The corollary follows by explicit computations that can be performed with [VSB⁺07]. \square

Similarly, one gets the number of self-dual representations.

Corollary 16. *Assume up to symmetry that $l \leq k$. The $\text{SL}_n(\mathbb{C})$ -module $V_n((2k)k^{n-2}) \otimes V_n((2l)l^{n-2})$ contains $(l+1)^2$ distinct self-dual isotypic components.*

Proof. By Proposition 3, one may assume that $n = 4$. Then, the set of self-dual isotypic components of $V_4((2k)k^2) \otimes V_4((2l)l^2)$ are obtained by adding the condition

$$\nu_1 + \nu_4 = \nu_2 + \nu_3$$

to the conditions (14). The corollary follows by explicit computations that can be performed with [VSB⁺07]. \square

6.3 Computation of $\text{Nb}(c_{\lambda\mu}^\bullet > c)$ for λ and μ near-rectangular

In this subsection, we report on the computation of the function

$$\begin{aligned} \text{Nb}_4(c_{\lambda\mu}^\bullet > c) : (\Lambda_4^{\text{nr}})^2 \times \mathbb{N} &\longrightarrow \mathbb{N} \\ (\lambda, \mu, c) &\longmapsto \#\{\nu \in \Lambda_4 : c_{\lambda\mu}^\nu > c\}. \end{aligned}$$

By Proposition 3, this function determines $\text{Nb}_n(c_{\lambda\mu}^\bullet > c)$ for any near-rectangular partitions λ and μ of length $n \geq 4$.

Since Propositions 9 and 3 give similar expressions for the Littlewood-Richardson coefficient, we can apply the strategy of Section 4.

We get that $\text{Nb}_4(c_{\lambda\mu}^\bullet > c)$ is the number of points $\nu \in \Lambda_4$ such that $\lambda_1 + 2\lambda_2 + \mu_1 + 2\mu_2 = \nu_1 + \nu_2 + \nu_3 + \nu_4$ and

$$\begin{array}{ll} -\lambda_2 - \mu_2 + \nu_2 \geq 0 & \lambda_2 + \mu_2 - \nu_3 \geq 0 \\ \lambda_1 - \lambda_2 \geq c & -\lambda_2 + \nu_3 \geq c \\ \nu_1 - \nu_2 \geq c & \nu_3 - \nu_4 \geq c \\ \lambda_1 + \mu_1 - \nu_1 \geq c & \lambda_2 + \mu_1 - \nu_2 \geq c \\ -\lambda_2 - \mu_2 + \nu_3 + \nu_4 \geq c & -\mu_2 + \nu_3 \geq c \\ \lambda_1 + \mu_2 - \nu_2 \geq c & \lambda_1 + \lambda_2 + \mu_2 - \nu_2 - \nu_4 \geq c \\ \lambda_1 + \mu_1 + \mu_2 - \nu_1 - \nu_4 \geq c & \lambda_2 + \mu_1 + \mu_2 - \nu_2 - \nu_4 \geq c \end{array}$$

In particular, it is the number of integer points in some polytope depending linearly on the data (λ, μ, c) . Then $\text{Nb}_4(c_{\lambda\mu}^\bullet > c)$ is piecewise quasi-polynomial, and can be computed using Barvinok's algorithm. Surprisingly, here $\Lambda = (\Lambda_4^{\text{nr}})^2 \times \mathbb{Z}$ and $\text{Nb}_4(c_{\lambda\mu}^\bullet > c)$ is piecewise polynomial.

As in Section 4, from this point on we use the basis of fundamental weights to write $\lambda = k_1\varpi_1 + k_2\varpi_1^*$ and $\mu = l_1\varpi_1 + l_2\varpi_1^*$. Thus the symmetry we want to observe is once again with respect to swapping k_1 and k_2 . Consider the function

$$\begin{aligned} \psi : \quad \mathbb{N}^5 &\longrightarrow \mathbb{N} \\ (k_1, k_2, l_1, l_2, c) &\longmapsto \#\{\nu \in \Lambda_4 \mid c_{k_1\varpi_1 + k_2\varpi_1^*, l_1\varpi_1 + l_2\varpi_1^*}^\nu > c\}. \end{aligned}$$

We now give details about the results in Proposition 4:

