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EXISTENCE AND UNIQUENESS OF MILD SOLUTIONS FOR QUASI-LINEAR FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT. In this present paper, we will discuss the existence and uniqueness of a class of mild solutions for quasi-linear fractional integrodifferential equations with impulsive conditions via Hausdorff measures of noncompactness and fixed point theory in Banach space Λ .

1. Introduction. The aim of this paper is to study the existence and uniqueness of a class mild solution of quasi-linear fractional integrodifferential equations with impulsive and nonlocal condition of the form

$$\left\{ \begin{array}{lcl} {}^C\mathcal{D}_{0+}^{\alpha}\theta(t) + \mathcal{A}(t, \theta(t))\theta(t) & = & \Phi(t, \theta(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(t, s, \theta(s)) ds, t \in [0, b], t \neq t_i \\ \theta(0) + \Xi(\theta) & = & \theta_0 \\ \Delta\theta(t_i) & = & I_i(\theta(t_i)), i = 1, \dots, n, 0 < t_1 < \dots < t_n < b \end{array} \right. \quad (1)$$

where ${}^C\mathcal{D}_{0+}^{\alpha}(\cdot)$ is the Caputo fractional derivative of order $0 < \alpha \leq 1, t \in [0, b]$, $\mathcal{A} : [0, b] \times \Lambda \rightarrow \Lambda$ be a closed and linear operator the generator of an (α, u) -resolvent family, $\theta_0 \in \Lambda$ (Λ is a Banach space), $\Phi : [0, b] \times \Lambda \rightarrow \Lambda$, $g : \Omega \times \Lambda \rightarrow \Lambda$, $\Xi : \mathcal{PC}([0, b], \Lambda) \times \Lambda \rightarrow \Lambda$ and $\Delta\theta(t_i) = \theta(t_i^+) - \theta(t_i^-)$ constitutes an impulsive condition. Here $\Omega = (t, s); 0 \leq s \leq t \leq b$.

Over the decades, the quest to discuss results about existence, uniqueness, controllability, among other properties of solutions of differential and integrodifferential equations, has been the target of an investigation of paramount importance in mathematics, among other areas of knowledge [4, 5, 6]. Tools, as a measure of non-compactness and fixed point theory, are very useful in discussing the theory of differential equations [7, 8, 9, 10].

On the other hand, in a special way, the theory of impulsive differential equations appears as a natural description of several real processes subject to certain disturbances whose duration is insignificant compared to the duration of the process. For

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many years, Lakshmikantham et al. [5], Ntouyas and Tsamatos [6], discussed numerous properties of their solutions. We do not restrict these authors here, but other important researchers who have developed and are developing other theories can be found in the papers and references therein [11, 12, 13].

In 2012, Arjunan et al. [38], investigated the existence of solutions impulsive differential equations with nonlocal conditions in a real Banach space X of the form

$$\begin{cases} u'(t) &= Au(t) + f(t, u(t)), t \in [0, b], t \neq t_i \\ u(0) &= g(u) \\ \Delta u(t_i) &= I_i(u(t_i)), i = 1, \dots, n, 0 < t_1 < \dots < t_n < b \end{cases}$$

where $A : D(A) \subset X$ is a non-densely defined operator, $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$, $u(t_i^-)$, $u(t_i^+)$ denote the right and left limit of u at t_i , respectively.

It is unquestionable the importance and relevance that fractional calculus over the years provides to numerous areas, in particular, physics, engineering, medicine, biology, among others [25, 26, 27, 28, 29, 30, 31, 32, 33]. Today, the theory of fractional calculus, is well established, and what has been noticed, is the growing number of researchers using tools discussed in fractional calculus, and applying in other areas, providing this important link, which grows exponentially over the years [44, 45, 46, 47, 48, 49]. Here, we will highlight the fractional differential equations, which has been the subject of study by several researchers [34, 35, 36, 37].

The theory of fractional differential equations is of paramount importance in both the theoretical and practical aspects. In a theoretical sense, we can highlight researchers such as: Trujillo, Nieto, Donal O'Regan, Benchohra, N'Guerekata, Debbouche, among other renowned researchers, who discussed important and relevant results that enabled the growth and strengthening of the area [39, 40, 41, 42, 43]. Recently, Sousa and Oliveira [15, 16, 17, 19, 20, 21], has discussed some results on existence, uniqueness, stability of solutions of fractional differential equations via ψ -Hilfer fractional derivative, obtaining a wide class of particular cases. Other works, involving the ψ -Hilfer fractional derivative, we suggest [22, 23, 24].

In 2010, Debbouche [2], discussed the existence and uniqueness of local mild and classic solutions of a class of nonlinear fractional evolution integro-differential systems with nonlocal conditions of the form

$$\begin{cases} \frac{d^\alpha}{dt^\alpha} u(t) + A(t)u(t) &= f(t, u(t)) + \int_0^t B(t-s)g(s, u(s))ds, \\ u(t_0) + h(u) &= u_0 \end{cases}$$

in a Banach space X , where $0 < \alpha \leq 1$, $0 \leq t_0 < t$, $-A(t)$ is a closed linear operator defined on a dense domain $D(A)$ in X into X such that $D(A)$ independent of t . It is assumed also that $-A(t)$ generates an evolution operator in the Banach space X , the function B is real valued and locally integrable on $[t_0, \infty]$, the nonlinear maps f and g are defined on $[t_0, \infty] \times X$ into X and $h : C(J, X) \rightarrow \overline{D(A)}$ is a given function. In addition to these important results, Debbouche and Baleanu [3], discussed results on controllability of fractional integrodifferential equations.

In 2017, Gou and Li [1], investigated the local and global existence of mild solutions to impulsive fractional semilinear integro-differential equation with non-compact semigroup

$$\begin{cases} u^\alpha(t) + \mathcal{A}u(t) &= f(t, u(t)) + \int_0^t q(t-s)g(s, u(s))ds, t > 0, \alpha \in (0, 1) \\ u(0) &= u_0 \in X \\ \Delta u|_{t=t_k} &= I_k(\theta(t_k)), k = 1, \dots, m \end{cases}$$

where the $u^\alpha(\cdot)$ is Caputo fractional derivative of order α ($\alpha \in (0, 1)$), $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ is a closed linear operator and $-\mathcal{A}$ generates a uniformly bounded c_0 -semigroup $T(t)$ ($t \geq 0$) in X , the nonlinear maps $f, g : [0, \infty] \times X \rightarrow X$, and $q : I \rightarrow X$ are continuous, $I = [0, T]$, $0 < T \leq \infty$, $u_0 \in X$. $I_k : X \rightarrow X$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^+) = \lim_{n \rightarrow \infty} u(t_k + n)$ and $u(t_k^-) = \lim_{n \rightarrow \infty} u(t_k - n)$ represent the right and left limits of $u(t)$ at $t = t_k$ respectively.

Motivated by the papers above, and also because the theory of fractional integrodifferential and differential equations is under construction, with numerous open questions, our goal is to contribute significantly, providing results on the existence and uniqueness of a class of mild solutions to the fractional problem (1) in the sense of Caputo fractional derivative. To make the development of the paper simple and clear, we highlight the main results discussed. It is also important to note that the results discussed in this paper are based on conditions that will be presented throughout the paper, which makes your discussions interesting and important. Basically, this paper is divided into two stages (main). The first step, we dedicate ourselves to investigate the existence of mild solutions for Eq.(1), i.e., the following results:

Theorem 1.1. *Assumptions \mathbf{EC}_1 - \mathbf{EC}_6 , below holds, then the impulsive nonlocal problem Eq.(1) has at least one mild solution.*

Theorem 1.2. *Suppose that the \mathbf{EC}_1 - \mathbf{EC}_2 and \mathbf{EC}_7 - \mathbf{EC}_8 are satisfied, then the Eq.(1) has at least one mild solution if*

$$\lim_{n \rightarrow \infty} \sup \frac{\mathbf{M}_0^\alpha}{r} \left(\phi(r) + \phi(r) \int_0^t p(s)ds + \frac{G_1 \chi(r)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s)ds + \sum_{i=1}^n d_i \right) < 1$$

where $\varphi(r) = \sup\{\|\Xi(\theta)\|, \|u\| < r\}$.

Theorem 1.3. *Let $0 < \alpha < 1$. Suppose that the assumptions \mathbf{EC}_1 - \mathbf{EC}_9 are satisfied, then the Eq.(1) has at least one mil solution provided that*

$$\mathbf{M}_0^\alpha \left[L_0 + 4 \int_0^t \left(k_1(s) + \frac{2G_0 k_3(s)(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right) ds + \sum_{i=1}^n l_i \right] < 1.$$

Theorem 1.4. *Suppose that the assumption \mathbf{EC}_1 - \mathbf{EC}_9 are satisfied, then the equation Eq.(1) has at least one mild solution if Eq.(28) and the following condition is satisfied,*

$$\mathbf{M}_0^\alpha L_0 + \lim_{n \rightarrow \infty} \frac{\mathbf{M}_0^\alpha}{r} \left(\phi(r) \int_0^t p(s)ds + \frac{\chi(r)G_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s)ds + \sum_{i=1}^n d_i \right) < 1. \quad (2)$$

The second step is dedicated to discussing the uniqueness of solutions for Eq.(1), i.e., it is to discuss the following result:

Theorem 1.5. *Let $\theta_0 \in \Lambda$ and let $\mathbf{B}_r = \{\theta \in \mathcal{PC}([0, b], \Lambda); \|\theta\| \leq r\}$, $r > 0$. If the assumptions \mathbf{UC}_1 - \mathbf{UC}_5 are satisfied, then Eq.(1) has a unique mild solution.*

The rest of the paper is divided as follows. In section 2, we present the fundamental concepts of Riemann-Liouville fractional integral and the Caputo fractional derivative. In this sense, we discussed some fundamental results on the noncompactness measure and the fundamental concept of the mild solution.

2. Preliminaries. In this section, we present fundamental concepts of integrals and fractional derivatives of Riemann-Liouville and Caputo, respectively. In this sense, the concept of noncompactness measure and some fundamental results that will be used throughout the paper are presented.

