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► **To cite this version:**

Vincent Guirardel, Camille Horbez, Jean Lécureux. Cocycle superrigidity from higher rank lattices to $\text{Out}(F_N)$. Journal of modern dynamics, 2022, 18, pp.291-344. 10.3934/jmd.2022010 . hal-02611060

HAL Id: hal-02611060

<https://hal.science/hal-02611060>

Submitted on 18 Nov 2020

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Cocycle superrigidity from higher rank lattices to $\text{Out}(F_N)$

Vincent Guirardel, Camille Horbez and Jean Lécureux

May 18, 2020

Abstract

We prove a rigidity result for cocycles from higher rank lattices to $\text{Out}(F_N)$ and more generally to the outer automorphism group of a torsion-free hyperbolic group. More precisely, let G be either a product of connected higher rank simple algebraic groups over local fields, or a lattice in such a product. Let $G \curvearrowright X$ be an ergodic measure-preserving action on a standard probability space, and let H be a torsion-free hyperbolic group. We prove that every Borel cocycle $G \times X \rightarrow \text{Out}(H)$ is cohomologous to a cocycle with values in a finite subgroup of $\text{Out}(H)$. This provides a dynamical version of theorems of Farb–Kaimanovich–Masur and Bridson–Wade asserting that every morphism from G to either the mapping class group of a finite-type surface or the outer automorphism group of a free group, has finite image.

The main new geometric tool is a barycenter map that associates to every triple of points in the boundary of the (relative) free factor graph a finite set of (relative) free splittings.

Introduction

A celebrated theorem of Farb–Kaimanovich–Masur states that every morphism from an irreducible lattice Γ in a higher rank semisimple real Lie group to the mapping class group of a finite-type surface has finite image [KM96, FM98]. Later, Bridson–Wade proved the analogous result with the mapping class group replaced with $\text{Out}(F_N)$, the outer automorphism group of a finitely generated free group [BW11] – their result actually applies to a more general class of groups Γ in the source, the only assumption being that no normal subgroup of Γ or of a finite-index subgroup of Γ surjects onto \mathbb{Z} .

These strong rigidity results have since led to further developments. Haettel recently proved that when Γ is an irreducible lattice in a (product of) higher rank simple connected algebraic groups over local fields, any action of Γ on a hyperbolic space is elementary, i.e. has either bounded orbits or fixes a finite set in the boundary [Hae16]. Since, on the other hand, (subgroups of) mapping class groups and $\text{Out}(F_N)$ have many interesting actions on hyperbolic spaces, this could be used to give a new proof of the Farb–Kaimanovich–Masur and Bridson–Wade results (for this more restrictive collection of source groups Γ), and actually tackle the more general situation where the target is $\text{Out}(G)$, where G is any torsion-free hyperbolic group. Among recent developments,

let us also mention Mimura’s extension of the Farb–Kaimanovich–Masur and Bridson–Wade’s theorems to the case where Γ is a non-arithmetic Chevalley group [Mim18].

Meanwhile, inspired by Margulis’ superrigidity theorem describing the representation theory of lattices in higher-rank semisimple Lie groups [Mar91], Zimmer proved a cocycle rigidity theorem for ergodic actions of these lattices [Zim80]. Indeed, given a lattice Γ in a locally compact group G , a natural induction procedure turns morphisms $\Gamma \rightarrow \Lambda$ into cocycles $G \times G/\Gamma \rightarrow \Lambda$, where one can take advantage of the rich structure of G . Zimmer’s cocycle superrigidity theorem turned out to have important consequences, for example to measure equivalence rigidity of such lattices [Fur99]. This paved the way to a whole cocycle rigidity theory: Spatzier–Zimmer [SZ91], Adams [Ada96], Monod–Shalom [MS04, MS03] and Hamenstädt [Ham09] established various strong cocycle rigidity results when the target group Λ satisfies certain negative curvature properties, and this was developed further more recently by Bader and Furman [BF14a, BF] who introduced a generalized notion of *Weyl group* towards tackling rigidity problems.

The main theorem of the present paper is the following.

Theorem 1. *Let G_0 be a product of connected higher rank simple algebraic groups over local fields. Let G be either G_0 , or a lattice in G_0 . Let X be a standard probability space equipped with an ergodic measure-preserving G -action. Let H be a torsion-free hyperbolic group.*

Then every cocycle $c : G \times X \rightarrow \text{Out}(H)$ is cohomologous to a cocycle that takes its values in a finite subgroup of $\text{Out}(H)$.

Here, we recall that a Borel map $c : G \times X \rightarrow \text{Out}(H)$ is a *cocycle* if for every $g_1, g_2 \in G$ and a.e. $x \in X$, one has $c(g_1g_2, x) = c(g_1, g_2x)c(g_2, x)$. Two cocycles $c_1, c_2 : G \times X \rightarrow \text{Out}(H)$ are *cohomologous* if there exists a Borel map $f : X \rightarrow \text{Out}(H)$ such that for all $g \in G$ and a.e. $x \in X$, one has $c_2(g, x) = f(gx)^{-1}c_1(g, x)f(x)$.

We would like to make a few comments on the statement. First, notice that the target can be in particular a mapping class group or $\text{Out}(F_N)$, so this yields yet another proof of the Farb–Kaimanovich–Masur and Bridson–Wade theorems, though with certain restrictions on the source group. Second, we would like to mention that our main theorem actually follows from a version for cocycles with values in the outer automorphism group of a free product (see Theorem 9.3), which is used to carry on inductive arguments – even to prove the result for $\text{Out}(F_N)$.

The proof of our main theorem follows a strategy initiated by Bader and Furman in [BF14a, BF]. We set up a geometric setting for the target group Λ which enables us to derive rigidity results for cocycles $G \times X \rightarrow \Lambda$, where G is as in the statement of the theorem. We now describe this geometric setting.

Geometric setting for cocycle superrigidity and inductive argument. Let Λ be a countable discrete group, and let \mathbb{D} be a countable discrete space equipped with a Λ -action. The following definition, which is central in the present work, is a form of negative curvature for Λ with respect to \mathbb{D} . One should think that stabilizers of points in \mathbb{D} will be simpler subgroups of Λ , which will enable us to carry inductive arguments. We denote by $\mathcal{P}_{<\infty}(\mathbb{D})$ the (countable) set of finite subsets of \mathbb{D} .

Definition. Let \mathbb{D} be a countable discrete space, let K be a compact metrizable space equipped with a Λ -action by homeomorphisms, and let Δ be a Polish space equipped with a Λ -action by Borel automorphisms. We say that the triple (\mathbb{D}, K, Δ) is geometrically rigid if the following properties hold:

- **Compatibility:** The space K has a Λ -invariant Borel partition $K = K_{\text{bdd}} \sqcup K_{\infty}$ coming with Λ -equivariant Borel maps $K_{\text{bdd}} \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$ and $K_{\infty} \rightarrow \Delta$.
- **Barycenter map:** There is a Λ -equivariant Borel barycenter map, assigning to every triple of pairwise distinct points of Δ a finite subset of \mathbb{D} .
- **Amenability:** The Λ -action on Δ is universally amenable.

We say that Λ is geometrically rigid with respect to \mathbb{D} if there exists a geometrically rigid triple (\mathbb{D}, K, Δ) .

Remark. In most examples we have in mind, \mathbb{D} will be the vertex set of a hyperbolic graph and Δ will be its Gromov boundary (see however Section 5.2 for a different setting concerning right-angled Artin groups).

This geometric setting is ubiquitous in geometric group theory. It applies for example when Λ is a hyperbolic group (with $\mathbb{D} = \Lambda$, with $K = \Lambda \sqcup \partial_{\infty}\Lambda$ and $\Delta = \partial_{\infty}\Lambda$), or more generally a relatively hyperbolic group using a construction of Bowditch [Bow98]. It also applies to the mapping class group of a finite-type surface acting on the vertex set \mathbb{D} of the curve graph – whose point stabilizers are related to mapping class groups of simpler subsurfaces: here K is the Thurston compactification of the Teichmüller space, K_{∞} is the subset of arational projective measured foliations, and Δ is the Gromov boundary of the curve graph. A crucial part of the present paper is to check that outer automorphisms of free groups, or more generally free products (or *relative automorphism groups*), also satisfy the above geometric setting in a natural way; we will say a word about this later in this introduction.

Theorem 2. Let Λ be a countable discrete group, and let \mathbb{D} be a countable discrete Λ -set. Assume that Λ is geometrically rigid with respect to \mathbb{D} .

Let G_0 be a product of connected higher rank simple algebraic groups over local fields. Let G be either G_0 , or a lattice in G_0 . Let $G \curvearrowright X$ be an ergodic measure-preserving action on a standard probability space.

Then every cocycle $G \times X \rightarrow \Lambda$ is cohomologous to a cocycle that takes its values in a subgroup that virtually fixes an element of \mathbb{D} .

As already mentioned above, this will be the basis for inductive arguments: in all situations we have in mind, stabilizers of points in \mathbb{D} will be simpler than the ambient group Λ .

Let us say a word about our proof of Theorem 2. By a standard argument of Zimmer, proving that a cocycle c is cohomologous to a cocycle with values in a subgroup having a finite orbit in \mathbb{D} amounts to constructing a c -equivariant Borel map $X \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$ with values in the set $\mathcal{P}_{<\infty}(\mathbb{D})$ of finite subsets of \mathbb{D} .

Compactness of K is crucial to construct probability measures. More precisely, let B be a Poisson–Furstenberg boundary of G and $\text{Prob}(K)$ be the space of probability measures on K . As the G -action on B is amenable in the sense of Zimmer, and K is compact and metrizable, there exists a c -equivariant map $p : X \times B \rightarrow \text{Prob}(K)$.

Assume first that for almost every $(x, b) \in X \times B$, the probability measure $p(x, b)$ is supported on K_{bdd} . Pushing it through the provided map $K_{\text{bdd}} \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$, one gets a c -equivariant map $X \times B \rightarrow \text{Prob}(\mathcal{P}_{<\infty}(\mathbb{D})) \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$, where the rightmost map $\text{Prob}(\mathcal{P}_{<\infty}(\mathbb{D})) \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$ consists in taking the union of all points with highest mass. Then, a strong ergodicity property of the Poisson boundary ensures that this map does not depend on the B -coordinate and gives the desired map $X \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$.

Assume now that for almost every $(x, b) \in X \times B$, the support of $p(x, b)$ contains at least three points of K_∞ , then we can use the barycenter map to get a map $X \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$ as above.

In the remaining cases, the map p yields an essentially unique equivariant map $p' : X \times B \rightarrow \Delta$ (or with values in the set of pairs of points in Δ). It may happen that p' does not depend on the B -coordinate in which case we have an equivariant map $X \rightarrow \Delta$. Using the fact that the G -action on Δ is universally amenable while G has property (T) and acts on X preserving a probability measure, an argument of Spatzier–Zimmer [SZ91] then ensures that the cocycle c is cohomologous to a cocycle with values in a finite subgroup of Λ .

If p' depends on the B -coordinate, we get a non-constant equivariant Borel map $q : B \rightarrow L^0(X, \Delta)$, where $L^0(X, \Delta)$ denotes the space of all equivalence classes of Lebesgue-measurable maps from X to Δ . This map q defines an equivariant quotient of B . Denoting by $\Delta^{(2)}$ the complement of the diagonal in Δ^2 , one then shows that there are exactly two Borel maps $F, F^* : B \times B \rightarrow L^0(X, \Delta^{(2)})$, namely

$$F : (b, b') \mapsto (x \mapsto (p'(x, b), p'(x, b'))) \text{ and } F^* : (b, b') \mapsto (x \mapsto (p'(x, b'), p'(x, b))).$$

We now use an argument of Bader–Furman relying on the analysis of the *Weyl group* of G , defined as the group W of all G -equivariant automorphisms of $B \times B$. It turns out that the Weyl group W is a Coxeter group, whose special subgroups (i.e. subgroups generated by subsets of the standard generating set) are related to measurable equivariant quotients $q : B \rightarrow Q$ as follows. From q and the first projection $\pi_1 : B \times B \rightarrow B$, one gets a map $q \circ \pi_1 : B \times B \rightarrow Q$, and the subgroup of all $\sigma \in W$ such that $q \circ \pi_1 \circ \sigma = q \circ \pi_1$ happens to be a special subgroup of W . In our situation W acts nontrivially on the 2-element set $\{F, F^*\}$; the kernel of this action is then a special subgroup of W of index 2, contradicting the assumption that all factors of G have rank at least 2.

The case of outer automorphism groups. We conclude this introduction by saying a few words about why $\text{Out}(F_N)$ has a geometrically rigid triple (\mathbb{D}, K, Δ) .

Here, \mathbb{D} is the set of conjugacy classes of proper free factors of F_N , the space K is the compactification of Culler–Vogtmann’s Outer space, and Δ is the Gromov boundary of the free factor graph. This boundary was described by Bestvina–Reynolds [BR15] and

Hamenstädt [Ham12] in terms of so-called *arational trees*, giving the compatibility property. Amenability of the $\text{Out}(F_N)$ -action on the boundary of this graph was essentially established by Bestvina and the first two named authors in [BGH17].

The difficulty in constructing the barycenter map comes from the lack of local finiteness of the free factor graph: hyperbolicity of this graph allows to associate to every triple of points in the boundary a bounded region of the graph, but this bounded region is still infinite. To bypass this difficulty, we take advantage of the local compactness of Outer space, and build the *barycenter* of three pairwise inequivalent arational trees in the boundary by looking at pencils of geodesics between them, as studied by Bestvina and Reynolds in [BR15].

However, more work is required to also extend this construction to a relative setting, where the corresponding Outer space is no longer locally compact. Working in a relative setting is crucial even when $\Lambda = \text{Out}(F_N)$, because when running the inductive arguments, stabilizers of points in \mathbb{D} are subgroups of $\text{Out}(F_N)$ that preserve the conjugacy class of a proper free factor.

Our general setting is the following. Let H_1, \dots, H_p be countable groups, let F_r be a free group of rank r , and let

$$H := H_1 * \dots * H_p * F_r.$$

The group $\text{Out}(H, \{H_i\})$ made of outer automorphisms of H that preserve the conjugacy classes of all subgroups H_i acts on a *relative free factor graph* FF , whose Gromov boundary $\partial_\infty \text{FF}$ was described in [GH19b] in terms of a notion of *relatively arational trees*.

Theorem 3. *Let H_1, \dots, H_p be countable groups, let F_r be a free group of rank r , and let*

$$H := H_1 * \dots * H_p * F_r.$$

There exists a Borel $\text{Out}(H, \{H_i\})$ -equivariant map assigning to every triple of pairwise distinct points in $\partial_\infty \text{FF}$ a finite set of free splittings of $(H, \{H_i\})$.

As before, the proof goes by associating to every such triple of trees a bounded region Y of the relative Outer space. But because the relative Outer space is not locally compact, we then need an additional trick to associate to Y a finite set of simplices in Outer space: the idea is to show that the subset of Y made of trees of maximal covolume and minimal number of H -orbits of edges always meets any finitely many simplices of the relative Outer space.

Organization of the paper. This article has two parts. The first part aims at proving the geometric rigidity criterion (Theorem 2). After a short preliminary section (Section 1), we compare various notions of amenability of group actions in Section 2. In Section 3, we establish various stability properties of cocycle rigidity, viewed as a property of the target group Λ ; we also explain that cocycle rigidity when the source is a lattice is equivalent to cocycle rigidity for the ambient algebraic group. Finally, we establish the

geometric rigidity criterion in Section 4, where we also explain how this criterion is used when Λ is a relatively hyperbolic group or a right-angled Artin group.

The second part of the paper is concerned with cocycle rigidity in the case of outer automorphism groups. Section 6 reviews some background about outer automorphisms of free products. In Section 7, we derive the amenability of the action on the boundary of the free factor graph from [BGH17]. The barycenter map is constructed in Section 8. Finally, the proofs of our main results are completed in Section 9.

Acknowledgments. We would like to thank Uri Bader and Roman Sauer for conversations regarding cocycles and in particular the stability properties established in Section 3. We are particularly indebted to Uri Bader for explaining us an alternative viewpoint on cocycles, described in Section 3.1, and how to use it to derive some stability properties. We are also grateful to Bruno Duchesne and Cyril Houdayer for useful discussions.

We acknowledge support from the Agence Nationale de la Recherche under Grants ANR-16-CE40-0006 DAGGER, ANR-14-CE25-0004 GAMME and ANR-16-CE40-0022-01 AGIRA. The first author thanks the Centre Henri Lebesgue ANR-11-LABX-0020-01 for creating an attractive mathematical environment.

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Part I

Cocycle rigidity: general setting

The goal of this first part is to establish a general geometric framework for proving cocycle rigidity statements.

1 Preliminaries

Let G be a locally compact group, equipped with its Borel σ -algebra. Let Δ be a standard Borel space. A *Borel action* of G on Δ is an action by Borel automorphisms of Δ such that the map $G \times \Delta \rightarrow \Delta$ is Borel.

When (X, μ) is a standard measured space, actions are understood in the following weaker sense. A *measure-class preserving action* of G on (X, μ) is a Borel map

$$\begin{aligned} G \times X &\rightarrow X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

such that for all $g, h \in G$ and μ -almost every $x \in X$, one has $(gh) \cdot x = g \cdot (h \cdot x)$ and $1 \cdot x = x$, and for every null set $A \subseteq X$, the set $g \cdot A$ is a null set. It is a *measure-preserving action* if for every Borel subset $A \subseteq X$ and every $g \in G$, one has $\mu(g \cdot A) = \mu(A)$.

Equivariance of maps between G -spaces has to be understood μ -almost everywhere. A Borel subset $A \subseteq X$ is *G -invariant* if for every $g \in G$, the sets $g \cdot A$ and A are equal up to null sets. The action is *ergodic* if any G -invariant subset is null or conull.

We say that an action is *strict* if for all $g, h \in G$ and **all** $x \in X$, one has $(gh) \cdot x = g \cdot (h \cdot x)$ and $1 \cdot x = x$. Forgetting the measure, a strict action on (X, μ) defines a Borel action on X .

By [Zim84, Theorem B.10], if G has a measure-class preserving action on (X, μ) , there exists a standard Borel space (X', μ') with a strict action of G and an isomorphism $\phi : X_0 \rightarrow X'_0$ between full measure subsets $X_0 \subseteq X$ and $X'_0 \subseteq X'$ such that for all $g \in G$ and almost every $x \in X_0$, one has $\phi(g \cdot x) = g \cdot \phi(x)$. Many results (including [Zim84]) are stated for strict actions, but this fact allows to use them in this broader context.

Let Λ be a countable discrete group. A Borel map $c : G \times X \rightarrow \Lambda$ is a *cocycle* if for every $g_1, g_2 \in G$ and a.e. $x \in X$, one has $c(g_1 g_2, x) = c(g_1, g_2 x) c(g_2, x)$. Two cocycles $c_1, c_2 : G \times X \rightarrow \Lambda$ are *cohomologous* if there exists a Borel map $f : X \rightarrow \Lambda$ such that for all $g \in G$ and a.e. $x \in X$, one has $c_2(g, x) = f(gx)^{-1} c_1(g, x) f(x)$.

Given a standard Borel space Δ equipped with a Λ -action by Borel automorphisms, we say that a Borel map $f : X \rightarrow \Delta$ is *c -equivariant* if for all $g \in G$ and a.e. $x \in X$, one has $f(gx) = c(g, x) f(x)$.

2 Various notions of amenability of a group action

2.1 Zimmer amenability of an action on a probability space

Let G be a locally compact group, and let (Ω, μ) be a standard probability space equipped with a measure-class preserving action of G .

Let B be a separable real Banach space. We denote by $\text{Isom}(B)$ the group of all linear isometries of B , equipped with the strong operator topology. We denote by B_1^* the unit ball of the dual Banach space B^* , equipped with the weak-* topology. Given a linear isometry T of B , we denote by T^* the restriction to B_1^* of the adjoint operator of T . The *adjoint cocycle* of a Borel cocycle $\rho : G \times \Omega \rightarrow \text{Isom}(B)$ is the cocycle $\rho^* : G \times \Omega \rightarrow \text{Homeo}(B_1^*)$ defined by letting $\rho^*(g, \omega) := \rho(g^{-1}, g\omega)^*$. We denote by $\text{Conv}(B_1^*)$ the set of all nonempty convex compact subsets of B_1^* . A *Borel convex field over Ω* is a map $K : \Omega \rightarrow \text{Conv}(B_1^*)$ such that

$$\{(\omega, k) \in \Omega \times B_1^* \mid k \in K(\omega)\}$$

is a Borel subset of $\Omega \times B_1^*$. A *Borel section* of a Borel convex field K over Ω is a Borel map $s : \Omega \rightarrow B_1^*$ such that for a.e. $\omega \in \Omega$, one has $s(\omega) \in K(\omega)$.

Definition 2.1 (Zimmer amenability of a group action [Zim78]). *Let G be a locally compact group, and let Ω be a standard probability space equipped with a measure-class preserving G -action. The G -action on Ω is Zimmer amenable if for every separable real Banach space B and every cocycle $\rho : G \times \Omega \rightarrow \text{Isom}(B)$, every ρ^* -equivariant Borel convex field over Ω admits a ρ^* -equivariant Borel section.*

Amenability of group actions will mostly be used through the following fact, which is established as in [Zim84, Proposition 4.3.9] (see e.g. the last paragraph of [Zim84, p. 103]).

Lemma 2.2 (Zimmer [Zim84]). *Let G be a locally compact group and let (Ω, μ) be a standard probability space equipped with an ergodic measure-preserving G -action. Assume that the G -action on (Ω, μ) is Zimmer amenable.*

Let Λ be a countable group, and let K be a compact metrizable space equipped with a Λ -action by homeomorphisms. Equip $\text{Prob}(K)$ with the weak- topology.*

Then for every cocycle $c : G \times \Omega \rightarrow \Lambda$, there exists a c -equivariant Borel map $\Omega \rightarrow \text{Prob}(K)$.

