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Non-integrability of the equal mass n -body problem with non-zero angular momentum

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Abstract We prove an integrability criterion and a partial integrability criterion for homogeneous potentials of degree -1 which are invariant by rotation. We then apply it to the proof of the meromorphic non-integrability of the n -body problem with Newtonian interaction in the plane on a surface of equation $(H, C) = (H_0, C_0)$ with $(H_0, C_0) \neq (0, 0)$ where C is the total angular momentum and H the Hamiltonian, in the case where the n masses are equal. Several other cases in the 3-body problem are also proved to be non integrable in the same way, and some examples displaying partial integrability are provided.

Keywords Non-integrability · Homogeneous potentials · Central configurations · Differential Galois theory · n -body problem

Mathematics Subject Classification 37J30 · 70F15

1 Introduction

In this article, our aim is to study dynamical systems of the form

$$\dot{\mathbf{q}}_i = \mathbf{p}_i \quad \dot{\mathbf{p}}_i = \frac{\partial}{\partial \mathbf{q}_i} V \quad i = 1, \dots, n \quad (1)$$

where V is a homogeneous function in $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$ meromorphic for $\mathbf{q} \in \mathbb{C}^n \setminus \{0\}$, and in particular the case of homogeneity degree -1 and its applications to Celestial Mechanics. In the following, we will call such a function a meromorphic homogeneous potential. One of the most important property in dynamical systems is integrability.

Definition 1 Let V be a meromorphic homogeneous potential. We say that the dynamical system associated to the potential V is meromorphically integrable if there exist n functions $(\mathbf{p}, \mathbf{q}) \longrightarrow (I_1(\mathbf{p}, \mathbf{q}), \dots, I_n(\mathbf{p}, \mathbf{q}))$ meromorphic for $(\mathbf{p}, \mathbf{q}) \in \mathbb{C}^{2n}$, $\mathbf{q} \neq 0$ such that

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- The functions I_i are constant along orbits, meaning that $\dot{I}_i = 0, i = 1, \dots, n$
- The functions I_i are in involution, meaning that for all $i, j = 1 \dots, n$, we have the Poisson bracket $\{I_i, I_j\} = 0$
- The functions I_i are independent almost everywhere, meaning that the Jacobian matrix of the application $(\mathbf{p}, \mathbf{q}) \longrightarrow (I_1(\mathbf{p}, \mathbf{q}), \dots, I_n(\mathbf{p}, \mathbf{q}))$ has maximal rank almost everywhere

Non-integrability of homogeneous potentials has been studied using mainly Morales-Ramis theorem (Morales Ruiz 1999) and Ziglin theory (Ziglin 1982). These methods require a particular algebraic orbit of the corresponding potential. With homogeneous potentials, generically there exist straight line orbits, corresponding to the Darboux points of the potentials.

Definition 2 (see for example Pina and Lonngi (2010); Lee and Santoprete (2009) on these equations) Let V be a meromorphic homogeneous potential. We say that $\mathbf{c} \in \mathbb{C}^n \setminus \{0\}$ is a Darboux point if there exists $\alpha \in \mathbb{C}$ such that

$$\frac{\partial}{\partial q_i} V(\mathbf{c}) = \alpha \mathbf{c}_i \quad \forall i = 1, \dots, n$$

We call α the multiplier, and we say that \mathbf{c} is non degenerated if $\alpha \neq 0$. A Darboux point \mathbf{c} is also called a central configuration in the case of the n -body problem.

To these Darboux points we can associate homothetic orbits (or more generally straight line orbits, Howard and Meiss 2009), which are explicit algebraic solutions of the differential equation (1) (note that the orbit is an algebraic curve, but not necessarily algebraic in its time parametrization). Using such orbits, Morales-Ramis method provides a mean to prove some facts about non-integrability, particularly in the case of homogeneous potentials.

Theorem 1 (Morales Ruiz (1999), Theorem 4.1.) *Let H be a Hamiltonian holomorphic on a complex symplectic manifold M of dimension $2n$, and $\Gamma \subset M$ a non-stationary orbit of H . If there are n meromorphic first integrals of H that are in involution and independent over a neighbourhood of Γ , then the identity component of Galois group of the variational equation near Γ is Abelian.*

Theorem 2 (Morales-Ruiz and Ramis (2001c), Theorem 3.) *Let V be a meromorphic homogeneous potential of degree -1 and \mathbf{c} a Darboux point with multiplier -1 . If V is meromorphically integrable, then*

$$Sp\left(\nabla^2 \mathbf{V}(\mathbf{c})\right) \subset \left\{ \frac{1}{2}(k-1)(k+2), k \in \mathbb{N} \right\}.$$

Early work on this subject has been done in Yoshida (1983, 1987), similar statements were made and applied in Morales-Ruiz and Ramis (2001a,b), generalizations were made for higher variational equations in Morales-Ruiz et al. (2007) and for non Hamiltonian cases in Ayoul and Zung (2010). Here we want to study variational equations and their Galois group near another type of particular orbit that we often encounter when the potential is invariant by rotation. In particular, if there exists a plane of Darboux points, invariant by the rotational symmetry of V (this case is not rare), then we can build particular orbits with non-zero angular momentum. Then we get a one parameter family of orbits on which we can apply Morales-Ramis theory. For all of them, the identity component of the Galois group of the variational equation should be Abelian, and thus we can expect a much stronger integrability criterion than Yoshida (1987). One difficulty is that the variational equation is intricate to study in the general case, thus we will make a complete analysis only in the case which we will call “partially decoupled”. We find very strong conditions, only two eigenvalues are

Table 1 Integrability table for homogeneous potential of degree -1 invariant by rotation

(C, H)	λ
$C = 0$	$\lambda \in \left\{ \frac{1}{2}(k-1)(k+2), k \in \mathbb{N} \right\}$
$C^2H = -1/2$	$\lambda \in \left\{ -k^2, k \in \mathbb{N} \right\}$
$H = 0$	$\lambda \in \left\{ \frac{1}{2}(k-1)(k+2), k \in \mathbb{N} \right\}$
$C^2H \notin \{0, -1/2\}$	$\lambda \in \{0, -1\}$
$(C, H) = (0, 0)$	$\lambda \in \mathbb{C}$

possible instead of an infinity. Moreover we will see that this type of orbit allows us to study a new type of partial integrability: the case where the potential would be integrable only for a fixed value of the Hamiltonian and angular momentum.

The main theorem of this article is the following.

Theorem 3 *The n -body problem with equal masses in the plane is neither meromorphically integrable on any hypersurface of the form $C^2H = \alpha$ with $\alpha \neq 0$ fixed, nor on the hypersurface $H = 0$, nor on the hypersurface $C = 0$ (H being the Hamiltonian, and C the total angular momentum).*

This result generalizes some already known non-integrability proofs as Boucher (2000) for $n = 3, C = 0$ (generalized in Maciejewski and Przybylska 2011), Morales-Ruiz and Simon (2009) for $n \geq 3, C = 0$ and Tsygvintsev (2001) for $n = 3, H = 0$ (later generalized in Tsygvintsev 2007). Along the proof of Theorem 3, we will prove more generally that if a homogeneous potential of degree -1 invariant by rotation is meromorphically integrable on a surface with fixed energy and “angular momentum” (as defined by Eq. 3), then the eigenvalue λ of the Hessian matrix of V on a Darboux point with multiplier -1 should belong to the Table 1 (which is found through the analysis of the Galois group of variational equations near conic orbits). The complete statement is Theorem 8, in which there is an additional a priori hypothesis, the “decoupling condition”. Partially integrable potential exists effectively as given in (13). This integrability table also gives indications on which particular level of energy and “angular momentum” we should focus when searching partial integrability. These cases correspond also to regular confluences of the variational equations (even without the “decoupling condition”), and the Galois group being typically smaller for these cases, it is always worthwhile to consider them in particular. The case $C^2H = -1/2$ leads for the restricted 3-body problem to an additional first integral on this level (the Jacobi integral), although this first integral is not valid everywhere on $C^2H = -1/2$. For the general 3-body problem, this case will correspond to the energy and total angular momentum of circular motions on Lagrange and Euler central configurations. Theorem 8 cannot solve all problems of this kind because of this “decoupling condition”. A complete analysis of the 3-body case gives all the masses which satisfy this condition in Theorem 17, which are not always symmetric. A non-integrability theorem like Theorem 3 is by the way immediate for these masses, except for $(m_1, m_2, m_3) = (1, 5, 1)$.

2 General properties

Definition 3 (see Gantmacher (1959); Craven (1969) for an overview of interesting properties.) We will call “norm” and scalar product the expressions

$$\|\mathbf{v}\|^2 = \sum_{i=1}^n v_i^2 \quad \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^n v_i w_i$$

even for complex \mathbf{v}, \mathbf{w} (in particular, the “norm” can vanish for non-zero \mathbf{v}). We will say moreover that a matrix is orthonormal complex if its columns $\mathbf{X}_1, \dots, \mathbf{X}_n$ are such that

$$\langle \mathbf{X}_i, \mathbf{X}_j \rangle = \sum_{k=1}^n (\mathbf{X}_i)_k (\mathbf{X}_j)_k = 0 \quad \forall i, j \quad \|\mathbf{X}_i\|^2 = \sum_{k=1}^n (\mathbf{X}_i)_k^2 = 1 \quad \forall i$$

We note $\mathbb{O}_n(\mathbb{C})$ the complex orthogonal group which is the group generated by these matrices, and $\mathbb{SO}_n(\mathbb{C})$ the subgroup of $\mathbb{O}_n(\mathbb{C})$ of matrices with determinant 1 (corresponding to rotations). In particular, the group $\mathbb{O}_n(\mathbb{C})$ conserve the “norm”.