Definition 2.1. [14, 18] Let (a, b) ($-\infty \leq a < b \leq \infty$) be a finite or infinite interval of the real line \mathbb{R} and let $\alpha > 0$. In addition, let $\psi(t)$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $\psi'(t)$ on (a, b) . The left-sided fractional integral of function θ with respect to function ψ on $[a, b]$ is defined by

$$I_{a+}^{\alpha; \psi} \theta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \theta(s) ds. \quad (3)$$

The right-sided fractional integral is defined in an analogous form [14, 18].

Choosing in particular $\psi(t) = t$, we have the Riemann-Liouville fractional integral given by [14, 18]

$$I_{a+}^{\alpha} \theta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \theta(s) ds, \quad (4)$$

where Γ is the gamma function and $f \in L^1([a, b], \mathbb{R})$.

If $a = 0$, we can write $I^{\alpha} \theta(t) = (g_{\alpha} * \theta)(t)$, where

$$g_{\alpha}(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1} & \text{se } t > 0 \\ 0 & \text{se } t \leq 0 \end{cases} \quad (5)$$

and as usual $*$ denotes convolution of functions, also we have $\lim_{\alpha \rightarrow 0} g_{\alpha}(t) = \delta(t)$, which the delta function. From choosing $\psi(\cdot)$, we have another fractional integrals.

Here, we restrict the Riemann-Liouville fractional integral to use to discuss the results of this paper. However, other formulations of fractional integrals can be obtained by choosing $\psi(\cdot)$ [14, 18].

On the other hand, we will present the definition of the ψ -Hilfer fractional derivative.

Definition 2.2. [14, 18] Let $n-1 < \alpha < n$, with $n \in \mathbb{N}$, let $I = [a, b]$ be an interval such that $-\infty \leq a < b \leq \infty$ and let $\theta, \psi \in C^n([a, b], \mathbb{R})$ be two functions, such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in I$. The left-sided ψ -Hilfer fractional derivative ${}^{\mathbf{H}}\mathbb{D}_{a+}^{\alpha, \beta; \psi}(\cdot)$ of a function θ , of order α and type $0 \leq \beta \leq 1$ is defined by

$${}^{\mathbf{H}}\mathbb{D}_{a+}^{\alpha, \beta; \psi} \theta(t) := I_{a+}^{\beta(h-\alpha); \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a+}^{(1-\beta)(n-\alpha); \psi} \theta(t). \quad (6)$$

The right-sided ψ -Hilfer fractional derivatives is defined in an analogous form [14, 18].

Choosing $\psi(t) = t$ and taking the limit with $\beta \rightarrow 1$, on both sides of the Eq.(6), we have the Caputo fractional derivative given by [14, 18]

$${}^C\mathcal{D}_{a+}^\alpha \theta(t) = I_{a+}^{(n-\alpha);\psi} \left(\frac{d}{dt} \right)^n \theta(t) = I_{a+}^{(n-\alpha);\psi} \theta^{(n)}(t) \quad (7)$$

To investigate our main result, we use Caputo fractional derivative Eq.(7).

Let Λ a Banach space with norm $\|\cdot\|$. Let $\mathcal{PC}([a, b]; \Lambda)$ consist of functions θ from $J = [0, b]$ into X , such that $\theta(t)$ is continuous at $t \neq t_i$ and let continuous at $t = t_i$ and the right $\lim \theta(t_0^+)$ exists for $i = 1, \dots, n$. Evidently $\mathcal{PC}([a, b]; \Lambda)$ is a Banach space, where

$$\|\theta\|_{\mathcal{PC}} = \sup_{t \in [0, b]} \|\theta(t)\|$$

and denoted $\mathcal{L}([0, b], \Lambda)$ by the space of Λ -valued Bochner integrable functions on $[0, b]$ with the norm [37]

$$\|\theta\|_{\mathcal{L}} = \int_0^b \|\theta(t)\| dt.$$

We will present below the definition of Hausdorff measure of noncompactness and some results of paramount importance in the discussion of the main results of this paper.

The Hausdorff measure of noncompactness $\mu_\Omega(\cdot)$ defined by

$$\mu(\mathbf{B}) = \inf\{r > 0, \mathbf{B} \text{ can be covered by a finite number of balls with radii } r\}$$

for bounded set \mathbf{B} in a Banach space Ω [37].

Lemma 2.3. [37] *Let Ω be a real Banach space and $\mathbf{B}, \mathbf{E} \subseteq \Omega$ be bounded, with the following properties:*

1. \mathbf{B} is precompact if and only if $\mu_\Omega(\mathbf{B}) = 0$;
2. $\mu_\Omega(\mathbf{B}) = \mu_\Omega(\bar{\mathbf{B}}) = \mu_\Omega(\text{con } \mathbf{B})$, where $\bar{\mathbf{B}}$ and $\text{con } \mathbf{B}$ mean the closure and convex hull of \mathbf{B} , respectively;
3. $\mu_\Omega(\mathbf{B}) \leq \mu_\Omega(\mathbf{E})$, where $\mathbf{B} \subseteq \mathbf{E}$;
4. $\mu_\Omega(\mathbf{B} + \mathbf{E}) \leq \mu_\Omega(\mathbf{B}) + \mu_\Omega(\mathbf{E})$, where $\mathbf{B} + \mathbf{E} = \{x + y; x \in \mathbf{B}, y \in \mathbf{E}\}$;
5. $\mu_\Omega(\mathbf{B} \cup \mathbf{E}) \leq \max\{\mu_\Omega(\mathbf{B}), \mu_\Omega(\mathbf{E})\}$;
6. $\mu_\Omega(\lambda \mathbf{B}) \leq |\lambda| \mu_\Omega(\mathbf{B})$, for any $\lambda \in \mathbb{R}$;
7. If the map $\Theta : D(\Theta) \subset \Omega \rightarrow \mathbb{Z}$ is a Lipschitz continuous with constant r , then $\mu_\mathbb{Z}(\Theta \mathbf{B}) \leq r \mu_\Omega(\mathbf{B})$ for any bounded subset $\mathbf{B} \subseteq D(\Theta)$, where \mathbb{Z} being Banach space;
8. $\mu_\Omega(\mathbf{B}) = \inf\{d_\Omega(\mathbf{B}, \mathbf{E}); \Theta \subseteq \Omega \text{ is precompact} = \inf\{d_\Omega(\mathbf{B}, \mathbf{E}); \mathbf{E} \subseteq \Omega \text{ is finite valued}\}$, where $d_\Omega(\mathbf{B}, \mathbf{E})$ means the non-symmetric (or symmetric) Hausdorff distance between \mathbf{B} and \mathbf{E} in Ω ;
9. If $\{\mathbb{W}_n\}_{n=1}^\infty$ is decreasing sequence of bounded closed nonempty subsets of Ω and $\lim_{n \rightarrow \infty} \mu_\Omega(\mathbb{W}_n) = 0$, then $\bigcap_{n=1}^\infty \mathbb{W}_n$ is nonempty and compact in Ω .

The map $\mathbb{W} \subseteq \Omega \rightarrow \Omega$ is said to be a μ_Ω -contraction if there exists a positive constant $r < 1$ such that $\mu_\Omega(\Theta(\mathbf{B})) \leq r \mu_\Omega(\mathbf{B})$ for any bounded closed subset $\mathbf{B} \subseteq \mathbb{W}$, where Ω if a Banach space.

Lemma 2.4. [37][Darbo-Sadovskii] *If $\mathbb{W} \subseteq \Omega$ is bounded and closed and convex, the continuous map $\Theta : \mathbb{W} \rightarrow \mathbb{W}$ is a μ_Ω -contraction, the map Θ has at least one fixed point in \mathbb{W} .*

We denote by μ the Hausdorff measure of noncompactness of Λ and denote μ_c by the Hausdorff measure of noncompactness of $\mathcal{PC}([a, b]; \Lambda)$.

Lemma 2.5. [37] *If W is bounded, then for each $\varepsilon > 0$, there is a sequence $\{\theta_n\}_{n=1}^\infty \subset \mathbb{W}$, such that*

$$\mu(\mathbb{W}) \leq 2\mu\left(\{\theta_n\}_{n=1}^\infty\right) + \varepsilon.$$

Lemma 2.6. [37] *If $\mathbb{W} \subseteq \mathcal{PC}([0, b], \Lambda)$ is bounded, then $\mu(\mathbb{W}(t)) \leq \mu_c(\mathbb{W})$, for all $t \in [0, b]$, where $\mathbb{W}(t) = \{\theta(k); \theta \in \mathbb{W}\} \subseteq \Lambda$. Furthermore, if \mathbb{W} is equicontinuous on $[0, b]$, then $\mu(\mathbb{W}(t))$ is continuous on $[a, b]$ and $\mu_c(\mathbb{W}) = \sup\{\mu(\mathbb{W}(t)), t \in [a, b]\}$.*

Lemma 2.7. [37] *If $\{\theta_n\}_{n=1}^\infty \subset \mathcal{L}^1([a, b], \Lambda)$ is uniformly integrable, then the function $\mu(\{\theta_n\}_{n=1}^\infty)$ is measurable and*

$$\mu\left(\left\{\int_0^t \theta_n(s) ds\right\}_{n=1}^\infty\right) \leq 2 \int_0^t \mu(\{\theta_n(s)\}_{n=1}^\infty) ds.$$

Lemma 2.8. [37] *If $\mathbb{W} \subseteq \mathcal{PC}([0, b], \Lambda)$ is bounded and equicontinuous, then $\mu(\mathbb{W}(t))$ is continuous and*

$$\mu\left(\int_0^t \mathbb{W}(s) ds\right) \leq \int_0^t \mu(\mathbb{W}(s)) ds,$$

for all $t \in [0, b]$, where

$$\int_0^t \mu(\mathbb{W}(s)) ds = \left\{ \int_0^t \theta(s) ds; \theta \in \mathbb{W} \right\}$$

The C_0 semigroup $U_\theta(t, s)$ is said to be equicontinuous if $(t, s) \rightarrow \left\{ U_\theta(t, s)\theta(s); \theta \in \mathbf{B} \right\}$ is equicontinuous for $t > 0$ for all bounded set \mathbf{B} in Λ . So, the following lemma is obvious.