2.2 Universal and Borel amenability of an action on a Borel space

Definition 2.3 (Universal amenability of a group action). *Let Λ be a countable group, and let Δ be a standard Borel space equipped with a Borel action of Λ . The Λ -action on Δ is universally amenable if for every probability measure μ on Δ whose measure class is Λ -invariant, the Λ -action on (Δ, μ) is Zimmer amenable.*

We denote by $\text{Prob}(\Lambda)$ the space of all probability measures on Λ , equipped with the topology of pointwise convergence. The left action of Λ on itself induces an action of Λ on $\text{Prob}(\Lambda)$ by $\lambda_*\nu(\lambda') = \nu(\lambda^{-1}\lambda')$ for all $\lambda, \lambda' \in \Lambda$.

Definition 2.4 (Borel amenability of a group action). Let Λ be a countable group, and let Δ be a standard Borel space equipped with a Borel Λ -action. The Λ -action on Δ is Borel amenable if there exists a sequence of Borel maps

$$\begin{aligned} \nu_n : \Delta &\rightarrow \text{Prob}(\Lambda) \\ \delta &\mapsto \nu_n^\delta \end{aligned}$$

such that for every $\delta \in \Delta$, the norm $\|\nu_n^{\lambda \cdot \delta} - \lambda_* \nu_n^\delta\|_1$ converges to 0 as n goes to $+\infty$.

A sequence $(\nu_n)_{n \in \mathbb{N}}$ as in the above definition is said to be *asymptotically Λ -equivariant*.

Proposition 2.5. Let Λ be a countable group, and let Δ be a standard Borel space equipped with a Borel Λ -action. If the Λ -action on Δ is Borel amenable, then it is universally amenable.

This proposition is probably well-known to the experts. However, we could not find this precise statement in the literature, so we provide a proof. We mention that a similar statement is proved in [ADR00, Theorem 3.3.7] for locally compact groups, under the extra assumption that the space Δ on which Λ is acting is also a locally compact topological space. Similar facts are also established in [JKL02].

Proof. We will follow Zimmer's proof of [Zim78, Proposition 2.2]. Let μ be a probability measure on Δ whose measure class is Λ -invariant. Let B be a separable real Banach space; we denote by B_1^* the closed unit ball of the dual Banach space B^* of B . Let $\rho : \Lambda \times \Delta \rightarrow \text{Isom}(B)$ be a cocycle, and let $K : \Delta \rightarrow \text{Conv}(B_1^*)$ be a ρ^* -equivariant Borel convex field over Δ . Starting from a Borel section $s : \Delta \rightarrow B_1^*$ of K (see [Zim78, Lemma 1.7] for the existence of such a section), we will build one which is ρ^* -equivariant.

Since the Λ -action on Δ is Borel amenable, we can find an asymptotically Λ -equivariant sequence of Borel maps

$$\begin{aligned} \nu_n : \Delta &\rightarrow \text{Prob}(\Lambda) \\ \delta &\mapsto \nu_n^\delta \end{aligned}$$

Notice that Λ acts on the set of all Borel sections of K by letting

$$\lambda \cdot s'(\delta) := \rho^*(\lambda, \lambda^{-1}\delta) s'(\lambda^{-1}\delta)$$

(notice indeed that $\lambda \cdot s'(\delta) \in K(\delta)$ by ρ^* -equivariance of K). Our goal is to find a Λ -invariant section, i.e. a section s' such that for every $\lambda \in \Lambda$, the maps s' and $\lambda \cdot s'$ coincide almost everywhere. For every $n \in \mathbb{N}$ and every $\delta \in \Delta$, we then let

$$s_n(\delta) := \sum_{\lambda \in \Lambda} \nu_n^\delta(\lambda) \lambda \cdot s(\delta),$$

which belongs to $K(\delta)$ by convexity.

Let $L^1(\Delta, B)$ be the space of all Lebesgue-measurable functions $f : \Delta \rightarrow B$ such that

$$\int_{\Delta} \|f(\delta)\| d\mu(\delta) < +\infty,$$

where as usual we identify two functions if their difference vanishes on a full measure subset. This is a Banach space, whose dual we now describe. A map $f : \Delta \rightarrow B^*$ is *scalarwise measurable* if for every $v \in B$, the function $\langle f(\cdot), v \rangle$ is Lebesgue-measurable. We denote by $L^\infty(\Delta, B^*)$ the space of all scalarwise measurable functions $f : \Delta \rightarrow B^*$ such that $\|f(\cdot)\|$ is bounded. Given $f \in L^1(\Delta, B)$ and $g \in L^\infty(\Delta, B^*)$, we let

$$\langle f, g \rangle := \int_{\Delta} \langle f(\delta), g(\delta) \rangle d\mu(\delta).$$

This determines a map $L^\infty(\Delta, B^*) \rightarrow (L^1(\Delta, B))^*$, which is an isomorphism [Edw65, Proposition 8.18.2]. The functions s_n all belong to the unit ball of $L^\infty(\Delta, B^*)$, which is compact for the weak-* topology. We will show that every accumulation point s_∞ of the sequence $(s_n)_{n \in \mathbb{N}}$ is ρ^* -equivariant. Since B is separable, it then follows from [Edw65, Proposition 8.15.3] that s_∞ is a.e. equal to some Borel map $\Delta \rightarrow B_1^*$.

To prove that s_∞ is ρ^* -equivariant, given $f \in L^1(\Delta, B)$ and $h \in \Lambda$, we compute

$$\langle h \cdot s_n - s_n, f \rangle = \int_{\Delta} \langle (h \cdot s_n - s_n)(\delta), f(\delta) \rangle d\mu(\delta),$$

and we aim to prove that this converges to 0. A quick computation shows that for every $\delta \in \Delta$, the difference $h \cdot s_n(\delta) - s_n(\delta)$ is equal to

$$\sum_{\lambda \in \Lambda} (\nu_n^{h^{-1}\delta}(h^{-1}\lambda) - \nu_n^\delta(\lambda)) \rho^*(\lambda, \lambda^{-1}\delta) s(\lambda^{-1}\delta),$$

whose norm is bounded by 2. As f is integrable, by applying Lebesgue's dominated convergence theorem, we deduce that $\langle h \cdot s_n - s_n, f \rangle$ converges to 0 as n goes to $+\infty$, as desired. \square

3 Stabilities

The goal of the present section is to establish some stability properties of cocycle rigidity. As a property of the target group, we will show in Section 3.2 that it is invariant under finite-index subgroups or overgroups and under group extensions. As a property of the source, we will show in Section 3.3 that cocycle rigidity for the lattice or its ambient algebraic group are equivalent. The proofs rely on an interpretation of cocycles in terms of their *developed actions* given in Section 3.1, which was explained to us by Uri Bader (see the MathOverflow answer [Bad]). We make the following definition.

Definition 3.1 (Cocycle-rigid pair). *Let G be a locally compact group, and let Λ be a countable discrete group. We say that the pair (G, Λ) is cocycle-rigid if for every standard probability space (X, μ) equipped with an ergodic measure-preserving G -action, every cocycle $G \times X \rightarrow \Lambda$ is cohomologous to a cocycle that takes its values in some finite subgroup of Λ .*

Remark 3.2. Using a decomposition into ergodic components [Var63, Theorem 4.4], a standard argument allows one to consider non-ergodic actions as follows. The pair (G, Λ) is cocycle-rigid if and only if for every action of G on a standard probability space X and every cocycle $c : G \times X \rightarrow \Lambda$, the space X has a countable almost G -invariant Borel partition $X = \sqcup_{i \in \mathbb{N}} X_i$ such that for every $i \in \mathbb{N}$, the cocycle $c|_{X_i}$ is cohomologous to a cocycle with values in a finite subgroup Λ_i of Λ .

3.1 Cocycles and developed actions

The goal of the present section is to characterize cocycle-rigidity of the pair (G, Λ) in terms of commuting actions of G and Λ : this is the contents of Corollary 3.8 below. The key notion towards this goal is the following.

Definition 3.3 (Developed action). *Let G be a locally compact group and let (X, μ) be a standard probability space equipped with a measure-preserving G -action. Let Λ be a countable discrete group. Let $c : G \times X \rightarrow \Lambda$ be a cocycle. Then $G \times \Lambda$ acts on $X \times \Lambda$ via*

$$(g, \lambda) \cdot (x, \lambda') := (gx, c(g, x)\lambda'\lambda^{-1}).$$

We equip $X \times \Lambda$ with the product measure ν of μ and of the counting measure on Λ . We call $G \times \Lambda \curvearrowright X \times \Lambda$ the developed action of the cocycle c .

Notice that the developed action of $G \times \Lambda$ on $X \times \Lambda$ preserves ν . Moreover, the Λ -action on $X \times \Lambda$ is free and has a finite measure fundamental domain, namely $X \times \{1\}$. We make the following definition, which is adapted to measure-preserving actions defined as in Section 1 (i.e. where the action relation is only required to be satisfied almost everywhere).

Definition 3.4. *Let (Y, ν) be a standard measured space equipped with a measure-preserving action of a countable group Λ . We say that the Λ -action on Y is essentially free with a finite measure fundamental domain if there exist a conull subset $Y_0 \subseteq Y$ and a Borel subset $X \subseteq Y$ of finite measure such that Y_0 decomposes as the countable disjoint union*

$$Y_0 = \bigsqcup_{\lambda \in \Lambda} \lambda X.$$

Proposition 3.5. *Let G be a locally compact group, and let Λ be a countable discrete group. Let (Y, ν) be a standard measured space equipped with a measure-preserving action of $G \times \Lambda$. Assume that the Λ -action on Y is essentially free with a finite measure fundamental domain X .*

Then there exist a measure-preserving G -action on X and a cocycle $c : G \times X \rightarrow \Lambda$ whose developed action is isomorphic to the $(G \times \Lambda)$ -action on Y .

Proof. Let $Y_0 \subseteq Y$ be a conull subset provided by Definition 3.4. Let $\phi : X \times \Lambda \rightarrow Y_0$ be the Borel isomorphism defined by $\phi(x, \lambda) = \lambda^{-1}x$. Let $\alpha : G \times X \rightarrow X$ and $c : G \times X \rightarrow \Lambda$ be such that for all $g \in G$ and a.e. $x \in X$, one has $\phi^{-1}(gx) = (\alpha_g(x), c(g, x))$. Thus we have $gx = c(g, x)^{-1}\alpha_g(x)$. Therefore, for all $g_1, g_2 \in G$ and a.e. $x \in X$, one has

$g_1(g_2x) = g_1c(g_2, x)^{-1}\alpha_{g_2}(x)$. Since the actions of G and Λ on Y commute, we deduce that for all $g_1, g_2 \in G$ and a.e. $x \in X$, one has

$$g_1g_2x = c(g_2, x)^{-1}g_1\alpha_{g_2}(x) = c(g_2, x)^{-1}c(g_1, \alpha_{g_2}(x))^{-1}\alpha_{g_1}(\alpha_{g_2}(x)),$$

and on the other hand

$$(g_1g_2)x = c(g_1g_2, x)^{-1}\alpha_{g_1g_2}(x).$$

Thus we get that for all $g_1, g_2 \in G$ and a.e. $x \in X$, one has $\alpha_{g_1}(\alpha_{g_2}(x)) = \alpha_{g_1g_2}(x)$, which means that $g \mapsto \alpha_g$ is indeed an action of G on X , and it is measure-preserving. Furthermore we also get that $c(g_1g_2, x)^{-1} = c(g_2, x)^{-1}c(g_1, \alpha_{g_2}(x))^{-1}$. Hence c is a cocycle for the α -action of G on X .

Now let β be the developed action of c , which is an action of $G \times \Lambda$ on $X \times \Lambda$. We claim that $\phi\beta\phi^{-1}$ coincides almost everywhere with the original action of $G \times \Lambda$ on Y . Indeed, for almost every $y \in Y$, letting $\phi(x, \lambda_1) = y$ we get

$$\begin{aligned} \phi\beta_{g,\lambda}\phi^{-1}(y) &= \phi\beta_{g,\lambda}(x, \lambda_1) \\ &= \phi(\alpha_g(x), c(g, x)\lambda_1\lambda^{-1}) \\ &= \lambda\lambda_1^{-1}c(g, x)^{-1}\alpha_g(x) \\ &= \lambda\lambda_1^{-1}gx \\ &= \lambda gy \end{aligned}$$

(where the last equality uses the fact that the actions of G and Λ on Y commute). \square

Remark 3.6. We can be more precise regarding the correspondence between cocycles and their developed actions. Let X be a standard probability space equipped with a measure-preserving G -action, and let Λ be a countable group. Identifying the fundamental domain X with $\overline{Y} = Y/\Lambda$ in Proposition 3.5 yields a cocycle $c : G \times \overline{Y} \rightarrow \Lambda$. If one changes the fundamental domain X to another fundamental domain X' , the resulting cocycle $G \times \overline{Y} \rightarrow \Lambda$ is cohomologous to c .

In fact the following holds. Define a *developing space* (Y, π) over X as a standard measure space Y with a measure-preserving action $G \times \Lambda \curvearrowright Y$ together with a G -equivariant map $\pi : Y \rightarrow X$ such that Λ acts essentially freely with a finite measure fundamental domain, and π induces an isomorphism $Y/\Lambda \rightarrow X$. Say that two developing spaces $(Y, \pi), (Y', \pi')$ over X are *isomorphic* if there is an equivariant isomorphism $\phi : Y \rightarrow Y'$ such that $\pi' \circ \phi = \pi$. Then the set of all cohomology classes of cocycles $G \times X \rightarrow \Lambda$ is in 1-1 correspondence with the set of isomorphism classes of developing spaces over X .

Lemma 3.7. *Let G be a locally compact group and let (X, μ) be a probability space equipped with an ergodic measure-preserving G -action. Let Λ be a countable discrete group. Let $c : G \times X \rightarrow \Lambda$ be a cocycle. Equip $X \times \Lambda$ with the developed action of $G \times \Lambda$. Let \mathbb{D} be a countable Λ -set. Then the following assertions are equivalent.*

- (i). *There exists $d_0 \in \mathbb{D}$ such that c is cohomologous to a cocycle which takes its values in $\text{Stab}_\Lambda(d_0)$ (the Λ -stabilizer of d_0).*

(ii). There exists a c -equivariant Borel map $f : X \rightarrow \mathbb{D}$.

(iii). There exists a Borel map $F : X \times \Lambda \rightarrow \mathbb{D}$ which is G -invariant and Λ -equivariant.

If $G \curvearrowright X$ is not ergodic, the statement remains true if one replaces (i) by

(i'). There exists a finite or countable almost G -invariant Borel partition $X = \sqcup X_i$, and a finite or countable subset $\{d_i\} \subseteq \mathbb{D}$, such that c is cohomologous to a cocycle c' such that for every i , the restriction $c'|_{G \times X_i}$ takes its values in $\text{Stab}_\Lambda(d_i)$.

Proof. If the action is ergodic, then (i') is equivalent to (i) so we prove the lemma without assuming ergodicity.

Assume (i'). Let $h : X \rightarrow \Lambda$ be a Borel map such that for every i and a.e. $x \in X_i$, one has $h(gx)^{-1}c(g, x)h(x) \in \text{Stab}_\Lambda(d_i)$. We claim that the Borel map $f : X \rightarrow \mathbb{D}$ defined by $f(x) = h(x) \cdot d_i$ for $x \in X_i$ is c -equivariant. Indeed, for all $g \in G$ and a.e. $x \in X_i$, one has (using the fact that $h(gx)^{-1}c(g, x)h(x) \in \text{Stab}_\Lambda(d_i)$ for the second equality):

$$f(gx) = h(gx)d_i = c(g, x)h(x)d_i = c(g, x)f(x),$$

so (ii) holds.

Now assume that (ii) holds and define $F : X \times \Lambda \rightarrow \mathbb{D}$ by $F(x, \lambda) = \lambda^{-1}f(x)$. Then for every $(g, \lambda) \in G \times \Lambda$ and a.e. $(x, \lambda') \in X \times \Lambda$, one has

$$F((g, \lambda).(x, \lambda')) = F(gx, c(g, x)\lambda'\lambda^{-1}) = \lambda\lambda'^{-1}c(g, x)^{-1}f(gx) = \lambda\lambda'^{-1}f(x) = \lambda F(x, \lambda')$$

and (iii) follows.

We now prove that (iii) implies (i'). Let $\{d_i\}$ be a family of representatives of the Λ -orbits of \mathbb{D} . Let X_i be the set of all $x \in X$ such that for every $\lambda \in \Lambda$, one has $F(x, \lambda) \in \Lambda d_i$. Note that X_i is G -invariant. Using sections $\Lambda d_i \rightarrow \Lambda$ one can define a Borel map $h : X \rightarrow \Lambda$ such that for almost every $x \in X_i$, one has $F(x, 1_\Lambda) = h(x)d_i$. Let $c'(g, x) = h(gx)^{-1}c(g, x)h(x)$. Then for all $g \in G$ and a.e. $x \in X_i$, one has

$$\begin{aligned} c'(g, x)d_i &= h(gx)^{-1}c(g, x)h(x)d_i \\ &= h(gx)^{-1}c(g, x)F(x, 1_\Lambda) \\ &= h(gx)^{-1}F(x, c(g, x)^{-1}). \end{aligned}$$

Since F is G -invariant, and since $g(x, c(g, x)^{-1}) = (gx, 1_\Lambda)$, one gets

$$c'(g, x)d_i = h(gx)^{-1}F(gx, 1_\Lambda) = d_i.$$

This concludes the proof. □

As a consequence of Lemma 3.7, we get the following characterization of cocycle rigidity.

Corollary 3.8. *Let G be a locally compact group and Λ be a countable group. Then the following assertions are equivalent.*

- (i) The pair (G, Λ) is cocycle-rigid.
- (ii) For every measure-preserving action $G \times \Lambda \curvearrowright Y$ on a standard measured space Y such that the action of Λ is essentially free with a fundamental domain of finite measure, there exists a countable Λ -set \mathbb{D} with finite point stabilizers and a G -invariant Λ -equivariant Borel map $F : Y \rightarrow \mathbb{D}$.

Proof. Assume that (G, Λ) is cocycle-rigid. Let $G \times \Lambda \curvearrowright Y$ be an action such that the action of Λ is essentially free with a fundamental domain X of finite measure. By Proposition 3.5, the action $G \times \Lambda \curvearrowright Y$ is the developed action of a cocycle $c : G \times X \rightarrow \Lambda$. By cocycle rigidity (as in Remark 3.2), the space X has a countable almost G -invariant Borel partition $X = \sqcup_i X_i$ such that $c|_{G \times X_i}$ is cohomologous to a cocycle with values in a finite group $\Lambda_i \subseteq \Lambda$. Applying Lemma 3.7 with $\mathbb{D} = \sqcup_i \Lambda/\Lambda_i$, one gets a Borel map $F : Y \rightarrow \mathbb{D}$ which is G -invariant and Λ -equivariant.

Conversely, consider a cocycle $c : G \times X \rightarrow \Lambda$, and let $Y = X \times \Lambda$ be its developed action. Our assumption gives us a countable set \mathbb{D} to which one can apply Lemma 3.7 and deduce that X has a countable almost G -invariant Borel partition $X = \sqcup_i X_i$ such that $c|_{X_i}$ is cohomologous to a cocycle with values in a finite group $\Lambda_i \subseteq \Lambda$. This concludes the proof. \square

3.2 Stabilities in the target: finite index overgroups and extensions

We now prove that, when viewed as a property of the target group Λ , cocycle rigidity of (G, Λ) is stable under finite-index subgroups, finite-index overgroups and group extensions.

Proposition 3.9. *Let G be a locally compact group, let Λ be a countable discrete group, and let Λ' be a finite index subgroup of Λ .*

Then (G, Λ) is cocycle-rigid if and only if (G, Λ') is cocycle-rigid.

Proof. We use the criterion given in Corollary 3.8. Assume that the pair (G, Λ) is cocycle-rigid, and let $G \times \Lambda' \curvearrowright Y$ be a measure-preserving action which is essentially free with a finite measure fundamental domain. We construct a larger space \hat{Y} endowed with a $(G \times \Lambda)$ -action so that Y embeds $(G \times \Lambda')$ -equivariantly in \hat{Y} as follows. We define $\hat{Y} := (Y \times \Lambda)/\sim_{\Lambda'}$, where $\sim_{\Lambda'}$ is the equivalence relation where $(g, \lambda_1) \sim_{\Lambda'} (\lambda'g, \lambda'\lambda_1)$ for every $\lambda' \in \Lambda'$. We will denote by $[y, \lambda_1]$ the equivalence class of (y, λ_1) . Then $G \times \Lambda$ acts on \hat{Y} via $(g, \lambda) \cdot [y, \lambda_1] := [gy, \lambda_1 \lambda^{-1}]$ and the embedding $\iota : Y \rightarrow \hat{Y}$ defined by $\iota(y) = [y, 1_\Lambda]$ is $(G \times \Lambda')$ -equivariant. The space \hat{Y} is isomorphic to $Y \times (\Lambda/\Lambda')$ so the measure on Y defines a measure on \hat{Y} which is $(G \times \Lambda)$ -invariant, and the action of Λ is essentially free with a finite measure fundamental domain.

Since (G, Λ) is cocycle-rigid, there exists a countable Λ -set \mathbb{D} with finite point stabilizers and a G -invariant Λ -equivariant Borel map $\hat{F} : \hat{Y} \rightarrow \mathbb{D}$. The map $\hat{F} \circ \iota : Y \rightarrow \mathbb{D}$ is G -invariant and Λ' -equivariant. This shows that (G, Λ') is cocycle-rigid.

Conversely, we now prove that cocycle-rigidity of (G, Λ') implies cocycle-rigidity of (G, Λ) , using again the characterization given in Corollary 3.8. Consider an action

$G \times \Lambda \curvearrowright Y$ such that the action of Λ is essentially free with a fundamental domain of finite measure. Since (G, Λ') is cocycle-rigid, and since Λ' acts on Y essentially freely with a fundamental domain of finite measure, there exists a countable Λ' -set \mathbb{D} and a Borel map $F : Y \rightarrow \mathbb{D}$ which is G -invariant and Λ' -equivariant.

