

Bounding Basis-Risk Using s-convex Orders on Beta-unimodal Distributions

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OICA

Cat model = 4 components

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Hazard: consists of a large number of catastrophe event scenarios that together provide a representation of possible loss-causing events, and an associated modeled rate of occurrence for each.

Let n be the number of stochastic events, and p_i be the occurrence rate of the event i , $i = 1, \dots, n$.

Cat model = 4 components



Hazard: catastrophe event scenarios + occurrence rate

Inventory: represents the exposure. In practice, the exposure is not perfectly known (uncertain even trajectory + uncertain exposure when the event occurs).

Let Y_i be a positive random variable representing the exposure of scenario i , $i = 1, \dots, n$. Assume $Y_i \leq m$ a.s., where for instance m stands for the total size of the portfolio.

Sometimes, $a_i \leq Y_i \leq b_i$ a.s.

Cat model = 4 components



Hazard: catastrophe event scenarios + occurrence rate

Inventory: uncertain exposure

Vulnerability: provides the intensity of the catastrophe event on the exposed portfolio. We use the damage ratio (aka destruction rate or loss proportion) represented by

$$S_i \sim \text{Beta}(\alpha_i, \beta_i), \quad S_i \perp Y_i.$$

The loss in scenario i is then $X_i = S_i Y_i$.

Cat model = 4 components



Hazard: catastrophe event scenarios + occurrence rate

Inventory: uncertain exposure

Vulnerability: intensity = destruction rate

$$S_i \sim \text{Beta}(\alpha_i, \beta_i), \quad S_i \perp Y_i.$$

Loss: translates the expected physical damage into monetary loss taking into account any insurance structures.

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Index-based transaction: the index in scenario i takes the value c_i .

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The final flow in scenario i is given by $S_i Y_i - c_i$.

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Identification of worst case scenarios using s -convex orders



Measurement of basis risk using penalty functions $\phi!$

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Identification of worst case scenarios using s -convex orders



Measurement of basis risk using penalty functions ϕ !

In a nutshell:

$$BR^{(s)} = \sum_{i=1}^n p_i \mathbb{E} \left[\phi \left(X_{i,max}^{(s)} - c_i \right) \right], \quad s = 2, \dots$$

Beta-unimodal Distributions

Definition

An \mathbb{R}_+ -valued rv X has a continuous Beta-unimodal distribution if it has the product representation

$$X =_d SY, \quad (1)$$

where Y is a positive continuous rv, and $S \sim \text{Beta}(\alpha, \beta)$ is a random contracting factor independent of Y .

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From Y to X : if $Y \sim F_Y$ and if X is Beta-unimodal

$$\bar{F}_X(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} x^\alpha (I_\beta \phi)(x), \quad (2)$$

where:

- $(I_\beta \phi)(x) = \int_x^\infty \frac{1}{\Gamma(\beta)} (t-x)^{\beta-1} \phi(t) dt$ is called the Weyl fractional-order integral operator,
- $\phi(t) = \bar{F}_Y(t) t^{-\alpha-\beta}$.

Beta-unimodal Distributions

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$$X =_d SY, \quad (3)$$

where Y is a positive continuous rv, and $S \sim \mathcal{B}(\alpha, \beta)$ is a random contracting factor independent of Y .



From X to Y : if $X \sim F_X$ and if X is Beta-unimodal

$$\bar{F}_Y(x) = (-1)^n x^{\alpha+\beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} (I_\delta D^n \psi)(x), \quad (4)$$

where:

- $\delta \in [0, 1]$ such as $\beta + \delta = n \in \mathbb{N}$,
- $\psi(t) = \bar{F}_X(t)t^{-\alpha}$,
- D^n denotes the n -fold derivative operator.

Identification: s-convex orders

Definition: s-convexity

A function ϕ defined on \mathcal{S} is said to be **s-convex** if the inequality

$$[x_0, \dots, x_s; \phi] \geq 0,$$

holds for any choice of distinct points x_0, \dots, x_s in \mathcal{S} .

$[x_0, \dots, x_s; \phi]$ denotes a divided difference of the function ϕ at the different points x_0, \dots, x_s .

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s differentiability condition: if $\phi^{(s)}$ exists in \mathcal{S} , then ϕ is s-convex if and only if $\phi^{(s)} \geq 0$.

Definition: s-convexity

Definition: s-increasing convexity

A function ϕ is said to be **s-increasing convex** on its domain \mathcal{S} if and only if for all choices of $k + 1$ distinct points $x_0 < x_1 < \dots < x_k$ in \mathcal{S} , we have

$$[x_0, x_1, \dots, x_k; \phi] \geq 0, \quad k = 2, \dots, s.$$

We denote by $\mathcal{U}_{s-icx}^{\mathcal{S}}$ the class of the s-increasing convex functions on \mathcal{S} .