Lemma 2.9. [37] *If the evolution family $\left\{ U_\theta(t, s) \right\}_{0 \leq s \leq t \leq b}$ is continuous and $\eta \in \mathcal{L}([a, b], \mathbb{R}^+)$, then the set $\left\{ \int_0^t U_\theta(t, s)\theta(s) ds \right\}$, $\|\theta(s)\| \leq \eta(s)$ for a.e. $s \in [0, b]$ is equicontinuous for $t \in [0, b]$.*

From Eq.(6) we know that for any fixed $u \in \mathcal{PC}([0, b], \Lambda)$ there exist a unique continuous function $U_\theta : [0, b] \times [0, b] \rightarrow B(\Lambda)$ defined on $[0, b] \times [0, b]$ such that [4]

$$U_\theta(t, s) = I + \int_s^t \mathcal{A}_\theta(w) U_\theta(w, s) dw, \quad (8)$$

where $\mathbf{B}(\Lambda)$ denote the Banach space of bounded linear operator from Λ to Λ with the norm $\|\Theta\| = \sup\{\|\Theta(\theta)\|; \|\theta\| = 1\}$ and I stands for the identity operator on Λ , $\mathcal{A}_\theta = \mathcal{A}(t, \theta(b))$, we have [4]

$$U_\theta(t, t) = I, \quad U_\theta(t, s) U_\theta(s, r) = U_\theta(t, r), \quad (t, s, r) \in [0, b] \times [0, b] \times [0, b]$$

and

$$\frac{\partial U_\theta(t, s)}{\partial t} = \mathcal{A}_\theta(t) U_\theta(t, s), \quad \text{for almost all } t, s \in [0, b].$$

Let \mathbf{E} be the Banach space formed from $D(\mathcal{A})$ with the graph norm. Since, $\mathcal{A}(t)$ is a closed operator, it follows that $\mathcal{A}(t)$ is in the set bounded from \mathbf{E} to Λ .

Definition 2.10. [2, 3] Let $\mathcal{A}(t, \theta)$ be a closed and linear operator with domain $D(\mathcal{A})$ defined on a Banach space Λ and $\alpha > 0$. Let $\rho(\mathcal{A}(t, \theta))$ be the resolvent set of $\mathcal{A}(t, \theta)$. We call $\mathcal{A}(t, \theta)$ the generator of an (α, θ) -resolvent family if there

exist $w \geq 0$ and a strongly continuous function $\mathcal{R}_{(\alpha, \theta)} : \mathbb{R}_+^2 \rightarrow \mathcal{L}(\Lambda)$ such that $\{\lambda^\alpha : R_e(\lambda) > w\} \subset \rho(\mathcal{A})$ and for $0 \leq s \leq t \leq \infty$,

$$\left(\lambda^\alpha I - \mathcal{A}(s, \theta)\right)^{-1} \nu = \int_0^\infty e^{-\lambda(t-s)} \mathcal{R}_{(\alpha, \theta)}(t, s) \nu dt, \quad R_e(\lambda) > w, \quad (\theta, \nu) \in X^2.$$

In this case, $\mathcal{R}_{(\alpha, \theta)}(t, s)$ is called the (α, θ) -resolvent family generated by $\mathcal{A}(t, \theta)$.

- Remark 1.**
1. We can deduce that Eq.(1) is well posed if and only if $\mathcal{A}(t, \theta)$ is the generator of (α, θ) -resolvent family.
 2. Here $\mathcal{R}_{(\alpha, \theta)}(t, s)$ can be extracted from the evolution operator of the generator $\mathcal{A}(t, \theta)$.
 3. The (α, θ) -resolvent family is similar to the evolution for non-autonomous differential equation in a Banach space.

3. The existence of mild solution. In this section, we will discuss the existence of a mild solution for the quasi-linear fractional integrodifferential equation with impulsive and nonlocal conditions in the sense of Caputo fractional derivative, using Hausdorff of noncompactness measure.

Definition 3.1. [2, 3] By a mild solution of Eq.(1) we mean a function $\theta \in \mathcal{PC}(J, \Lambda)$ with values in Ω satisfying the integral equation

$$\begin{aligned} \theta(t) &= \mathcal{R}_{(\alpha, \theta)}(t, 0)[\theta_0 - \Xi(\theta)] + \sum_{0 < t_i < t} \mathcal{R}_{(\alpha, \theta)}(t, s)(t, t_i) I_i(\theta(t_0)) \\ &\quad + \int_0^t \mathcal{R}_{(\alpha, \theta)}(t, s) \left(\Phi(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} g(s, x, \theta(x)) dx \right) ds \end{aligned} \quad (9)$$

$t \in J$ for all $\theta_0 \in \Lambda$ and $0 < \alpha < 1$.

In this work, we denote

$$\mathbf{M}_0^\alpha = \sup\{\mathcal{R}_{(\alpha, \theta)}(t, s); (t, s) \in [0, b] \times [0, b], 0 < \alpha < 1\}$$

for all $\theta \in \Lambda$. Without loss of generality, we let $\theta_0 = 0$.

To investigate the main result of this paper, we need some conditions, namely:

- EC₁.** The evolution family $\{\mathcal{R}_{(\alpha, \theta)}(t, s)\}_{0 \leq s \leq t \leq b}$ is called the (α, θ) -resolvent generated by $\mathcal{A}(t, \theta(t))$ if it is equicontinuous and $\|\mathcal{R}_{(\alpha, \theta)}(t, s)\| \leq \mathbf{M}_0^\alpha$ for almost every $t, s \in [0, b]$ and $0 < \alpha < 1$.
- EC_{2.1}** The function $\Xi : \mathcal{PC}([0, b], \Lambda) \rightarrow \Lambda$ is continuous and compact;
- EC_{2.2}** There exist $\mathbf{N}_0 > 0$ such that $\|\Xi(\theta)\| \leq \mathbf{N}_0$.
- EC_{3.1}.** The nonlinear function $\Phi : [0, b] \times \Lambda \rightarrow \Lambda$ satisfies the Caratheodory-type conditions, i.e., $\Phi(\cdot, \theta)$ is measurable for all $\theta \in \Lambda$ and $\Phi(t, \cdot)$ is continuous for almost every $t \in [0, b]$;
- EC_{3.2}.** There exist a function $\xi \in \mathcal{L}([0, b]; \mathbb{R}^+)$ such that for every $\theta \in \Lambda$, we have $\|\Phi(t, \theta)\| \leq \xi(t)(1 + \|\theta\|)$, a.e. $t \in [0, b]$;
- EC_{3.3}.** There exists a function $K_1 \in \mathcal{L}([0, b], \mathbb{R}^+)$ such that, for every bounded $D \subset \Lambda$, we have

$$\mu(\Phi(t, D)) \leq k_1(t) \mu(D), \quad \text{a.e. } t \in [0, b]. \quad (10)$$

- EC_{4.1}.** The nonlinear function $g : [0, b] \times [0, b] \times \Lambda \rightarrow \Lambda$ satisfies the Caratheodory-type conditions, i.e., $g(\cdot, \cdot, \theta)$ is continuous for a.e. $t \in [0, b]$;

EC_{4.2}. There exist two functions $\beta_1 \in \mathcal{L}([0, b], \mathbb{R}^+)$ and $\beta_2 \in \mathcal{L}([0, b], \mathbb{R}^+)$ such that for every $\theta \in \Lambda$, we have

$$\|g(t, s, \theta(s))\| \leq \beta_1(t)\beta_2(t)(1 + \|\theta(s)\|), \text{ a.e. } t \in [0, b]; \quad (11)$$

EC_{4.3}. There exist functions $k_2, k_3 \in \mathcal{L}([0, b], \mathbb{R}^+)$ such that, for every bounded $D \subset \Lambda$, we have

$$\mu(g(t, s, D)) \leq k_2(t)k_3(t) \mu(D), \text{ a.e. } t \in [0, b]. \quad (12)$$

EC_{5.1}. Assume that the finite bound of $\int_0^t k_2(t)ds$ is G_0 .

EC_{5.2}. For every $t \in [0, b]$ there exist positive constants \mathbf{N}_1 and \mathbf{N}_2 , the scalar equation

$$m^\alpha(t) = \mathbf{M}_0^\alpha \mathbf{N}_1 + \mathbf{M}_0^\alpha \mathbf{N}_2 \left[\frac{1}{\Gamma(\alpha)} \int_0^t (\xi(t) + C_0(t-s)^{\alpha-1} \beta_2(s)) (1 + m(s)) ds + \sum_{i=1}^n d_i \right], \quad (13)$$

for $0 < \alpha < 1$.

EC_{6.1}. $I_i : \Lambda \rightarrow \Lambda$ is continuous. There exist a constant $d_i > 0$, $i = 1, \dots, n$ such that

$$\|I_i(\theta(t_i))\| \leq \sum_{i=1}^n d_i, \quad i = 1, \dots, n. \quad (14)$$

For any bounded subset $D \subset \Lambda$, and there is a constant $l_i > 0$ such that

$$\mu(I_i(D)) \leq \sum_{i=1}^n l_i \mu(D), \quad i = 1, \dots, n. \quad (15)$$

The first result that we will discuss, is Theorem 3.2 with some conditions as previously presented. In this sense, the other results that we will discuss in this section, we use the conditions **EC₁** - **EC₆** and impose other conditions.

Theorem 3.2. *Assumptions **EC₁**-**EC₆**, hold then the impulsive nonlocal problem Eq.(1) has at least one mild solution.*

Proof. Consider

$$m^\alpha(t) = \mathbf{M}_0^\alpha \mathbf{N}_0 + \mathbf{M}_0^\alpha \left(\frac{1}{\Gamma(\alpha)} \int_0^t (\xi(t) + C_0(t-s)^{\alpha-1} \beta_2(s)) (1 + m(s)) ds + \sum_{i=1}^n d_i \right)$$

be a solution and assume that the finite bound of $\int_0^t \beta_1(s) ds$ is C_0 for $t \in [0, b]$.