We define $\hat{\mathbb{D}} = \text{Maps}_{\Lambda'}(\Lambda, \Lambda \times \mathbb{D})$ as the countable set of Λ' -equivariant maps from Λ to $\Lambda \times \mathbb{D}$ for the standard left actions, i.e. the set of maps $\phi = (\phi_1, \phi_2)$ such that

$$\phi_1(\lambda'\lambda) = \lambda'\phi_1(\lambda) \text{ and } \phi_2(\lambda'\lambda) = \lambda'\phi_2(\lambda).$$

The group Λ acts on $\hat{\mathbb{D}}$ by

$$\lambda_0 \cdot (\phi_1, \phi_2) = [\lambda \mapsto (\phi_1(\lambda\lambda_0)\lambda_0^{-1}, \phi_2(\lambda\lambda_0))].$$

To check that the action of Λ on $\hat{\mathbb{D}}$ has finite stabilizers, it suffices to check that the Λ' -stabilizer of any (ϕ_1, ϕ_2) is finite. If $\lambda'_0 \in \Lambda'$ fixes (ϕ_1, ϕ_2) then

$$(\phi_1(1_\Lambda), \phi_2(1_\Lambda)) = [\lambda'_0 \cdot (\phi_1, \phi_2)](1_\Lambda) = (\lambda'_0\phi_1(1_\Lambda)(\lambda'_0)^{-1}, \lambda'_0\phi_2(1_\Lambda))$$

so λ'_0 fixes the point $\phi_2(1_\Lambda) \in \mathbb{D}$, and we conclude since \mathbb{D} has finite stabilizers.

We finally define $\hat{F} : Y \rightarrow \hat{\mathbb{D}}$ by

$$\hat{F}(y) = [\lambda \mapsto (\lambda, F(\lambda y))].$$

This map is Λ -equivariant and G -invariant because the actions of G and Λ on Y commute. This concludes the proof. \square

Proposition 3.10. *Let G be a locally compact group, and let Λ, N, Q be countable discrete groups sitting in a short exact sequence*

$$1 \rightarrow N \rightarrow \Lambda \rightarrow Q \rightarrow 1.$$

If (G, N) and (G, Q) are cocycle-rigid, then so is (G, Λ) .

Proof. Let (X, μ) be a standard probability space equipped with an ergodic measure-preserving action of G , and let $c : G \times X \rightarrow \Lambda$ be a cocycle. Using the projection map $\Lambda \rightarrow Q$, we get a cocycle $G \times X \rightarrow Q$, which is cohomologous by assumption to a cocycle that takes its values in a finite subgroup of Q . In other words c is cohomologous to a cocycle that takes its values in a finite extension N^1 of N (inside Λ). Since (G, N) is cocycle-rigid, Lemma 3.9 implies that (G, N^1) is cocycle-rigid. Therefore c is cohomologous to a cocycle that essentially takes its values in a finite subgroup of Λ . \square

3.3 Stabilities in the source: groups and their lattices

The goal of the present section is to prove the following proposition. This is probably well-known to the experts, but we could not find the statement given as such in the literature, so we include a proof.

Proposition 3.11. *Let G be a locally compact second countable group, let Γ be a lattice in G , and let Λ be a countable discrete group.*

Then (G, Λ) is cocycle-rigid if and only if (Γ, Λ) is cocycle-rigid.

The proof of Proposition 3.11 will be carried through Lemmas 3.12 and 3.13 below.

3.3.1 From the lattice to the ambient group

Lemma 3.12. *Let G be a locally compact second countable group, let Γ be a lattice in G , and let Λ be a countable discrete group.*

If (Γ, Λ) is cocycle-rigid, then (G, Λ) is cocycle-rigid.

Proof. We use the characterization of cocycle rigidity given in Corollary 3.8.

Consider an action of $G \times \Lambda$ on a standard measure space Y such that Λ acts essentially freely with a fundamental domain of finite measure. Restricting the action to $\Gamma \times \Lambda$, cocycle rigidity of (Γ, Λ) provides a countable Λ -set \mathbb{D} with finite point stabilizers and a Γ -invariant Λ -equivariant map $F : Y \rightarrow \mathbb{D}$.

We denote by $\text{Maps}(\Gamma \backslash G, \mathbb{D})$ the collection of all Borel maps from $\Gamma \backslash G$ to \mathbb{D} : this is endowed with a $(G \times \Lambda)$ -action given by

$$(g, \lambda) \cdot \theta = (\Gamma h \mapsto \lambda \theta(\Gamma h g)).$$

The map $\tilde{F} : Y \rightarrow \text{Maps}(\Gamma \backslash G, \mathbb{D})$ defined by letting

$$\tilde{F}(y) := (\Gamma h \mapsto F(hy))$$

is well-defined (because F is Γ -invariant) and $(G \times \Lambda)$ -equivariant.

Let μ be a (finite) Haar measure on $\Gamma \backslash G$. We denote by $\mathcal{P}_{< \infty}(\mathbb{D})$ the countable set of all finite subsets of \mathbb{D} . Consider the map

$$\Psi : \text{Maps}(\Gamma \backslash G, \mathbb{D}) \rightarrow \mathcal{P}_{< \infty}(\mathbb{D}),$$

sending a map θ to the collection of all elements of \mathbb{D} with maximal $\theta_* \mu$ -measure (which is finite because \mathbb{D} is countable and μ is a finite measure). The map Ψ is G -invariant and Λ -equivariant. The composition $\Psi \circ \tilde{F}$ is a G -invariant Λ -equivariant Borel map $Y \rightarrow \mathcal{P}_{< \infty}(\mathbb{D})$. The space $\mathcal{P}_{< \infty}(\mathbb{D})$ is countable, and the Λ -action on $\mathcal{P}_{< \infty}(\mathbb{D})$ has finite point stabilizers (because the Λ -action on \mathbb{D} has finite point stabilizers). This concludes the proof. \square

3.3.2 From the ambient group to the lattice

We now prove the converse to Lemma 3.12.

Lemma 3.13. *Let G be a locally compact second countable group, let Γ be a lattice in G , and let Λ be a countable discrete group.*

If (G, Λ) is cocycle-rigid, then (Γ, Λ) is cocycle-rigid.

Proof. We use the characterization of cocycle rigidity given in Corollary 3.8.

Consider a measure preserving action of $\Gamma \times \Lambda$ on a standard measured space Y such that the action of Λ is essentially free with a fundamental domain of finite measure. Let $\hat{Y} = (G \times Y) / \sim_\Gamma$ where \sim_Γ is the equivalence relation defined by

$$(g, \gamma y) \sim_\Gamma (g\gamma, y)$$

for $g \in G$, $y \in Y$ and $\gamma \in \Gamma$. We denote by $[g, y]$ the class of (g, y) in \hat{Y} . The space \hat{Y} can also be viewed as $(G/\Gamma) \times Y$, and thus be endowed with the product measure of the Haar measure with the measure of Y . The group $G \times \Lambda$ acts on \hat{Y} by

$$(g_0, \lambda_0) \cdot [h, y] = [g_0 h, \lambda_0 y]$$

and preserves the measure of \hat{Y} . Denote by X a fundamental domain for the action of Λ on Y . Then the image $(G \times X)/\sim_\Gamma$ is isomorphic to $G/\Gamma \times X$ and is a fundamental domain of finite measure for the action of Λ on \hat{Y} .

By cocycle rigidity for (G, Λ) , there exists a countable Λ -set \mathbb{D} with finite point stabilizers and a G -invariant Λ -equivariant map $F : \hat{Y} \rightarrow \mathbb{D}$. Recall that this means that for all $(g_0, \lambda_0) \in G \times \Lambda$ and a.e. $y \in \hat{Y}$, one has $F((g_0, \lambda_0) \cdot y) = \lambda_0 F(y)$. It would be natural to consider the restriction of F to the subset $\{1\} \times Y \subseteq \hat{Y}$ but since this is a null set, we do not know that this restriction is Γ -invariant and Λ -equivariant.

By Fubini's Theorem, for almost every $y \in Y$, the map

$$\begin{aligned} G &\rightarrow \mathbb{D} \\ g &\mapsto F([g, y]) \end{aligned}$$

is essentially constant. Denote by $\tilde{F}(y) \in \mathbb{D}$ the corresponding constant. Extending this definition arbitrarily on a null set, we get a Borel map $\tilde{F} : Y \rightarrow \mathbb{D}$. This map is clearly Γ -invariant and Λ -equivariant, thus concluding our proof. \square

4 Cocycle rigidity: a general criterion

In this section, we use an argument due to Bader and Furman (see e.g. [BF14a] or the preprint [BF]) to establish a general criterion ensuring rigidity phenomena (Theorem 4.17 below, which is Theorem 2 from the introduction).

4.1 A digression on measurable functions

We start by recalling some facts about convergence in measure that we will need throughout the section. Let (X, μ) be a standard probability space. Let Y be a Polish space, and let d_Y be a metric on Y which is compatible with the topology. Let $L^0(X, Y)$ be the space of equivalence classes of Borel-measurable functions from X to Y , where two functions are equivalent if they coincide on a full measure subset. We equip $L^0(X, Y)$ with the topology of convergence in measure: this is the topology for which a sequence $(f_n)_{n \in \mathbb{N}}$ converges to f if and only if for every $\varepsilon > 0$, the μ -measure of $\{x \in X \mid d_Y(f_n(x), f(x)) > \varepsilon\}$ converges to 0 as n goes to $+\infty$. This topology can be metrized through the metric δ defined by letting

$$\delta(\phi, \phi') = \int_X \min\{d_Y(\phi(x), \phi'(x)), 1\} d\mu(x).$$

The set $L^0(X, Y)$, with this topology, is again a Polish space. The following basic lemma is taken from [Mar91, VII.1.3]; it is essentially an application of Fubini's theorem.

Lemma 4.1. *Let X and Ω be standard probability spaces, and let Y be a Polish space. Let $q : \Omega \times X \rightarrow Y$ be a Borel map. Then the formula $\bar{q}(\omega)(x) = q(\omega, x)$ defines a map $\bar{q} \in L^0(\Omega, L^0(X, Y))$.*

Assume now that a locally compact group G acts on X and Ω in a measure-preserving way, that a countable group Λ acts on Y , and that we have a cocycle $c : G \times X \rightarrow \Lambda$. Let $c_1 : G \times X \times \Omega \rightarrow \Lambda$ be the cocycle defined by letting $c_1(g, x, \omega) := c(g, x)$. Then G acts on $L^0(X, Y)$ via $g \cdot f(x) = c(g, g^{-1}x)f(g^{-1}x)$. Under the identification provided by Lemma 4.1, c_1 -equivariant Borel maps $X \times \Omega \rightarrow Y$ correspond to G -equivariant Borel maps $\Omega \rightarrow L^0(X, Y)$. Note also that if Λ acts isometrically on Y and G preserves the measure on X then G acts isometrically on $L^0(X, Y)$.

4.2 Strong boundaries and Weyl groups

A useful tool is the notion of a *strong boundary*. A slightly weaker notion first appeared in the work of Burger–Monod [BM99] (see also [MS04, Definition 2.3]). A stronger definition was then coined by Bader and Furman in [BF14a, Definition 2.3]. We use the following version.

Definition 4.2 (Isometrically ergodic action). *Let G be a locally compact group, and let (B, μ) be a standard probability space equipped with a measure class preserving G -action. The action of G on (B, μ) is isometrically ergodic if for every separable metric space Y equipped with a continuous isometric G -action, every G -equivariant Borel map $B \rightarrow Y$ is essentially constant.*

Remark 4.3. If the action of G on (B, μ) is isometrically ergodic, then it is ergodic. Indeed, it suffices to consider $Y = \{0, 1\}$ with the trivial action of G . More generally, if (X, ν) is a standard probability space, and G acts ergodically on X in a measure-preserving way, then the diagonal action of G on $B \times X$ is again ergodic. Indeed, G acts isometrically on the separable metric space $Y = L^0(X, \mathbb{R})$, and if $A \subseteq B \times X$ is G -invariant, then the function $f : B \rightarrow L^0(X, \mathbb{R})$ defined by $f(\omega)(x) = \mathbb{1}_A(\omega, x)$ is a G -equivariant function. Isometric ergodicity implies that f must be essentially constant, which means that A is null or conull. Note that the fact that G preserves the measure ν on X is crucial to ensure that G acts by isometries on $L^0(X, \mathbb{R})$.

Definition 4.4 (Strong boundary). *Let G be a locally compact group, and let (B, μ) be a standard probability space equipped with a measure class preserving G -action. The space (B, μ) is a strong boundary for G if*

- (1) *the G -action on (B, μ) is Zimmer amenable, and*
- (2) *the G -action on $(B \times B, \mu \times \mu)$ is isometrically ergodic.*

Remark 4.5. Notice that if the G -action on $B \times B$ is isometrically ergodic, then so is the G -action on B .

We mention that every locally compact second countable group G admits a strong boundary, which can be obtained as the Poisson boundary of (G, μ) for a suitable measure μ on G (see [Kai03] or [BF14a, Theorem 2.7]). The following definition first appeared in a preprint of Bader–Furman–Shaker [BFS06], see also [BF14a, §4].

Definition 4.6 (Weyl group). *Let G be a locally compact group, and let (B, μ) be a strong boundary for G . The Weyl group $W_{G,B}$ is the group of all measure class preserving Borel automorphisms of $(B \times B, \mu \times \mu)$ which commute with the G -action.*

A subgroup $W' \subseteq W_{G,B}$ is special if there exists a standard Borel space C equipped with a Borel G -action, and a G -equivariant Borel map $\pi : B \rightarrow C$ such that $W' = \{w \in W_{G,B} \mid \pi \circ p_1 \circ w = \pi \circ p_1\}$, where $p_1 : B \times B \rightarrow B$ denotes the first projection (we write $W' = W_{G,B}(\pi)$ in this situation).

In other words $W_{G,B}(\pi)$ is the subgroup of $W_{G,B}$ made of all automorphisms w of $B \times B$ that commute with the G -action, and make the following diagram commute:

$$\begin{array}{ccc} & w & \\ & \curvearrowright & \\ B \times B & \xrightarrow{p_1} & B \xrightarrow{\pi} C \end{array}$$

The main example of boundaries and Weyl groups comes from the realm of simple Lie groups. The following proposition was established by Bader–Furman [BF14a, Theorem 2.5] in the context of real Lie groups, and by Bader–Furman–Shaker [BFS06, Lemma 2.8 and Proposition 2.13] in the more general context of algebraic groups over local fields but with a slightly different notion of strong boundary. We therefore include a proof for completeness of our argument. The reader less familiar with the formalism of algebraic groups is invited to read the example of $\mathrm{SL}_3(\mathbf{R})$ (Example 4.8) before Proposition 4.7.

Before stating the proposition, let us introduce a few notations. Let I be a finite set. For each $i \in I$, choose a local field k_i , a connected simple algebraic group \mathbf{G}_i defined over k_i of k_i -rank at least 2. Let \mathbf{S}_i be a maximal k_i -split torus, let \mathbf{Z}_i the centralizer of \mathbf{S}_i , and \mathbf{P}_i be a minimal parabolic subgroup containing \mathbf{Z}_i . Let $G_i = \mathbf{G}_i(k_i)$, $P_i = \mathbf{P}_i(k_i)$, $Z_i = \mathbf{Z}_i(k_i)$, $S_i = \mathbf{S}_i(k_i)$ and $B_i = G_i/P_i$. Let $W_i = N_{G_i}(S_i)/Z_i$.

Finally, let $G = \prod_{i \in I} G_i$, $P = \prod_{i \in I} P_i$, $W = \prod_{i \in I} W_i$, $S = \prod_{i \in I} S_i$ and $Z = \prod_{i \in I} Z_i$. The group $W = N_G(S)/Z$ is the so-called *restricted Weyl group* of G . It is a Coxeter group, generated by some set $R = \sqcup_{i \in I} R_i$ of simple reflections.

Proposition 4.7. *The space $B := G/P$ (with the Haar measure class) is a strong boundary for G . The Weyl group of $(G, G/P)$ is isomorphic to W . Furthermore, $W_{G,G/P}$ does not have any nontrivial special normal subgroup of index 2.*

Proof. For each i , the group P_i is an extension of its unipotent radical (which is amenable) by a k_i -anisotropic reductive group, which is therefore compact (see [Rou77] or [Pra82]). Hence P_i is amenable, and $P = \prod P_i$ also is. Therefore B is an amenable G -space.

The Bruhat decomposition [BT65, 5.15] says that every element of g can be written as $bn_w b'$, for some $b, b' \in P$, and n_w is a lift of some $w \in W$ in G . Furthermore, the

element $w \in W$ is unique. This means that $P \backslash G/P$ is in bijection with W . The G -orbits of the space $G/P \times G/P$ are in bijection (via the map $G.(hP, gP) \mapsto Ph^{-1}gP$) with $P \backslash G/P$, hence with W .

Among these orbits, there is one which is open and dense (in the Zariski and the Hausdorff topology): it consists in the set of opposite parabolic points (the longest element of W) [Bor91, 21.26, 21.28]. The stabilizer of a point in this orbit is conjugate to Z [BT65, 4.8]. All the other orbits are of lower dimension, and therefore of Haar measure 0. It follows that $B \times B$ is G -equivariantly measurably isomorphic to G/Z , and we can (and shall) choose this isomorphism so that the composition with the first projection $G/Z \simeq G/P \times G/P \rightarrow G/P$ is the map $G/Z \rightarrow G/P$ induced by the inclusion $Z \subseteq P$. The isometric ergodicity of $B \times B$ follows from [BG17, Theorem 6.6] (see [BF14a, Lemma 2.6] for an alternative argument).

The Weyl group of B is $\text{Aut}_G(B \times B) = \text{Aut}_G(G/Z)$. An automorphism of G/Z is uniquely determined by the image of eZ , which is g_0Z for some $g_0 \in N_G(Z)$, where $N_G(Z)$ is the normalizer of Z . Hence we have $\text{Aut}_G(G/Z) \simeq N_G(Z)/Z$ (see [BFS06, Lemma 2.14] for more details). So the Weyl group of B is exactly W .

We can also describe the special subgroups of W . A quotient of G/P is of the form G/Q , with $P < Q$. By [BT65, 5.18, 5.20] the groups containing P are in bijection with the subsets $T \subseteq R$, the bijection being given by $T \mapsto P_T = PW_T P$, where $W_T = \langle T \rangle$, $T \subseteq R$. Then the special subgroup associated to the quotient $\pi_T : G/P \rightarrow G/P_T$ is exactly W_T . Indeed, if $w \in W_T$, lift w to $n_w \in N_G(S)$. Then $n_w \in P_T$ and therefore for every $g \in G$, gP and $gn_w P$ have the same image in G/P_T . Hence $W_T \subseteq W_{G,B}(\pi_T)$. Conversely, if $w \notin W_T$, lift again w to an element n_w of $N_G(S)$. By disjointness in the Bruhat decomposition we have $P_T \cap n_w P_T = \emptyset$. Hence $n_w P_T$ and P_T are distinct cosets, which means that $w \notin W_{G,B}(\pi_T)$. Therefore $W_T = W_{G,B}(\pi_T)$ as claimed.

Since all G_i are simple, each W_i is irreducible, hence no non-trivial proper special subgroup $W_T \subseteq W_i$ is normal. Hence the special subgroups of W are all products of the form $\prod_{i \in J} W_i$, for some $J \subseteq I$. Since each G_i has rank at least 2, we get that none of these subgroups have index 2. \square

Example 4.8. Let $G = \text{SL}_3(\mathbf{R})$. Then the previous definitions unfold as follows: P is the subgroup of upper-triangular matrices, $Z = S$ is the subgroup of diagonal matrices, and W is the group of permutations S_3 . The space $B = G/P$ is the set of flags (line \subset plane) in \mathbf{R}^3 . The set $B \times B$ of pairs of flags has a G -orbit of full measure: the set of pairs of flags which are transverse (in the sense that the line of one of them is in direct sum with the plane of the other). A pair of transverse flags ($d \subset P$) and ($d' \subset P'$) determines a frame (i.e. a triple of lines (d_1, d_2, d_3) such that $d_1 \oplus d_2 \oplus d_3 = \mathbf{R}^3$): take for example $d_1 = d$, $d_2 = P \cap P'$, $d_3 = d'$. The group W acts by permuting the lines of the frame. If an element of $B \times B$ is represented by the frame (d_1, d_2, d_3) then the first projection $p_1 : B \times B \rightarrow B$ is given by the flag $d_1 \subset d_1 \oplus d_2$, and the second projection by $d_3 \subset d_3 \oplus d_2$.

There are exactly two non-trivial proper G -quotients of G/P , namely G/Q_1 and G/Q_2 , where Q_1 is the stabilizer of a line and Q_2 is the stabilizer of a plane; they correspond to maps $(d \subset P) \mapsto d$ and $(d \subset P) \mapsto P$. The special subgroup associated

with the quotient $G/P \rightarrow G/Q_1$ is the group $\{\text{id}, (2, 3)\}$ whereas the special subgroup associated to the quotient $G/P \rightarrow G/Q_2$ is the group $\{\text{id}, (1, 2)\}$. None of them is normal in W .

4.3 Geometrically rigid actions

Given a standard Borel space Δ and $i \in \mathbb{N}$, we denote by $\Delta^{(i)}$ the subset of Δ^i made of all tuples having pairwise distinct coordinates, which is again a standard Borel space. Given a set \mathbb{D} , we denote by $\mathcal{P}_{<\infty}(\mathbb{D})$ the set of all finite subsets of \mathbb{D} . The following property is the central definition of the present work; it will be used to derive rigidity statements regarding cocycles ranging in the group Λ .