Proposition 17. *We have $\psi(k_1, k_2, l_1, l_2, c) = 0$ unless*

$$c \leq \min(k_1, k_2, l_1, l_2).$$

Moreover, this cone decomposes into 36 cones C_1, \dots, C_{36} on which ψ is given by polynomial functions P_1, \dots, P_{36} . 12 of these pairs (C_i, P_i) are kept unchanged by swapping k_1 and k_2 (namely (C_1, P_1) to (C_{12}, P_{12})). The 24 other such pairs are pairwise swapped by this operation (for all $i \in \{7, \dots, 18\}$, (C_{2i-1}, P_{2i-1}) and (C_{2i}, P_{2i}) are swapped).

In particular, Conjecture 1 holds for GL_4 and λ, μ near-rectangular.

Now to present as clearly as possible these cones and polynomials without writing all of them, let us use the two following involutions: s_1 corresponding to the exchange of k_1 and k_2 , and s_2 corresponding to swapping (k_1, k_2) and (l_1, l_2) . Then $\langle s_1, s_2 \rangle$ acts on the set of all pairs (C_i, P_i) with 8 orbits. Let us give below one representative for each one of these. The labelling is the one of the complete list [Res20, pol_and_cones_SL4nr2.txt], chosen so that the stability when swapping k_1 and k_2 is easier to see:

$$C_1 : \quad l_1 + l_2 \leq k_1 + c, \quad l_1 + l_2 \leq k_2 + c,$$

$$P_1 = \left(-\frac{1}{2}\right) \cdot (-l_2 + c - 1) \cdot (-l_1 + c - 1) \cdot (-l_1 - l_2 + 2c - 2)$$

has a $\langle s_1, s_2 \rangle$ -orbit of size 2;

$$C_{16} : \quad l_1 + l_2 \leq k_1 + c, \quad l_1 + l_2 \geq k_2 + c, \quad k_2 \geq l_1, \quad k_2 \geq l_2,$$

$$P_{16} = P_1 - \binom{-k_2 + l_1 + l_2 - c + 2}{3}$$

has an orbit of size 4;

$$C_2 : \quad l_1 + l_2 \geq k_1 + c, \quad l_1 + l_2 \geq k_2 + c, \quad k_1 \geq l_1, \quad k_1 \geq l_2, \quad k_2 \geq l_1, \quad k_2 \geq l_2,$$

$$P_2 = P_{16} - \binom{-k_1 + l_1 + l_2 - c + 2}{3}$$

has an orbit of size 2;

$$C_{19} : \quad l_1 + l_2 \geq k_1 + c, \quad k_1 \geq l_1, \quad k_2 \leq l_1, \quad k_2 \geq l_2,$$

$$P_{19} = P_2 + \binom{-k_2 + l_1 + 1}{3}$$

has an orbit of size 8;

$$C_{21} : \quad l_1 + l_2 \leq k_1 + c, \quad k_2 \leq l_1, \quad k_2 \geq l_2,$$

$$P_{21} = P_{16} + \binom{-k_2 + l_1 + 1}{3}$$

has an orbit of size 8;

$$C_{29} : \quad k_1 + k_2 \geq l_1 + l_2, \quad l_1 + l_2 \geq k_1 + c, \quad k_2 \leq l_1, \quad k_2 \leq l_2,$$

$$P_{29} = P_{19} + \binom{-k_2 + l_2 + 1}{3}$$

has an orbit of size 4;

$$C_{27} : \quad k_1 + k_2 \leq l_1 + l_2, \quad k_1 \geq l_1, \quad k_1 \geq l_2,$$

$$P_{27} = P_{29} + \binom{-k_1 - k_2 + l_1 + l_2 + 1}{3}$$

has an orbit of size 4; finally,

$$C_{36} : \quad l_1 + l_2 \leq k_1 + c, \quad k_2 \leq l_1, \quad k_2 \leq l_2,$$

$$P_{36} = P_{21} + \binom{-k_2 + l_2 + 1}{3}$$

also has an orbit of size 4.

Remark. One can observe that the polynomials P_i are expressed using each other. We exploit here the fact that the difference between two polynomials associated to two adjacent cones has a simple expression theoretically given by the Paradan formula [Par04, BV09].