Definition 4 Let V be a homogeneous meromorphic potential of degree -1 in dimension $n \geq 2$. We note

$$G = \{ \mathbf{g} \in \mathbb{O}_n(\mathbb{C}), \quad V(\mathbf{g}\cdot\mathbf{x}) = V(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{C}^n \} \tag{2}$$

We will call G the symmetry group of V . For $\mathbf{v} \in \mathbb{C}^n$, we note

$$G\mathbf{v} = \{ \alpha \mathbf{g}\cdot\mathbf{v}, \alpha \in \mathbb{C}, \mathbf{g} \in G \}$$

We will say that V is invariant by rotation if G contains at least a subgroup isomorphic to $\mathbb{SO}_2(\mathbb{C})$. We will say that \mathbf{v} is an eigenvector of G if for all $\mathbf{g} \in G$, \mathbf{v} is an eigenvector of \mathbf{g} .

In the following, we will have to consider three notions of angular momentum. The first one (and the most general) is a (non constant) first integral of a potential V on \mathbb{C}^n of the form

$$C = \sum_{0 \leq i < j \leq n} a_{i,j} (\mathbf{p}_i \mathbf{q}_j - \mathbf{p}_j \mathbf{q}_i) \quad a_{i,j} \in \mathbb{C} \tag{3}$$

that we will call an “angular momentum” (with quotes). The second one is the canonical angular momentum on a plane $P \subset \mathbb{C}^n$ which equals to $p_2 q_1 - p_1 q_2$ for a direct orthonormal complex basis of P . Eventually, in the last part about the n -body problem, we will consider the total angular momentum which is the sum of the canonical angular momentum of each body with respect to the center of mass.

Theorem 4 Let V be a homogeneous potential of degree -1 in dimension $n \geq 2$ and G its symmetry group. Suppose there exists a Darboux point \mathbf{c} with $\|\mathbf{c}\| \neq 0$ and \tilde{G} a subgroup of G such that $P = \tilde{G}\mathbf{c}$ is a plane. Then it exists a conic orbit on P and the variational equation near this conic orbit with parameters $(C\|\mathbf{c}\|^2, H\|\mathbf{c}\|^2) \in \mathbb{C}^2$ (canonical angular momentum on P and energy) is given by

$$t(-C^2 + 2t + 2Ht^2)\ddot{\mathbf{X}} + (-t + C^2)\dot{\mathbf{X}} = \mathbf{R}_{\theta(t)}^{-1} \nabla^2 \mathbf{V}(\mathbf{c}) \mathbf{R}_{\theta(t)} \mathbf{X} \tag{4}$$

where $\mathbf{R}_{\theta(t)} \in \tilde{G}$ with coefficients in $\mathbb{C} \left(t, \sqrt{2H - C^2 t^{-2} + 2t^{-1}} \right)$

Proof Let c be a Darboux point of V with multiplier -1 and \tilde{G} a subgroup of G . After rotation of the coordinates, we can suppose that $\mathbf{c} = (\gamma, 0, \dots, 0)$ and that the plane $\tilde{G}\mathbf{c}$ is generated by $(\gamma, 0, \dots, 0), (0, \gamma, 0, \dots, 0)$. A conic orbit for the Darboux point \mathbf{c} corresponds to the orbit given by

$$(\mathbf{q}_1, \mathbf{q}_2) = \varphi_t(1, 0) \quad \mathbf{q}_i = 0 \quad i = 3, \dots, n$$

where φ_t is given by

$$\varphi_t(x, y) = \phi(t) \begin{pmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

Replacing this in the Hamiltonian and canonical angular momentum (the potential V restricted to the plane P is invariant by rotation), we get

$$\frac{1}{2}\dot{\phi}^2\gamma^2 + \frac{1}{2}\gamma^2\phi^2\dot{\theta}^2 + \frac{V(\mathbf{c})}{\phi} = H\gamma^2 \quad C\gamma^2 = \gamma^2\phi^2\dot{\theta}$$

And after replacing, we get

$$\frac{1}{2}\dot{\phi}^2\gamma^2 + \gamma^2\frac{C^2}{2\phi^2} + \frac{V(\mathbf{c})}{\phi} = H\gamma^2$$

Knowing that the multiplier is -1 , we use Euler equation for V

$$\gamma^2 = -V(\mathbf{c}) \quad \frac{1}{2}\dot{\phi}^2 + \frac{C^2}{2\phi^2} - \frac{1}{\phi} = H$$

Therefore, the variational equation is of the form

$$\ddot{\mathbf{X}} = \frac{1}{\phi(t)^3}\nabla^2V(\mathbf{R}_{\theta(t)}\mathbf{c})\mathbf{X}$$

with $\mathbf{R}_{\theta(t)}$ a rotation matrix. We also know that we are on some conic orbit (due to the fact that the homogeneity degree is -1)

$$\phi(t) = \frac{p}{1 + e \cos(\theta)}$$

with p and e some parameters depending on C, H . We get that $\cos(\theta), \sin(\theta)$ are rational fractions in $\phi, \dot{\phi}$. Then, with variable change $\phi \rightarrow t$ we get the following expression

$$t(-C^2 + 2t + 2Ht^2)\ddot{\mathbf{X}} + (-t + C^2)\dot{\mathbf{X}} = \nabla^2V(\mathbf{R}_{\theta(t)}\mathbf{c})\mathbf{X} \tag{6}$$

We know that the potential V is invariant by rotation. Then the matrix $\nabla^2V(\mathbf{R}_{\theta(t)}\mathbf{c})$ corresponds to $\nabla^2V(\mathbf{c})$ after a basis change and gives

$$\nabla^2V(\mathbf{R}_{\theta(t)}\mathbf{c}) = \mathbf{R}_{\theta(t)}^{-1}\nabla^2V(\mathbf{c})\mathbf{R}_{\theta(t)} \tag{7}$$

Replacing this in (6) gives us the Eq. (4). □

Remark 1 The main difficulty of this variational equation is that it does not decouple after basis change. Indeed, we can make a basis change with some matrix \mathbf{P} , but this matrix \mathbf{P} should commute with the rotations $\mathbf{R}_{\theta(t)}$.

Let us now give a proper definition of what we will call integrable on some level of first integrals, as given in Theorem 3.

Theorem 5 *Let V be a homogeneous meromorphic potential of degree -1 in dimension $n \geq 2$. Let I_1, \dots, I_k be meromorphic first integrals such that*

$$\{I_i, I_j\} = 0 \quad \forall i, j$$

where $\{, \}$ is the Poisson bracket. We pose

$$\mathcal{O} = \{(\mathbf{p}, \mathbf{q}) \rightarrow g(\mathbf{p}, \mathbf{q}, I_1(\mathbf{p}, \mathbf{q}), \dots, I_k(\mathbf{p}, \mathbf{q})) \mid g \text{ holomorphic for } \mathbf{q} \neq 0\}$$

the ring of holomorphic functions in $(\mathbf{p}, \mathbf{q}, I_1, \dots, I_k)$, $\mathbf{q} \neq 0$. We suppose that $\langle I_1, \dots, I_k \rangle$ is a prime ideal on \mathcal{O} and we pose $K = \text{Frac}(\mathcal{O} / \langle I_1, \dots, I_k \rangle)$ the corresponding fraction field. Then the following functions are well defined

- For all $i = 1, \dots, k$, the functions $\varphi_i : K \rightarrow K, f \rightarrow \{f, I_i\}$.
- The function

$$\Psi : \left(\bigcap_{i=1}^k \varphi_i^{-1}(0) \right)^2 \rightarrow K, f, g \rightarrow \{f, g\}$$

- The functions $K^{n-k} \rightarrow K$ which associate to f_1, \dots, f_{n-k} a subdeterminant of size $n \times n$ of the Jacobian matrix (a matrix of size $2n \times n$) of $I_1, \dots, I_k, f_1, \dots, f_{n-k}$.

Proof Let us write a representant of $f \in K$ as $P/Q, P, Q \in \mathcal{O}$. We just need to check that the value of the function φ_i does not depend on the choice of the representant. We consider $h_1, \dots, h_k, g_1, \dots, g_k \in \mathcal{O}$ and we have

$$\begin{aligned} \left\{ \frac{P + \sum_{s=1}^k h_s I_s}{Q + \sum_{s=1}^k g_s I_s}, I_i \right\} &= \left(Q + \sum_{s=1}^k g_s I_s \right)^{-1} \left\{ P + \sum_{s=1}^k h_s I_s, I_i \right\} \\ &\quad - \frac{P + \sum_{s=1}^k h_s I_s}{(Q + \sum_{s=1}^k g_s I_s)^2} \left\{ Q + \sum_{s=1}^k g_s I_s, I_i \right\} \\ &= \left(Q + \sum_{s=1}^k g_s I_s \right)^{-1} \{P, I_i\} - \frac{P + \sum_{s=1}^k h_s I_s}{(Q + \sum_{s=1}^k g_s I_s)^2} \{Q, I_i\} \\ &= Q^{-1} \{P, I_i\} - P Q^{-2} \{Q, I_i\} = \left\{ \frac{P}{Q}, I_i \right\} \end{aligned}$$

so the function is well defined on K .