Definition 4.9 (Geometrically rigid). *Let Λ be a countable discrete group. Let \mathbb{D} be a countable discrete Λ -space, let K be a compact metric space equipped with a Λ -action by homeomorphisms, and let Δ be a Polish space equipped with a Λ -action by Borel automorphisms. We say that the triple (\mathbb{D}, K, Δ) is geometrically rigid if the following hold:*

- (1) *The space K admits a Borel Λ -invariant partition $K = K_{\text{bdd}} \sqcup K_{\infty}$ such that*
 - *there exists a Borel Λ -equivariant map $\theta_{\text{bdd}} : K_{\text{bdd}} \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$,*
 - *there exists a Borel Λ -equivariant map $\theta_{\infty} : K_{\infty} \rightarrow \Delta$.*
- (2) *There exists a Borel Λ -equivariant map $\text{bar} : \Delta^{(3)} \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$.*
- (3) *The Λ -action on Δ is universally amenable.*

We say that Λ is geometrically rigid with respect to \mathbb{D} if there exists a compact metric Λ -space K and a Polish Λ -space Δ such that the triple (\mathbb{D}, K, Δ) is geometrically rigid.

Some examples. A first example is the following: every Gromov hyperbolic group Λ is geometrically rigid with respect to itself. Indeed, let Δ be the Gromov boundary of Λ , and $K := \Lambda \cup \Delta$. Then Assumption (1) clearly holds true by letting $K_{\text{bdd}} := \Lambda$ and $K_{\infty} := \Delta$, and letting θ_{bdd} and θ_{∞} be the identity maps. Assertion (2) states that we can associate a ‘barycenter’ (which is a finite subset of Λ) to any triple of pairwise distinct points in $\Delta = \partial_{\infty}\Lambda$: this was established by Adams in [Ada94a, Section 6]. The Borel amenability of the Λ -action on Δ was also established by Adams in [Ada94a].

More generally, we will see in Section 5.1 that if Λ is hyperbolic relative to a finite set \mathcal{P} of subgroups, then Λ is geometrically rigid with respect to the vertex set of a hyperbolic graph whose point stabilizers are all parabolic.

We will also see (Proposition 9.2) that the mapping class group $\text{Mod}(\Sigma)$ of a connected, orientable hyperbolic surface Σ of finite type is geometrically rigid with respect to the collection \mathbb{D} of isotopy classes of essential simple closed curves, by taking for K the Thurston compactification of the Teichmüller space, and for Δ the Gromov boundary of the curve graph.

Finally, a key point of the present paper is to show that $\text{Out}(F_N)$, or more generally the group $\text{Out}(G, \mathcal{F}^{(t)})$ in the context of free products (see Section 6 for definitions), is geometrically rigid with respect to the countable collection \mathbb{D} of all conjugacy classes of proper free factors (taking for K the compactification of the projectivized Outer space, and for Δ the boundary of the free factor graph): this will be completed in Proposition 9.7. A crucial task in the sequel of the paper will be to build the barycenter map in this setting (in particular for free products).

4.4 A Bader–Furman type argument

Our goal is now to prove the following proposition, following a method developed by Bader and Furman in [BF14a] or [BF].

Proposition 4.10. *Let G be a locally compact group, and let B be a strong boundary for G . Let X be a standard probability space equipped with an ergodic measure-preserving G -action. Let Λ be a countable discrete group, let \mathbb{D} be a countable discrete Λ -space, and assume that Λ is geometrically rigid with respect to \mathbb{D} .*

Then there exists a Polish space Δ equipped with a universally amenable Λ -action, such that for every cocycle $c : G \times X \rightarrow \Lambda$, either

1. *the cocycle c is cohomologous to a cocycle that takes its values in the setwise stabilizer of a finite subset of \mathbb{D} , or*
2. *there exists a c -equivariant Borel map $X \rightarrow \Delta$ or a c -equivariant Borel map $X \rightarrow \Delta^{(2)}/\mathfrak{S}_2$, or else*
3. *there exists a nontrivial homomorphism $W_{G,B} \rightarrow \mathbb{Z}/2\mathbb{Z}$ whose kernel is a special subgroup of the Weyl group $W_{G,B}$.*

In the sequel of this section, we will adopt the notations from Proposition 4.10. We let Δ be a Polish space equipped with a universally amenable Λ -action coming from the definition of geometric rigidity of Λ with respect to \mathbb{D} . We let $\text{Prob}(\Delta)$ be the set of all Borel probability measures on Δ equipped with the topology generated by the maps $\mu \mapsto \int f d\mu$, where f varies over the set of bounded continuous real-valued functions on Δ , again a Polish space [Kec95, Theorem 17.23]. We denote by $\text{Prob}_{\geq 3}(\Delta)$ the space of all probability measures on Δ whose support contains at least 3 points: this is a Borel subset of $\text{Prob}(\Delta)$. For every $i \in \mathbb{N}$, we let

$$c_i : \begin{array}{ccc} G \times X \times B^i & \rightarrow & \Lambda \\ (g, x, b_1, \dots, b_i) & \mapsto & c(g, x) \end{array}$$

which is again a cocycle.

Lemma 4.11. *One of the following holds.*

1. *There exists a c_1 -equivariant Borel map $X \times B \rightarrow \mathcal{P}_{< \infty}(\mathbb{D})$.*
2. *There exists a c_1 -equivariant Borel map $X \times B \rightarrow \text{Prob}_{\geq 3}(\Delta)$.*

3. There exist $i \in \{1, 2\}$ and a c_1 -equivariant Borel map $X \times B \rightarrow \Delta^{(i)}/\mathfrak{S}_i$.

Proof. Since B is a strong boundary for G , the G -action on B is Zimmer amenable, so by [Zim84, Proposition 4.3.4], the G -action on $X \times B$ is also Zimmer amenable. As K is a compact metric space, Lemma 2.2 ensures that there exists a c_1 -equivariant Borel map

$$\begin{aligned} \nu : X \times B &\rightarrow \text{Prob}(K) \\ (x, b) &\mapsto \nu_{x,b} \end{aligned}$$

Let

$$V_{\text{bdd}} := \{(x, b) \in X \times B \mid \nu_{x,b}(K_\infty) = 0\}$$

and

$$V_\infty := \{(x, b) \in X \times B \mid \nu_{x,b}(K_{\text{bdd}}) = 0\}.$$

Then V_{bdd} and V_∞ are Borel subsets of $X \times B$, see e.g. [Kec95, Theorem 17.25]. In addition, as K_{bdd} and K_∞ are both Λ -invariant, and ν is c_1 -equivariant, the sets V_{bdd} and V_∞ are both G -invariant. Since B is a strong boundary for G , the G -action on $X \times B$ is ergodic (Remark 4.3). Therefore, either V_{bdd} or V_∞ (or both) has measure 0. By replacing ν by (the renormalization of) its restriction to either K_∞ or K_{bdd} , we can thus assume that either

- (1) the probability measure $\nu_{x,b}$ gives full measure to K_{bdd} for a.e. $(x, b) \in X \times B$, or
- (2) the probability measure $\nu_{x,b}$ gives full measure to K_∞ for a.e. $(x, b) \in X \times B$.

We first assume that (1) holds. Since there is a Λ -equivariant Borel map $\theta_{\text{bdd}} : K_{\text{bdd}} \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$ (by the first assumption from Definition 4.9 of geometric rigidity), we deduce that there exists a c_1 -equivariant Borel map $X \times B \rightarrow \text{Prob}(\mathcal{P}_{<\infty}(\mathbb{D}))$. Since $\mathcal{P}_{<\infty}(\mathbb{D})$ is countable, there is a Λ -equivariant Borel map $\text{Prob}(\mathcal{P}_{<\infty}(\mathbb{D})) \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$, sending a probability measure λ on $\mathcal{P}_{<\infty}(\mathbb{D})$ to the union of all finite subsets of \mathbb{D} having maximal λ -measure. Hence the first conclusion of the lemma holds.

We now assume that (2) holds. Since there is a Borel Λ -equivariant map $\theta_\infty : K_\infty \rightarrow \Delta$, we deduce that there exists a c_1 -equivariant Borel map $X \times B \rightarrow \text{Prob}(\Delta)$. Since the cardinality of the support of the measure $\nu_{x,b}$ is G -invariant, using again the ergodicity of the G -action on $X \times B$, we deduce that one of the last three conclusions of the lemma holds. \square

The following lemma deals with the first case of Lemma 4.11: when applied to $\hat{\mathbb{D}} = \mathcal{P}_{<\infty}(\mathbb{D})$ and $i = 1$, it shows that the first case from Lemma 4.11 yields the first conclusion of Proposition 4.10.

Lemma 4.12. *Let $\hat{\mathbb{D}}$ be a countable discrete Λ -space. Assume that for some $i \in \{1, 2\}$, there exists a c_i -equivariant Borel map $f : X \times B^i \rightarrow \hat{\mathbb{D}}$.*

Then c is cohomologous to a cocycle that takes its values in the stabilizer of an element of $\hat{\mathbb{D}}$.

Proof. In view of Lemma 3.7, it is enough to construct a c -equivariant Borel map $X \rightarrow \hat{\mathbb{D}}$.

We equip $\hat{\mathbb{D}}$ with the discrete metric (i.e. $d_{\hat{\mathbb{D}}}(x, y) = 1$ if $x \neq y$), which is Λ -invariant. Let $L := L^0(X, \hat{\mathbb{D}})$, equipped with the metric δ and the isometric action of G given in Section 4.1.

By Lemma 4.1, the map f gives rise to a Lebesgue-measurable map $\bar{f} : B^i \rightarrow L$, defined by $\bar{f}(b)(x) = f(x, b)$ for all $b \in B^i$ and all $x \in X$. Furthermore, the fact that f is c_i -equivariant translates into the fact that \bar{f} is G -equivariant.

Since B is a strong boundary for G , the G -actions on both B and $B \times B$ are isometrically ergodic. It follows that the map \bar{f} is essentially constant. Its essential image is therefore a G -invariant measurable function $s_0 \in L$. In other words $s_0 : X \rightarrow \hat{\mathbb{D}}$ satisfies that, for a.e. $x \in X$ and all $g \in G$, we have $s_0(gx) = c(g, x)s_0(x)$. That is, s_0 is a c -equivariant measurable function, and it is therefore almost everywhere equal to some c -equivariant Borel map $X \rightarrow \hat{\mathbb{D}}$, as desired. \square

The following lemma deals with the second case from Lemma 4.11.

Lemma 4.13. *Assume that for some $i \in \{1, 2\}$, there exists a c_i -equivariant Borel map $\nu : X \times B^i \rightarrow \text{Prob}_{\geq 3}(\Delta)$.*

Then c is cohomologous to a cocycle that takes its values in the setwise stabilizer of a finite subset of \mathbb{D} .

Proof. For a.e. $(x, b) \in X \times B^i$, the product measure $\nu_{x,b} \times \nu_{x,b} \times \nu_{x,b}$ on Δ^3 gives positive measure to $\Delta^{(3)}$, so it induces a probability measure on $\Delta^{(3)}$ (after renormalizing). Pushing it forward through the Λ -equivariant map $\text{bar} : \Delta^{(3)} \rightarrow \mathcal{P}_{< \infty}(\mathbb{D})$, this yields a c_i -equivariant Borel map $X \times B^i \rightarrow \text{Prob}(\mathcal{P}_{< \infty}(\mathbb{D}))$. Since $\mathcal{P}_{< \infty}(\mathbb{D})$ is countable, there is a Borel Λ -equivariant map $\text{Prob}(\mathcal{P}_{< \infty}(\mathbb{D})) \rightarrow \mathcal{P}_{< \infty}(\mathbb{D})$ (sending a measure λ to the union of all elements of $\mathcal{P}_{< \infty}(\mathbb{D})$ with maximal λ -measure). The conclusion then follows from Lemma 4.12 applied to $\hat{\mathbb{D}} := \mathcal{P}_{< \infty}(\mathbb{D})$. \square

In order to tackle the third case from Lemma 4.11, we start with the following lemma.

Lemma 4.14. *Assume that for some $i \in \{1, 2\}$, there exists a c_i -equivariant Borel map $X \times B \rightarrow \Delta^{(i)}/\mathfrak{S}_i$. Then either*

1. *c is cohomologous to a cocycle that takes its values in the stabilizer of a finite subset of \mathbb{D} , or*
2. *there exist $i \in \{1, 2\}$ and a c -equivariant Borel map $X \rightarrow \Delta^{(i)}/\mathfrak{S}_i$, or else*
3. *there is a c_1 -equivariant Borel map $X \times B \rightarrow \Delta$, and an essentially unique c_2 -equivariant Borel map $X \times B \times B \rightarrow \Delta^{(2)}/\mathfrak{S}_2$.*

Proof. Let $f_1 : X \times B \rightarrow \Delta^{(i)}/\mathfrak{S}_i$ be a c_i -equivariant Borel map. Then the sets

$$V_{=} := \{(x, b, b') \in X \times B \times B \mid f_1(x, b) = f_1(x, b')\}$$

and

$$V_{\neq} := \{(x, b, b') \in X \times B \times B \mid f_1(x, b) \neq f_1(x, b')\}$$

are G -invariant (up to null sets). By ergodicity of the G -action on $X \times B \times B$, it follows that one of these sets is null and the other is conull.

If V_- is conull, then the map f_1 essentially does not depend on its second coordinate. Therefore it yields a c -equivariant Borel map $X \rightarrow \Delta^{(i)}/\mathfrak{S}_i$, so the second conclusion of the lemma holds.

We now assume that V_{\neq} is conull. If $i = 2$, then there is a c_2 -equivariant Borel map f' from a subset of full measure of $X \times B \times B$ to $\Delta^{(3)}/\mathfrak{S}_3 \cup \Delta^{(4)}/\mathfrak{S}_4$, defined by letting $f'(x, b, b') = f_1(x, b) \cup f_1(x, b')$. In this case Lemma 4.13 shows that the first conclusion of the lemma holds.

We now assume that $i = 1$. As above, we have a c_2 -equivariant map

$$\begin{aligned} f_2 : X \times B \times B &\rightarrow \Delta^{(2)}/\mathfrak{S}_2 \\ (x, b, b') &\mapsto \{f_1(x, b), f_1(x, b')\} \end{aligned}$$

If such a map is essentially unique, then the third conclusion of the lemma holds.

We can therefore assume that there exists another c_2 -equivariant map $f'_2 : X \times B \times B \rightarrow \Delta^{(2)}/\mathfrak{S}_2$ which is not a.e. equal to f_2 . By ergodicity of the G -action on $X \times B \times B$, we have $f_2(x, b, b') \neq f'_2(x, b, b')$ for a.e. $(x, b, b') \in X \times B \times B$. Hence the map $(x, b, b') \mapsto f_2(x, b, b') \cup f'_2(x, b, b')$ yields a c_2 -equivariant Borel map from $X \times B \times B$ to $\Delta^{(3)}/\mathfrak{S}_3 \cup \Delta^{(4)}/\mathfrak{S}_4$. Lemma 4.13 therefore implies that c is cohomologous to a cocycle that takes its values in the stabilizer of a finite subset of \mathbb{D} , so the first conclusion of the lemma holds. \square

The following lemma completes our proof of Proposition 4.10.

Lemma 4.15. *Assume that there is no c -equivariant Borel map $X \rightarrow \Delta$. Assume in addition that there exist a c_1 -equivariant Borel map $f_1 : X \times B \rightarrow \Delta$ and an essentially unique c_2 -equivariant Borel map $X \times B \times B \rightarrow \Delta^{(2)}/\mathfrak{S}_2$.*

Then there exists a nontrivial homomorphism $W_{G,B} \rightarrow \mathbb{Z}/2\mathbb{Z}$ whose kernel is a special subgroup of the Weyl group $W_{G,B}$.

Proof. Consider the c_2 -equivariant Borel maps $X \times B \times B \rightarrow \Delta^{(2)}$ defined by

$$f_1^2 : (x, b, b') \mapsto (f_1(x, b), f_1(x, b'))$$

and

$$(f_1^2)^* : (x, b, b') \mapsto (f_1(x, b'), f_1(x, b)).$$

We note that f_1^2 and $(f_1^2)^*$ essentially take their values in $\Delta^{(2)}$: otherwise, by ergodicity, one has $f_1(x, b) = f_1(x, b')$ for almost every $(x, b, b') \in X \times B \times B$ so f_1 defines a c -equivariant map $f : X \rightarrow \Delta$, contradicting our hypothesis.

We claim that the maps f_1^2 and $(f_1^2)^*$ are the only c_2 -equivariant Borel maps $X \times B \times B \rightarrow \Delta^{(2)}$. Indeed, if $\psi : X \times B \times B \rightarrow \Delta^{(2)}$ is a c_2 -equivariant Borel map, by uniqueness of the map $X \times B \times B \rightarrow \Delta^{(2)}/\mathfrak{S}_2$, we get that for a.e. (x, b, b') , either $\psi(x, b, b') = f_1^2(x, b, b')$ or $\psi(x, b, b') = (f_1^2)^*(x, b, b')$. By ergodicity of the G -action on $X \times B \times B$, one of these equalities holds almost everywhere. This proves our claim.

Let $F_1 : B \rightarrow L^0(X, \Delta)$ be the map induced by f_1 , and let $F_1^2, (F_1^2)^* : B \times B \rightarrow L^0(X, \Delta^{(2)})$ be the maps induced by f_1^2 and $(f_1^2)^*$. Thus, the set of G -equivariant Borel maps $B \times B \rightarrow L^0(X, \Delta^{(2)})$ is a set consisting of exactly two elements, namely F_1^2 and $(F_1^2)^*$. The Weyl group $W_{G,B}$ acts by precomposition on this set, thus defining a homomorphism $m : W_{G,B} \rightarrow \mathbb{Z}/2\mathbb{Z}$. The flip $(b, b') \mapsto (b', b)$ is not in the kernel of m , as otherwise F_1 would be essentially constant, contradicting that there is no c -equivariant Borel map $X \rightarrow \Delta$. Thus m is nontrivial and $\ker m$ is a normal subgroup of index 2 of $W_{G,B}$.

There remains to prove that $\ker m$ is a special subgroup of $W_{G,B}$ (see Definition 4.6). Taking $C = L^0(X, \Delta)$ and $\pi = F_1 : B \rightarrow C$, it suffices to check that $\ker m = W_{G,B}(\pi)$ i.e. that $w \in \ker m$ if and only if $F_1 \circ p_1 = F_1 \circ p_1 \circ w$. Write $w \in W_{G,B}$ as $w(b, b') = (w_1(b, b'), w_2(b, b'))$. If $w \in \ker m$, then $F_1^2 \circ w = F_1^2$ so $F_1(b) = F_1(w_1(b, b'))$ for almost every $(b, b') \in B^2$, which precisely means that $F_1 \circ p_1 = F_1 \circ p_1 \circ w$.

Conversely, assume that $F_1 \circ p_1 = F_1 \circ p_1 \circ w$. If $w \notin \ker m$, then $F_1^2 \circ w = (F_1^2)^*$ so $F_1(b') = F_1(w_1(b, b'))$ for almost every $(b, b') \in B^2$, i.e. $F_1 \circ p_2 = F_1 \circ p_1 \circ w$. It follows that $F_1 \circ p_1 = F_1 \circ p_2$, hence F_1 is essentially constant. Its image is therefore a G -invariant function in $L^0(X, \Delta)$, in other words, a c -equivariant measurable map from X to Δ , a contradiction. This proves that $\ker m = W_{G,B}(\pi)$ is a special subgroup of $W_{G,B}$. \square

4.5 Property (T) versus amenability of the action

In the case where the group G has property (T), and in fact more generally as soon as every cocycle from $G \times X$ to an amenable group is cohomologous to one with finite values, the second conclusion from Proposition 4.10 yields cocycle superrigidity, as was established by Spatzier and Zimmer in [SZ91].

Lemma 4.16 (Spatzier–Zimmer [SZ91]). *Let G be a locally compact second countable group with property (T), and let X be a standard Borel probability space equipped with an ergodic measure-preserving G -action. Let Λ be a countable group acting on a standard Borel space Δ , so that the Λ -action on Δ is universally amenable.*

Let $c : G \times X \rightarrow \Lambda$ be a measurable cocycle, and assume that there exists a c -equivariant Borel map from X to Δ .

Then c is cohomologous to a cocycle with values in a finite subgroup of Λ .

Proof. The argument appears in the proof of [SZ91, Lemma 4.1], we recall it here. Let $G \times \Lambda \curvearrowright Y = X \times \Lambda$ be the developed action. The c -equivariant Borel map $X \rightarrow \Delta$ yields a G -invariant Λ -equivariant map $F : Y \rightarrow \Delta$. Let M be the Mackey range of c , i.e. the space of ergodic components of $G \curvearrowright Y$. It is naturally equipped with a Λ -action, and the map F factors through a Λ -equivariant map $M \rightarrow \Delta$. As Δ is universally amenable, it follows that the Λ -action on M is amenable in the sense of Zimmer, see e.g. [AEG94, Corollary C].

We first assume that the Λ -action on M is essentially transitive. By [Zim84, Proposition 4.3.2], there exists an amenable subgroup $\Lambda^1 \subseteq \Lambda$ and a Λ -equivariant isomorphism $M \rightarrow \Lambda/\Lambda^1$. By Lemma 3.7, this implies that c is cohomologous to a cocycle with val-

ues in Λ^1 . The conclusion then follows from Zimmer’s cocycle rigidity theorem [Zim84, Theorem 9.1.1].

We now assume that the Λ -action on M is properly ergodic. By [OW80], it is orbit-equivalent to an action of \mathbb{Z} . Let $\alpha : \Lambda \times M \rightarrow \mathbb{Z}$ be a cocycle coming from the orbit equivalence. Consider the cocycle $\beta : G \times X \rightarrow \mathbb{Z}$ defined by letting $\beta(g, x) := \alpha(c(g, x)^{-1}, [gx, e])$ (where $[gx, e]$ denotes the image of $(gx, e) \in X \times \Lambda$ in M). Then the Mackey range $\mathbb{Z} \curvearrowright M'$ of β is orbit equivalent to $\Lambda \curvearrowright M$, and is properly ergodic in particular. But since G has Property (T) and \mathbb{Z} is amenable, it follows from [Zim84, Theorem 9.1.1] that the cocycle β is cohomologically trivial, so that $\mathbb{Z} \curvearrowright M'$ has trivial orbits. This yields a contradiction. \square

4.6 Specialization to the case where G is an algebraic group

We now specialize Proposition 4.10 to the case where G is a product of higher rank simple algebraic groups. The following is Theorem 2 from the introduction.