6.4 Computation of $\text{Nb}_4(c_{\lambda\mu}^\bullet > 0)$ for λ near-rectangular

In this section, we report on the computation of the function

$$\begin{aligned} \text{Nb}_4(c_{\lambda\mu}^\bullet > 0) : \quad \Lambda_4^{\text{nr}} \times \Lambda_4 &\longrightarrow \mathbb{N} \\ (\lambda, \mu) &\longmapsto \#\{\nu \in \Lambda_4 : c_{\lambda\mu}^\nu > 0\}. \end{aligned}$$

As we recalled in Proposition 7, $\text{Nb}_4(c_{\lambda\mu}^\bullet > 0)$ is the number of integer points in an affine section of the Horn cone. The inequalities defining this cone are explicitly given in Proposition 13. Then, one can compute explicitly the quasi-polynomial function with the program [VSB⁺07]. The output is too big (even using symmetries) to be collected there. The interested reader can get details from [Res20, Supplementary material].

Proposition 18. *The cone $\Lambda_4^{\text{nr}} \times \Lambda_4$ decomposes into 205 cones of non empty interior. On 151 of them $\text{Nb}_4(c_{\lambda\mu}^\bullet > 0)$ is polynomial of degree 3, and on the other 54 it is quasi-polynomial. The only congruence occurring is the parity of $\lambda_1 + |\mu|$.*

Moreover, for any pair (C, P) where C is one of the 205 cones and P the corresponding function, one can see that in this list there is also a pair (C', P') obtained by replacing λ by λ^ (in 57 cases, $(C', P') = (C, P)$). In particular, Conjecture 1 holds.*

Under the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ there are 61 orbits of actual polynomials and 22 orbits of quasi-polynomials.

Here we give three examples illustrating some of the variety of cases that one can observe. The function $\text{Nb}_4(c_{\lambda\mu}^\bullet > 0)$ for $\lambda = k_1\varpi_1 + k_2\varpi_1^* \in \Lambda_4^{\text{nr}} \cap \Lambda_4^0$ and $\mu = \mu_1\mu_2\mu_3 \in \Lambda_4^0$ is given:

- on the cone defined by $\mu_1 \geq k_1 + \mu_3$, $\mu_1 \geq k_2 + \mu_3$, $\mu_2 \geq \mu_3$, $k_1 + k_2 + \mu_3 \geq \mu_1 + \mu_2$, $\mu_3 \geq 0$, by the polynomial

$$P = \frac{\mu_3}{2} \cdot \left(\mu_2(2\mu_1 - \mu_2 + 1) + 2(\mu_1 + 1) - (\mu_3 + 1)(k_1 + k_2 + \mu_1 - \mu_2 + 2) \right) - \frac{\mu_2 + 1}{6} \cdot \left(3(k_1^2 + k_2^2) - 3(k_1 + k_2)(2\mu_1 + 1) + 3\mu_1^2 + 2\mu_2^2 - 3\mu_1 + 4\mu_2 - 6 \right),$$

symmetric in k_1, k_2 .

- on the cone defined by $\mu_1 + \mu_2 \geq k_1 + k_2 + \mu_3$, $k_2 + \mu_1 \geq k_1 + \mu_2 + \mu_3$, $k_2 + \mu_3 \geq \mu_2$, $k_1 + \mu_1 \geq k_2 + \mu_2 + \mu_3$, $k_1 + \mu_3 \geq \mu_2$, $k_1 + k_2 \geq \mu_1$, $\mu_3 \geq 0$ (adjacent to the previous one), by the quasi-polynomial

$$\begin{cases} P + \frac{1}{24} (k_1 + k_2 - \mu_1 - \mu_2 + \mu_3 - 1) \\ \cdot (k_1 + k_2 - \mu_1 - \mu_2 + \mu_3 + 1) & \text{if } k_1 + k_2 + \mu_1 + \mu_2 + \mu_3 \text{ is odd} \\ \cdot (-2k_1 - 2k_2 + 2\mu_1 + 2\mu_2 + 4\mu_3 + 3) \\ \\ P + \frac{1}{24} (k_1 + k_2 - \mu_1 - \mu_2 + \mu_3) \\ \cdot (2 + (k_1 + k_2 - \mu_1 - \mu_2 + \mu_3) & \text{if } k_1 + k_2 + \mu_1 + \mu_2 + \mu_3 \text{ is even,} \\ \cdot (-2k_1 - 2k_2 + 2\mu_1 + 2\mu_2 + 4\mu_3 + 3) \end{cases}$$

also symmetric in k_1, k_2 .