Let us consider $f_1, f_2 \in \bigcap_{i=1}^k \varphi_i^{-1}(0)$ and we write P/Q a representant of f_1 . Using the fact that $\{I_i, f_2\} = 0$, we can do exactly the same calculations as before just replacing $\{, I_i\}$ by $\{, f_2\}$. Using the fact that the Poisson bracket is symmetric, we can do the same interverting the indices 1, 2. So the function Ψ is well defined.

Let us consider x one of the $2n$ variables $\mathbf{p}, \mathbf{q}, f \in K$ and P/Q a representant. We have the classical formula

$$\partial_x \left(\frac{P}{Q} \right) = Q^{-1} \partial_x P - P Q^{-2} \partial_x Q$$

So adding elements of $\langle I_1, \dots, I_k \rangle$ to P and Q corresponds to add in the determinant a linear combination of $\partial_x I_1, \dots, \partial_x I_k$. The determinant being multilinear, this will not change the value of the determinant because it contains the columns of the derivatives of $I_i, i = 1, \dots, k$. □

One needs to be extremely cautious when manipulating these derivatives, because K is not a differential field, so we cannot conclude directly that all notions we will need (Poisson brackets, independence) are well defined. For example, the Jacobian matrix itself is not well defined on K , only its sub-determinants of size $n \times n$ are. Remark that, in the following, we will always consider a representant and will forget the field K , and so this complicated definition will have no impact. This theorem is here to give a proper definition of integrability on some particular level of first integrals, and this complicated presentation has some

advantage as it also includes all singular levels thanks to the prime ideal condition (we never ask for example the first integrals I_i to be independent).

Definition 5 Let V be a homogeneous meromorphic potential of degree -1 in dimension $n \geq 2$. Let I_1, \dots, I_k be meromorphic first integrals satisfying the hypotheses of Theorem 5. We say that V is meromorphically integrable on the manifold $(I_1, \dots, I_k) = 0$ if there exists $F_1, \dots, F_{n-k} \in K$ (K is defined as in Theorem 5) such that

$$\{H, F_i\} = 0 \in K \quad \forall i \quad \{I_i, F_j\} = 0 \in K \quad \forall i, j \quad \{F_i, F_j\} = 0 \in K \quad \forall i, j$$

and such that at least one of the sub determinants of size $n \times n$ of the Jacobian matrix of $I_1, \dots, I_k, F_1, \dots, F_{n-k}$ is not 0 in K (this corresponds to the condition of independence almost everywhere).

Remark 2 This definition of partial integrability implies the integrability of the differential system restricted to $\cap_{i=1}^k I_i^{-1}(0)$ in the Bogoyavlensky sense (1998). Indeed in the case of a partially integrable potential with Definition 5, the restricted corresponding differential system is of dimension $2n - k$, there are $n - k$ first integrals F_i and n commuting vector fields given by $J\nabla I_i, i = 1, \dots, k, J\nabla F_i, i = 1, \dots, n - k$ (J is the matrix of the canonical symplectic form). Moreover, for this broader definition of integrability, we have the following generalization of Theorem 1.

Theorem 6 (Ayoul and Zung 2010) *Assume that a dynamical system given by a holomorphic vector field X on a complex analytic variety M is meromorphically integrable in the Bogoyavlensky sense, and let $\Gamma \subset M$ a non-stationary solution of X . Then for any natural number $n \geq 1$, the identity component of the differential Galois group of the variational equation of order n of X along Γ is Abelian.*

Theorem 7 *Let $V \neq 0$ be a homogeneous meromorphic potential of degree -1 in dimension $n \geq 2$ and suppose it exists a non trivial first integral C of V of the form (3). Let us fix the value of the Hamiltonian $H = H_0 \neq 0$ and $C = C_0 \neq 0$. If V is integrable on this manifold of codimension 2, then V is integrable on the hypersurface $C^2H = C_0^2H_0$.*

Proof We consider the following transformation

$$\varphi \mathbb{C}^{2n} \longrightarrow \mathbb{C}^{2n} \quad (\mathbf{p}, \mathbf{q}) \longrightarrow (\alpha\mathbf{p}, \alpha^{-2}\mathbf{q}) \tag{8}$$

We see that the transformation φ just multiplies the Hamiltonian $H \longrightarrow \alpha^2H$, and this does not change the integrability of H . Let us suppose that H be integrable on the manifold $(H, C) = (H_0, C_0)$. We have

$$H(\varphi(\mathbf{p}, \mathbf{q})) = \alpha^2H \quad C(\varphi(\mathbf{p}, \mathbf{q})) = \alpha^{-1}C \quad (C^2H)(\varphi(\mathbf{p}, \mathbf{q})) = C^2H$$

Then H is also integrable on the manifold $(H, C) = (\alpha^2H_0, \alpha^{-1}C_0)$. We also have

$$\bigcup_{\alpha \in \mathbb{C}^*} (H^{-1}(\alpha^2H_0) \cap C^{-1}(\alpha^{-1}C_0)) = \{(\mathbf{p}, \mathbf{q}) \in \mathbb{C}^{2n}, C(\mathbf{p}, \mathbf{q})^2H(\mathbf{p}, \mathbf{q}) = C_0^2H_0\}$$

because $C_0^2H_0 \neq 0$. This gives the theorem. □

Remark 3 Using Noether theorem (a simple statement can be found in Sarlet and Cantrijn 1981), when a potential is invariant by rotation, we can build a first integral of the form (3). We can see that the study of integrability on a specific manifold makes sense only if this manifold is invariant by φ , because if it is not the case, then our potential will be integrable on a

manifold with higher dimension. Remark that the ideals $\langle C - C_0, H - H_0 \rangle, \langle C^2 H - C_0^2 H_0 \rangle$ are always prime for $C_0^2 H_0 \neq 0$ and $V \neq 0$, so integrability on these manifolds is well defined, contrary to the case $\langle C^2 H \rangle$ which will need to be split in two parts $\langle C \rangle, \langle H \rangle$.

3 Integrability table

Theorem 8 *Let V be a homogeneous meromorphic potential of degree -1 in dimension $n \geq 2$ and G its symmetry group. We consider \mathbf{c} a Darboux point of V with $\|\mathbf{c}\| \neq 0$ and multiplier -1 , and E an eigenspace of $\nabla^2 \mathbf{V}(\mathbf{c})$. Suppose it exists \tilde{G} a subgroup of G such that $P = \tilde{G}\mathbf{c}$ is a plane and that E is invariant by \tilde{G} . Considering an “angular momentum” whose restriction to P is the canonical angular momentum on P , if V is meromorphically integrable (respectively on some specific level $(H_0, C_0) \in \mathbb{C}^2$ of the Hamiltonian and “angular momentum”), then the following equation possesses a Galois group whose identity component is Abelian (respectively for the parameters $(H, C) = \|\mathbf{c}\|^{-2}(H_0, C_0)$)*

$$t(-C^2 + 2t + 2Ht^2)\ddot{X} + (-t + C^2)\dot{X} = \lambda X \quad H, C, \lambda \in \mathbb{C} \tag{9}$$

where λ is the eigenvalue of $\nabla^2 \mathbf{V}(\mathbf{c})$ associated to the eigenspace E .

Proof This is a direct application of Theorem 4. We have a plane $P = \tilde{G}\mathbf{c}$ and all vectors in this plane are Darboux points. The potential restricted to this plane is invariant by rotation (because \tilde{G} is a subgroup of the symmetry group of V). On the eigenspace E , the matrix $\nabla^2 \mathbf{V}(\mathbf{c})$ corresponds to $\lambda \mathbf{Id}_E$. Moreover we know that the space E is invariant by the rotations $\mathbf{R}_{\theta(t)}$ which correspond to elements of \tilde{G} . Thus we have

$$\mathbf{R}_{\theta(t)}^{-1} \nabla^2 \mathbf{V}(\mathbf{c}) \mathbf{R}_{\theta(t)} \Big|_E = \lambda \mathbf{Id}_E$$

So the Eq. (4) on the eigenspace E simplifies and becomes Eq. (9). The condition on the Galois group of Eq. (9) comes either from Theorem 1 in the case of complete integrability, or from Theorem 6 in the partially integrable case (in this case X is the Hamiltonian vector field restricted to the level (H_0, C_0) , and the manifold M is an open neighbourhood of the conic orbit on this level). □

Remark 4 The Theorem 8 has lots of hypotheses, but in fact only one of them is really restrictive. The existence of an invariant plane $\tilde{G}\mathbf{c}$ on which the potential is invariant by rotation is common in practical cases. This often results from the symmetry of the system. This is for example always the case in the n -body problem. The restrictive condition is the existence of E invariant by \tilde{G} . In fact, this is a condition very similar to the codiagonalization constraint that Maciejewski–Przybylska found when studying potentials which are the sum of two homogeneous potentials. In fact, a potential invariant by rotation in dimension n can also be reduced to become a potential in dimension $n - 1$ which will be a sum of a homogeneous potential and the potential C^2/r^2 . This new potential is not homogeneous and our condition corresponds to the commutation of the Hessian matrices (at least on some non trivial subspace).