Theorem 4.17. *Let Λ be a countable group. Let \mathbb{D} be a countable Λ -set. Assume that Λ is geometrically rigid with respect to \mathbb{D} .*

Let G be a product of connected higher rank simple algebraic groups over local fields. Let X be a standard probability space equipped with an ergodic measure-preserving G -action.

Then every cocycle $c : G \times X \rightarrow \Lambda$ is cohomologous to a cocycle that takes its values in a subgroup of Λ that virtually fixes an element of \mathbb{D} .

Proof. We apply Proposition 4.10 to the cocycle c . If the first conclusion of Proposition 4.10 holds, then c is cohomologous to a cocycle with values in the setwise stabilizer of a finite subset of \mathbb{D} . Assume that the second conclusion of Proposition 4.10 holds, i.e. there is a c -equivariant Borel map from X to either Δ or $\Delta^{(2)}/\mathfrak{S}_2$. The Λ -action on Δ is universally amenable by assumption, and therefore so is the Λ -action on $\Delta^{(2)}/\mathfrak{S}_2$ (see the argument in [Ada94b, Lemma 5.2]). Since G has Property (T), Lemma 4.16 therefore ensures that c is cohomologous to a cocycle taking its values in a finite subgroup of Λ and the conclusion holds. Finally, by Proposition 4.7, the Weyl group $W_{G,B}$ does not contain any normal special subgroup of index 2, which excludes the third conclusion from Proposition 4.10. \square

5 First examples of cocycle rigidity results

5.1 Cocycle rigidity with target a relatively hyperbolic group

The following theorem was announced by Bader–Furman, see the unpublished paper [BF]. Since, to our knowledge, its statement does not appear elsewhere in the literature, we include a proof in the present section.

Theorem 5.1 (Bader–Furman [BF]). *Let Λ be a group which is hyperbolic relative to a finite collection \mathcal{P} of subgroups. Let G_0 be a product of connected higher rank simple*

algebraic groups defined over local fields. Let G be either G_0 , or a lattice in G_0 . Let X be a standard probability space equipped with an ergodic measure-preserving action of G .

Then every cocycle $G \times X \rightarrow \Lambda$ is cohomologous to a cocycle that takes its values in a parabolic or in a finite subgroup of Λ .

We recall here that a subgroup of G is *parabolic* if it is conjugate into one of the subgroups in \mathcal{P} . In particular, if we assume that for every $P \in \mathcal{P}$, the pair (G, P) is cocycle-rigid, then (G, Λ) is cocycle-rigid. In the particular case where $\mathcal{P} = \emptyset$, we recover the following result originally due to Adams.

Corollary 5.2 (Adams [Ada96]). *Let Λ be a hyperbolic group. Let G_0 be a product of connected higher rank simple algebraic groups defined over local fields. Let G be either G_0 , or a lattice in G_0 . Let X be a standard probability space equipped with an ergodic measure-preserving action of G .*

Then every cocycle $G \times X \rightarrow \Lambda$ is cohomologous to a cocycle that takes its values in a finite subgroup of Λ .

Proof of Theorem 5.1. By work of Bowditch [Bow98], there exists a *uniformly fine* hyperbolic graph Y on countably many vertices equipped with an isometric Λ -action whose nontrivial point stabilizers are all conjugate to subgroups in \mathcal{P} . Here, we recall that uniform fineness means that for every $n \in \mathbb{N}$, there exists $C_n > 0$ such that for every edge $e \subseteq Y$, there are at most C_n closed circuits of length n that contain e .

We let \mathbb{D} be the vertex set of Y . We will prove that Λ is geometrically rigid with respect to \mathbb{D} ; Theorem 4.17 then shows that the cocycle is cohomologous to a cocycle with values in a subgroup $K \subseteq \Lambda$ where some finite index subgroup of K fixes a point in \mathbb{D} . Thus if K is infinite, then K has a finite index subgroup which is contained in a parabolic subgroup P . Since parabolic subgroups are almost malnormal (i.e. $P^g \cap P$ is finite for $g \notin P$), it follows that H is in fact contained in P .

Let Δ be the Gromov boundary of Y , and let $K := \mathbb{D} \cup \Delta$. Then K has a natural structure of a compact metric space (usually called the *Bowditch boundary* of (Λ, \mathcal{P})) in which Δ is a Borel subset, and Λ acts on K by homeomorphisms [Bow98, Sections 8 and 9]. In particular Δ is a Polish space.

The ‘barycenter map’ was essentially constructed by Bowditch in [Bow98, Section 8] (see e.g. the comment right after [Bow98, Lemma 8.2]), with an argument of Kida [Kid08, Section 4.1.2] for its measurability. More precisely, using [Bow98, Lemma 8.2] (which relies on the fineness on Y), one sees that the graph Y has the following two properties:

1. for every $r > 0$, there exists $C_1 > 0$ such that given any two points $x, y \in Y$, and any point z lying on a geodesic from x to y , there are at most C_1 points that lie on some geodesic from x to y and are contained in $B(z, r)$;
2. for every $r > 0$, there exist $C_2 > 0$ and $\kappa > 0$ such that for every $r' > 0$, given any two points $x, y \in Y$, and any point $z \in Y$ lying on a geodesic from x to y and such that $d(z, \{x, y\}) \geq r + \kappa$, the intersection of $B(z, r)$ with the collection of all geodesics in Y with origin in $B(x, r')$ and endpoint in $B(y, r')$ has cardinality at most C_2 .

These are precisely the conditions required by Kida to ensure the existence of a Λ -equivariant Borel map $\Delta^{(3)} \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$, see [Kid08, Section 4.1.2].

Finally, universal amenability of the Λ -action on Δ was established by Ozawa in [Oza06, Lemma 8]. \square

5.2 Cocycle rigidity with target a right-angled Artin group

In order to illustrate the ubiquity of our notion of geometric rigidity in geometric group theory, we will now use it to prove cocycle rigidity with target a right-angled Artin group. The results of the present section will not be used later in the paper. The analogous rigidity statement for homomorphisms from higher-rank lattices to right-angled Artin groups was first established by Behrstock and Charney in [BC12]. We also mention that rigidity in the realm of group actions on cube complexes has been extensively studied by Chatterji, Fernós and Iozzi in [CFI16] using bounded cohomology methods, with applications to orbit equivalence rigidity through work of Monod and Shalom [MS06].

Given a finite simple graph Γ (i.e. Γ has no loop-edge and no multiple edges joining two vertices), recall that the *right-angled Artin group* A_Γ associated to Γ is the group defined by the following presentation: it has one generator per vertex of Γ , and two generators commute whenever the corresponding vertices of Γ are joined by an edge.

Theorem 5.3. *Let Γ be a finite simple graph, and let A_Γ be the associated right-angled Artin group.*

Let G_0 be a product of connected higher rank simple algebraic groups defined over local fields. Let G be either G_0 , or a lattice in G_0 . Let X be a standard probability space equipped with an ergodic measure-preserving action of G .

Then every cocycle $G \times X \rightarrow A_\Gamma$ is cohomologous to the trivial cocycle.

Given a full subgraph $\Lambda \subseteq \Gamma$ (i.e. two vertices of Λ are joined by an edge if and only if they are joined by an edge in Γ), the Salvetti complex S_Λ naturally embeds as a subcomplex of S_Γ . A *standard subcomplex* of type Λ of the universal cover \tilde{S}_Γ is a connected component of the preimage of S_Λ under the covering map. Notice that its stabilizer is a conjugate of A_Λ . In particular, the vertices of \tilde{S}_Γ are the standard subcomplexes associated to $\Lambda = \emptyset$. We denote by \mathbb{D}_Γ the countable set of all proper (i.e. with $\Lambda \neq \Gamma$) standard subcomplexes of \tilde{S}_Γ .

Lemma 5.4. *Let Γ be a finite simple graph that does not split nontrivially as a join. Then A_Γ is geometrically rigid with respect to \mathbb{D}_Γ .*

Proof. Let \tilde{S}_Γ be the universal cover of the Salvetti complex of A_Γ . Let K be the Roller boundary of \tilde{S}_Γ , a compact metrizable space. Let $K_\infty \subseteq K$ be the Borel subset that consists of all *regular* points as defined by Fernós in [Fer18, Definition 3.7], and we let $K_{\text{bdd}} = K \setminus K_\infty$ and $\Delta = K_\infty$.

By [HH, Theorem 2.1], as Γ does not split nontrivially as a join, there exists a Borel A_Γ -equivariant map $K_{\text{bdd}} \rightarrow \mathbb{D}_\Gamma$ that associates a proper standard subcomplex to every nonregular point of the Roller boundary.

The barycenter map, associating a vertex in \tilde{S}_Γ to every triple of pairwise distinct points in K_∞ , is provided by [FLM18, Lemma 5.14].

Finally, as the A_Γ -action on the Roller boundary of \tilde{S}_Γ is Borel amenable [Duc18, Theorem 5.11] whence universally amenable, so is the A_Γ -action on K_∞ (alternatively, we may use [NS13, Theorem 7.2] and the fact that regular points of the Roller boundary correspond to non-terminating ultrafilters). \square

We are now in position to complete our proof of Theorem 5.3.

Proof of Theorem 5.3. The proof is by induction on the number of vertices of the defining graph Γ . When Γ is reduced to a point, the group A_Γ is isomorphic to \mathbb{Z} , so the conclusion follows from [Zim84, Theorem 9.1.1].

If Γ is a join of two subgraphs Γ_1 and Γ_2 , then A_Γ decomposes as a direct product $A_{\Gamma_1} \times A_{\Gamma_2}$, and the conclusion follows by induction using the fact that cocycle super-rigidity is stable under direct products as a property of the target group.

We now assume that Γ is not a join. As A_Γ is geometrically rigid with respect to \mathbb{D}_Γ (Lemma 5.4), it follows from Theorem 4.17 that every cocycle $G \times X \rightarrow A_\Gamma$ is cohomologous to a cocycle that takes its values in the stabilizer of a proper standard subcomplex of \tilde{S}_Γ , which is a conjugate of some proper parabolic subgroup A_Λ . The theorem follows by induction. \square

Part II

Outer automorphism groups as targets

6 Background on outer automorphisms of free products

General context. Let $\mathcal{F} := \{G_1, \dots, G_k\}$ be a finite collection of countable groups, let F_N be a free group of rank N , and let

$$G := G_1 * \dots * G_k * F_N.$$

We define the *complexity* of (G, \mathcal{F}) as the pair $\xi(G, \mathcal{F}) = (k + N, N)$; complexities will be ordered lexicographically. We say that (G, \mathcal{F}) is *sporadic* if $\xi(G, \mathcal{F}) \leq (2, 1)$, and *nonsporadic* otherwise. More concretely, sporadic cases correspond to the following five possibilities:

- $G = \{1\}$ and $\mathcal{F} = \emptyset$ (equivalently $\xi(G, \mathcal{F}) = (0, 0)$),
- $G = G_1$ and $\mathcal{F} = \{G_1\}$ (equivalently $\xi(G, \mathcal{F}) = (1, 0)$),
- $G = \mathbb{Z}$ and $\mathcal{F} = \emptyset$ (equivalently $\xi(G, \mathcal{F}) = (1, 1)$),
- $G = G_1 * G_2$ and $\mathcal{F} = \{G_1, G_2\}$ (equivalently $\xi(G, \mathcal{F}) = (2, 0)$),
- $G = G_1 * \mathbb{Z}$ and $\mathcal{F} = \{G_1\}$ (equivalently $\xi(G, \mathcal{F}) = (2, 1)$).

Subgroups of G that are conjugate into one of the subgroups in \mathcal{F} are called \mathcal{F} -peripheral. We denote by $\text{Out}(G, \mathcal{F})$ the subgroup of $\text{Out}(G)$ made of all automorphisms that preserve the conjugacy class of each of the subgroups G_i . We denote by $\text{Out}(G, \mathcal{F}^{(t)})$ the subgroup made of all automorphisms whose restriction to each of the factors G_i is the conjugation by an element $g_i \in G$.

Trees, free splittings, free factors of (G, \mathcal{F}) . A (G, \mathcal{F}) -tree is an \mathbb{R} -tree T equipped with a minimal G -action by isometries, such that every subgroup in \mathcal{F} fixes a point in T (we recall that the G -action on T is said to be *minimal* if T does not contain any proper G -invariant subtree). It is *relatively free* if every subgroup of G that fixes a point is \mathcal{F} -peripheral.

A (G, \mathcal{F}) -free splitting is a simplicial (non-metric) (G, \mathcal{F}) -tree in which all edge stabilizers are trivial. We will always consider (G, \mathcal{F}) -free splittings up to G -equivariant homeomorphisms. We denote by FS the countable set of all (equivariant homeomorphism classes of) (G, \mathcal{F}) -free splittings.

A (G, \mathcal{F}) -free factor is a subgroup of G which is equal to a point stabilizer in some (G, \mathcal{F}) -free splitting. A (G, \mathcal{F}) -free factor is *proper* if it is neither \mathcal{F} -peripheral nor equal to G . More generally, a collection of conjugacy classes of proper (G, \mathcal{F}) -free factors is a (G, \mathcal{F}) -free factor system if it coincides with the collection of conjugacy classes of nontrivial point stabilizers in some (G, \mathcal{F}) -free splitting.

Every proper (G, \mathcal{F}) -free factor A inherits a decomposition as a free product $A = A_1 * \cdots * A_l * F_r$ from the splitting of G : the factors A_i that arise in this decomposition form a set of representatives of the nontrivial point stabilizers in the minimal A -invariant subtree of any relatively free (G, \mathcal{F}) -free splitting. We will denote by $\mathcal{F}|_A$ the collection of all the A -conjugacy classes of the subgroups A_i from the above decomposition of A : this is a free factor system of A .

Outer space and its closure. A *Grushko (G, \mathcal{F}) -tree* is a minimal simplicial metric relatively free (G, \mathcal{F}) -tree. The *unprojectivized relative Outer space* \mathcal{O} is the space of all G -equivariant isometry classes of Grushko (G, \mathcal{F}) -trees [CV86, GL07]. We denote by $\mathbb{P}\mathcal{O}$ the projectivized Outer space, where trees are considered up to homothety instead of isometry. In the sequel, when \mathcal{F} is clear from context, we will simply say ‘Grushko tree’ instead of ‘Grushko (G, \mathcal{F}) -tree’.

The closure $\overline{\mathcal{O}}$ of \mathcal{O} in the space of all nontrivial (G, \mathcal{F}) -trees (equipped with the equivariant Gromov–Hausdorff topology introduced in [Pau88]) is equal to the space of all *very small* (G, \mathcal{F}) -trees, i.e. those trees T for which stabilizers of nondegenerate tripods are trivial, and stabilizers of nondegenerate arcs are either trivial, or isomorphic to \mathbb{Z} , root-closed, and nonperipheral [CL95, BF94, Hor17]. We let $\partial\mathcal{O} := \overline{\mathcal{O}} \setminus \mathcal{O}$.

Arational trees. The notion of an arational tree was first introduced by Reynolds in the context of free groups [Rey12]. In the context of free products, a tree $T \in \overline{\mathcal{O}}$ is *arational* if $T \in \partial\mathcal{O}$, no proper (G, \mathcal{F}) -free factor A fixes a point in T , and for every proper (G, \mathcal{F}) -free factor A , the A -minimal invariant subtree $T_A \subseteq T$ is a Grushko

$(A, \mathcal{F}|_A)$ -tree. We denote by \mathcal{AT} the subspace of $\overline{\mathcal{O}}$ made of all arational (G, \mathcal{F}) -trees, and by $\mathbb{P}\mathcal{AT}$ its projectivized version (a subspace of $\mathbb{P}\overline{\mathcal{O}}$). These are Borel subsets of $\overline{\mathcal{O}}$ and $\mathbb{P}\overline{\mathcal{O}}$, respectively [Hor14, Lemma 5.5].

Two arational trees T and T' are *equivalent* (denoted by $T \sim T'$) if they are G -equivariantly homeomorphic when equipped with the *observers' topology* introduced in [CHL07]: this is the topology on a tree T for which a basis of open sets is given by the connected components of complements of points in T .

The free factor graph and its boundary. We adopt the definition of the free factor graph from [GH19b], see [GH19b, Section 2.2] for a discussion of its quasi-isometry with other models of the graph found in the literature. We recall that two free splittings S, S' of (G, \mathcal{F}) are *compatible* if there exists a free splitting S'' of (G, \mathcal{F}) such that both S and S' are obtained from S'' by collapsing every edge from a G -invariant subset of edges to a point. The *free factor graph* FF is the graph whose vertices are the G -equivariant homeomorphism classes of free splittings of (G, \mathcal{F}) , where two vertices are joined by an edge if the corresponding splittings are either compatible or have a common nonperipheral elliptic element.

With this definition, there is a natural $\text{Out}(G, \mathcal{F})$ -equivariant map $\pi : \mathcal{O} \rightarrow \text{FF}$, which simply consists in forgetting the metric on the trees.

The free factor graph is hyperbolic whenever (G, \mathcal{F}) is nonsporadic: this was first proved by Bestvina–Feighn [BF14b] for free groups (i.e. when $G = F_N$ and $\mathcal{F} = \emptyset$), and extended by Handel–Mosher in [HM14] to the context of free products, see also [GH19b, Section 2.2]. The Gromov boundary of FF was described by Bestvina–Reynolds [BR15] and Hamenstädt [Ham09] in the context of free groups, with an extension to free products in [GH19b], as the space of equivalence classes of arational (G, \mathcal{F}) -trees; in fact these descriptions provide a continuous extension of the map π to the set of arational (G, \mathcal{F}) -trees in $\partial\mathcal{O}$.

Algebraic laminations. The notion of algebraic laminations for free groups was introduced and extensively studied by Coulbois, Hilion and Lustig in [CHL08a, CHL08b]; it was generalized to the context of free products in [GH19a].

Given a Grushko tree R , we denote by $\partial_\infty R$ the Gromov boundary of R and by $V_\infty(R)$ the set of vertices of infinite valence in R . We then let $\partial R := \partial_\infty R \cup V_\infty(R)$. The space $R \cup \partial R$ is compact when equipped with the observers' topology. The boundary ∂R is closed in $R \cup \partial R$. Up to natural identifications (see [GH19a, Section 1]), the spaces $\partial_\infty R, V_\infty(R)$ and ∂R do not depend on the choice of a Grushko tree R ; we thus denote them by $\partial_\infty(G, \mathcal{F}), V_\infty(G, \mathcal{F})$ and $\partial(G, \mathcal{F})$. We let $\partial^2(G, \mathcal{F}) := \partial(G, \mathcal{F}) \times \partial(G, \mathcal{F}) \setminus \Delta$, where Δ denotes the diagonal. Given any nonperipheral element $g \in G$, we denote by $(g^{-\infty}, g^{+\infty})_R$ the oriented axis of g in R , which determines a pair $(g^{-\infty}, g^{+\infty}) \in \partial^2(G, \mathcal{F})$. An element $(\alpha, \omega) \in \partial^2(G, \mathcal{F})$ is *simple* if it is the limit of a sequence $(g_n^{-\infty}, g_n^{+\infty})$, where for every $n \in \mathbb{N}$, the element g_n is simple, i.e. contained in a proper free factor of (G, \mathcal{F}) .

We let $i : \partial^2(G, \mathcal{F}) \rightarrow \partial^2(G, \mathcal{F})$ be the flip map, defined by $i(x, y) = (y, x)$. An *algebraic lamination* is a closed G -invariant i -invariant subset of $\partial^2(G, \mathcal{F})$.

Given a tree $T \in \overline{\mathcal{O}}$ and $\varepsilon > 0$, we let $\Lambda_\varepsilon^2(T)$ be the closure in $\partial^2(G, \mathcal{F})$ of the set $\{(g^{-\infty}, g^{+\infty}) \mid \|g\|_T < \varepsilon\}$. The *dual lamination* of T is then defined as

$$\Lambda^2(T) := \bigcap_{\varepsilon > 0} \Lambda_\varepsilon^2(T).$$

The *one-sided dual lamination* $\Lambda^1(T)$ is defined to be the set of all $\xi \in \partial_\infty(G, \mathcal{F})$ such that for every Grushko tree R , every G -equivariant Lipschitz map $f : R \rightarrow T$, and every basepoint $x_0 \in R$, the image $f([x_0, \xi)_R)$ has bounded diameter in T (this property does not actually depend on the choices of R , f and x_0).

The following statement gives a unique duality property for arational trees; it will be of crucial importance in the construction of the barycenter map in Section 8.

Theorem 6.1 ([GH19a]). *Let $T, T' \in \overline{\mathcal{O}}$, with T arational.*

Assume that there exists a simple pair $(\alpha, \omega) \in \partial^2(G, \mathcal{F})$ which belongs to $\Lambda^2(T) \cap \Lambda^2(T')$. Then T' is arational and $T' \sim T$.

7 Amenability of the action on the boundary of the free factor graph

In the present section, we will show that, under a condition on centralizers of \mathcal{F} -peripheral elements, the $\text{Out}(G, \mathcal{F}^{(t)})$ -action on the set of equivalence classes of arational (G, \mathcal{F}) -trees (equivalently, on the boundary of the associated free factor graph) is Borel amenable (Proposition 7.2 below). This will be derived as a consequence of the following theorem, established by Bestvina and the first two named authors in an earlier work – the only remaining task for us being to pass from arational trees to equivalence classes of arational trees.