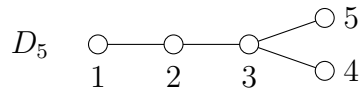
- on the cone defined by $\mu_1 \geq k_1$, $\mu_1 \geq k_2 + \mu_3$, $\mu_2 \geq \mu_3$, $k_2 \geq \mu_2$, $k_1 + \mu_3 \geq \mu_1$ (also adjacent to the first one), by the non-symmetric polynomial

$$P + \binom{k_1 - \mu_1 + \mu_3 + 1}{3}.$$

7 Related questions

7.1 In type D_n

Apart from type A_n , it is only in types D_n and E_6 that the irreducible representations of simple Lie algebras are not all self-dual. Consider here type D_5 .



Let $(\varpi_1, \dots, \varpi_5)$ be the list of fundamental weights. Then $V(\varpi_4)^* \simeq V(\varpi_5)$ whereas $V(\varpi_1)$, $V(\varpi_2)$ and $V(\varpi_3)$ are self-dual. The natural generalization of near-rectangular

partitions is then to consider the dominant weights in $\mathbb{N}\varpi_4 \oplus \mathbb{N}\varpi_5$. A natural generalization of Conjecture 2 would be: for $\lambda = a\varpi_4 + b\varpi_5 \in \mathbb{N}\varpi_4 \oplus \mathbb{N}\varpi_5$ and μ a dominant weight of D_5 , do the two tensor products

$$V_{D_5}(a\varpi_4 + b\varpi_5) \otimes V_{D_5}(\mu) \quad \text{and} \quad V_{D_5}(b\varpi_4 + a\varpi_5) \otimes V_{D_5}(\mu)$$

contain the same number of isotypic components?

The answer is NO, even assuming that $\mu \in \mathbb{N}\varpi_4 \oplus \mathbb{N}\varpi_5$ too. An example is $\lambda = 2\varpi_4 + \varpi_5$ and $\mu = \varpi_4 + 2\varpi_5$. The two tensor products have respectively 31 and 30 isotypic components as checked using SageMath [The20]:

```
sage: D5=WeylCharacterRing("D5",style="coroots")
sage: len(D5(0,0,0,2,1)*D5(0,0,0,1,2))
31
sage: len(D5(0,0,0,1,2)*D5(0,0,0,1,2))
30
```

7.2 In type A_n

The representations of $SL_n(\mathbb{C})$ corresponding to near-rectangular partitions are of the form $V(a\varpi_1 + b\varpi_{n-1})$. Observe that (ϖ_1, ϖ_{n-1}) is a pair of mutually dual fundamental weights. One could hope that Conjecture 1 or 2 hold for any linear combination of a given pair of mutually dual fundamental weights. This is not true even for (ϖ_2, ϖ_3) and $n = 5$. Indeed, for $\lambda = \varpi_2 + 2\varpi_3$ and $\mu = 3\varpi_2 + \varpi_3$, the numbers of isotypic components in $V(\lambda) \otimes V(\mu)$ and $V(\lambda)^* \otimes V(\mu)$ differ:

```
sage: len(lrcalc.mult([3,3,2],[4,4,1],5))
34
sage: len(lrcalc.mult([3,3,1],[4,4,1],5))
33
```

Mention finally that we checked Conjecture 1 on examples, using SageMath. See [Res20, test_Conj1.sage]:

- Conjecture 1 holds for GL_4 if $\max(\lambda_1 - \lambda_2, \lambda_2) \leq 20$ and $|\mu| \leq 40$.
- Conjecture 1 holds for GL_5 if $\max(\lambda_1 - \lambda_2, \lambda_2) \leq 20$ and $|\mu| \leq 30$.
- Conjecture 1 holds for GL_6 if $\max(\lambda_1 - \lambda_2, \lambda_2) \leq 10$ and $|\mu| \leq 30$.
- Conjecture 1 holds for GL_{10} if $\max(\lambda_1 - \lambda_2, \lambda_2) \leq 10$ and $|\mu| \leq 15$.

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