Definition: s-convexity

Definition: s-increasing convexity

Definition: s-convex order

Let X_1 and X_2 be two random variables that take on values in \mathcal{S} . Then X_1 is said to be smaller than X_2 in the s-convex order, denoted by $X_1 \leq_{s-cx}^{\mathcal{S}} X_2$ if

$$\mathbb{E}[\phi(X_1)] \leq \mathbb{E}[\phi(X_2)] \text{ for all } \phi \in \mathcal{U}_{s-cx}^{\mathcal{S}}, \quad (5)$$

where $\mathcal{U}_{s-cx}^{\mathcal{S}}$ is the class of all the s-convex functions $\phi : \mathcal{S} \rightarrow \mathbb{R}$.

Definition: moment space

We denote by $\mathcal{B}_s([a, b], \mu_1, \mu_2, \dots, \mu_{s-1})$ the moment space of all the random variables valued in $[a, b]$ and with known $s - 1$ moments μ_1, \dots, μ_{s-1} .

Identification: s-convex extrema

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Theorem: s-convex extrema

Let $Y \in \mathcal{B}_s([a, b], \mu_1, \mu_2, \dots, \mu_{s-1})$.

Within $\mathcal{B}_s([a, b], \mu_1, \mu_2, \dots, \mu_{s-1})$, there exist two unique random variables $Y_{min}^{(s)}$ and $Y_{max}^{(s)}$ such that


$$Y_{min}^{(s)} \leq_{s-cx} Y \leq_{s-cx} Y_{max}^{(s)}.$$

Proof. See Denuit et al. (1999).


s-convex extrema are actually extremal distributions built from the s-1 first moments of Y .

3-convex extremal distributions

Let $Y \in \mathcal{B}_3([a, b], \mu_1, \mu_2)$.


$$Y_{\min}^{(3)} = \begin{cases} a & \text{with proba } \frac{\mu_2 - \mu_1^2}{(a - \mu_1)^2 + \mu_2 - \mu_1^2}, \\ b & \text{with proba } \frac{(a - \mu_1)^2}{(a - \mu_1)^2 + \mu_2 - \mu_1^2}, \end{cases}$$

and


$$Y_{\max}^{(3)} = \begin{cases} \mu_1 - \frac{\mu_2 - \mu_1^2}{b - \mu_1} & \text{with proba } \frac{(b - \mu_1)^2}{(b - \mu_1)^2 + \mu_2 - \mu_1^2}, \\ b & \text{with proba } \frac{\mu_2 - \mu_1^2}{(b - \mu_1)^2 + \mu_2 - \mu_1^2}. \end{cases}$$

Generalization to Beta-unimodal distributions

Proposition (Beta-unimodal s -convex extrema)

Let $Y \in \mathcal{B}_s(\mathcal{S}, \mu_1, \mu_2, \dots, \mu_{s-1})$, and let $\phi \in \mathcal{U}_{s-cx}^{\mathcal{S}}$. If X is Beta-unimodal, then $X \in \mathcal{B}_s(\mathcal{S}, \nu_1, \nu_2, \dots, \nu_{s-1})$ and the s -convex extrema in this set are

$$X_{min}^{(s)} = SY_{min}^{(s)}, \quad \text{and} \quad X_{max}^{(s)} = SY_{max}^{(s)}.$$

Besides, if $\phi \in \mathcal{U}_{s-icx}^{\mathcal{S}}$, then

$$\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(SY_{max}^{(s)})] \leq \mathbb{E}[\phi(SY_{max}^{(s-1)})] \leq \dots \leq \mathbb{E}[\phi(SY_{max}^{(2)})],$$

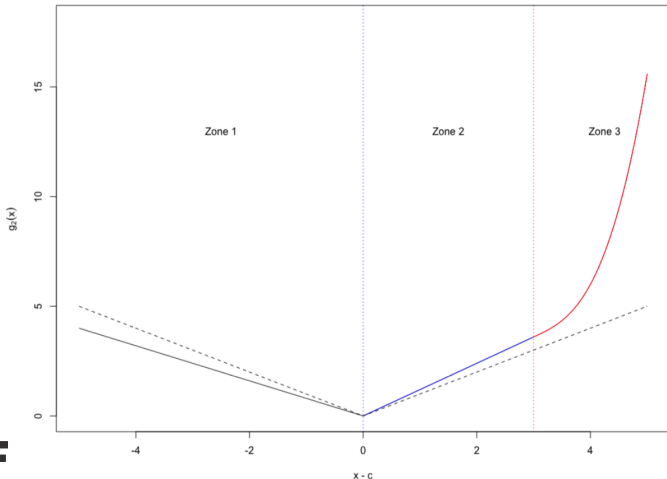
which can be written as,

$$\forall k \in \llbracket 2, s \rrbracket, X \leq_{k-cx} X_{max}^{(s)} \leq_{k-cx} X_{max}^{(s-1)} \leq_{k-cx} \dots \leq_{k-cx} X_{max}^{(2)}.$$

When $\phi \in \mathcal{U}_{s-icx}^{\mathcal{S}}$, the more moments, the sharper the bounds !

Penalty functions

A penalty function g allows us to represent the consequences of a positive or negative difference between the Loss and the Index.