Consider the map $\Theta : \mathcal{PC}([0, b], \Lambda) \rightarrow \mathcal{PC}([0, b], \Lambda)$ defined by

$$\begin{aligned} (F\theta)(t) &= \mathcal{R}_{(\alpha, \theta)}(t, 0)[\theta_0 - \Xi(\theta)] + \sum_{0 < t_i < t} \mathcal{R}_{(\alpha, \theta)}(t, t_i) I_i(\theta(t_i)) \\ &\quad + \int_0^t \mathcal{R}_{(\alpha, \theta)}(t, s) \left(\Phi(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} g(s, x, \theta(x)) dx \right) ds, \end{aligned} \quad (16)$$

for all $\theta \in \mathcal{PC}([0, b], \Lambda)$ and $\mathbb{W}_0 = \{\theta \in \mathcal{PC}([0, b], \Lambda), \|\theta(t)\| \leq m(t), \text{ for all } t \in [0, b]\}$. Then, $\mathbb{W}_0 \subseteq \mathcal{PC}([0, b], \Lambda)$ is bounded and convex.

On the other hand, consider we define $\mathbb{W}_1 = \overline{\text{cov}}K(w_0)$, where $\overline{\text{cov}}$ means the closure of the convex hull in $\mathcal{PC}([0, b], X)$ is bounded due to Lemma 2.6, using the assumptions, $\mathbb{W}_1 \subseteq \mathcal{PC}([0, b], \Lambda)$ is bounded closed convex nonempty and equicontinuous on $[0, b]$.

Now, for $\theta \in \Theta(w_0)$, yields

$$\begin{aligned}
\|\theta(t)\| &\leq \|\mathcal{R}_{(\alpha,\theta)}(t,0)\Xi(\theta)\| + \sum_{0 < t_i < t} \|\mathcal{R}_{(\alpha,\theta)}(t,t_i)I_i(\theta(t_i))\| \\
&\quad + \int_0^t \left\| \mathcal{R}_{(\alpha,\theta)}(t,s) \left(\Phi(s,\theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} g(s,x,\theta(x)) dx \right) \right\| ds \\
&\leq \|\mathcal{R}_{(\alpha,\theta)}(t,0)\| \|\Xi(\theta)\| + \sum_{0 < t_i < t} \|\mathcal{R}_{(\alpha,\theta)}(t,t_i)\| \|I_i(\theta(t_i))\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s \|\mathcal{R}_{(\alpha,\theta)}(t,s)\| (s-x)^{\alpha-1} \|g(s,x,\theta(x))\| dx ds \\
&\quad + \int_0^t \|\mathcal{R}_{(\alpha,\theta)}(t,s)\| \|\Phi(s,\theta(s))\| ds.
\end{aligned}$$

Using the conditions **EC**_{2.1}, **EC**_{2.2}, **EC**_{3.2}, **EC**_{4.2} and **EC**_{6.1}, we have

$$\begin{aligned}
&\|\theta(t)\| \\
&\leq \mathbf{M}_0 \mathbf{N}_0 + \mathbf{M}_0^\alpha \int_0^t \xi(s)(1+m(s))ds + \mathbf{M}_0^\alpha C_0 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \beta_2(s)(1+\|\theta(s)\|)ds + \mathbf{M}_0^\alpha \sum_{i=1}^n d_i \\
&\leq \mathbf{M}_0^\alpha \mathbf{N}_0 + \mathbf{M}_0^\alpha \left(\frac{1}{\Gamma(\alpha)} \int_0^t (\xi(s) + C_0(t-s)^{\alpha-1} \beta_2(s)) (1+m(s))ds + \sum_{i=1}^n d_i \right) \\
&= m_\alpha(t).
\end{aligned} \tag{17}$$

It follows that $\mathbb{W}_1 \subset \mathbb{W}_0$. We define $\mathbb{W}_{n+1} = \overline{\text{con}} \Theta(w_n)$, for $n = 1, 2, 3, \dots$. From above we know that $\mathbb{W} \{w_n\}_{n=1}^\infty$ is a decreasing sequence of bounded, closed, convex, equicontinuous on $[0, b]$ and nonempty subsets in $\mathcal{PC}([0, b], \Lambda)$.

Now for $n \geq 1$ and $t \in [0, b]$, $w_n(t)$ and $\Theta(w_n(t))$ are bounded subsets of Λ , hence, for any $\varepsilon > 0$, there is a sequence $\{\theta_n\}_{n=1}^\infty \subseteq w_n$, using the Lemma 2.5, Lemma 2.6, Lemma 2.7 and Lemma 2.8, such that

$$\begin{aligned}
&\mu(w_{n+1}(t)) \\
&= \mu(\Theta(w_n(t))) \\
&\leq 2\mu \left(\int_0^t \mathcal{R}_{(\alpha,\theta_k)}(t,s) \Phi(s, \{\theta_k\}_{k=1}^\infty) ds \right) + 2\mu \left(\sum_{i=1}^n \mathcal{R}_{(\alpha,\theta_k)}(t,t_i) I_i(\{\theta_k(t_i)\}_{n=1}^\infty) \right) \\
&\quad + 2\mu \left(\int_0^t \int_0^s \mathcal{R}_{(\alpha,\theta_k)}(t,s) \frac{1}{\Gamma(\alpha)} (s-x)^{\alpha-1} g(s,x, \{\theta_k(x)\}_{k=1}^\infty) dx ds \right) + \varepsilon \\
&\leq 4\mathbf{M}_0^\alpha \int_0^t k_1(s) \mu(\{\theta_k\}_{k=1}^\infty) ds + \frac{8\mathbf{M}_0}{\Gamma(\alpha)} \int_0^t \int_0^s (s-x)^{\alpha-1} k_2(s) k_3(x) \mu(\{\theta_k(x)\}_{k=1}^\infty) dx ds \\
&\quad + 4\mathbf{M}_0^\alpha \sum_{i=1}^\infty l_i \mu(\{\theta_k(t_i)\}_{k=1}^\infty) + \varepsilon \\
&\leq 4\mathbf{M}_0^\alpha \left(\int_0^t k_1(s) \mu(w_n(s)) ds + \frac{2G_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} k_3(s) \mu(w_n(s)) ds + \sum_{i=1}^\infty l_i \mu(w_n(t_0)) \right) + \varepsilon.
\end{aligned} \tag{18}$$

Since $\varepsilon > 0$ is arbitrary, from inequality (18), we have

$$\mu(w_{n+1}(t)) \leq 4\mathbf{M}_0^\alpha \left(\int_0^t k_1(s) \mu(w_n(s)) ds + \frac{2G_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} k_3(s) \mu(w_n(s)) ds + \sum_{i=1}^{\infty} l_i \mu(w_n(t_i)) \right) \quad (19)$$

for all $t \in [0, b]$ and $0 < \alpha \leq 1$. Note that w_n is decreasing for n , yields

$$\mathbf{H}(t) = \lim_{n \rightarrow \infty} \mu(w_n(t)) \quad (20)$$

for all $t \in [0, b]$. So, by means of the inequalities (19) and (20), we have

$$\mathbf{H}(t) = \lim_{n \rightarrow \infty} \mu(w_n(t)) \leq 4\mathbf{M}_0^\alpha \left(\int_0^t \left(k_1(s) + \frac{2G_0 k_3(s)}{\Gamma(\alpha)} \right) \mathbf{H}(s) ds + \sum_{i=1}^n l_i \mathbf{H}(t_i) \right) \quad (21)$$

for $t \in [0, b]$, which implies that $\mathbf{H}(t) = 0$ for all $t_i \in [0, b]$.

Using the Lemma 2.6, we have $\lim_{n \rightarrow \infty} \mu(w_n(t)) = 0$. On the other hand, by means of the Lemma 2.3, we have $\mathbb{W} = \bigcap_{n=1}^{\infty} w_n$ is convex compact and nonempty in $\mathcal{PC}([0, b], \Lambda)$ and $\mathbb{W} \subset \mathbb{W}$. Finally, making use the Schauder fixed point theorem, there exist at least one mild solution u of the initial value problem Eq.(1), where $\theta \in \mathbb{W}$ is a fixed point of the continuous map Θ . \square

The natural question that arises is: will it be possible to obtain the existence of solutions to the problem, excluding some conditions \mathbf{EC}_1 - \mathbf{EC}_6 and imposing? At first the answer is yes. So, let's change the terms \mathbf{EC}_3 - \mathbf{EC}_4 , for the following conditions:

1. **EC_{7.1}**. There exists a function $p \in \mathcal{L}([0, b], \mathbb{R}^+)$ and an increasing function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\|t, \theta\| \leq p(t) \phi(\|\theta\|)$$

for a.e. $t \in [0, b]$ and for all $u \in \mathcal{PC}([0, b], \Lambda)$.

2. **EC_{8.1}**. There exist two functions $q \in \mathcal{L}([0, b], \mathbb{R}^+)$ and $\hat{q} \in \mathcal{L}([0, b], \mathbb{R}^+)$ and a increasing function $\chi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\|g(t, s, \theta)\| \leq q(t) \hat{q}(s) \chi(\|\theta\|),$$

for a.e. $t \in [0, b]$ and for all $\theta \in \mathcal{PC}([0, b], \Lambda)$. Assume that the finite bound of

$$\int_0^t g(s) ds \text{ is } G_1.$$

Theorem 3.3. *Suppose that the \mathbf{EC}_1 - \mathbf{EC}_2 and \mathbf{EC}_7 - \mathbf{EC}_8 are satisfied, then the Eq.(1) has at least one mild solution if*

$$\lim_{n \rightarrow \infty} \sup \frac{\mathbf{M}_0^\alpha}{r} \left(\varphi(r) + \phi(r) \int_0^t p(s) ds + \frac{G_1 \chi(r)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s) ds + \sum_{i=1}^n d_i \right) < 1 \quad (22)$$

where $\varphi(r) = \sup\{\|\Xi(\theta)\|, \|u\| < r\}$.