Theorem 7.1 ([BGH17, Theorem 6.4]). *Let G_1, \dots, G_k be countable groups, let F_N be a finitely generated free group, and let*

$$G := G_1 * \dots * G_k * F_N.$$

Let \mathcal{F} be the collection of all G -conjugacy classes of the groups G_i . Assume that for each $i \in \{1, \dots, k\}$, the centralizer in G_i of every element of G_i is amenable.

Then the $\text{Out}(G, \mathcal{F}^{(t)})$ -action on $\mathbb{P}\mathcal{AT}$ is Borel amenable.

We now aim to deduce the following consequence regarding the amenability of the action on the quotient $\mathbb{P}\mathcal{AT}/\sim$.

Proposition 7.2. *Let G_1, \dots, G_k be countable groups, let F_N be a finitely generated free group, and let*

$$G := G_1 * \dots * G_k * F_N.$$

Let \mathcal{F} be the collection of all G -conjugacy classes of the groups G_i . Assume that for each $i \in \{1, \dots, k\}$, the centralizer in G_i of every element of G_i is amenable.

Then the $\text{Out}(G, \mathcal{F}^{(t)})$ -action on $\mathbb{P}\mathcal{AT}/\sim$ is Borel amenable.

The set of all projective classes of arational trees that are equivalent to a given arational tree T is a finite-dimensional simplex, see [Gui00, Corollary 5.4] or [GH19a, Proposition 13.5]. Trees in \mathcal{AT} that project in $\mathbb{P}\mathcal{AT}$ to an extremal point of the simplex of an arational tree are called *ergometric*.

Lemma 7.3. *Let $T \in \mathcal{AT}$. Then T is ergometric if and only if for every tree $T' \in \overline{\mathcal{O}}$ that is not homothetic to T , there is no Lipschitz G -equivariant map from T to T' .*

Proof. Since $T \in \mathcal{AT}$, it has dense orbits, so every Lipschitz G -equivariant map $T \rightarrow T'$ is alignment-preserving, see e.g. [BGH17, Lemma 2.2]. It then follows from [GH19a, Corollary 13.4] that T' is equivalent to T whenever such a map exists. The characterization of ergometric trees inside the (cone on a) simplex of T then follows from [Gui00, Section 5.1]. \square

We denote by $\mathbb{P}\mathcal{AT}^{\text{erg}}$ the subset of $\mathbb{P}\mathcal{AT}$ made of all projective classes of ergometric arational trees.

Lemma 7.4. *The set $\mathbb{P}\mathcal{AT}^{\text{erg}}$ is a Borel subset of $\mathbb{P}\mathcal{AT}$.*

Proof. By choosing a continuous section $\mathbb{P}\overline{\mathcal{O}} \rightarrow \overline{\mathcal{O}}$, we identify $\mathbb{P}\overline{\mathcal{O}}$ with a Borel subset of $\overline{\mathcal{O}}$. We let $\mathbb{P}\overline{\mathcal{O}}^{(2)} := (\mathbb{P}\overline{\mathcal{O}} \times \mathbb{P}\overline{\mathcal{O}}) \setminus \Delta$, where Δ denotes the diagonal.

Since $\mathbb{P}\overline{\mathcal{O}}$ is a compact metrizable space, there exists an increasing sequence of compact subsets K_n of $\mathbb{P}\overline{\mathcal{O}}^{(2)}$ that exhausts $\mathbb{P}\overline{\mathcal{O}}^{(2)}$.

For every $n \in \mathbb{N}$, let Lip_n be the set of all pairs $(T, T') \in K_n$ such that there exists an n -Lipschitz G -equivariant map from T to T' (this makes sense using the above identification of $\mathbb{P}\overline{\mathcal{O}}$ as a subset of $\overline{\mathcal{O}}$). The set Lip_n is a closed subset of K_n (see e.g. [Gui00, Proposition 5.5]), and therefore it is compact. We denote by $p_1(\text{Lip}_n)$ the first projection of Lip_n , which is a closed (whence Borel) subset of $\mathbb{P}\overline{\mathcal{O}}$.

Notice that the Borel set $\mathbb{P}\overline{\mathcal{O}} \setminus \bigcup_{n \in \mathbb{N}} p_1(\text{Lip}_n)$ is equal to the set of all trees $T \in \mathbb{P}\overline{\mathcal{O}}$ such that for every $T' \in \mathbb{P}\overline{\mathcal{O}}$ distinct from T , there is no Lipschitz G -equivariant map from T to T' . By Lemma 7.3, the set $\mathbb{P}\mathcal{AT}^{\text{erg}}$ is equal to the intersection of this set with $\mathbb{P}\mathcal{AT}$, so it is Borel. \square

Lemma 7.5. *There exists a sequence of Borel maps $f_n : \mathbb{P}\mathcal{AT} \rightarrow \mathbb{P}\mathcal{AT}^{\text{erg}}$ such that for every $T \in \mathbb{P}\mathcal{AT}$, the set $\{f_n(T)\}_{n \in \mathbb{N}}$ is equal to the (finite) set of all projective classes of ergometric trees that are equivalent to T .*

Proof. We identify $\mathbb{P}\overline{\mathcal{O}}$ to a Borel subset of $\overline{\mathcal{O}}$ by choosing a continuous section. Notice that the subset $\text{Comp} \subseteq \mathbb{P}\overline{\mathcal{O}} \times \mathbb{P}\overline{\mathcal{O}}$ made of pairs of trees that are compatible is closed, whence Borel: indeed, two trees are compatible if and only if the sum of their length functions is again the length function of a G -action on an \mathbb{R} -tree [GL17, Theorem A.10], and by [CM87] the space of length functions of G -actions on \mathbb{R} -trees is closed in \mathbb{R}^G . Using Lemma 7.4, we deduce that $\text{Comp} \cap (\mathbb{P}\mathcal{AT} \times \mathbb{P}\mathcal{AT}^{\text{erg}})$ is a Borel subset of $\mathbb{P}\overline{\mathcal{O}}^2$, and it is made of all pairs $(T, T') \in \mathbb{P}\mathcal{AT} \times \mathbb{P}\mathcal{AT}^{\text{erg}}$ with $T' \sim T$. The conclusion then follows by using a measurable selection theorem [Sri81]. More precisely, letting $C := \{0, 1\}^{\mathbb{N}}$ be the Cantor space, it follows from [Sri81, Theorem 1.2] that there exists

a map $f : \mathbb{P}\mathcal{AT} \times (\mathbb{N} \times C) \rightarrow \mathbb{P}\mathcal{AT}^{\text{erg}}$ such that for every $T \in \mathbb{P}\mathcal{AT}$, the map $f(T, \cdot)$ is continuous and its image equals the set of all ergometric trees that are equivalent to T , and for every $(n, c) \in \mathbb{N} \times C$, the map $f_{n,c} := f(\cdot, (n, c))$ is Borel. Let $(n_i, c_i)_{i \in \mathbb{N}}$ be a dense sequence in $\mathbb{N} \times C$. Then the maps f_{n_i, c_i} satisfy the required conclusion. \square

Proof of Proposition 7.2. By Theorem 7.1, there exists a sequence of Borel maps

$$\nu_n : \mathbb{P}\mathcal{AT} \rightarrow \text{Prob}(\text{Out}(G, \mathcal{F}^{(t)}))$$

which is asymptotically $\text{Out}(G, \mathcal{F}^{(t)})$ -equivariant. We will explain how to modify the maps ν_n so that they become constant on every \sim -class.

Let $f_n : \mathbb{P}\mathcal{AT} \rightarrow \mathbb{P}\mathcal{AT}^{\text{erg}}$ be a sequence of Borel maps provided by Lemma 7.5. For all $T \in \mathbb{P}\mathcal{AT}$, we let $k(T)$ be the number of ergometric trees that are \sim -equivalent to T , and we let $n_1(T), \dots, n_{k(T)}(T)$ be integers such that the trees $f_{n_i(T)}(T)$ are pairwise distinct, and $\{f_{n_i(T)}(T)\}$ is the set of all ergometric trees that are \sim -equivalent to T , with $(n_1(T), \dots, n_{k(T)}(T))$ minimal for the lexicographic order among tuples satisfying the above properties. Note that the maps $T \mapsto k(T)$ and $T \mapsto n_i(T)$ are Borel. We then define a sequence of Borel maps

$$\tilde{\nu}_n : \mathbb{P}\mathcal{AT} \rightarrow \text{Prob}(\text{Out}(G, \mathcal{F}^{(t)}))$$

by letting

$$\tilde{\nu}_n(T) := \frac{1}{k(T)} \sum_{i=1}^{k(T)} \nu_n(f_{n_i(T)}(T)).$$

Then the sequence $\tilde{\nu}_n$ is asymptotically $\text{Out}(G, \mathcal{F}^{(t)})$ -equivariant, and the maps $\tilde{\nu}_n$ are constant on every \sim -class. Therefore the $\tilde{\nu}_n$ induce a sequence of Borel maps $\mathbb{P}\mathcal{AT}/\sim \rightarrow \text{Prob}(\text{Out}(G, \mathcal{F}^{(t)}))$ which is asymptotically $\text{Out}(G, \mathcal{F}^{(t)})$ -equivariant. This completes our proof. \square

8 A barycenter map for $\text{Out}(G, \mathcal{F})$

Let G_1, \dots, G_k be countable groups, let F_N be a free group of rank N , and let $G := G_1 * \dots * G_k * F_N$. We denote by \mathcal{F} the finite collection made of the conjugacy classes in G of the subgroups G_i . We assume throughout the section that (G, \mathcal{F}) is not sporadic. We recall that FS denotes the countable set of all free splittings of (G, \mathcal{F}) , and that $\mathcal{P}_{< \infty}(\text{FS})$ denotes the (countable) set of finite subsets of FS. The goal of the present section is to prove the following theorem.

Theorem 8.1. *There exists a Borel $\text{Out}(G, \mathcal{F})$ -equivariant map*

$$\text{bar} : (\mathbb{P}\mathcal{AT}/\sim)^{(3)} \rightarrow \mathcal{P}_{< \infty}(\text{FS}).$$

Remark 8.2. The main point in the statement is the finiteness of the subset of FS associated to every triple of pairwise distinct equivalence classes of arational trees. By construction, one can check that there is a uniform bound of the diameter of this subset in the free factor graph FF.

8.1 Unique duality for arational trees

Throughout the section, we fix a Grushko tree R , and given $g \in G$, we write $|g| := \|g\|_R$. The goal of the present section is to strengthen the following unique duality statement for arational trees obtained in [CHR15, GH19a] (it essentially follows from [GH19a, Proposition 4.26 and Theorem 13.1]): our stronger statement is Lemma 8.5 below.

Recall that an element $g \in G$ is *simple* if it is contained in some proper relative free factor of G (equivalently, g is elliptic in some free splitting of (G, \mathcal{F})).

Lemma 8.3 ([GH19a, Proposition 4.26 and Theorem 13.1]). *Let $T, T' \in \overline{\mathcal{O}}$, with T arational. Let $(T_n)_{n \in \mathbb{N}}, (T'_n)_{n \in \mathbb{N}} \in \mathcal{O}^{\mathbb{N}}$ be sequences that converge (non-projectively) to T, T' respectively.*

Assume that there exists a sequence of nonperipheral simple elements $(g_n)_{n \in \mathbb{N}} \in G^{\mathbb{N}}$ such that $\|g_n\|_{T_n} \rightarrow 0$ and $\|g_n\|_{T'_n} \rightarrow 0$.

Then T' is arational and $T' \sim T$.

We recall that given a G -equivariant map $f : R \rightarrow T$ between two (G, \mathcal{F}) -trees, the *bounded backtracking constant* $\text{BBT}(f)$ is defined as the smallest real number K such that for all $x, y, z \in R$ aligned in this order, we have $d_T(f(y), [f(x), f(z)]) \leq K$. Notice in particular that if g is a nonperipheral element, then for every point x that belongs to the axis of g in R , we have $d_T(f(x), \text{Char}_T(g)) \leq K$ (where $\text{Char}_T(g)$ denotes the characteristic set of g in T , i.e. either its axis if g acts hyperbolically on T , or else its fix point set).

Lemma 8.4. *Let $(S_n)_{n \in \mathbb{N}} \in \mathcal{O}^{\mathbb{N}}$, let $(g_n)_{n \in \mathbb{N}} \in G^{\mathbb{N}}$ be a sequence of nonperipheral elements, and let $B > 0$. Assume that*

1. *for every $n \in \mathbb{N}$, there exists a G -equivariant map $f_n : R \rightarrow S_n$ with $\text{BBT}(f_n) \leq B$, and*

2. *we have*

$$\frac{\|g_n\|_{S_n}}{|g_n|} \rightarrow 0.$$

Then up to passing to a subsequence, there exist segments $I_n \subseteq (g_n^{-\infty}, g_n^{+\infty})_R$, with $|I_n| \rightarrow +\infty$, such that $\text{diam}_{S_n}(f_n(I_n)) \leq 5B$.

Proof. For every $n \in \mathbb{N}$, one has $\text{Axis}_{S_n}(g_n) \subseteq f_n(\text{Axis}_R(g_n))$. We thus choose $a_n \in \text{Axis}_R(g_n)$ with $f_n(a_n) \in \text{Axis}_{S_n}(g_n)$.

First assume that for all $n \geq 0$, one has $\|g_n\|_{S_n} \leq B$. Let $m_n = \lfloor \frac{B}{\|g_n\|_{S_n}} \rfloor$, let $b_n = g_n^{m_n} a_n$, and let $I_n = [a_n, b_n] \subseteq R$. Using the second assumption from the lemma, we see that $|I_n| = m_n |g_n|$ tends to $+\infty$. On the other hand $d(f_n(a_n), f_n(b_n)) = m_n \|g_n\|_{S_n} \leq B$. Since $\text{BBT}(f_n) \leq B$, the image $f_n(I_n)$ is contained in the B -neighborhood of $[f_n(a_n), f_n(b_n)]$ so $\text{diam}(f_n(I_n)) \leq 3B$.

Up to taking a subsequence, we can therefore assume that for all $n \geq 0$, one has $\|g_n\|_{S_n} > B$. Consider the fundamental domain $J_n = [a_n, c_n]$ of $\text{Axis}_R(g_n)$ where $c_n = g_n a_n$. The segment $K_n = [f_n(a_n), f_n(c_n)]$ is a fundamental domain of $\text{Axis}_{S_n}(g_n)$. Let

$N = \lceil \frac{\|g_n\|_{S_n}}{B} \rceil$, and subdivide the segment $[a_n, c_n]$ into $t_0 = a_n, t_1, \dots, t_N = c_n$ with $d(t_i, t_{i+1}) = \frac{\|g_n\|}{N}$ for all $i < N$. We claim that we can take I_n to be one of the intervals $[t_i, t_{i+1}]$. Note that their length is at least $B \frac{N-1}{N} \frac{\|g_n\|}{\|g_n\|_{S_n}}$ which tends to infinity with n by assumption.

For $i < N$, let $q_i = f_n(t_i)$ and p_i be the projection of q_i on K_n . Since $\text{BBT}(f_n) \leq B$, one has $d(p_i, q_i) \leq B$. Using again that $\text{BBT}(f_n) \leq B$, we see that $\text{diam}(f_n([t_i, t_{i+1}])) \leq 2B + d(q_i, q_{i+1})$, hence it suffices to find $i < N$ such that $d(q_i, q_{i+1}) \leq 3B$.

We consider the natural ordering on $K_n = [p_0, p_N]$ such that $p_0 < p_N$. First assume that there exists $i < N$ such that $p_{i+1} < p_i$ (i.e. there is some backtracking among the p_i). As $\text{BBT}(f_n) \leq B$, this implies that $B \geq d(q_i, [p_0, q_{i+1}]) = d(q_i, p_{i+1}) \geq d(q_i, q_{i+1}) - B$ and we are done.

We can therefore assume that $p_0 \leq p_1 \leq \dots \leq p_N$. Let $m = \min\{d(p_i, p_{i+1}) | i < N\}$. It suffices to prove that $m \leq B$. If not, then

$$\|g_n\|_{S_n} = d(p_0, p_N) \geq Nm > \lceil \frac{\|g_n\|_{S_n}}{B} \rceil B \geq \|g_n\|_{S_n},$$

a contradiction. □

Lemma 8.5. *Let $T, T' \in \overline{\mathcal{O}}$, with T arational. Let $(T_n)_{n \in \mathbb{N}}, (T'_n)_{n \in \mathbb{N}} \in \mathcal{O}^{\mathbb{N}}$ be sequences that converge (non-projectively) to T, T' respectively.*

Assume that there exists a sequence of nonperipheral simple elements $(g_n)_{n \in \mathbb{N}} \in G^{\mathbb{N}}$ such that

$$\|g_n\|_{T_n} \rightarrow 0 \quad \text{and} \quad \frac{\|g_n\|_{T'_n}}{\|g_n\|} \rightarrow 0.$$

Then T' is arational and $T' \sim T$.

Proof. We denote by C the covolume of our fixed Grushko tree R , i.e. the sum of the lengths of all edges of the quotient graph R/G . Let $f' : R \rightarrow T'$ be a G -equivariant Lipschitz map. Since $(T'_n)_{n \in \mathbb{N}}$ converges non-projectively to T' , we can find G -equivariant Lipschitz maps $f'_n : R \rightarrow T'_n$ converging to f' in the Gromov–Hausdorff equivariant topology on the set of morphisms introduced in [GL07]. In particular, there exists $K \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, the map f'_n is K -Lipschitz. By [BFH97, Lemma 4.1] (see [Hor16, Proposition 3.12] in the context of free products), the map f'_n has BBT at most KC .

We claim that up to passing to a subsequence, there exists a sequence $(h_n)_{n \in \mathbb{N}} \in G^{\mathbb{N}}$ such that $h_n \cdot (g_n^{-\infty}, g_n^{+\infty})$ converges to $(\alpha, \omega) \in \partial^2(G, \mathcal{F})$, with $(\alpha, \omega) \in (\Lambda^1(T') \cup V_\infty(G, \mathcal{F}))^2$.

We now prove our claim. Applying Lemma 8.4 to the trees T'_n and the maps f'_n whose bounded backtracking constant is bounded from above by KC , we obtain subsegments $I_n \subseteq (g_n^{-\infty}, g_n^{+\infty})_R$ such that $|I_n| \rightarrow +\infty$ and $\text{diam}_{T'_n}(f'_n(I_n)) \leq 5KC$. Let $e_n = [u_n, v_n]$ be an edge in I_n such that both connected components of $I_n \setminus e_n$ have length diverging to $+\infty$ as n goes to $+\infty$. Since the G -action on R is cocompact, up to passing to a subsequence, we can assume that all edges e_n lie in the G -orbit of a common edge

$e = [u, v] \subseteq R$, and choose $h_n \in G$ such that $h_n \cdot e_n = e$. By compactness of $\partial(G, \mathcal{F})$, up to passing to a further subsequence, we can assume that $h_n \cdot (g_n^{-\infty}, g_n^{+\infty})$ converges to (α, ω) , and $\alpha \neq \omega$ because they are separated by the edge e . We are left proving that $\alpha, \omega \in \Lambda^1(T') \cup V_\infty(G, \mathcal{F})$. We prove it for α , by symmetry the result also holds for ω . Assume that $\alpha \notin V_\infty(G, \mathcal{F})$; we aim to show that $f'([u, \alpha]_R)$ has bounded diameter in T' . Let $[u, u']_R$ be an initial segment of $[u, \alpha]_R$. Then there exists $n \in \mathbb{N}$ such that $[u, u']_R \subseteq h_n I_n$. Therefore $f'_n([u, u']_R)$ has diameter at most $5KC$. Since f'_n converges to f' , this implies that the diameter of $f'([u, u']_R)$ is uniformly bounded (with a bound not depending on the point u'). Therefore $f'([u, \alpha]_R)$ is bounded, which concludes the proof of our claim.

We now finish the proof of the lemma. Since $\|g_n\|_{T_n}$ converges to 0 and T_n converges to the tree T which is arational (and therefore has dense F_N -orbits), it follows from [GH19a, Proposition 4.26] that $(\alpha, \omega) \in \Lambda^2(T)$. Since T is arational, we have $(\alpha, \omega) \notin V_\infty(G, \mathcal{F})^2$, as otherwise the subgroup of G generated by the stabilizers of α and ω would fix a point in T , while point stabilizers in arational trees are either peripheral or cyclic (see [Hor14, Section 4]). So either α or ω (say α) belongs to $\Lambda^1(T) \cap \Lambda^1(T')$. By [GH19a, Proposition 4.19], we thus have $\Lambda^2(\alpha) \subseteq \Lambda^2(T) \cap \Lambda^2(T')$, where $\Lambda^2(\alpha)$ is the *limit set* defined in [GH19a, Definition 4.9]. This limit set is nonempty [GH19a, Lemma 4.1], and it consists entirely of simple pairs [GH19a, Lemma 5.6]. We thus have found a simple pair in $\Lambda^2(T) \cap \Lambda^2(T')$. Theorem 6.1 thus implies that T' is arational, and $T' \sim T$. \square

8.2 Pencils between two arational trees

We recall that given two trees $T, T' \in \overline{\mathcal{O}}$, a map $f : T \rightarrow T'$ is a *morphism* if it is F_N -equivariant and every segment $I \subseteq T$ can be subdivided into finitely many subsegments I_1, \dots, I_k such that for every $i \in \{1, \dots, k\}$, the restriction $f|_{I_i}$ is an isometry onto its image. A morphism $f : T \rightarrow T'$ is *optimal* if every point $x \in T$ is contained in an open segment I such that $f|_I$ is an isometry onto its image.

The following lemma generalizes a theorem of Bestvina–Reynolds [BR15, Theorem 6.6] to the context of free products.