Theorem 9 *The differential equation (9) is a Fuchsian equation with 4 singularities, of Heun type (Ronveaux 1995). The Galois group is $SL_2(\mathbb{C})$ except if the values of (C, H, λ) belong to the Table 1.*

Proof We first remark that in the case $H = C = 0$, Eq. (9) simplifies to

$$2t^2 \ddot{X} - t \dot{X} = \lambda X$$

which always has an Abelian Galois group. Thus from now we will suppose $(H, C) \neq (0, 0)$. Using a linear variable change $t \rightarrow \alpha t$ in Eq. (9), we get the following equation

$$t(2H\alpha^2 t^2 + 2\alpha t - C^2)\ddot{X} - (\alpha t - C^2)\dot{X} = \alpha\lambda X \tag{10}$$

For $C^2H \neq 0$, we can choose $\alpha = (\sqrt{1 + 2C^2H} - 1)/(2H)$ which gives the equation

$$2t(t - 1) \left(\left(1 - \sqrt{1 + 2C^2H} + C^2H \right) t + C^2H \right) \ddot{X} + \left(\left(1 - \sqrt{1 + 2C^2H} \right) t + 2C^2H \right) \dot{X} = \left(-1 + \sqrt{1 + 2C^2H} \right) \lambda X \tag{11}$$

We begin by the case $C^2H \neq -1/2$. The Eq. (11) has exactly 4 regular singularities on

$$0, 1, \frac{C^2H}{\gamma - 1 - C^2H}, \infty$$

where $\gamma = \sqrt{1 + 2C^2H}$. We make Frobenius expansion on these 4 singularities, and we find a logarithmic term for $t = 0$ and for $t = \infty$. More precisely, we get

$$\begin{aligned} X(t) &= c_1 t^2 \left(1 - \frac{(\lambda - 2)(\gamma - 1)}{6C^2H} t + O(t^2) \right) \\ &\quad + c_2 \left(\ln t \left(\frac{\lambda(1 + \lambda)(1 - \gamma + C^2H)}{2C^4H^2} t^2 + O(t^3) \right) - 2 - \frac{(\gamma - 1)\lambda}{C^2H} t + O(t^2) \right) \\ X(t) &= c_1 \left(1 + \frac{(\gamma - 1)\lambda}{4(1 - \gamma + C^2H)t} + O\left(\frac{1}{t^2}\right) \right) + c_2 \left(\ln t \left(\frac{(1 + \lambda)(\gamma - 1)}{2(1 - \gamma + C^2H)} + O\left(\frac{1}{t}\right) \right) \right. \\ &\quad \left. - t \left(1 - \frac{(\gamma - 1)}{2(1 - \gamma + C^2H)t} + O\left(\frac{1}{t^2}\right) \right) \right) \end{aligned}$$

These expansions are valid for $\lambda \neq -1, 0$. In the case $\lambda = -1$, we can compute explicitly the solutions and we find

$$X(t) = c_1(t - \gamma - 1) + c_2\sqrt{(t - 1)(\gamma + 1 + C^2H(t + 1))}$$

The Galois group is then $\mathbb{Z}/2\mathbb{Z}$, Abelian. In the case $\lambda = 0$, we find the solution

$$\begin{aligned} X(t) &= c_1 + c_2\sqrt{(t - 1)(2t - 2t\gamma + 2C^2H(t + 1))} \\ &\quad + c_2 \ln \left(-1 - t\gamma + t + \sqrt{(t - 1)(2t - 2t\gamma + 2C^2H(t + 1))} \right) \end{aligned}$$

The identity component of the Galois group is then \mathbb{C} , thus Abelian. Let us consider the case $\lambda \neq -1, 0$. Among the three solvable cases of Kovacic’s algorithm (Kovacic 1986), the only possible one with a logarithmic term requires the existence of a solution of the form

$$X(t) = \exp \left(\int F(s) ds \right) \quad F \in \mathbb{C}(t)$$

If F has singularities of order more than 2 then X does not have a Puiseux expansion near this singularity. This is impossible because all singularities are regular. If the degree of F is positive, then the expansion at infinity is not a Puiseux series. Then the particular solution $X(t)$ should be of the following form

$$X(t) = \prod_{i=1}^k (t - t_i)^{m_i}$$

If m_i is not a non-negative integer, then t_i is a singularity of X and equals to one of the singularities of the equation. This gives even more constraints on the m_i because the Frobenius exponents on $1, C^2H/(\gamma - 1 - C^2H)$ are $0, 1/2$. On 0 , the possible exponent is 2 , and on infinity it is 0 (the other ones correspond to the logarithmic behavior). This implies that the sum of the m_i is zero. The m_i being all non-negative, all of them are zero. The only left possibility is then $X(t) = 1$. We replace in Eq. (11) and we find $\lambda = 0$, case already done. Then the Galois group is $SL_2(\mathbb{C})$.

The cases $C = 0, H = 0, C^2H = -1/2$ correspond to confluences. These confluences are all regular (this has probably something to do with the fact that the system comes from a variational equation of a Hamiltonian system). The case $C = 0$ has already been treated by Yoshida (1987) and Morales Ruiz (1999). Let us study the case $H = 0$. This corresponds to the parabolic case (some study of this case has already been done by Tsygvintsev (2001)). Putting $\alpha = C^2/2$, Eq. (10) becomes

$$2t(t - 1)\ddot{X} - (t - 2)\dot{X} = \lambda X$$

There is a logarithmic term for the singularity $t = 0$

$$X(t) = c_1 t^2 \left(1 + \left(\frac{1}{3} - \frac{1}{6}\lambda \right) t + O(t^2) \right) + c_2 \left(\ln t \left(\left(\frac{1}{4}\lambda^2 + \frac{1}{4}\lambda \right) t^2 + O(t^3) \right) - 2 - \lambda t - \left(\frac{1}{2}\lambda + \frac{1}{4} \right) t^2 + O(t^3) \right)$$

for $\lambda \neq 0, -1$. In the cases $\lambda \in \{0, -1\}$, we find the solutions

$$X(t) = c_1 + c_2 \sqrt{t - 1}(2 + t) \quad X(t) = c_1(t - 2) + c_2 \sqrt{t - 1}$$

The Galois group is then $\mathbb{Z}/2\mathbb{Z}$ in both cases, then Abelian. We now look at the case $\lambda \neq 0, -1$. The possible exponents are $\{2\}$ at 0 , $\{0, 1/2\}$ at 1 and

$$\left\{ -\frac{3}{4} + \frac{1}{4}\sqrt{8\lambda + 9}, -\frac{3}{4} - \frac{1}{4}\sqrt{8\lambda + 9} \right\}$$

at ∞ . As before, we prove that we need a solution of the form

$$X(t) = \prod_{i=1}^k (t - t_i)^{m_i}$$

All possible exponents outside infinity are integers or half integers, non-negative, thus the sum of the m_i is a non-negative integer or half integer. Then

$$-\frac{3}{4} + \frac{1}{4}\sqrt{8\lambda + 9} = \frac{1}{2}(k - 1) \quad k \in \mathbb{N}^*$$

We solve this equation and we find

$$\lambda = \frac{1}{2}(k - 1)(k + 2) \quad k \in \mathbb{N}^*$$

This is exactly the condition of Theorem 9. We now want to compute the Galois group for these remaining cases. We write the solutions of the equation in the following form (it is a hypergeometric equation, and the solutions can be written using hypergeometric series ${}_2F_1$)

$$\begin{aligned}
 X(t) = & c_1 {}_2F_1 \left(\left[1 - \frac{1}{2}k, \frac{1}{2}k + \frac{3}{2} \right], \left[\frac{1}{2} \right], -t + 1 \right) t^2 \\
 & + c_2 {}_2F_1 \left(\left[2 + \frac{1}{2}k, \frac{3}{2} - \frac{1}{2}k \right], \left[\frac{3}{2} \right], -t + 1 \right) \sqrt{t-1} t^2
 \end{aligned}$$

These hypergeometric series are finite if the first bracket in ${}_2F_1$ contains a non-positive integer. For $k \geq 2$, we see that either $1 - \frac{1}{2}k$ or $\frac{3}{2} - \frac{1}{2}k$ is a non-positive integer. Then one of the two functions is a polynomial. We always have a solution in $\mathbb{C}[t, \sqrt{t-1}]$, and then the identity component of the Galois group is Abelian.

Let us now study the case $C^2H = -1/2$. The Eq. (11) becomes

$$-t(t-1)^2 \ddot{X} - (t-1)\dot{X} = \lambda X$$

The expansion on 0 is the following

$$\begin{aligned}
 X(t) = & c_1 t^2 \left(1 + \left(\frac{1}{3} - \frac{1}{6}\lambda \right) t + \left(\frac{1}{96}\lambda^2 - \frac{11}{96}\lambda + \frac{3}{16} \right) t^2 + O(t^3) \right) \\
 & + c_2 \left(\ln t \left(\left(\frac{1}{4}\lambda^2 + \frac{1}{4}\lambda \right) t^2 + O(t^3) \right) - 2 - \lambda t - \left(\frac{1}{2}\lambda + \frac{1}{4} \right) t^2 + O(t^3) \right)
 \end{aligned}$$

and possesses a logarithmic term for $\lambda \neq 0, -1$. The expansion at infinity is

$$\begin{aligned}
 X(t) = & c_1 \left(1 - \frac{\lambda}{2t} + \frac{\lambda(\lambda-5)}{12t^2} + O(t^{-3}) \right) \\
 & + c_2 \left(\ln t \left(\lambda + 1 - \frac{\lambda(\lambda+1)}{2t} + O(t^{-2}) \right) + t + 1 - \frac{4 + 11\lambda + 3\lambda^2}{4t} + O(t^{-2}) \right)
 \end{aligned}$$