Example

$$g(x - c) = f_1(x) + f_2(x) + \phi_n(x), \quad x \in \mathbb{R}$$

where

$$f_1(x) = \eta(c - x)_+, \quad f_2(x) = \gamma(x - c)_+, \quad \phi_n(x) = \gamma(x - (c + d))_+^n.$$

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f_1 and f_2 are only 2-cx whereas ϕ_n is n -icx. Consequently,

$$\begin{aligned} & \mathbb{E}[f_1(X_{min}^{(2)}) + f_2(X_{min}^{(2)})] + \mathbb{E}[\phi_n(X_{min}^{(n-1)})] \\ & \leq \mathbb{E}[f_1(X_{min}^{(2)}) + f_2(X_{min}^{(2)})] + \mathbb{E}[\phi_n(X_{min}^{(n)})] \\ & \leq \mathbb{E}[g(X)] \\ & \leq \mathbb{E}[f_1(X_{max}^{(2)}) + f_2(X_{max}^{(2)})] + \mathbb{E}[\phi_n(X_{max}^{(n)})] \\ & \leq \mathbb{E}[f_1(X_{max}^{(2)}) + f_2(X_{max}^{(2)})] + \mathbb{E}[\phi_n(X_{max}^{(n-1)})] \end{aligned}$$

Numerical illustrations with ϕ_4

Ex 1:

$$Y \sim \text{Beta}([60, 80], 0.5, 0.5), \quad c + d = 69.6, \quad S \sim \text{Beta}(7, 2)$$

Numerical illustrations with ϕ_4

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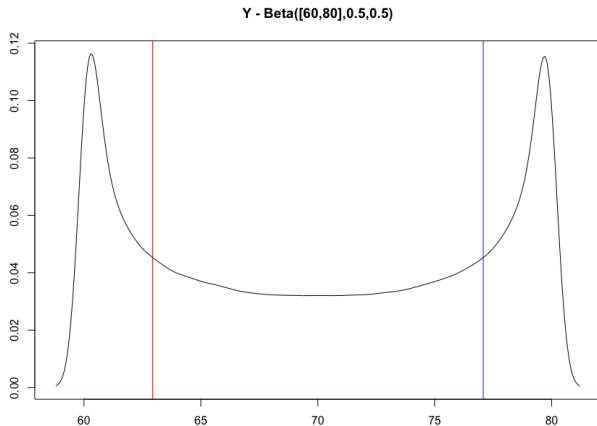
Convex order used	Values
2,min	1.8470e-06
3,min	5.8227
4,min	27.8081
$\mathbb{E}[\phi_4(X)]$	58.3232
4,max	88.8281
3,max	118.4375
2,max	177.6562

Table: Values of $\mathbb{E}[\phi_4(X_{\min/\max}^{(s)})]$ for $s = 2, 3, 4$.

Numerical illustrations with ϕ_4

Ex 1:

$$Y_{min}^{(4)} = \begin{cases} 62.9289 & \text{with proba } 0.5 \text{ (red)} \\ 77.0711 & \text{with proba } 0.5 \text{ (blue)}, \end{cases}$$



Numerical illustrations with ϕ_4

Ex 2:

$Y \sim \text{Beta}([60, 80], 2, 2)$, $c + d = 63.036$, $S \sim \text{Beta}(2, 2)$

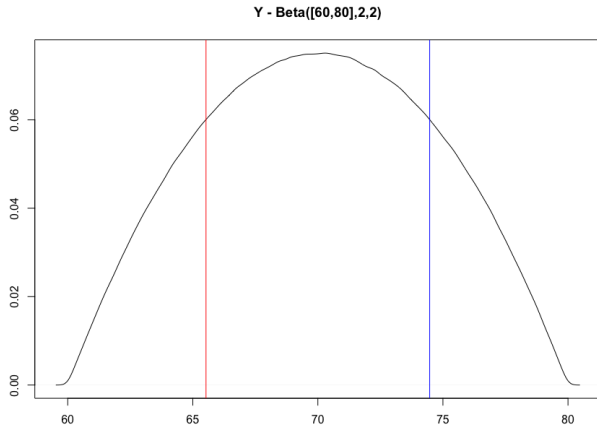
Convex order used	Values
2,min	5.4281
3,min	19.3039
4,min	46.2856
$\mathbb{E}[\phi_4(X)]$	58.0398

Table: Values of $\mathbb{E}[\phi_4(X_{min}^{(s)})]$ for $s = 2, 3, 4$.

Numerical illustrations with ϕ_4

Ex 2:

$$Y_{min}^{(4)} = \begin{cases} 65.5279 & \text{with proba } 0.5 \text{ (red)} \\ 74.4721 & \text{with proba } 0.5 \text{ (blue)}, \end{cases}$$



Conclusion

Promise kept ?



Identification of s-convex extrema for Beta-Unimodal random variables,

Promise kept ?



Identification of s-convex extrema for Beta-Unimodal random variables,



Measurement of basis risk, and show the impact of information (in terms of moments) on basis risk assessment,

Is basis risk always bounded ? $S \perp Y$?