Lemma 8.6. *Let $S, T \in \mathcal{AT}$. Let $(S_n)_{n \in \mathbb{N}} \in \mathcal{O}^{\mathbb{N}}$ be a sequence which converges projectively to S , and let $(T_n)_{n \in \mathbb{N}} \in \mathcal{O}^{\mathbb{N}}$ be a sequence that converges non-projectively to T .*

Assume that for every $n \in \mathbb{N}$, there exists an optimal morphism $f_n : S_n \rightarrow T_n$. Let $U \in \overline{\mathcal{O}}$, and assume that there exists a sequence $(U_n)_{n \in \mathbb{N}} \in \mathcal{O}^{\mathbb{N}}$ converging projectively to U , such that for every $n \in \mathbb{N}$, the morphism f_n factors through optimal morphisms $S_n \rightarrow U_n$ and $U_n \rightarrow T_n$.

Then either U is equivalent to S , or U is equivalent to T , or U belongs to \mathcal{O} (as opposed to $\partial\mathcal{O}$). If $U \in \mathcal{O}$, then $(U_n)_{n \in \mathbb{N}}$ accumulates non-projectively to λU for some $\lambda \in \mathbb{R}_+^$.*

Proof. Up to passing to a subsequence, we can find sequences $(|S_n|), (|U_n|) \in \mathbb{R}^{\mathbb{N}}$ such that $\tilde{S}_n := S_n/|S_n|$ converges to S , and $\tilde{U}_n := U_n/|U_n|$ converges to U . Also $(T_n)_{n \in \mathbb{N}}$ converges to T by assumption, without renormalizing.

For all $n \in \mathbb{N}$, there are morphisms $S_n \rightarrow U_n \rightarrow T_n$. If $|U_n|$ is bounded, then there is a G -equivariant Lipschitz map from U to T . By [BGH17, Lemma 2.3], either U belongs to \mathcal{O} (in which case we are done) or else U is a tree with dense G -orbits. In the latter case, it follows from [BGH17, Lemma 2.2] that there exists a G -equivariant alignment-preserving map from U to T , and from [GH19a, Corollary 13.4] we deduce that U is equivalent to T .

We can therefore assume that $|U_n|$ diverges to $+\infty$. We aim to prove that U is arational and equivalent to S ; the lemma will follow from this fact.

Since \tilde{S}_n converges to S which has dense orbits, the covolume $\text{covol}(\tilde{S}_n)$ of \tilde{S}_n tends to 0. For every $n \in \mathbb{N}$, as f_n is optimal, we can find a nonperipheral element $g_n \in G$, whose axis is legal (i.e. f_n is an isometry when restricted to the axis of g_n) and with $\|g_n\|_{\tilde{S}_n} \leq 2\text{covol}(\tilde{S}_n)$, so $\|g_n\|_{\tilde{S}_n}$ tends to 0. Since g_n is legal, $\|g_n\|_{S_n} = \|g_n\|_{U_n} = \|g_n\|_{T_n}$.

Since T_n converges non-projectively to T , there exists a constant C such that for every $n \in \mathbb{N}$, the Lipschitz constant of any optimal map from R to T_n is at most C . It follows that $\|g_n\|_{T_n} \leq C|g_n|$.

We thus deduce that

$$\frac{\|g_n\|_{\tilde{U}_n}}{|g_n|} = \frac{\|g_n\|_{U_n}}{|g_n| \cdot |U_n|} = \frac{\|g_n\|_{T_n}}{|g_n| \cdot |U_n|} \leq \frac{C}{|U_n|} \rightarrow 0.$$

Since in addition $\|g_n\|_{\tilde{S}_n} \rightarrow 0$ and \tilde{S}_n converges to S , Lemma 8.5 implies that U is arational and equivalent to S . \square

8.3 Barycenter of a triple of pairwise inequivalent arational trees

Given two inequivalent arational trees $S, T \in \mathcal{AT}$ and $U \in \mathcal{O}$, we say that the triple (S, U, T) has an *aligned approximation* if there exist

- a sequence $(S_n)_{n \in \mathbb{N}} \in \mathcal{O}^{\mathbb{N}}$ converging projectively to a tree $S' \sim S$,
- a sequence $(U_n)_{n \in \mathbb{N}} \in \mathcal{O}^{\mathbb{N}}$ converging non-projectively to U ,
- a sequence $(T_n)_{n \in \mathbb{N}} \in \mathcal{O}^{\mathbb{N}}$ converging non-projectively to T ,

such that for every $n \in \mathbb{N}$, there exists an optimal morphism $S_n \rightarrow T_n$ which factors through optimal morphisms $S_n \rightarrow U_n$ and $U_n \rightarrow T_n$. In this case, we say that $((S_n)_{n \in \mathbb{N}}, (U_n)_{n \in \mathbb{N}}, (T_n)_{n \in \mathbb{N}})$ is an *aligned approximation* of (S, U, T) .

Remark 8.7. If (S, U, T) has an aligned approximation, then for any $\lambda, \lambda' \in \mathbb{R}_+^*$, so does $(\lambda'S, \lambda U, \lambda'T)$.

Let now $L \geq 1$ and $C \geq 0$ be such that the projection to FF of any optimal folding path is an (unparametrized) (L, C) -quasigeodesic, see e.g. [HM14] or [GH19b, Proposition 2.11]. Let δ be an integer such that for every triple (ξ_1, ξ_2, ξ_3) of pairwise distinct points of $\partial_\infty \text{FF}$, there exist neighborhoods V_1, V_2, V_3 of ξ_1, ξ_2, ξ_3 in $\text{FF} \cup \partial_\infty \text{FF}$ with the following property: given any three (L, C) -quasigeodesics $\gamma_1, \gamma_2, \gamma_3$, where γ_i has its

origin in V_i and its extremity in V_{i+1} (with indices taken mod 3), there exists a triple of points (x_1, x_2, x_3) , with x_i lying on the image of γ_i , such that for every $i, j \in \{1, 2, 3\}$, we have $d_{\text{FF}}(x_i, x_j) \leq \delta$.

Let $\pi : \mathcal{O} \rightarrow \text{FF}$ be the natural map consisting in forgetting the metric on the tree (recall that the vertex set of FF consists of non-metric free splittings). Let \mathcal{T} be the set of all triples $(U_1, U_2, U_3) \in \mathcal{O}^3$ such that for all $i, j \in \{1, 2, 3\}$, one has $d_{\text{FF}}(\pi(U_i), \pi(U_j)) \leq \delta$.

Given a triple $(T_1, T_2, T_3) \in \mathcal{AT}^3$ of pairwise inequivalent arational trees, we next define $B(T_1, T_2, T_3)$ as the set of all trees $U_{12} \in \mathcal{O}$ such that there exist $U_{23}, U_{31} \in \mathcal{O}$, and for every $i \in \{1, 2, 3\}$ (taken modulo 3), an aligned approximation $(S_{i,i+1}^n, U_{i,i+1}^n, T_{i,i+1}^n)$ of $(T_i, U_{i,i+1}, T_{i+1})$, such that for every $n \in \mathbb{N}$, we have $(U_{12}^n, U_{23}^n, U_{31}^n) \in \mathcal{T}$.

We note that for all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_+^*$, one has $B(\lambda_1 T_1, \lambda_2 T_2, \lambda_3 T_3) = \lambda_2 B(T_1, T_2, T_3)$.

Lemma 8.8. *For every triple $(T_1, T_2, T_3) \in \mathcal{AT}^3$ of pairwise inequivalent arational trees, the set $B(T_1, T_2, T_3)$ is a non-empty compact subset of \mathcal{O} .*

Proof. We first prove that $B(T_1, T_2, T_3)$ is non-empty. For every $i \in \{1, 2, 3\}$, let $(T_i^n)_{n \in \mathbb{N}} \in \mathcal{O}^{\mathbb{N}}$ be a sequence that converges nonprojectively to T_i . For every $i \in \{1, 2, 3\}$, we let $T_{i,i+1}^n := T_{i+1}^n$ and we let $S_{i,i+1}^n$ be a tree obtained from T_i^n by changing the edge lengths and admitting an optimal morphism towards T_{i+1}^n . Then up to taking a subsequence, $(S_{i,i+1}^n)_{n \in \mathbb{N}}$ converges projectively to a tree T_i' which is arational and equivalent to T_i . Let ξ_1, ξ_2, ξ_3 be the points in $\partial_{\infty} \text{FF}$ corresponding to the arational trees T_1, T_2, T_3 . Let V_1, V_2, V_3 be neighborhoods of ξ_1, ξ_2, ξ_3 in $\text{FF} \cup \partial_{\infty} \text{FF}$ coming from the definition of δ . By [GH19b, Theorem 3], for all sufficiently large $n \in \mathbb{N}$ and all $i \in \{1, 2, 3\}$, the projection $\pi(S_{i,i+1}^n)$ belongs to V_i , and $\pi(T_{i,i+1}^n)$ belongs to V_{i+1} . For every such n and every $i \in \{1, 2, 3\}$, we then consider an optimal folding path from $S_{i,i+1}^n$ to $T_{i,i+1}^n$. Since optimal folding paths project to unparametrized (L, C) -quasigeodesics in FF, it follows from the definition of δ that we can find a triple $(U_{12}^n, U_{23}^n, U_{31}^n) \in \mathcal{O}^3$, with $U_{i,i+1}^n$ on the folding path from $S_{i,i+1}^n$ to $T_{i,i+1}^n$, such that $(U_{12}^n, U_{23}^n, U_{31}^n) \in \mathcal{T}$. As $\mathbb{P}\overline{\mathcal{O}}$ is compact, up to passing to a subsequence, we can assume that the trees U_{12}^n, U_{23}^n and U_{31}^n converge projectively to trees $U_{12}^{\infty}, U_{23}^{\infty}, U_{31}^{\infty} \in \mathbb{P}\overline{\mathcal{O}}$. Since the π -images of these trees lie in a bounded region of FF, it follows from [GH19b, Theorem 3] that the limits $U_{i,i+1}^{\infty}$ are all non-arational, and in particular not equivalent to T_1, T_2, T_3 . It then follows from Lemma 8.6 that all trees $U_{i,i+1}^{\infty}$ belong to $\mathbb{P}\mathcal{O}$ (as opposed to belonging to the boundary), and they have representatives $U_{12}, U_{23}, U_{31} \in \mathcal{O}$ such that for every $i \in \{1, 2, 3\}$, the sequence $(U_{i,i+1}^n)_{n \in \mathbb{N}}$ accumulates nonprojectively to $U_{i,i+1}$. Therefore $U_{12} \in B(T_1, T_2, T_3)$ and $B(T_1, T_2, T_3) \neq \emptyset$.

We now prove that $B(T_1, T_2, T_3)$ is compact. Let $(U(n))_{n \in \mathbb{N}}$ be a sequence of points in $B(T_1, T_2, T_3)$. Since $\mathbb{P}\overline{\mathcal{O}}$ is compact, up to passing to a subsequence, we can assume that $(U(n))_{n \in \mathbb{N}}$ converges projectively to a tree $U(\infty)$. From the definition of $B(T_1, T_2, T_3)$, a diagonal argument together with Lemma 8.6 shows that either $U(n)$ accumulates nonprojectively to a tree homothetic to $U(\infty)$ that lies in $B(T_1, T_2, T_3)$ or else $U(\infty)$ is equivalent to either T_1 or T_2 . But all the approximations of $U(n)$ used in the definition project to a common bounded subset of FF, so $U(\infty)$ is not an arational tree. Therefore

$U(n)$ accumulates nonprojectively to a tree in $B(T_1, T_2, T_3)$, showing that $B(T_1, T_2, T_3)$ is a compact subset of \mathcal{O} . \square

We have thus associated a compact set in an equivariant way to any triple of pairwise inequivalent arational trees. We now explain how to associate a canonical finite set of simplices of $\mathbb{P}\mathcal{O}$ from this compact subset of \mathcal{O} – this is not obvious as Outer space is not locally compact in general.

Given a compact subset K of \mathcal{O} , we let $\text{covol}_{\max}(K)$ be the maximal covolume of a tree in K . As the covolume varies upper semicontinuously with the tree, the set K_{\max} of all trees in K whose covolume equals $\text{covol}_{\max}(K)$ is nonempty and compact. We then let $K_{\max, \min}$ be the subset of K_{\max} made of all trees having the minimal possible number of G -orbits of edges.

Lemma 8.9. *Let K be a compact subset of \mathcal{O} . Then $\pi(K_{\max, \min})$ is finite.*

In other words, after forgetting the metric on the trees, there are only finitely many underlying simplicial trees in $K_{\max, \min}$.

Proof. Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of trees in $K_{\max, \min}$; we aim to show that the underlying simplicial trees of the trees U_n take only finitely many values. Since K is compact, up to passing to a subsequence, we can assume that $(U_n)_{n \in \mathbb{N}}$ converges to a tree $U_\infty \in K$. Since the covolume is upper semicontinuous, the tree U_∞ has maximal covolume (equal to the covolume of the trees U_n). We will now prove that for all sufficiently large $n \in \mathbb{N}$, all trees U_n and U_∞ have the same underlying simplicial tree. To prove this fact, let $\varepsilon > 0$ be very small compared to the shortest edge of U_∞ . There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exists a $(1 + \varepsilon)$ -Lipschitz map from U_∞ to U_n : this is proved by comparing the lengths of a finite set of candidate loops in U_∞ and in U_n and applying [Hor16, Theorem 3.7]. This map folds several edges together, and we can assume that ε has been chosen small enough so that every edge is only partially folded by this map. We observe that there exists a vertex in U_n such that all edges based at that vertex but one are folded together: otherwise, folding would create at least one new orbit of edges, which would imply that U_n has strictly more orbits of edges than U_∞ , a contradiction. This implies that U_n and U_∞ belong to the same standard cone, as claimed. \square

Since $B(\lambda_1 T_1, \lambda_2 T_2, \lambda_3 T_3) = \lambda_2 B(T_1, T_2, T_3)$ for all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_+^*$, the finite set $\pi(B(T_1, T_2, T_3)_{\max, \min})$ only depends on the projective classes of T_1 , T_2 , and T_3 . The map $(T_1, T_2, T_3) \mapsto \pi(B(T_1, T_2, T_3)_{\max, \min})$ thus induces an $\text{Out}(G, \mathcal{F})$ -equivariant map assigning to every triple of pairwise inequivalent projective arational trees, a finite set of free splittings of (G, \mathcal{F}) .

In the next section, we will prove that this map is Borel.

8.4 Measurability considerations

We denote by $\mathcal{AT}^{(3)}$ the space of triples of pairwise inequivalent arational trees, and by $\mathbb{P}\mathcal{AT}^{(3)}$ the space of triples of pairwise inequivalent projective arational trees. The goal of the present section is to prove the following proposition.

Proposition 8.10. *The map*

$$\begin{aligned} \mathcal{AT}^{(3)} &\rightarrow \mathcal{P}_{<\infty}(\text{FS}) \\ (T_1, T_2, T_3) &\mapsto \pi(B(T_1, T_2, T_3)_{\max, \min}) \end{aligned}$$

is Borel, and therefore induces a Borel map $\mathbb{P}\mathcal{AT}^{(3)} \rightarrow \mathcal{P}_{<\infty}(\text{FS})$.

We start with a technical lemma. We denote by $\mathcal{O}_{\mathbb{Q}}$ be the subset of \mathcal{O} made of all trees whose edge lengths are all rational. Aligned approximations were defined at the beginning of Section 8.3.

Lemma 8.11. *If (S, U, T) has an aligned approximation, then it has an aligned approximation $(S'_n, U'_n, T'_n)_{n \in \mathbb{N}}$ with $S'_n, U'_n, T'_n \in \mathcal{O}_{\mathbb{Q}}$.*

Proof. Consider an aligned approximation of (S, U, T) given by $S_n \xrightarrow{f_n} U_n \xrightarrow{g_n} T_n$ with f_n, g_n and $g_n \circ f_n$ optimal morphisms. Up to subdividing U_n and T_n , one may assume that f_n and g_n map vertices to vertices. Now let $T'_n \in \mathcal{O}_{\mathbb{Q}}$ be obtained by approximating the edge lengths of T_n by rational numbers. Consider U'_n obtained from U_n by changing the metric on the edges so that the map $g_n : U'_n \rightarrow T'_n$ is isometric on edges. Construct S'_n similarly so that $f_n : S'_n \rightarrow U'_n$ is isometric on edges. Then S'_n, U'_n, T'_n are in $\mathcal{O}_{\mathbb{Q}}$. Since one can choose the rational approximations so that S'_n, U'_n , and T'_n converge to the same limits as S_n, U_n , and T_n , the lemma follows. \square

Proof of Proposition 8.10. Given a subset $E \subseteq \mathcal{O}$, let B_E be the set of all triples (T_1, T_2, T_3) of pairwise inequivalent arational trees such that $B(T_1, T_2, T_3)$ intersects E . We will first prove that if E is compact, then B_E is a Borel subset of $\mathcal{AT}^{(3)}$.

Let $d_{\mathcal{O}}, d_{\overline{\mathcal{O}}}, d_{\mathbb{P}\overline{\mathcal{O}}}$ be metrics that induce the Gromov–Hausdorff equivariant topology on the corresponding spaces. Given $\varepsilon > 0$, we define $\tilde{B}_{E, \varepsilon}$ to be the subset of $\mathcal{AT}^{(3)} \times \mathcal{O}_{\mathbb{Q}}^9$ made of all tuples

$$(T_1, T_2, T_3, S_{12}, S_{23}, S_{31}, U_{12}, U_{23}, U_{31}, T_{12}, T_{23}, T_{31})$$

such that for all $i, j \in \{1, 2, 3\}$, denoting by $[T_i]$ the set of all projective classes of arational trees that are equivalent to T_i , the following hold true:

- $d_{\mathbb{P}\overline{\mathcal{O}}}(S_{i, i+1}, [T_i]) < \varepsilon$,
- $d_{\overline{\mathcal{O}}}(T_{i, i+1}, T_{i+1}) < \varepsilon$,
- there exists an optimal morphism $S_{i, i+1} \rightarrow T_{i, i+1}$ which factors through optimal morphisms $S_{i, i+1} \rightarrow U_{i, i+1}$ and $U_{i, i+1} \rightarrow T_{i, i+1}$,
- $(U_{12}, U_{23}, U_{13}) \in \mathcal{T}$,
- $d_{\mathcal{O}}(U_{12}, E) < \varepsilon$.

We then let $\pi : \mathcal{AT}^{(3)} \times \mathcal{O}_{\mathbb{Q}}^9 \rightarrow \mathcal{AT}^{(3)}$ be the projection to the first three coordinates. Since $\mathcal{O}_{\mathbb{Q}}$ is countable, the π -image of any Borel subset is again Borel. Since, in view of Lemma 8.11, we have

$$B_E = \bigcap_{n>0} \pi(\tilde{B}_{E, \frac{1}{n}}),$$

we deduce that B_E is a Borel subset of \mathcal{O} . For all integers i, q with $q \neq 0$, define $E^{(i,q)}$ as the subset of E consisting of trees of covolume in the interval $[\frac{i}{q}, \frac{i+1}{q}]$. Let

$$\text{covol}_{\max}^E(T_1, T_2, T_3) = \sup_{q>0} \sup_{i \geq 0} \frac{i}{q} \mathbb{1}_{B_{E^{(i,q)}}}(T_1, T_2, T_3)$$

be the maximal covolume of a tree in $B(T_1, T_2, T_3) \cap E$. Write \mathcal{O} as a countable union of compact sets E_n . Then

$$\text{covol}_{\max}^E(T_1, T_2, T_3) = \sup_n \text{covol}_{\max}^{E_n}(T_1, T_2, T_3)$$

is the maximal covolume of all trees in $B(T_1, T_2, T_3)$. Thus $\text{covol}_{\max}^E(T_1, T_2, T_3)$ and $\text{covol}_{\max}(T_1, T_2, T_3)$ are two Borel functions of (T_1, T_2, T_3) .

The Borel set

$$B_{\max}^E = \{(T_1, T_2, T_3) \in B_E \mid \text{covol}_{\max}^E(T_1, T_2, T_3) = \text{covol}_{\max}(T_1, T_2, T_3)\}$$

is the set of all triples (T_1, T_2, T_3) such that $B(T_1, T_2, T_3) \cap E$ contains a tree of maximal covolume in $B(T_1, T_2, T_3)$.

Given a standard open cone σ of \mathcal{O} , write it as a countable union of compact sets $E_{\sigma, n}$. The Borel set

$$B_{\max}^{\sigma} = \cup_n B_{\max}^{E_{\sigma, n}}$$

is the set of all triples (T_1, T_2, T_3) such that $B(T_1, T_2, T_3) \cap \sigma$ contains a tree of maximal covolume in $B(T_1, T_2, T_3)$.

Finally, we define $a(\sigma)$ to be the number of orbits of edges of any tree in σ . Then σ intersects $B(T_1, T_2, T_3)_{\max, \min}$ if and only if (T_1, T_2, T_3) lies in the set

$$B_{\max, \min}^{\sigma} = B_{\max}^{\sigma} \setminus \bigcup_{a(\sigma') > a(\sigma)} B_{\max}^{\sigma'}.$$

Since this set is Borel, this concludes the proof. \square

8.5 Conclusion

Proof of Theorem 8.1. By Proposition 8.10, the map

$$\beta : (T_1, T_2, T_3) \mapsto \pi(B(T_1, T_2, T_3)_{\max, \min})$$

can be viewed as a Borel $\text{Out}(G, \mathcal{F})$ -equivariant map from $\mathbb{P}\mathcal{AT}^{(3)}$ to $\mathcal{P}_{< \infty}(\text{FS})$. But this map might fail to pass to the quotient under equivalence of arational trees. To

solve this problem, for every projective arational tree $T \in \mathbb{P}\mathcal{AT}$, define $[T]_{\text{erg}}$ as the finite set of all projective classes of ergometric trees that are equivalent to T . For all $(T_1, T_2, T_3) \in \mathbb{P}\mathcal{AT}^{(3)}$, we then define

$$\widetilde{\text{bar}}(T_1, T_2, T_3) = \bigcup_{T'_1 \in [T_1]_{\text{erg}}, T'_2 \in [T_2]_{\text{erg}}, T'_3 \in [T_3]_{\text{erg}}} \beta(T'_1, T'_2, T'_3),$$

which is a finite subset of FS. To prove that this map is Borel, consider the sequence of Borel maps $f_n : \mathbb{P}\mathcal{AT} \rightarrow \mathbb{P}\mathcal{AT}^{\text{erg}}$ given in Lemma 7.5. We then note that

$$\widetilde{\text{bar}}(T_1, T_2, T_3) = \bigcup_{n_1, n_2, n_3} \beta(f_{n_1}(T_1), f_{n_2}(T_2), f_{n_3}(T_3)),$$

showing that $\widetilde{\text{bar}} : \mathbb{P}\mathcal{AT}^{(3)} \rightarrow \mathcal{P}_{<\infty}(\text{FS})$ is Borel. The map $\widetilde{\text{bar}}$ then descends to a Borel $\text{Out}(G, \mathcal{F})$ -equivariant map

$$\text{bar} : (\mathbb{P}\mathcal{AT}/\sim)^{(3)} \rightarrow \mathcal{P}_{<\infty}(\text{FS})$$

as required. \square

9 Cocycle rigidity for outer automorphism groups

In this section, we complete the proofs of our main theorems. All the rigidity theorems are stated with a product of higher rank simple algebraic groups G as the source of the cocycle, but in view of the induction statement (Proposition 3.11), one can also replace G by a lattice in G .