Then there is always at least one logarithmic term for $\lambda \neq -1$. Remark that we already know that this equation has an Abelian Galois group for $\lambda = 0, -1$ (either using the limiting process of the generic solution for all C , or running Kovacic’s algorithm for these specific cases). So now we will suppose that $\lambda \neq 0, -1$. We know that if the Galois group is not $SL_2(\mathbb{C})$, then it exists a solution of the form

$$X(t) = \exp \left(\int F(s) ds \right) \quad F \in \mathbb{C}(t)$$

The equation is Fuchsian and then $X(t)$ can be written

$$X(t) = \prod_{i=1}^k (t - t_i)^{m_i}$$

The m_i need to be non-negative integers except maybe at singularities. The exponents at 1 are $+\sqrt{-\lambda}, -\sqrt{-\lambda}$. Then one of the following equation is satisfied

$$2 + \sqrt{-\lambda} + k = 0 \quad \text{or} \quad 2 - \sqrt{-\lambda} + k = 0 \quad k \in \mathbb{N}$$

Then

$$\lambda = -(k + 2)^2 \quad k \in \mathbb{N}$$

We add the cases $\lambda = 0, -1$ and this gives exactly the condition given by Theorem 9. We now need to compute the Galois group for these specific cases. We write the solutions of

the equation in the following form (it is a hypergeometric equation, and the solutions can be written using hypergeometric series)

$$X(t) = c_1 {}_2F_1 \left(\left[2 - i\sqrt{\lambda}, 1 - i\sqrt{\lambda} \right], \left[1 - 2i\sqrt{\lambda} \right], 1 - t \right) t^2(t - 1)^{-i\sqrt{\lambda}} + c_2 {}_2F_1 \left(\left[1 + i\sqrt{\lambda}, 2 + i\sqrt{\lambda} \right], \left[1 + 2i\sqrt{\lambda} \right], 1 - t \right) t^2(t - 1)^{i\sqrt{\lambda}}$$

These hypergeometric series are finite if the first bracket in the hypergeometric series ${}_2F_1$ contains a non-positive integer. We see that for $\lambda = -k^2$ $k \in \mathbb{N}^*$, it is the case for the solution in c_1 . There is always a polynomial solution and then the Galois group is always Abelian. □

Remark that such a work can also be done using the classification of hypergeometric equation which are solvable by quadrature in [Kimura \(1969\)](#).

4 Algebraic potentials

In the following sections, we will often need to consider algebraic potentials instead of meromorphic ones. This is a problem because [Theorem 2](#) deals only with meromorphic potentials. This problem is often not addressed, except in [Ziglin \(2000\)](#), but in fact his procedure does not work. This is because making cuts in the complex plane does not allow afterwards to make all possible monodromy paths. Then, the monodromy group will be reduced. It could have no consequences, but here there are important consequences because we absolutely need to be able to turn around the point 0 in the variational equation (this is because for the two other singularities, the exponents are 0, 1/2, thus if we restrict ourselves to these ones, the monodromy group will always be Abelian). Let us now make a precise statement

Definition 6 Let V be a meromorphic function on an algebraic complex manifold S of dimension n . We define the critical set of V by

$$\Sigma(V) = \{ \mathbf{x} \in S, \quad V \text{ is not } C^\infty \text{ on } \mathbf{x} \}$$

Theorem 10 *Let V be a meromorphic homogeneous potential of degree -1 on an algebraic complex manifold S of dimension $n \geq 2$ and G its symmetry group. We consider $\mathbf{c} \notin \Sigma(V)$ a Darboux point of V with $\|\mathbf{c}\| \neq 0$ and multiplier -1 , and E an eigenspace of $\nabla^2 \mathbf{V}(\mathbf{c})$. Suppose it exists \tilde{G} a subgroup of G such that $P = \tilde{G}\mathbf{c}$ is a plane and that E is invariant by \tilde{G} . Considering an “angular momentum” whose restriction to P is the canonical angular momentum on P , if V is integrable (respectively on some specific level $(H_0, C_0) \in \mathbb{C}^2$ of the Hamiltonian and “angular momentum”) with first integrals meromorphic on S , then the identity component of the Galois group of Eq. (9) is Abelian (respectively for the parameters $(H, C) = \|\mathbf{c}\|^{-2}(H_0, C_0)$).*

Proof This is almost the statement of [Theorem 8](#), and the only difference is that we consider V on an algebraic complex manifold S . We know that V is homogeneous and invariant by \tilde{G} , then so is the critical set $\Sigma(V) \subset S$. Therefore, if $\mathbf{c} \notin \Sigma(V)$, then the whole conic orbit is not in $\Sigma(V)$ except maybe for $\mathbf{q} = 0$. Noting Γ the algebraic curve corresponding to the conic orbit (without the singular point $\mathbf{q} = 0$), we pose $M \subset S$ an open neighbourhood of Γ on which the potential is holomorphic (recall that for each $x \in S \setminus \Sigma(V)$, V is holomorphic on a neighbourhood of x). The first integrals of V are meromorphic on S , and so are meromorphic on M . We can then apply [Theorem 1](#) with the manifold M (respectively [Theorem](#)

6 with X the Hamiltonian vector field of H restricted to the level (H_0, C_0) and M restricted to this level). Following the proof of Theorem 8, Eq. (9) is the variational equation near Γ restricted to the eigenspace E , and so has a virtually Abelian Galois group.

To conclude, we now need to precise that the Galois group in Theorems 1, 6 is over the field of meromorphic functions on Γ , which corresponds after variable change to meromorphic functions in $t, \sqrt{2E - C^2t^{-2} + 2t^{-1}}, t \neq 0$. As Morales and Ramis using Kimura table Kimura (1969) have done, we are in fact computing the Galois group of (9) over the base field $\mathbb{C}(t)$. This is not a problem here because in all cases, the variational equation (9) is Fuchsian and so is regular at zero and infinity, and then so are the first integrals. We know that a meromorphic function on $\bar{\mathbb{C}}$ is in fact rational, so the field we should consider is in fact

$$\mathbb{C} \left(t, \sqrt{2E - C^2t^{-2} + 2t^{-1}} \right)$$

This is just an extension of degree 2 of the field $\mathbb{C}(t)$, so the identity component of the Galois group will be the same.

5 The case of dimension 3

In the particular case of dimension 3, we get

Theorem 11 *Let V be a potential meromorphic in $\sqrt{x^2 + y^2}, \sqrt{x^2 + y^2 + z^2}, z^2$ and homogeneous of degree -1 in dimension 3 (this implies that the symmetry group of V contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{O}_2$). Suppose that $V(1, 0, 0) \neq 0, \infty$. If V is meromorphically integrable, then it belongs to one of the following families*

$$V = \frac{a}{\sqrt{x^2 + y^2 + z^2}} \quad a \in \mathbb{C}^* \quad V = \frac{b}{\sqrt{x^2 + y^2}} \quad b \in \mathbb{C}^* \tag{12}$$

This theorem is almost the best we can have (with a reasonable statement). To apply our previous theory, we need an invariant plane on which the potential is invariant by rotation. Such an invariant plane comes here from the symmetry in z . The constraint $V(1, 0, 0) \neq \infty$ cannot be removed, but the constraint $V(1, 0, 0) \neq 0$ could maybe be removed with a lot of additional work. There are two keys which allow us to give such a complete statement, which are the fact that the decoupling condition is always satisfied, and then that the potential can be reduced on a plane for which an almost complete classification is already done in arXiv:1110.6130 (for a finite number of eigenvalues).

Proof The potential V possesses a symmetry group G such that

$$G \supset \left\langle \left(\begin{array}{ccc} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right) \right\rangle$$

We consider P the plane $z = 0$. This is an invariant plane because $\partial_z V(x, y, 0) = 0$ thanks to parity in z . Using the hypotheses, the restriction of V to the plane P is not zero or infinite. Therefore the point $\mathbf{c} = (1, 0, 0)$ is a non degenerated Darboux point. The matrix $\nabla^2 \mathbf{V}(\mathbf{c})$ contains a stable subspace of dimension 2 associated to P . Then the supplementary space generated by the vector $(0, 0, 1)$ is also an eigenspace. The rotation group generated by the rotations around the z -axis let the vector $(0, 0, 1)$ invariant. The conditions of Theorem 10 are satisfied and the ‘‘vertical’’ (normal to the plane P) variational equation is then

$$t(-C^2 + 2t + 2Et^2)\ddot{X} + (-t + C^2)\dot{X} = \partial_{zz} V(\mathbf{c})X$$

This equation is integrable for all values of C only if

$$\partial_{zz} V(\mathbf{c}) \in \{0, -1\}$$

with \mathbf{c} with multiplier equal to -1 . We now restrict our potential to the plane $\tilde{P} : y = 0$. The potential V is invariant by rotation around the z -axis, then \tilde{P} is an invariant plane and we consider the restriction $\tilde{V} : \tilde{P} \mapsto \tilde{\mathbb{C}}$.