Proof. The inequality (22) implies that these exist a constant $r > 0$ such that

$$\mathbf{M}_0^\alpha \left[\varphi(r) + \phi(r) \int_0^t p(s) ds + \frac{G_1 \chi(r)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s) ds + \sum_{i=1}^n d_i \right] < r. \quad (23)$$

Now, we consider the following set $\mathbb{W}_0 = \{\theta \in \mathcal{PC}([0, b], \Lambda), \|\theta\| \leq r\}$ and $\mathbb{W}_1 = \text{con}\Theta(\mathbb{W}_0)$. Then for any $\theta \in \mathbb{W}_1$, and using the conditions **EC**₁-**EC**₂ and **EC**₇-**EC**₈, yields

$$\begin{aligned} \|\theta(t)\| &\leq \|\mathcal{R}_{(\alpha, \theta)}(t, 0)\| \|\Xi(\theta)\| + \int_0^t \|R_{(\alpha, \theta)}(t, s)\| \|\Phi(s, \theta(s))\| ds + \sum_{0 < t_i < t} \|\mathcal{R}_{(\alpha, \theta)}(t, t_i)\| \|I_i(\theta(t_i))\| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s \|\mathcal{R}_{(\alpha, \theta)}(t, s)\| (s-x)^{\alpha-1} \|g(s, x, \theta(x))\| dx ds \\ &\leq \mathbf{M}_0^\alpha \varphi(r) + \mathbf{M}_0^\alpha \int_0^t p(s) \phi(\|\theta(s)\|) ds + \frac{\mathbf{M}_0^\alpha G_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s) \chi(\|\theta(s)\|) ds + \mathbf{M}_0^\alpha \sum_{i=1}^n d_i \\ &\leq \mathbf{M}_0^\alpha \varphi(r) + \mathbf{M}_0^\alpha \phi(r) \int_0^t p(s) ds + \frac{\mathbf{M}_0^\alpha G_1 \chi(r)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s) ds + \mathbf{M}_0^\alpha \sum_{i=1}^n d_i. \end{aligned}$$

In this sense, we have

$$\|\theta(t)\| \leq \left[\varphi(r) + \phi(r) \int_0^t p(s) ds + \frac{G_1 \chi(r)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s) ds + \mathbf{M}_0 \sum_{i=1}^n d_i \right] < r$$

for $t \in [0, b]$. It means that $\mathbb{W}_1 \subset \mathbb{W}_2$. So we can complete the proof similarly to Theorem 3.2. \square

Now let's investigate the existence of the mild solution for the function Ξ being Lipschitz, however ϕ, g and I_i are not Lipschitz. First, let's admit the following condition:

1. **EC**_{9.1}. The function Ξ is Lipschitz continuous in Λ , there exists a constant $L_0 > 0$ such that

$$\|\Xi(\theta) - \Xi(\nu)\| \leq L_0 \|\theta - \nu\|, \quad \theta, \nu \in \mathcal{PC}([0, b], \Lambda).$$

Theorem 3.4. *Let $0 < \alpha < 1$. Suppose that the assumptions **EC**₁-**EC**₉ are satisfied, then the Eq.(1) has at least one mil solution provided that*

$$\mathbf{M}_0^\alpha \left[L_0 + 4 \int_0^t \left(k_1(s) + \frac{2G_0 k_3(s)(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right) ds + \sum_{i=1}^n l_i \right] < 1.$$

Proof. For the proof, we consider the map $\Theta : \mathcal{PC}([0, b], \Lambda) \rightarrow \mathcal{PC}([0, b], \Lambda)$ is defined by $\Theta = \Theta_1 + \Theta_2$, where

$$(\Theta_1 \theta)(t) = \mathcal{R}_{(\alpha, \theta)}(t, 0) \Xi(\theta)$$

and

$$(\Theta_2 \theta)(t) = \int_0^t \mathcal{R}_{(\alpha, \theta)}(t, s) \Phi \left((s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} g(s, x, \theta(x)) dx \right) ds + \sum_{0 < t_i < t} \mathcal{R}_{(\alpha, \theta)}(t, t_i) I_i(\theta(t_i))$$

for $\theta \in \mathcal{PC}([0, b], \Lambda)$.

As introduced in the proof of Theorem 3.2, we consider $\mathbb{W}_0 = \{\theta \in \mathcal{PC}([0, b], \Lambda); \|\theta(t)\| \leq m(t), \text{ for all } t \in [0, b]\}$ and let $\mathbb{W} = \text{con}\Theta w_0$. Then from the proof of Theorem 3.2 we know that \mathbb{W} is a bounded, closed, convex and equicontinuous subset of $\mathcal{PC}([0, b], \Lambda)$ and $\Theta \mathbb{W} \subset \mathbb{W}$. We shall prove that Θ is μ_c -contraction on \mathbb{W} . Then Darbo-Sadovskii's fixed point theorem can be used to get a fixed point of Θ in \mathbb{W} ,

which is a mil solution of Eq.(1). First, for every bounded subset $\mathbf{B} \subset \mathbb{W}$, from the Eq.(1). and by Lemma 2.3, we have

$$\begin{aligned} \mu_c(\Theta_1 \mathbf{B}) &= \mu_c(\mathcal{R}_{(\alpha, \mathbf{B})}(t, 0)\Xi(\mathbf{B})) \\ &\leq \mathbf{M}_0 \mu_c(\Xi(\mathbf{B})) \\ &\leq \mathbf{M}_0 L_0 \mu_c(\mathbf{B}). \end{aligned} \quad (24)$$

For every bounded subset $\mathbf{B} \subset \mathbb{W}$, for $t \in [0, b]$ and every $\varepsilon > 0$, there exists the sequence $\{\theta_k\}_{k=1}^\infty \subset \mathbf{B}$, such that

$$\mu(\Theta_2(\mathbf{B}(t))) \leq 2\mu(\{\Theta_2 \theta_k(t)\}_{k=1}^\infty) + \varepsilon. \quad (25)$$

Note that \mathbf{B} and $\Theta_2 \mathbf{B}$ are equicontinuous. So, by means of the Lemmas 2.3, 2.5, 2.7, 2.8 and using the conditions \mathbf{EC}_1 - \mathbf{EC}_9 , yields

$$\begin{aligned} &\mu(\Theta_2(\mathbf{B}(t))) \\ &\leq 2\mu\left(\int_0^t \mathcal{R}_{(\alpha, \theta)}(t, s)\Phi(s, \{\theta_k(s)\}_{k=1}^\infty)ds\right) + 2\mu\left(\sum_{0 < t_i < t} \mathcal{R}_{(\alpha, \theta)}(t, t_i)I_i(\{\theta_k(t_i)\}_{k=1}^\infty)\right) \\ &\quad + \frac{2\mu}{\Gamma(\alpha)}\left(\int_0^t \int_0^s \mathcal{R}_{(\alpha, \theta)}(t, s)(x-s)^{\alpha-1}g(s, x, \{\theta_k(x)\}_{k=1}^\infty)dx ds\right) \\ &\leq 4\mathbf{M}_0^\alpha \int_0^t k_1(s)\mu(\{\theta_k(s)\}_{k=1}^\infty)ds + \frac{8\mathbf{M}_0^\alpha}{\Gamma(\alpha)} \int_0^t \int_0^s (s-x)^{\alpha-1}k_2(s)k_3(x)\mu(\{\theta_k(x)\}_{k=1}^\infty)dx ds \\ &\quad + 4\mathbf{M}_0^\alpha \sum_{0 < t_i < t} l_i \mu(\{\theta_k(t_i)\}_{k=1}^\infty) \\ &\leq 4\mathbf{M}_0^\alpha \left(\int_0^t k_1(s)\mu(\mathbf{B})ds + \frac{2G_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}k_3(s)\mu(\mathbf{B})ds + \sum_{i=1}^n l_i \mu(\mathbf{B})\right) + \varepsilon. \end{aligned} \quad (26)$$

Since $\varepsilon > 0$ is arbitrary, using above inequality (note that $\mu_c(\mathbb{W}) = \sup\{\mu(w(t)), t \in [0, b]\}$), we have

$$\mu_c(\Theta_2(\mathbf{B}(t))) \leq 4\mathbf{M}_0 \left[\int_0^t \left(k_1(s) + \frac{2G_0}{\Gamma(\alpha)}(t-s)^{\alpha-1}k_3(s)\right)ds + \sum_{i=1}^n l_i\right] \mu_c(\mathbf{B}) \quad (27)$$

for any bounded $\mathbf{B} \subset \mathbb{W}$.

Now, for any subset $\mathbf{B} \subset \mathbb{W}$, using the Lemma 2.4 and the inequalities (24) and (27), we have

$$\begin{aligned} \mu_c(\Theta \mathbf{B}) &= \mu_c(\Theta_1 \mathbf{B} + \Theta_2 \mathbf{B}) \leq \mu_c(\Theta_1 \mathbf{B}) + \mu_c(\Theta_2 \mathbf{B}) \\ &\leq \mathbf{M}_0^\alpha L_0 \mu_c(\mathbf{B}) + 4\mathbf{M}_0^\alpha \left(\int_0^t \left(k_1(s) + \frac{2G_0}{\Gamma(\alpha)}(t-s)^{\alpha-1}k_3(s)\right)ds + \sum_{i=1}^n l_i\right) \mu_c(\mathbf{B}) \\ &= \mathbf{M}_0^\alpha \left(L_0 + 4 \int_0^t \left(\frac{2G_0}{\Gamma(\alpha)}(t-s)^{\alpha-1}k_3(s)\right)ds + \sum_{i=1}^n l_i\right) \mu_c(\mathbf{B}). \end{aligned} \quad (28)$$

So, from Eq.(28), we obtain that Θ is a μ_c -contraction on \mathbb{W} . Therefore, by Lemma 2.4, there is a fixed point θ of Θ in \mathbb{W} , which is a solution of Eq.(1). \square

Finally, the next Theorem 3.5, also aims to discuss the existence of mild solution using the conditions \mathbf{EC}_1 - \mathbf{EC}_9 .