9.1 Cocycle rigidity for mapping class groups

A *surface of finite type* is a (possibly disconnected, possibly non-orientable) surface Σ obtained from a boundaryless compact surface by possibly removing finitely many points and the interior of finitely many disks. Its *extended mapping class group* $\text{Mod}^*(\Sigma)$ is the group of all isotopy classes of homeomorphisms of Σ which restrict to the identity on every boundary component (where isotopies are required to fix the boundary at all times). The main result of the present section is the following theorem, compare with [Ham09, Section 9].

Theorem 9.1. *Let Σ be a surface of finite type. Let G be a product of connected higher rank simple algebraic groups over local fields. Let X be a standard probability space equipped with a measure-preserving ergodic G -action.*

Then every cocycle $G \times X \rightarrow \text{Mod}^(\Sigma)$ is cohomologous to a cocycle that takes its values in a finite subgroup of $\text{Mod}^*(\Sigma)$.*

In order to prove Theorem 9.1, we will apply the strategy developed in Section 4. The *topological complexity* $\xi(\Sigma)$ of a connected, boundaryless, orientable surface Σ of finite type is defined as $\xi(\Sigma) = 3g(\Sigma) + s(\Sigma) - 3$, where $g(\Sigma)$ denotes the genus of Σ and $s(\Sigma)$ is the number of punctures. We first establish the following fact.

Proposition 9.2. *Let Σ be a connected boundaryless orientable surface of finite type with $\xi(\Sigma) > 0$. Then $\text{Mod}^*(\Sigma)$ is geometrically rigid with respect to the countable set of all isotopy classes of essential simple closed curves on Σ .*

Proof. Let $K := \text{PML}$ be the space of projective measured laminations on Σ , which is compact and metrizable (in fact homeomorphic to a finite-dimensional sphere). Let $K_\infty \subseteq \text{PML}$ be the subspace made of minimal filling laminations: this is a Borel subset of PML , because it is also equal to the subset of PML made of all laminations that have positive intersection number with every essential simple closed curve on Σ . Let $K_{\text{bdd}} := K \setminus K_\infty$. Let Δ be the Gromov boundary of the curve graph of Σ (whose hyperbolicity was proved by Masur and Minsky in [MM99]). Being the Gromov boundary of a hyperbolic graph, the space Δ is completely metrizable (see e.g. [Vä05, Proposition 5.31]); using the fact that the vertex set of the curve graph is countable, it is easy to see that Δ is also separable, whence a Polish space. We will denote by \mathbb{D} the countable set of all isotopy classes of essential simple closed curves on Σ .

By a theorem of Klarreich [Kla99], there exists a continuous (whence Borel) $\text{Mod}^*(\Sigma)$ -equivariant map $K_\infty \rightarrow \Delta$. By [KM96, Section 1.1] or [Kid08, Lemma 4.38], there exists a Borel $\text{Mod}^*(\Sigma)$ -equivariant map $K \setminus K_\infty \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$. The existence of a Borel $\text{Mod}^*(\Sigma)$ -equivariant ‘barycenter map’ $\Delta^{(3)} \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$ was proved by Kida in [Kid08, Section 4.1.2]. Finally, universal amenability of the $\text{Mod}^*(\Sigma)$ -action on Δ was established by Hamenstädt in [Ham09, Corollary 2] and independently by Kida in [Kid08, Theorem 3.19]. \square

Proof of Theorem 9.1. As Σ is a surface of finite type, it has finitely many connected components. Therefore $\text{Mod}^*(\Sigma)$ has a finite-index subgroup isomorphic to the direct product of finitely many extended mapping class groups of connected surfaces. Using the fact that cocycle-rigidity, as a property of the target group, is stable under extensions (Proposition 3.10), we can therefore assume that Σ is connected.

By attaching a once-punctured disk on every boundary component of Σ , we get a boundaryless surface $\widehat{\Sigma}$ of finite type, and the inclusion $\Sigma \hookrightarrow \widehat{\Sigma}$ induces a homomorphism $\text{Mod}^*(\Sigma) \rightarrow \text{Mod}^*(\widehat{\Sigma})$ onto a finite-index subgroup of $\text{Mod}^*(\widehat{\Sigma})$, whose kernel is abelian (see [FM12, Proposition 3.19]). Since (G, A) is cocycle-rigid for every abelian group A by Zimmer’s theorem [Zim84, Theorem 9.1.1], and cocycle-rigidity, as a property of the target group, is stable under group extensions and finite-index overgroups, we can therefore assume that Σ is boundaryless.

We first assume that Σ is orientable. The proof is by induction on the topological complexity $\xi(\Sigma)$. If $\xi(\Sigma) \leq 0$, then $\text{Mod}^*(\Sigma)$ is finite if Σ is a sphere with at most 3 punctures, and isomorphic to $\text{GL}(2, \mathbb{Z})$ whence virtually free if Σ is a closed torus. In both cases $(G, \text{Mod}^*(\Sigma))$ is cocycle-rigid.

We now assume that $\xi(\Sigma) > 0$. Let $c : G \times X \rightarrow \text{Mod}^*(\Sigma)$ be a cocycle. By Proposition 9.2, the group $\text{Mod}^*(\Sigma)$ is geometrically rigid with respect to the countable set \mathbb{D} of all isotopy classes of essential simple closed curves on Σ . Theorem 4.17 thus shows that c is cohomologous to a cocycle with values in a subgroup H of $\text{Mod}^*(\Sigma)$ that virtually fixes the isotopy class of simple closed curve c on Σ . Let Σ^c be the (possibly

disconnected) surface obtained by cutting Σ along c .

If Σ^c is connected, then H has a finite index subgroup H^0 (fixing c) for which there is a morphism $H^0 \rightarrow \text{Mod}^*(\Sigma^c)$, whose kernel is at most cyclic (generated by a twist about c). Since $g(\Sigma^c) = g(\Sigma) - 1$ and $s(\Sigma^c) = s(\Sigma) + 2$, we get $\xi(\Sigma^c) < \xi(\Sigma)$, so we know by induction that $(G, \text{Mod}^*(\Sigma^c))$ is cocycle-rigid. Since (G, \mathbb{Z}) is cocycle-rigid [Zim84, Theorem 9.1.1], and cocycle-rigidity is stable under finite-index overgroups and group extensions (Propositions 3.9 and 3.10), it follows that $(G, \text{Mod}^*(\Sigma))$ is cocycle-rigid, which concludes our proof in this case.

If Σ^c is disconnected, denoting by Σ_1 and Σ_2 its connected components, then H has a finite index subgroup H^0 (fixing c without reversing its orientation) for which there is a morphism $H^0 \rightarrow \text{Mod}^*(\Sigma_1) \times \text{Mod}^*(\Sigma_2)$, whose kernel is at most cyclic (generated by a twist about c). The proof then goes by induction as in the previous case using Proposition 3.10 and the fact that $\xi(\Sigma_1), \xi(\Sigma_2) < \xi(\Sigma)$.

We now assume that Σ is non-orientable. If Σ is closed, then by [Fuj02, Lemma 4], denoting by $\hat{\Sigma}$ an orientable double cover of Σ , the group $\text{Mod}^*(\Sigma)$ has a finite-index subgroup that embeds in $\text{Mod}^*(\hat{\Sigma})$. The conclusion then follows from the orientable case. The case where Σ is non-closed can be derived from the case where Σ is closed using the short exact sequence given in [Fuj02, Lemma 6] and Proposition 3.10.¹ \square

9.2 Cocycle rigidity for automorphisms of free products

The main result of the present paper is the following theorem. Given a group G , we denote by $Z(G)$ the center of G .

Theorem 9.3. *Let H_1, \dots, H_p be countable groups, let F_N be a free group of rank N , and let*

$$H := H_1 * \dots * H_p * F_N.$$

Let G be a product of connected higher rank simple algebraic groups over local fields. Assume that

- *for each $i \in \{1, \dots, k\}$, the pair $(G, H_i/Z(H_i))$ is cocycle-rigid, and*
- *for each $i \in \{1, \dots, k\}$, the centralizer in H_i of every nontrivial element is amenable.*

Then $(G, \text{Out}(H, \{H_i\}^{(t)}))$ is cocycle-rigid.

Remark 9.4. If we assume in addition that $(G, \text{Out}(H_i))$ is cocycle-rigid for all $i \in \{1, \dots, k\}$, then we can conclude that $(G, \text{Out}(H, \{H_i\}))$ is cocycle-rigid. This follows from the short exact sequence

$$1 \rightarrow \text{Out}(H, \{H_i\}^{(t)}) \rightarrow \text{Out}(H, \{H_i\}) \rightarrow \prod_{i=1}^k \text{Out}(H_i) \rightarrow 1$$

and the fact that cocycle-rigidity is stable under extensions as a property of the target group (Proposition 3.10).

¹Notice that when Σ is non-closed, the mapping class group $\text{Mod}^*(\Sigma)$ is also a subgroup of $\text{Out}(F_N)$, so the conclusion also follows from Corollary 9.6 in this case.

Remark 9.5. As shown in [BGH17, Theorem 6.4 and Remark 6.5], the second assumption in Theorem 9.3 is exactly the required assumption to ensure that the action of $\text{Out}(H, \{H_i\}^{(t)})$ on the space of all arational $(H, \{H_i\})$ -trees is universally amenable.

An immediate consequence of Theorem 9.3 (applied with $p = 0$) is the following.

Corollary 9.6. *Let $N \geq 2$. Let G be a product of connected higher rank simple algebraic groups over local fields. Let X be a standard probability space equipped with an ergodic measure-preserving G -action.*

Then every cocycle $G \times X \rightarrow \text{Out}(F_N)$ is cohomologous to a cocycle that takes its values in some finite subgroup of $\text{Out}(F_N)$. \square

Our proof of Theorem 9.3 will follow the strategy established in Section 4. We start with the following fact.

Proposition 9.7. *Let H_1, \dots, H_k be countable groups, let F_N be a free group of rank N , and let*

$$H := H_1 * \dots * H_k * F_N.$$

Assume that for each $i \in \{1, \dots, k\}$, the centralizer in H_i of every nontrivial element is amenable.

Then $\text{Out}(H, \{H_i\}^{(t)})$ is geometrically rigid with respect to the countable set \mathbb{D} of all proper free factors of $(H, \{H_i\})$.

Proof. Denote by \mathcal{F} the collection of all H -conjugacy classes of the subgroup H_i . Let $K := \mathbb{P}\overline{\mathcal{O}}(H, \mathcal{F})$ be the compactification of the corresponding Outer space, which is metrizable. Let Δ be the Gromov boundary of the free factor graph of (H, \mathcal{F}) , which is a Polish space.

Let $K_\infty \subseteq K$ be the subset made of all arational (H, \mathcal{F}) -trees: this is a Borel subset of K by [Hor14, Lemma 5.5]. Let $K_{\text{bdd}} := K \setminus K_\infty$. By [GH19b, Theorem 3], there exists a continuous (whence Borel) $\text{Out}(H, \mathcal{F})$ -equivariant map $K_\infty \rightarrow \Delta$. By [Rey12, Hor14], there exists an $\text{Out}(H, \mathcal{F})$ -equivariant Borel map $K_{\text{bdd}} \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$. The existence of an $\text{Out}(H, \mathcal{F})$ -equivariant Borel map $\Delta^{(3)} \rightarrow \mathcal{P}_{<\infty}(\mathbb{D})$ was established in Theorem 8.1. Finally, universal amenability of the $\text{Out}(H, \mathcal{F}^{(t)})$ -action on Δ was established in Proposition 7.2. \square

Proof of Theorem 9.3. We denote by \mathcal{F} the (finite) collection of all H -conjugacy classes of the subgroups H_i . The proof is by induction on the complexity $\xi(H, \mathcal{F}) = (p + N, N)$ (complexities are ordered lexicographically). If $\xi(H, \mathcal{F}) \leq (2, 1)$, then we are in one of the following four cases.

1. If $\xi(H, \mathcal{F}) \leq (1, 0)$, then $H = \{1\}$ or $H = H_1$. In this case $\text{Out}(H, \mathcal{F}^{(t)})$ is trivial, so the conclusion is obvious.

2. If $\xi(H, \mathcal{F}) = (1, 1)$, then $H = \mathbb{Z}$, so $\text{Out}(H, \mathcal{F}^{(t)}) = \mathbb{Z}/2\mathbb{Z}$, and again the conclusion is obvious.

3. If $\xi(H, \mathcal{F}) = (2, 0)$, then $H = H_1 * H_2$, and $\text{Out}(H, \mathcal{F}^{(t)})$ is isomorphic to $H_1/Z(H_1) \times H_2/Z(H_2)$ (see [Lev05]). As $(G, H_i/Z(H_i))$ is cocycle-rigid for every $i \in \{1, 2\}$, we deduce that $(G, \text{Out}(H, \mathcal{F}^{(t)}))$ is cocycle-rigid.

4. If $\xi(H, \mathcal{F}) = (2, 1)$, then $H = H_1 * \mathbb{Z}$ and $\mathcal{F} = \{[H_1]\}$, and $\text{Out}(H, \mathcal{F}^{(t)})$ has an index 2 subgroup which is isomorphic to $(H_1 \times H_1)/Z(H_1)$, where $Z(H_1)$ is diagonally embedded in $H_1 \times H_1$, see [Lev05]. The group $(H_1 \times H_1)/Z(H_1)$ maps onto $H_1/Z(H_1) \times H_1/Z(H_1)$ with abelian kernel. Since for every abelian group A , the pair (G, A) is cocycle-rigid [Zim84, Theorem 9.1.1], we deduce (using stability under extensions and finite index overgroups) that $(G, \text{Out}(H, \mathcal{F}^{(t)}))$ is cocycle-rigid.

We now assume that $\xi(H, \mathcal{F}) > (2, 1)$, i.e. (H, \mathcal{F}) is nonsporadic. Let X be a standard probability space equipped with a measure-preserving G -action, and let $c : G \times X \rightarrow H$ be a cocycle. By Proposition 9.7, the group $\text{Out}(H, \mathcal{F}^{(t)})$ is geometrically rigid with respect to the collection of all conjugacy classes of proper (H, \mathcal{F}) -free factors. Therefore, Theorem 4.17 ensures that c is cohomologous to a cocycle c' that takes its values in a subgroup of $\text{Out}(H, \mathcal{F}^{(t)})$ that virtually fixes a proper (H, \mathcal{F}) -free factor A . Let $\text{Out}(H, \mathcal{F}^{(t)}, A)$ be the subgroup of $\text{Out}(H, \mathcal{F}^{(t)})$ made of all automorphisms that preserve the conjugacy class of A . It is enough to prove that $(G, \text{Out}(H, \mathcal{F}^{(t)}, A))$ is cocycle-rigid.

Recall that $\mathcal{F}|_A$ denotes the free factor system of A induced by \mathcal{F} ; we also denote by $\tilde{\mathcal{F}}$ the smallest (H, \mathcal{F}) -free factor system that contains both the conjugacy classes in \mathcal{F} and the conjugacy class of A . There is a short exact sequence

$$1 \rightarrow \text{Out}(H, \tilde{\mathcal{F}}^{(t)}) \rightarrow \text{Out}(H, \mathcal{F}^{(t)}, A) \rightarrow \text{Out}(A, \mathcal{F}|_A^{(t)}) \rightarrow 1.$$

Since $\xi(A, \mathcal{F}|_A) < \xi(H, \mathcal{F})$ and $\xi(H, \tilde{\mathcal{F}}) < \xi(H, \mathcal{F})$ (see e.g. the proof of [Hor14, Theorem 6.3]), by induction the pairs $(G, \text{Out}(A, \mathcal{F}|_A^{(t)}))$ and $(G, \text{Out}(H, \tilde{\mathcal{F}}^{(t)}))$ are cocycle-rigid. Using stability under extensions, we deduce that the pair $(G, \text{Out}(H, \mathcal{F}^{(t)}, A))$ is cocycle-rigid, which concludes our proof. \square

9.3 Cocycle rigidity for automorphisms of relatively hyperbolic groups

We now give an extension of Theorem 9.3 to the context of torsion-free relatively hyperbolic groups.

Theorem 9.8. *Let H be a torsion-free group which is hyperbolic relative to a finite collection \mathcal{P} of finitely generated subgroups. Let G be a product of connected higher rank simple algebraic groups over local fields. Assume that*

- *for every $P \in \mathcal{P}$, the pair $(G, P/Z(P))$ is cocycle-rigid, and*
- *for every $P \in \mathcal{P}$, the centralizer in P of every nontrivial element is amenable.*

Then $(G, \text{Out}(H, \mathcal{P}^{(t)}))$ is cocycle-rigid.

In particular, by applying Theorem 9.8 to the case where $\mathcal{P} = \emptyset$ (in which case H is hyperbolic), we get the following extension of Corollary 9.6.

Corollary 9.9. *Let G be a product of connected higher rank simple algebraic groups over local fields. Let X be a standard probability space equipped with an ergodic measure-preserving G -action. Let H be a torsion-free hyperbolic group.*

Then every cocycle $G \times X \rightarrow \text{Out}(H)$ is cohomologous to a cocycle that takes its values in some finite subgroup of $\text{Out}(H)$. \square

Our proof of Theorem 9.8 is similar to the proof of [Hae16, Appendix, Corollary 3].

Proof of Theorem 9.8. We first assume that H is freely indecomposable relative to \mathcal{P} . Let S be the canonical elementary JSJ decomposition of H relative to \mathcal{P} (see [GL11, Theorem 4]). Then $\text{Out}(H, \mathcal{P}^{(t)})$ has a finite-index subgroup which surjects onto a finite direct product of mapping class groups of surfaces of finite type, with kernel the group of twists \mathcal{T} of S . By [Hae16, Appendix, Lemma 5], the group \mathcal{T} maps with abelian kernel to a direct product of finitely many groups, each of which is isomorphic to $P/Z(P)$ for some $P \in \mathcal{P}$. Using Theorem 9.1 for mapping class groups and the fact that cocycle-rigidity, as a property of the target group, is stable under finite index overgroups and extensions, we deduce that the pair $(G, \text{Out}(H, \mathcal{P}^{(t)}))$ is cocycle-rigid.

We now consider the general case, and let

$$H = H_1 * \cdots * H_k * F_N$$

be a Grushko decomposition of H relative to \mathcal{P} . Each subgroup H_i is torsion-free, hyperbolic relative to $\mathcal{P}_{|H_i}$, and freely indecomposable relative to $\mathcal{P}_{|H_i}$. Let $\text{Out}^0(H, \mathcal{P}^{(t)})$ be the finite-index subgroup of $\text{Out}(H, \mathcal{P}^{(t)})$ consisting of all automorphisms that preserve the conjugacy class of each subgroup H_i (as opposed to permuting them). There is a morphism $\text{Out}^0(H, \mathcal{P}^{(t)}) \rightarrow \prod_{i=1}^k \text{Out}(H_i, \mathcal{P}_{|H_i}^{(t)})$, whose kernel is equal to $\text{Out}(H, \{H_i\}^{(t)})$. By the previous case, for every $i \in \{1, \dots, k\}$, the pair $(G, \text{Out}(H_i, \mathcal{P}_{|H_i}^{(t)}))$ is cocycle-rigid. We will apply Theorem 9.3 to show that $(G, \text{Out}(H, \{H_i\}^{(t)}))$ is cocycle-rigid, which will conclude our proof.

By assumption on centralizers in parabolic subgroups, the centralizer in the relatively hyperbolic group H of every nontrivial element is amenable. Therefore, in order to apply Theorem 9.3, we only need to check that for every $i \leq k$, the pair $(G, H_i/Z(H_i))$ is cocycle-rigid. This holds by assumption if H_i is equal to one of the subgroups in \mathcal{P} . Otherwise $Z(H_i)$ is trivial, and H_i is hyperbolic relative to $\mathcal{P}_{|H_i}$ so we apply cocycle rigidity for relatively hyperbolic groups (Theorem 5.1), and for that it suffices to check that the pair (G, P) is cocycle rigid for every parabolic group $P \in \mathcal{P}$. Using the stability of cocycle rigidity under extensions (Proposition 3.10), this is indeed the case because $(G, P/Z(P))$ is cocycle-rigid by assumption and $(G, Z(P))$ is cocycle-rigid because $Z(P)$ is abelian. \square

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Vincent Guirardel
 Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France
e-mail:vincent.guirardel@univ-rennes1.fr

Camille Horbez
 CNRS
 Université Paris-Saclay, CNRS, Laboratoire de mathématiques d’Orsay, 91405, Orsay, France
e-mail:camille.horbez@universite-paris-saclay.fr

Jean Lécureux
 Université Paris-Saclay, CNRS, Laboratoire de mathématiques d’Orsay, 91405, Orsay, France
e-mail:jean.lecureux@universite-paris-saclay.fr