The restriction of the function $\sqrt{x^2 + y^2}$ to $y = 0$ gives a bivaluated function whose values are $+x$ or $-x$. We can choose either for the restriction \tilde{V} (because \tilde{V} should be integrable for both possibilities anyway), and we choose arbitrary

$$\sqrt{x^2 + y^2} \Big|_{y=0} = x$$

The function $\tilde{V}(x, z)$ is then meromorphic in $x, \sqrt{x^2 + z^2}, z$. It possesses a Darboux point $\mathbf{c} = (1, 0)$, and it is non degenerated. There are only two possible eigenvalues for the Hessian matrix on this Darboux point, and so we have a complete classification of such integrable potentials. Then, applying the symmetry group, we find that if V is meromorphically integrable, then it should be of the form (12). These potentials effectively possess an additional first integral, respectively $I = p_z x - p_x z$ and $I = p_z$, which are functionally independent with the Hamiltonian and the canonical angular momentum $p_x y - p_y x$. □

Remark 5 We can see here the importance of the symmetry group structure in the study of integrability. Here, the vertical variational equation is simple because in dimension 3, a group of rotations (except $\mathbb{S}\mathbb{O}_3$) always possesses a common eigenvector. This is no more the case in dimension 4 and higher. In particular, the complexity of the variational equation is closely linked to the symmetry group, and if it is too complicated, we will need additional properties on the matrix $\nabla^2 V(c)$. As we will see afterwards, in the n -body problem, an explicit decoupling condition appear because the symmetry group contains the rotations

$$\mathbf{R}_\theta = \begin{pmatrix} \cos \theta \mathbf{I}_n & -\sin \theta \mathbf{I}_n \\ \sin \theta \mathbf{I}_n & \cos \theta \mathbf{I}_n \end{pmatrix}$$

and this group does not possess a common eigenvector of eigenvalue 1.

Now we can wonder whether the Theorems 8, 9, 10 are really “useful” and not only purely theoretical possibilities without any examples. Does it exist effectively some potentials that would be integrable only for a specific energy and value of an “angular momentum”? Using Hietarinta’s (1983) direct method and then our non-integrability approach, we find the following potentials (the second one probably being new).

Theorem 12 *We consider the potentials*

$$V_1 = \frac{\sqrt{x^2 + y^2}}{x^2 + y^2 - z^2} \quad V_2 = \frac{x^2 + y^2 + z^2}{(x^2 + y^2)^{3/2}} \tag{13}$$

The potential V_1 is integrable for zero canonical angular momentum $C = p_x y - p_y x = 0$, but not on any other hypersurface $C^2 H = \alpha, \alpha \in \mathbb{C}^$ (the question about integrability on $H = 0$ is still open).*

The potential V_2 is integrable on the hypersurface $H = 0$ of zero energy, but neither on any other hypersurface $C^2 H = \alpha, \alpha \in \mathbb{C}^$, nor on the hypersurface $C = 0$.*

Proof The first integral of V_1 is given by

$$I_1 = \frac{(xp_x + yp_y)p_z}{\sqrt{x^2 + y^2}} - \frac{z}{x^2 + y^2 - z^2}$$

The potential V_1 possesses a Darboux point $(1, 0, 0)$, and the associated eigenvalue is $\lambda = 2$. Using the integrability Table 1, this value is the only possible one for the hypersurfaces $C = 0$ and $H = 0$. Then V_1 is not meromorphically integrable on any hypersurface of the form $C^2H = \alpha$, $\alpha \in \mathbb{C}^*$.

The first integral of V_2 is given by

$$I_2 = (x^2 + y^2 - z^2)^2 p_z^2 - 4z(x^2 + y^2 - z^2)p_z(xp_x + yp_y) + 4z^2(xp_x + yp_y)^2$$

The potential V_2 possesses a Darboux point $(1, 0, 0)$, and the associated eigenvalue is $\lambda = 2$. Using the integrability Table 1, this value is the only possible one for the hypersurfaces $C = 0$ and $H = 0$. We know it is integrable for $H = 0$. Suppose it is integrable for $C = 0$. Then, we could reduce the potential by rotation and we would obtain the following potential (on the plane $y = 0$)

$$\tilde{V}_2 = \frac{x^2 + z^2}{x^3}$$

This potential possesses a Darboux point $(1, 0)$ and the associated eigenvalue is $\lambda = 2$. But in this case, it already has been proved that the potential should belong to one of the following families (after rotation)

$$V = \frac{a}{x} + \frac{b}{z} \quad a, b \in \mathbb{C}^* \quad V = \frac{a(x^2 + z^2)}{(x + \epsilon iz)^3} + \frac{a}{x + \epsilon iz} \quad a \in \mathbb{C}^*, \epsilon = \pm 1 \quad (14)$$

The second case is impossible because it is always complex. For the first one, we apply a rotation to \tilde{V}_2 of angle θ

$$\tilde{V}_{2\theta} = \frac{x^2 + z^2}{(\cos(\theta)x + \sin(\theta)z)^3}$$

and this never coincide with expression (14). Then V_2 is not integrable on the hypersurface $C = 0$. □

6 Application to the n -body problem

We consider V the potential of the n -body problem in the plane

$$V = \sum_{i>j} \frac{m_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|} \quad (15)$$

with positive masses m_i , $\mathbf{q}_i \in \mathbb{C}^2$. The symmetry group is (at least)

$$G = \left\langle \left(\begin{array}{cc} \cos \theta \mathbf{I}_n & -\sin \theta \mathbf{I}_n \\ \sin \theta \mathbf{I}_n & \cos \theta \mathbf{I}_n \end{array} \right), \theta \in \mathbb{C} \right\rangle$$

Let \mathbf{c} be a Darboux point with multiplier -1 and such that $\|\mathbf{c}\|^2 \neq 0$. Then $G\mathbf{c}$ is a plane and inside this plane we can build a conic orbit (by definition, the mutual distances between the bodies are not zero). For the following, we will pose

$$\mathbf{W}_{i,j} = \frac{1}{m_i} \left(\nabla^2 \mathbf{V}(\mathbf{c}) \right)_{i,j} \quad \mathbf{W} \in M_{2n}(\mathbb{C}) \tag{16}$$

using notation $m_{i+n} = m_i$. Remark for the following that the potential of the n -body problem as given by (15) is not reduced at all. This means in particular that the kinetic part is

$$\sum_{i=1}^n \frac{\|p_i\|^2}{2m_i}$$

and so does not correspond exactly to the case we studied before. Still, it is almost the same and we just have to make a variable change $p_i \rightarrow p_i \sqrt{m_i}$. The matrix $\nabla^2 \mathbf{V}(\mathbf{c})$ becomes notably the matrix given by (16).

6.1 General properties

Definition 7 Let V be the potential of the n -body problem with positive masses m_i , \mathbf{c} a Darboux point with multiplier -1 . We will say that the variational equation near a conic orbit is partially decoupled if it exists a non trivial vector space \tilde{V} and $\lambda \in \mathbb{C}$ such that

$$\mathbf{W}\mathbf{v} = \lambda\mathbf{v} \quad \forall \mathbf{v} \in \tilde{V}$$

and \tilde{V} is stable by the rotations

$$\mathbf{R}_\theta = \begin{pmatrix} \cos \theta \mathbf{I}_n & -\sin \theta \mathbf{I}_n \\ \sin \theta \mathbf{I}_n & \cos \theta \mathbf{I}_n \end{pmatrix}$$

Remark 6 This definition corresponds exactly to the existence of a non trivial eigenspace E satisfying Theorem 10.

Theorem 13 Let V be the potential of the n -body problem with positive masses m_i , \mathbf{c} a Darboux point with multiplier -1 and $\mathbf{W} \in M_{2n}(\mathbb{C})$ the associated matrix (given by Eq. 16). The variational equation near a conic orbit is partially decoupled if and only if it exists a vector $\mathbf{v} \in \mathbb{C}^{2n} \setminus \{0\}$ and $\lambda \in \mathbb{C}$ such that

$$\mathbf{W}\mathbf{v} = \mathbf{J}^{-1} \mathbf{W} \mathbf{J} \mathbf{v} = \lambda \mathbf{v} \tag{17}$$

where $\mathbf{J} \in M_{2n}(\mathbb{C})$ is matrix of the canonical symplectic form.

Proof Suppose at first that \mathbf{v} is not an eigenvector of \mathbf{R}_θ (these matrices commute so they have the same eigenvectors). We just have to take $\tilde{V} = \text{Span}(\mathbf{v}, \mathbf{J}\mathbf{v})$ because the space generated by $\mathbf{R}_\theta \mathbf{v}$, $\forall \theta$ is a 2-dimensional space which contains $\mathbf{v}, \mathbf{J}\mathbf{v}$ (\tilde{V} is always 2-dimensional because \mathbf{v} is not an eigenvector of $\mathbf{J} = \mathbf{R}_{\pi/2}$). Using the hypotheses, \mathbf{v} and $\mathbf{J}\mathbf{v}$ are eigenvectors of \mathbf{W} with the same eigenvalue, so \tilde{V} is an eigenspace of \mathbf{W} stable by the rotations \mathbf{R}_θ . If \mathbf{v} is an eigenvector of \mathbf{R}_θ , then we take $\tilde{V} = \mathbb{C} \cdot \mathbf{v}$ and \tilde{V} is an eigenspace of \mathbf{W} stable by the rotations \mathbf{R}_θ .

Conversely, if we have an eigenspace \tilde{V} stable by the rotations \mathbf{R}_θ , we take any vector $\mathbf{v} \in \tilde{V}$ and it satisfies (17) because $\mathbf{J} = \mathbf{R}_{\pi/2}$ and then $\mathbf{J}\mathbf{v} \in \tilde{V}$, and so it is likewise an eigenvector of eigenvalue λ . □

Theorem 14 Let V be the potential of the n -body problem with positive masses m_i , \mathbf{c} a Darboux point with multiplier -1 and $\mathbf{W} \in M_{2n}(\mathbb{C})$ the associated matrix (given by Eq. 16). If the variational equation near a conic orbit is partially decoupled then the matrix \mathbf{W} of (16) has a double eigenvalue.