Theorem 3.5. *Suppose that the assumption \mathbf{EC}_1 - \mathbf{EC}_9 are satisfied, then the equation Eq.(1) has at least one mild solution if Eq.(28) and the following condition are satisfied,*

$$\mathbf{M}_0^\alpha L_0 + \lim_{n \rightarrow \infty} \frac{\mathbf{M}_0^\alpha}{r} \left(\phi(r) \int_0^t p(s) ds + \frac{\chi(r)G_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s) ds + \sum_{i=1}^n d_i \right) < 1. \quad (29)$$

Proof. Using the condition (29) and the fact that $L_0 < 1$, there exist a constant $r > 0$ such that

$$\mathbf{M}_0^\alpha \left(L_0 r + \|\Xi(0)\| + \phi(r) \int_0^t p(s) ds + \frac{\chi(r)G_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s) ds + \sum_{i=1}^n d_i \right) < r.$$

Now, we consider $\mathbb{W}_0 = \{\theta \in \mathcal{PC}([0, b], \Lambda), \|\theta(t)\| \leq r\}$ for all $t \in [0, b]$. Then for every $\theta \in \mathbb{W}_0$ and using the conditions \mathbf{EC}_9 , yields

$$\begin{aligned} & \|\Theta\theta(t)\| \\ & \leq \left\| \mathcal{R}_{(\alpha, \theta)}(t, 0)\Xi(\theta) + \sum_{0 < t_i < t} \mathcal{R}_{(\alpha, \theta)}(t, t_i)I_i(\theta(t_i)) \right\| \\ & \quad + \left\| \int_0^t \mathcal{R}_{(\alpha, \theta)}(t, s) \left(\Phi(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} g(s, x, \theta(x)) dx \right) ds \right\| \\ & \leq \mathbf{M}_0^\alpha \|\Xi(\theta) - \Xi(0) + \Xi(0)\| + \mathbf{M}_0^\alpha \int_0^t p(s) \phi(\|\theta(s)\|) ds \\ & \quad + \frac{\mathbf{M}_0^\alpha}{\Gamma(\alpha)} \int_0^t \int_0^s (s-x)^{\alpha-1} q(s) \hat{q}(x) \chi(\|\theta(x)\|) dx ds + \mathbf{M}_0^\alpha \sum_{i=1}^n d_i \\ & \leq \mathbf{M}_0^\alpha \left(L_0 r + \|\Xi(0)\| + \int_0^t p(s) \phi(\|\theta(s)\|) ds + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s (s-x)^{\alpha-1} q(s) \hat{q}(x) \chi(\|\theta(x)\|) dx ds + \sum_{i=1}^n d_i \right) \\ & \leq \mathbf{M}_0^\alpha \left(L_0 r + \|\Xi(0)\| + \phi(r) \int_0^t p(s) ds + \frac{1}{\Gamma(\alpha)} \chi(r) G_1 \int_0^t (t-s)^{\alpha-1} \hat{q}(s) ds + \sum_{i=1}^n d_i \right). \end{aligned} \quad (30)$$

Thus, we have

$$\|\Theta\theta(t)\| \leq \mathbf{M}_0^\alpha \left[L_0 r + \|\Xi(0)\| + \phi(r) \int_0^t p(s) ds + \frac{\chi(r)G_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s) ds + \sum_{i=1}^n d_i \right] < r,$$

for all $t \in [0, b]$. This means that $\Theta\mathbb{W}_0 \subset \mathbb{W}_0$.

Define $\mathbb{W} = \overline{\text{con}}\Theta(\mathbb{W}_0)$. The above proof also implies that $\Theta\mathbb{W} \subset \mathbb{W}$. So we can prove the theorem similar with Theorem 3.4 and hence we omit. \square

One of the direct consequences of fractional calculus, in particular, the discussion of fractional differential equations, is to discuss the special case $\alpha = 1$. So, in this sense, we will present the results for the entire case, for the problem Eq.(1). Then, taking the limit $\alpha \rightarrow 1$ in the Eq.(1), we have

$$\begin{cases} \theta'(t) + A(t, \theta(t))\theta(t) &= \Phi(t, \theta(t)) + \int_0^t g(t, s, \theta(s)) ds, t \in [0, b], t \neq t_i \\ \theta(0) + \Xi(\theta) &= \theta_0 \\ \Delta\theta(t_i) &= I_i(\theta(t_i)), i = 1, \dots, n, 0 < t_1 < \dots < t_n < b \end{cases} \quad (31)$$

Consequently, taking the limit $\alpha \rightarrow 1$ in the Eq.(9), we have the mild solution for the Eq.(31), given by

$$\begin{aligned} \theta(t) &= U_\theta(t, 0)[\theta_0 - \Xi(\theta)] + \int_0^t U_\theta(t, s) \left(\Phi(s, \theta(s)) + \int_0^s g(s, x, \theta(x)) dx \right) ds \\ &\quad + \sum_{0 < t_i < t} U_\theta(t, t_i) I_i(\theta(t_i)) \end{aligned}$$

$0 \leq t \leq b$.

The following results are direct consequences of the existence results previously investigated. In this sense, their respective statements are omitted.

Theorem 3.6. *Assumptions \mathbf{EC}_1 - \mathbf{EC}_6 holds, then the impulsive nonlocal problem Eq.(31) has at least one mild solution.*

Proof. Direct consequence of the Theorem 3.2. \square

Theorem 3.7. *Suppose that the assumptions \mathbf{EC}_1 - \mathbf{EC}_2 and \mathbf{EC}_7 - \mathbf{EC}_8 , are satisfied, then the Eq.(31) has at least are mild solution if*

$$\lim_{r \rightarrow \infty} \sup \frac{\mathbf{M}_0}{r} \left(\varphi(r) + \phi(r) \int_0^t p(s) ds + G_1 \chi(r) \int_0^t \hat{q}(s) ds + \sum_{i=1}^n d_i \right) < 1,$$

where $\varphi(r) = \sup\{\|\Xi(\theta)\|, \|u\| < r\}$.

Proof. Direct consequence of the Theorem 3.3. \square

Theorem 3.8. *Suppose that the \mathbf{EC}_1 - \mathbf{EC}_9 , are satisfied, then the Eq.(31) has at least one mild solution provided that*

$$\mathbf{M}_0 \left(L_0 + 4 \int_0^t (k_1(s) + 2G_0 k_3(s)) ds + \sum_{i=1}^n l_i \right) < 1.$$

Proof. Direct consequence of the Theorem 3.4. \square

Theorem 3.9. *Suppose that the \mathbf{EC}_1 - \mathbf{EC}_9 , are satisfied, then the Eq.(31) has at least one mild solution if Eq.(28) (with $\alpha \rightarrow 1$) and the following condition is satisfied*

$$\mathbf{M}_0 L_0 + \lim_{r \rightarrow \infty} \frac{\mathbf{M}_0}{r} \left(\varphi(r) \int_0^t p(s) ds + \chi(r) G_1 \int_0^t \hat{q}(s) ds + \sum_{i=1}^n d_i \right) < 1.$$

Proof. Direct consequence of the Theorem 3.5. \square

We conclude the section on existence of mild solutions for the quasi-linear fractional integrodifferential equations with impulsive and nonlocal conditions given by Eq.(1), discussing some necessary and sufficient conditions for discussing it. In addition, as a particular case, we discuss the entire case. In this sense, the next section is intended to investigate the uniqueness of mild solutions for Eq.(1).

4. **The uniqueness of mild solution.** In this section, we will discuss the uniqueness of mild solutions for Eq.(1), using the fixed point technique. However, first, we need some conditions before we attack the main purpose of this section. So, consider:

1. **UC₁.** $\Phi : [0, b] \times \Lambda \rightarrow \Lambda$ is continuous and there exists constant $\Theta_{\mathcal{A}} > 0$, and $\Theta_0 > 0$ such that

$$\|\Phi(t, \theta) - \Phi(t, \nu)\| \leq \Theta_{\mathcal{A}}(\|\theta - \nu\|), \quad \theta, \nu \in \Lambda \text{ and } \Theta_0 = \max_{t \in [0, b]} \|\Phi(t, 0)\|.$$

2. **UC₂.** $g : [0, b] \times \Omega \rightarrow \Lambda$ is continuous and there exists constant $\mathbf{H}_{\mathcal{A}} > 0$, and $\mathbf{H}_0 > 0$ such that

$$\int_0^t \|g(s, \theta) - g(s, \nu)\| ds \leq \mathbf{H}_{\mathcal{A}} \|\theta - \nu\|, \quad \Theta_0 = \max \left\{ \int_0^t \|g(s, 0)\| ds; \quad t \in [0, b] \right\}.$$

3. **UC₃.** $h : \mathcal{PC}([0, b], \Lambda) \rightarrow \Omega$ is Lipschitz continuous in Λ and there a constant $l_i > 0$, $i = 1, \dots, n$ such that

$$\|\Xi(\theta) - \Xi(\nu)\| \leq G_{\mathcal{A}} \|\theta - \nu\|_{\mathcal{PC}}, \quad \theta, \nu \in \mathcal{PC}([0, b], \Lambda).$$

4. **UC₄.** $I_i : \Omega \rightarrow \Omega$ is continuous and there exist a constant $l_i > 0$, $i = 1, \dots, n$ such that

$$\|I_i(\theta) - I_i(\nu)\| \leq l_i \|\theta - \nu\|, \quad \theta, \nu \in \Lambda.$$

5. **UC₅.** Let

$$\rho = \left\{ k_0 a r + 2k_0 a (G_{\mathcal{A}} r + \|g(0)\|) + \mathbf{M}_0 F_{\mathcal{A}} a + \frac{\mathbf{M}_0}{\Gamma(\alpha)} H_A a + k_0 a r \sum_{i=1}^n l_i + \mathbf{M}_0 \sum_{i=1}^n l_i \right\}$$

be such that $0 < \rho < 1$.

Further there exists a constant k_0 such that for every $\theta, v \in \mathcal{PC}([0, b], \Lambda)$ and $y \in \Lambda$ we have,

$$\|\mathcal{R}_{(\alpha, \theta)}(t, s)y - \mathcal{R}_{(\alpha, v)}(t, s)y\| \leq k_0 a \|y\|_{\Lambda} \|\theta - v\|_{\mathcal{PC}},$$

for $0 < \alpha < 1$.

Theorem 4.1. *Let $\theta_0 \in \Lambda$ and let $\mathbf{B}_r = \{\theta \in \mathcal{PC}([0, b], \Lambda); \|\theta\| \leq r\}$, $r > 0$. If the assumptions **UC₁-UC₅** are satisfied, then Eq.(1) has a unique mild solution.*

Proof. Consider $\theta_0 \in \Lambda$ (fixed) and the operator Θ on $\mathcal{PC}([0, b], \Lambda)$, given by

$$\begin{aligned} (\Theta\theta)(t) &= \mathcal{R}_{(\alpha, \theta)}(t, 0)[\theta_0 - \Xi(\theta)] + \sum_{0 < t_i < t} \mathcal{R}_{(\alpha, \theta)}(t, t_i) I_i(\theta(t_i)) \\ &\quad + \int_0^t \mathcal{R}_{(\alpha, \theta)}(t, s) \left(\Phi(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} g(s, x, \theta(x)) dx \right) ds. \end{aligned}$$

Note that $\Theta : \mathcal{PC}([0, b], \Lambda) \longrightarrow \mathcal{PC}([0, b], \Lambda)$. Also, we have

$$\begin{aligned}
 & \|(\Theta\theta)(t) - (\Theta\nu)(t)\| \\
 \leq & \|(\mathcal{R}_{(\alpha,\theta)}(t, 0) - \mathcal{R}_{(\alpha,\nu)}(t, 0))\theta_0\| + \|\mathcal{R}_{(\alpha,\theta)}(t, 0)\Xi(\theta) - \mathcal{R}_{(\alpha,\nu)}(t, 0)\Xi(\nu)\| \\
 & + \int_0^t \left\| \mathcal{R}_{(\alpha,\theta)}(t, s) \left(\Phi(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} g(s, x, \theta(x)) dx \right) \right\| ds \\
 & - \int_0^t \left\| \mathcal{R}_{(\alpha,\nu)}(t, s) \left(\Phi(s, \nu(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} g(s, x, \nu(x)) dx \right) \right\| ds \\
 & + \sum_{0 < t_i < t} \|\mathcal{R}_{(\alpha,\theta)}(t, t_i) I_i(\theta(t_i))\| - \sum_{0 < t_i < t} \|\mathcal{R}_{(\alpha,\nu)}(t, t_i) I_i(\nu(t_i))\| \\
 \leq & \|(\mathcal{R}_{(\alpha,\theta)}(t, 0) - \mathcal{R}_{(\alpha,\nu)}(t, 0))\theta_0\| + \|\mathcal{R}_{(\alpha,\theta)}(t, 0)\Xi(\theta) - \mathcal{R}_{(\alpha,\nu)}(t, 0)\Xi(\nu)\| \\
 & + \int_0^t \|\mathcal{R}_{(\alpha,\theta)}(t, s)\| \|\Phi(s, \theta(s))\| ds + \frac{1}{\Gamma(\alpha)} \int_0^t \|\mathcal{R}_{(\alpha,\theta)}(t, s)\| \int_0^s (s-x)^{\alpha-1} \|g(s, x, \theta(x))\| dx ds \\
 & - \int_0^t \|\mathcal{R}_{(\alpha,\nu)}(t, s)\| \|\Phi(s, \nu(s))\| ds - \frac{1}{\Gamma(\alpha)} \int_0^t \|\mathcal{R}_{(\alpha,\nu)}(t, s)\| \int_0^s (s-x)^{\alpha-1} \|g(s, x, \nu(x))\| dx ds \\
 & + \sum_{0 < t_i < t} \|\mathcal{R}_{(\alpha,\theta)}(t, t_i) I_i(\theta(t_i))\| - \sum_{0 < t_i < t} \|\mathcal{R}_{(\alpha,\nu)}(t, t_i) I_i(\nu(t_i))\|. \tag{32}
 \end{aligned}$$

Using

$$\|\mathcal{R}_{(\alpha,\theta)}(t, s)\| \leq \mathbf{M}_0^\alpha, \quad \|g(\theta)\| \leq G_{\mathcal{A}} \|\theta\|_{\mathcal{PC}}$$

and

$$\|\mathcal{R}_{(\alpha,\theta)}(t, s)y - \mathcal{R}_{(\alpha,\nu)}(t, s)y\| \leq k_0 a \|y\|_X \|\theta - \nu\|_{\mathcal{PC}}$$

with k_0 constant, yields

$$\begin{aligned}
 \|\mathcal{R}_{(\alpha,\nu)}(t, 0)g(\nu) - \mathcal{R}_{(\alpha,\theta)}(t, 0)g(\theta)\| & \leq \|\mathcal{R}_{(\alpha,\nu)}(t, 0)\| \|g(\nu)\| + \|\mathcal{R}_{(\alpha,\theta)}(t, 0)\| \|g(\theta)\| \\
 & \leq \mathbf{M}_0^\alpha \|g(\nu)\| + \mathbf{M}_0^\alpha \|g(\theta)\| \\
 & \leq \mathbf{M}_0^\alpha G_{\mathcal{A}} \|\theta - \nu\|. \tag{33}
 \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
 \|\mathcal{R}_{(\alpha,\theta)}(t, 0)g(\theta) - \mathcal{R}_{(\alpha,\nu)}(t, 0)g(\theta)\| & \leq k_0 a \|g(\theta)\| \|\theta - \nu\| \\
 & \leq k_0 a G_{\mathcal{A}} \|\theta\| (\|\theta\| + \|\nu\|). \tag{34}
 \end{aligned}$$

Using the Eq.(33)-Eq.(34), yields

$$\mathbf{M}_0^\alpha G_{\mathcal{A}} (\|\theta\| + \|\nu\|) \leq a k_0 G_{\mathcal{A}} \|\theta\| (\|\theta\| + \|\nu\|).$$

So, then we conclude that

$$\|\mathcal{R}_{(\alpha,\nu)}(t, 0)g(\nu) - \mathcal{R}_{(\alpha,\theta)}(t, 0)g(\theta)\| \leq \|\mathcal{R}_{(\alpha,\theta)}(t, 0)g(\theta) - \mathcal{R}_{(\alpha,\nu)}(t, 0)g(\nu)\|. \tag{35}$$

Now, using the inequality (35), we obtain

$$\begin{aligned}
 & \|\mathcal{R}_{(\alpha,\theta)}(t, 0)g(\theta) - \mathcal{R}_{(\alpha,\nu)}(t, 0)g(\nu)\| \\
 \leq & \|\mathcal{R}_{(\alpha,\theta)}(t, 0)g(\theta) - \mathcal{R}_{(\alpha,\nu)}(t, 0)g(\nu)\| + \|\mathcal{R}_{(\alpha,\nu)}(t, 0)g(\nu) - \mathcal{R}_{(\alpha,\theta)}(t, 0)g(\theta)\| \\
 & + \|\mathcal{R}_{(\alpha,\theta)}(t, 0)g(0) - \mathcal{R}_{(\alpha,\nu)}(t, 0)g(0)\| + \|\mathcal{R}_{(\alpha,\theta)}(t, 0)g(0) - \mathcal{R}_{(\alpha,\nu)}(t, 0)g(0)\| \\
 \leq & 2 \|\mathcal{R}_{(\alpha,\theta)}(t, 0)g(\theta) - \mathcal{R}_{(\alpha,\nu)}(t, 0)g(\nu)\| + 2 \|\mathcal{R}_{(\alpha,\theta)}(t, 0)g(0) - \mathcal{R}_{(\alpha,\nu)}(t, 0)g(0)\| \\
 \leq & 2k_0 a \|g(\theta)\|_X \|\theta - \nu\|_{\mathcal{PC}} + 2k_0 a \|g(0)\|_X \|\theta - \nu\|_{\mathcal{PC}} \\
 \leq & 2k_0 a G_{\mathcal{A}} \|\theta\|_{\mathcal{PC}} \|\theta - \nu\|_{\mathcal{PC}} + 2k_0 a G_{\mathcal{A}} \|g(0)\|_X \|\theta - \nu\|_{\mathcal{PC}} \\
 = & 2k_0 a (G_{\mathcal{A}} \|\theta\| + \|g(0)\|) \|\theta - \nu\|_{\mathcal{PC}}
 \end{aligned}$$

Now, returning to the inequality (32), we have

$$\begin{aligned}
& \|(\Theta\theta)(t) - (\Theta\nu)(t)\| \\
\leq & k_0 a \|\theta\| \|\theta - \nu\|_{\mathcal{PC}} + 2k_0 a (G_{\mathcal{A}} \|\theta\| + \|g(0)\|) \|\theta - \nu\|_{\mathcal{PC}} \\
& + \mathbf{M}_0^\alpha \int_0^t (\|\Phi(s, \theta(s)) - \Phi(s, 0)\| + \|\Phi(s, 0)\|) ds \\
& - \frac{\mathbf{M}_0^\alpha}{\Gamma(\alpha)} \int_0^t \int_0^s (s-x)^{\alpha-1} (\|g(s, x, \theta(x)) - g(s, 0, 0)\| + \|g(s, 0, 0)\|) dx ds \\
& - \mathbf{M}_0^\alpha \int_0^t (\|\Phi(s, \theta(s)) - \Phi(s, 0)\| + \|\Phi(s, 0)\|) ds \\
& - \frac{\mathbf{M}_0^\alpha}{\Gamma(\alpha)} \int_0^t \int_0^s (s-x)^{\alpha-1} (\|g(s, x, \nu(x)) - g(s, 0, 0)\| + \|g(s, 0, 0)\|) dx ds \\
& + \sum_{i=1}^n k_0 a \|\theta - \nu\|_{\mathcal{PC}} l_i \|\theta(t_i)\| + \mathbf{M}_0^\alpha \sum_{i=1}^n l_i \|\theta - \nu\|_{\mathcal{PC}} \\
\leq & k_0 a \|\theta\| \|\theta - \nu\|_{\mathcal{PC}} + 2k_0 a (G_{\mathcal{A}} \|\theta\| + \|g(0)\|) \|\theta - \nu\|_{\mathcal{PC}} \\
& + \mathbf{M}_0^\alpha \Theta_{\mathcal{A}} \int_0^t (\|\theta(s)\| - \|\nu(s)\|) ds + \frac{\mathbf{M}_0^\alpha}{\Gamma(\alpha)} H_{\mathcal{A}} \int_0^t (\|\theta\| - \|\nu\|) ds \\
& + \sum_{i=1}^n k_0 a \|\theta - \nu\|_{\mathcal{PC}} l_i \|\theta(t_i)\| + \mathbf{M}_0^\alpha \sum_{i=1}^n l_i \|\theta - \nu\|_{\mathcal{PC}} \\
\leq & k_0 a r \|\theta - \nu\|_{\mathcal{PC}} + 2k_0 a (G_{\mathcal{A}} r + \|g(0)\|) \|\theta - \nu\|_{\mathcal{PC}} \\
& + \mathbf{M}_0^\alpha \Theta_{\mathcal{A}} \int_0^t \|\theta - \nu\|_{\mathcal{PC}} ds + \frac{\mathbf{M}_0^\alpha}{\Gamma(\alpha)} H_{\mathcal{A}} \int_0^t \|\theta - \nu\|_{\mathcal{PC}} ds \\
& + \sum_{i=1}^n k_0 a \|\theta - \nu\|_{\mathcal{PC}} l_i r + \mathbf{M}_0^\alpha \sum_{i=1}^n l_i \|\theta - \nu\|_{\mathcal{PC}} \\
= & \left[k_0 a r + 2k_0 a (G_{\mathcal{A}} r + \|g(0)\|) + \mathbf{M}_0^\alpha \Theta_{\mathcal{A}} a + \frac{\mathbf{M}_0^\alpha}{\Gamma(\alpha)} H_{\mathcal{A}} a + k_0 a r \sum_{i=1}^n l_i + \mathbf{M}_0^\alpha \sum_{i=1}^n l_i \right] \|\theta - \nu\|_{\mathcal{PC}} \\
= & \rho \|\theta - \nu\|_{\mathcal{PC}}
\end{aligned}$$

From this inequality it follows that for any $t \in [0, b]$, we have

$$\|(\Theta\theta)(t) - (\Theta\nu)(t)\| \leq \rho \|\theta - \nu\|_{\mathcal{PC}}.$$

□

As presented in the previous section, here we also have the special case, when we choose $\alpha = 1$, given by the following Theorem.