Proof If $\dim(\tilde{V}) \geq 2$ then by definition the matrix \mathbf{W} has a double eigenvalue. Let us consider the case $\dim(\tilde{V}) = 1$. The corresponding vector has to be a common eigenvector of \mathbf{J} and \mathbf{W} . The eigenvectors J are of the form $(\mathbf{w}, i\mathbf{w})$, $\mathbf{w} \in \mathbb{C}^n$. In particular, they have zero “norm”. But if \mathbf{W} has only simple eigenvalues, then it is diagonalizable, so diagonalizable in an orthonormal complex basis (this is a small linear algebra proof done in Craven (1969) Theorem 1, 3). So if \mathbf{W} has an eigenvector with zero “norm”, then this eigenvector is a linear combination of two eigenvectors and this implies the existence of an eigenspace of dimension ≥ 2 , so a double eigenvalue. \square

Theorem 15 *Let V be the potential of the n -body problem with positive masses m_i , \mathbf{c} a Darboux point with multiplier -1 such that the bodies are aligned and $\mathbf{W} \in M_{2n}(\mathbb{C})$ the associated matrix (given by Eq. (16)). We suppose that \mathbf{W} is diagonalizable. Then the variational equation near a conic orbit has a Galois group G such that*

$$G \sim \tilde{G} \text{ with } \tilde{G} \subset \mathbb{C} \times Sp(2)^{n-2}$$

where $Sp(2)$ is the 4 dimensional symplectic group.

Proof For an aligned Darboux point, we have the following property (found by direct computation)

$$\mathbf{W} = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -\frac{1}{2}\mathbf{A} \end{pmatrix} \quad \mathbf{J}^{-1}\mathbf{W}\mathbf{J} = \begin{pmatrix} -\frac{1}{2}\mathbf{A} & 0 \\ 0 & \mathbf{A} \end{pmatrix} \tag{18}$$

Then \mathbf{W} and $\mathbf{J}^{-1}\mathbf{W}\mathbf{J}$ commute. Then it exists a common eigenvector basis of \mathbf{W} and $\mathbf{J}^{-1}\mathbf{W}\mathbf{J}$. Then there exists a decomposition of \mathbb{C}^{2n} in spaces V_i of dimension 2 with the V_i stable by rotations \mathbf{R}_θ . We can then write the variational equation under the following form

$$t(-C^2 + 2t + 2Ht^2)\ddot{\mathbf{X}} + (-t + C^2)\dot{\mathbf{X}} = \mathbf{R}_{\theta(t)}^{-1}\mathbf{A}_i\mathbf{R}_{\theta(t)}\mathbf{X} \quad i = 1 \dots n$$

with \mathbf{A}_i a 2×2 matrix (we can choose \mathbf{A}_i diagonal after a basis change). Among the matrices \mathbf{A}_i , there is one corresponding to the motion of the center of mass and this gives $\mathbf{A}_1 = 0$. There is also a matrix corresponding to the Hamiltonian and total angular momentum (which are first integrals), and this corresponds to $\mathbf{A}_2 = \text{diag}(2, -1)$. The other matrices do not have a priori special properties. Then the Galois group for the cases $i = 1, 2$ is \mathbb{C} , and for the others, it is at most $Sp(2)$. \square

Theorem 16 *Let V be the potential of the n -body problem with positive masses m_i , \mathbf{c} an aligned Darboux point with multiplier -1 and $\mathbf{W} \in M_{2n}(\mathbb{C})$ the associated matrix (given by Eq. (16)). The variational equation near a conic orbit is partially decoupled if and only if $\det(\mathbf{W}) = 0$.*

Proof For an aligned Darboux point, we have the equalities (18). We pose $\mathbf{v} = (\mathbf{w}_1, \mathbf{w}_2)$. If \mathbf{v} is an eigenvector of \mathbf{W} , then \mathbf{w}_1 is an eigenvector of \mathbf{A} and $-\frac{1}{2}\mathbf{A}$ with the same eigenvalue. Then $\det(\mathbf{A}) = 0$. Conversely, if $\det(\mathbf{W}) = 0$, then it exists an eigenvector \mathbf{w} of eigenvalue 0 of \mathbf{A} , and then $\mathbf{v} = (\mathbf{w}, \mathbf{w})$ is admissible. \square

6.2 The 3-body problem and some specific cases

We already know that in all cases, the matrix \mathbf{W} should have a double eigenvalue. Our approach will be the following. We search masses and Darboux points such that \mathbf{W} has a double eigenvalue. Then for the corresponding eigenvector \mathbf{v} , there are two possibilities.

- Either \mathbf{Jv} is also an eigenvector of \mathbf{W} with the same eigenvalue. This corresponds to the case where the associated eigenspace is of dimension ≥ 2 .
- Or \mathbf{v} can be written $\mathbf{v} = (\mathbf{w}, i\mathbf{w})$, and the matrix \mathbf{W} is not diagonalizable.

For the aligned case, it is easier because we just have to look at the determinant. But in fact for the real ones, there is no zero eigenvalue if the Darboux point is real (this is due to the result of Pacella 1987), so we need to look at complex cases. But, even there, this constraint is much stronger than expected. We find the following theorem

Theorem 17 *Let V be the potential of the 3-body problem with positive masses m_1, m_2, m_3 such that $m_1 + m_2 + m_3 = 1$. Then V possesses a Darboux point such that the variational equation near a conic orbit is partially decoupled if and only if*

$$(m_1, m_2, m_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{7}, \frac{5}{7}, \frac{1}{7}\right), \left(\frac{1}{4} + \frac{\sqrt{21} + \sqrt{126 + 42\sqrt{21}}}{84}, \frac{1}{2} - \frac{\sqrt{21}}{42}, \frac{1}{4} + \frac{\sqrt{21} - \sqrt{126 + 42\sqrt{21}}}{84}\right) \tag{19}$$

or permutation of these cases.

Proof Let us begin with aligned case. After renormalization, we can take $\mathbf{c} = (-1, 0, \rho)$ with $\rho \neq 0, -1$ and we have the Euler quintic equation

$$L = (-m_1 - m_2) \rho^5 + (-3m_1 - 2m_2) \rho^4 + (-3m_1 - m_2) \rho^3 + (3m_3 + m_2) \rho^2 + (3m_3 + 2m_2) \rho + m_2 + m_3 = 0$$

We search the eigenvalues of \mathbf{W} , and we find that $\det(\mathbf{W}) = 0$ if and only if

$$2\rho^2 + 3\rho + 2 = 0$$

After taking the resultant, we have

$$\text{Res}(2\rho^2 + 3\rho + 2, L, \rho) = 7m_2^2 - 35m_1m_2 - 35m_2m_3 + 56m_1^2 + 63m_1m_3 + 56m_3^2$$

We want this resultant to vanish, and the only possibility for real positive masses is

$$(m_1, m_2, m_3) = \left(\frac{1}{7}, \frac{5}{7}, \frac{1}{7}\right)$$

We can permute the masses in the equation and this gives all the possible permutations of this solution. But there is still a ‘‘complex order’’ and the corresponding potential is the following

$$V = \frac{m_1m_2}{q_1 - q_2} - \frac{m_1m_3}{q_1 - q_3} + \frac{m_2m_3}{q_2 - q_3}$$

The Darboux point equation leads to

$$L = (-m_1 - m_2) \rho^5 + (-3m_1 - 2m_2) \rho^4 + (-3m_1 + 2m_3 - m_2) \rho^3 + (-2m_1 + 3m_3 + m_2) \rho^2 + (3m_3 + 2m_2) \rho + m_2 + m_3 = 0$$

The eigenvalues of \mathbf{W} never vanish in this case. Let us look now at the Lagrange configuration. For complex coordinates, this corresponds to the case

$$r_1^3 = r_2^3 = r_3^3$$

where r_1, r_2, r_3 are the mutual distances between the bodies. We begin by the case $r_1 = r_2 = r_3$. We need a double eigenvalue and we find the condition

$$3m_2^2 - 3m_2m_3 - 3m_1m_2 + 3m_3^2 - 3m_1m_3 + 3m_1^2 = 0$$

whose unique solution is

$$(m_1, m_2, m_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

We check that the associated eigenspace of eigenvalue $1/2$ is invariant by \mathbf{J} , and it is the case. Let us look now at the complex cases. Among the $27 - 1$ possibilities lots of them are in fact the same after dilatation-permutation. After these reductions, we find that there are only 3 essentially different cases

$$(r_1, r_2, r_3) = (1, 1, j), (1, 1, j^2), (1, j, j^2) \quad j = e^{\frac{2i\pi}{3}}$$

The last one is also an aligned Darboux point (it is both Lagrange and Euler configuration), therefore it has already been treated. First we search for masses such that \mathbf{W} has a double eigenvalue. We find for $(1, 1, j)$ and $(1, 1, j^2)$ a single real positive solution, which is the last one of (19). This is the same for both Darboux points because they are conjugated. We look at the corresponding eigenspace (the double eigenvalue is $1/2$), and we find that the eigenspace is only 1-dimensional. This is not enough for the case $\dim(\tilde{V}) \geq 2$. In the case $\dim(\tilde{V}) = 1$, we know that \mathbf{W} should be non-diagonalizable. Moreover, the eigenvector should be of the form $\mathbf{v} = (\mathbf{w}, i\mathbf{w})$. We check these properties and they are satisfied. \square

Remark 7 The last case of (19) is very interesting for many reasons. We can study the variational equations and we see that the structure of the equations is not so degenerated as in the other cases. Because of this, a far deeper analysis should be possible. For example, another notion of partial integrability is considered in Maciejewski et al. (2007), Morales-Ruiz and Simon (2009) about the existence of a single additional first integral. For this last masses case, the two notions could probably be fused together to prove the non existence of a single additional first integral restricted to a single level of the Hamiltonian and total angular momentum. This is because the variational equation on the characteristic space associated to the eigenvalue $1/2$ is simple enough to allow complete study, but is not trivial. Moreover, the fact that these masses do not possess any symmetry will avoid to consider special invariant sub manifold as the isosceles 3-body problem in, for example, the complete search of algebraic invariant manifold for the 3-body problem with these masses.

Theorem 18 (see Corbera et al. 2009 for examples) *We consider V the potential of the n body problem in the plane with positive masses, and \mathbf{c} a real central configuration such that there exists a rotation*

$$\mathbf{R}_\theta, \quad \theta \notin \{k\pi, k \in \mathbb{Z}\}$$

in the plane which sends the configuration on itself (conserving also the masses). Then the matrix $\mathbf{W}(\mathbf{c})$ has a double eigenvalue and the associated eigenspace is of dimension ≥ 2 .

Proof Let \mathbf{R}_θ be a rotation such that $\theta \notin \{k\pi, k \in \mathbb{Z}\}$ which sends the configuration \mathbf{c} on itself and conserves the masses. We pose $\mathbf{W}(\mathbf{c})$ as in (16) and we have then the identities

$$\mathbf{W}(\mathbf{R}_\theta \mathbf{c}) = \mathbf{R}_{-\theta} \mathbf{W}(\mathbf{c}) \mathbf{R}_\theta \quad \mathbf{W}(\mathbf{R}_\theta \mathbf{c}) = \mathbf{W}(\mathbf{Pc}) = \mathbf{P}^{-1} \mathbf{W}(\mathbf{c}) \mathbf{P}$$

with \mathbf{P} a permutation matrix (the rotation conserves the configuration and the masses of the bodies, but not the numeration of the bodies). Then

$$\mathbf{W}(\mathbf{c}) = (\mathbf{R}_\theta \mathbf{P}^{-1})^{-1} \mathbf{W}(\mathbf{c}) \mathbf{R}_\theta \mathbf{P}^{-1}$$

Let \mathbf{v} be an eigenvector of $\mathbf{W}(\mathbf{c})$. Then $\mathbf{R}_\theta \mathbf{P}^{-1} \mathbf{v}$ is also an eigenvector with the same eigenvalue. We just have to prove it is not the same. We can write in a good basis

$$\mathbf{R}_\theta = \begin{pmatrix} \cos \theta \mathbf{I}_n & -\sin \theta \mathbf{I}_n \\ \sin \theta \mathbf{I}_n & \cos \theta \mathbf{I}_n \end{pmatrix} \quad \mathbf{P} = \begin{pmatrix} \mathbf{P}_\sigma & 0 \\ 0 & \mathbf{P}_\sigma \end{pmatrix}$$

with \mathbf{P}_σ a permutation matrix. We find then that \mathbf{P} and \mathbf{R}_θ commute. We know that the rotation \mathbf{R}_θ is of finite order (because there are only a finite number of bodies and that the configuration is real). Then $\theta = 2\pi/k$ with $k \in \mathbb{N}^*$, and $k \geq 3$. The matrix \mathbf{P} is then also of order k .

Let us consider the body number i with coordinates \mathbf{q}_i . We look at the orbit $\mathbf{R}_\theta^j \mathbf{q}_i$, $j = 0, \dots, k - 1$. This orbit contains either k elements or a single one (and this case could only happen once, for a body placed on the center of mass). We conclude that the permutation matrix should be of the following form

$$\mathbf{P}_\sigma = \begin{pmatrix} \mathbf{T} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & \mathbf{T} & 0 \\ 0 & \dots & 0 & \mathbf{Id} \end{pmatrix} \quad \text{or} \quad \mathbf{P}_\sigma = \begin{pmatrix} \mathbf{T} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & \mathbf{T} & 0 \\ 0 & \dots & 0 & \mathbf{T} \end{pmatrix} \quad \text{with} \quad \mathbf{T} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

We conclude that the matrix $\mathbf{R}_\theta \mathbf{P}$ can be diagonalized in the form

$$\begin{aligned} \mathbf{R}_\theta \mathbf{P} &\sim \text{diag} \left(e^{i\theta}, e^{-i\theta}, \left(e^{i(j+1)\theta}, \dots, e^{i(j+1)\theta}, e^{i(j-1)\theta}, \dots, e^{i(j-1)\theta} \right)_{j=0..,k-1} \right) \\ \text{or} \quad \mathbf{R}_\theta \mathbf{P} &\sim \text{diag} \left(\left(e^{i(j+1)\theta}, \dots, e^{i(j+1)\theta}, e^{i(j-1)\theta}, \dots, e^{i(j-1)\theta} \right)_{j=0..,k-1} \right) \end{aligned}$$

We know that the masses are positive and that the Darboux point is real, then all eigenvectors \mathbf{v} of $\mathbf{W}(\mathbf{c})$ are real. Suppose that $\mathbf{W}(\mathbf{c})$ does not have any eigenspace of dimension ≥ 2 . Then all its eigenvectors are eigenvectors of $\mathbf{R}_\theta \mathbf{P}$. As $\mathbf{R}_\theta \mathbf{P}$ is real, if \mathbf{v} is a real eigenvector of $\mathbf{R}_\theta \mathbf{P}$, then the associated eigenvalue is real and so the associated eigenvalue is ± 1 . This would mean that

$$Sp(\mathbf{R}_\theta \mathbf{P}) \subset \{-1, 1\}$$

This is impossible because $k \geq 3$.

6.3 The equal masses case

Theorem 19 *Let V be the potential of the n -body problem in the plane with equal masses, \mathbf{c} the Darboux point given by the following*

$$\mathbf{c}_i = \alpha \cos \left(\frac{2\pi(i-1)}{n} \right) \quad \mathbf{c}_{i+n} = \alpha \sin \left(\frac{2\pi(i-1)}{n} \right) \quad i = 1, \dots, n$$

where α is such that the multiplier equals to -1 . Let \mathbf{v} be the vector given by

$$\mathbf{v}_i = \cos \left(\frac{4\pi(i-1)}{n} \right) \quad \mathbf{v}_{i+n} = \sin \left(\frac{4\pi(i-1)}{n} \right) \quad i = 1, \dots, n$$

Then (17) is satisfied with

$$\lambda = 2 - \frac{2 \sin\left(\frac{\pi}{n}\right)}{1 - \cos\left(\frac{\pi}{n}\right)} \left(\sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{\pi j}{n}\right)} \right)^{-1} \tag{20}$$

Proof The proof is only a direct computation of the matrix \mathbf{W} and then of $\mathbf{W}\mathbf{v}$ and the use of (lots of) trigonometric formulas. \square

We can now eventually prove Theorem 3.

Proof Using Theorem 9, one just need to avoid specific values for λ . We will then build a majoration and minoration for λ given by formula (20). First of all, we remark that for $n \geq 3$

$$\frac{2 \sin\left(\frac{\pi}{n}\right)}{1 - \cos\left(\frac{\pi}{n}\right)} \left(\sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{\pi j}{n}\right)} \right)^{-1} > 0$$

Then $\lambda < 2$. Let us prove now that $\lambda > 0$. First we prove the following inequality

$$\sin(z) < \frac{1}{\sin(z)} \quad \forall z \in]0, \pi/2[\cup]\pi/2, \pi[$$

and we compute the formula

$$\frac{\sin\left(\frac{\pi}{n}\right)}{1 - \cos\left(\frac{\pi}{n}\right)} = \sum_{j=1}^{n-1} \sin\left(\frac{\pi j}{n}\right)$$

Using both of them, this gives for $n \geq 3$

$$\frac{\sin\left(\frac{\pi}{n}\right)}{1 - \cos\left(\frac{\pi}{n}\right)} < \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{\pi j}{n}\right)}$$

So we get that $\lambda > 0$. Using the integrability table of Theorem 9, there are no exceptional values in $]0, 2[$. \square

The case $C^2H = 0$ is special. In this case, we have either $C = 0$ or $H = 0$ (or both). The case $H = 0$ corresponds to parabolic orbits. These orbits are used by Tsygvinsev in Tsygvinsev (2001), and he solves the case for 3 bodies with equal masses (he also studies the existence of a single additional first integral in Tsygvinsev (2007) that we do not consider). In the case of $C = 0$, the problem is solved by Morales-Ruiz and Simon (2009) for the n equal masses. Our reasoning is also valid for all these cases.

Remark 8 The only left case is $H = C = 0$. Here, the variational equation is always integrable and in fact it is always the case at all orders. This is linked to the fact that we can reduce the system using homogeneity and rotation, allowing to diminish the dimension of 4. We obtain then a “direction” field (we loose notion of time after reduction, but not the integrability notions) on a manifold of dimension $4n - 8$. This, however, destroy Hamiltonian structure, and moreover, the Darboux points correspond now to fixed points of this field. One would need a new particular orbit (explicit) to apply Morales-Ramis method, but no such orbit is known.

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