Theorem 4.2. *Let $\theta_0 \in \Lambda$ and let $\mathbf{B}_r = \{\theta \in \mathcal{PC}([0, b], \Lambda); \|\theta\| \leq r\}$, $r > 0$. If the assumptions \mathbf{UC}_1 - \mathbf{UC}_5 are satisfied, then Eq.(1) has a unique mild solution.*

Proof. Follows straight **Theorem 4.1**. □

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REFERENCES

- [1] Gou, H., and B. Li. "Local and global existence of mild solution to impulsive fractional semilinear integro-differential equation with noncompact semigroup." *Commun. Nonlinear Sci. Numer. Simul.* 42 (2017): 204-214.
- [2] Debbouche, A.. "Fractional evolution integro-differential systems with nonlocal conditions." *Adv. Dyn. Syst. Appl* 5.1 (2010): 49-60.
- [3] Debbouche, A., and D. Baleanu. "Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems." *Comput. Math. Appl.* 62.3 (2011): 1442-1450.
- [4] Samuel, F. P., and K. Balachandran. "Existence of solutions for quasi-linear impulsive functional integrodifferential equations in Banach spaces." *J. Nonlinear Sci. Appl* 7 (2014): 115-125.
- [5] Byszewski, L., and V. Lakshmikantham, Theorems about the existence and uniqueness of solutions of a nonlocal Cauchy problem in Banach spaces, *Applicable Anal.*, 40 (1990), 11-19.
- [6] Ntouyas, S. K., P. Ch. Tsamatos, Global existence for semilinear evolution equations with nonlocal conditions, *J. Math. Anal. Appl.*, 210 (1997), 679–687.
- [7] Banas, J., and K. Goebel, Measure of noncompactness in Banach spaces, *Lecture Notes in Pure and Applied Mathematics*, vol. 60, Marcle Dekker, New York, 1980.
- [8] Banas, J. and W. G. El-Sayed, Measures of noncompactness and solvability of an integral equation in the class of functions of locally bounded variation, *J. Math. Anal. Appl.*, 167 (1992), 133–151.
- [9] Ayerbe, J.M., T.D. Benavides and G.L. Acedo, Measure of noncompactness in metric fixed point theorem, *Birkhauser, Basel*, 1997.
- [10] Emmanuele, G. Measures of weak noncompactness and fixed point theorems, *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, 25 (1981), 353–358.
- [11] Fan, Z., and G. Li. "Existence results for semilinear differential equations with nonlocal and impulsive conditions." *J. Funct. Anal.* 258.5 (2010): 1709-1727.
- [12] Hernández, E., and Donal O'Regan. "On a new class of abstract impulsive differential equations." *Proc. Amer. Math. Soc.* 141.5 (2013): 1641-1649.
- [13] Ji, S., and G. Li. "Existence results for impulsive differential inclusions with nonlocal conditions." *Comput. Math. Appl.* 62.4 (2011): 1908-1915.
- [14] Sousa, J. Vanterler da C., and E. Capelas de Oliveira. "On the ψ -Hilfer fractional derivative." *Commun. Nonlinear Sci. Numer. Simul.* 60 (2018): 72-91.
- [15] Sousa, J. Vanterler da C., and E. Capelas De Oliveira. "Ulam–Hyers stability of a nonlinear fractional Volterra integro-differential equation." *Appl. Math. Lett.* 81 (2018): 50-56.
- [16] Sousa, J. Vanterler da C., Kishor D. Kucche, and E. Capelas De Oliveira. "Stability of ψ -Hilfer impulsive fractional differential equations." *Appl. Math. Lett.* 88 (2019): 73-80.
- [17] Sousa, J. Vanterler da C., and E. Capelas de Oliveira. "On the Ulam–Hyers–Rassias stability for nonlinear fractional differential equations using the ψ -Hilfer operator." *J. Fixed Point Theory Appl.* 20.3 (2018): 96.
- [18] Sousa, J. Vanterler da C., and E. Capelas de Oliveira. "Leibniz type rule: ψ -Hilfer fractional operator." *Commun. Nonlinear Sci. Numer. Simul.* 77 (2019): 305-311.
- [19] Sousa, J. Vanterler da C., M. Magnun, L. A. Magna.,and E. Capelas de Oliveira. "Validation of a fractional model for erythrocyte sedimentation rate." *Comput. Appl. Math.* 37.5 (2018): 6903-6919.
- [20] Sousa, J. Vanterler da C., and E. Capelas de Oliveira. "On the Ψ -fractional integral and applications." *Comput. Appl. Math.* 38.1 (2019): 4.
- [21] Sousa, J. Vanterler da C., E. Capelas de Oliveira, and L. A. Magna. "Fractional calculus and the ESR test." *AIMS Math.* 2.4 (2017): 692.
- [22] Sousa, J. Vanterler da C., D. S. Oliveira, and E. Capelas de Oliveira. "A note on the mild solutions of Hilfer impulsive fractional differential equations." *arXiv:1811.09256* (2018).
- [23] Sousa, J. Vanterler da C., D. Ferreira Gomes, and E. Capelas de Oliveira. "A new class of mild and strong solutions of integro-differential equation of arbitrary order in Banach space." *arXiv:1812.11197* (2018).
- [24] Sousa, J. Vanterler da C. "Existence of mild solutions to Hilfer fractional evolution equations in Banach space." *arXiv:1812.02213* (2018).
- [25] Teodoro, G. Sales, J. A Tenreiro Machado, and E. Capelas De Oliveira. "A review of definitions of fractional derivatives and other operators." *J. Comput. Phys.* 388 (2019): 195-208.

- [26] Yang, X.-J., F. Gao, J. A. Tenreiro Machado, D. Baleanu "Exact travelling wave solutions for local fractional partial differential equations in mathematical physics." *Math. Meth. Engineering*. Springer, Cham, 2019. 175-191.
- [27] Moghaddam, B. P., J. A. Tenreiro Machado, and M. L. Morgado. "Numerical approach for a class of distributed order time fractional partial differential equations." *Appl. Numer. Math.* 136 (2019): 152-162.
- [28] Ortigueira, M. D., D. Valério, and J. Tenreiro Machado. "Variable order fractional systems." *Commun. Nonlinear Sci. Numer. Simul.* 71 (2019): 231-243.
- [29] Almeida, R., A. M. C. Brito da Cruz, N. Martins, and M. Teresa T. Monteiro "An epidemiological MSEIR model described by the Caputo fractional derivative." *Inter. J. Dyn. Control* 7.2 (2019): 776-784.
- [30] Almeida, R., Agnieszka B. Malinowska, and Tatiana Odziejewicz. "On systems of fractional differential equations with the ψ -Caputo derivative and their applications." *Math. Meth. Appl. Sci.* (2019).
- [31] Almeida, R. "A Caputo fractional derivative of a function with respect to another function." *Commun. Nonlinear Sci. Numer. Simul.* 44 (2017): 460-481.
- [32] Nemati, S., Pedro M. Lima, and Delfim F. M. Torres. "A numerical approach for solving fractional optimal control problems using modified hat functions." *Commun. Nonlinear Sci. Numer. Simul.* 78 (2019): 104849.
- [33] Ammi, M. R. S., and Delfim F. M. Torres. "Optimal control of a nonlocal thermistor problem with ABC fractional time derivatives." *Comput. Math. Appl.* 78.5 (2019): 1507-1516.
- [34] Diethelm, K., and N. J. Ford. "Analysis of fractional differential equations." *J. Math. Anal. Appl.* 265.2 (2002): 229-248.
- [35] Lakshmikantham, V., and A. S. Vatsala. "Basic theory of fractional differential equations." *Nonlinear Analysis: Theory, Meth. Appl.* 69.8 (2008): 2677-2682.
- [36] Kilbas, A. A., H. M. Srivastava, and J. J. Trujillo. *Theory and applications of fractional differential equations*. North-Holland, New York (2006).
- [37] Yong, Z., W. Jinrong, and Z. Lu. *Basic theory of fractional differential equations*. World Scientific, 2016.
- [38] Arjunan, M. Mallika, V. Kavitha, and S. Selvi. "Existence results for impulsive differential equations with nonlocal conditions via measures of noncompactness." *J. Nonlinear Sci. Appl* 5 (2012): 195-205.
- [39] Benchohra, M., S. Litimein, and J. J. Nieto. "Semilinear fractional differential equations with infinite delay and non-instantaneous impulses." *J. Fixed Point Theory Appl.* 21.1 (2019): 21.
- [40] Tuan, N., Nane, E., O'Regan, D., Phuong, N. "Approximation of mild solutions of a semilinear fractional differential equation with random noise." *Proc. Amer. Math. Soc.* (2020).
- [41] Tuan, N. H., A. Debbouche, and T. B. Ngoc. "Existence and regularity of final value problems for time fractional wave equations." *Comput. Math. Appl.* 78.5 (2019): 1396-1414.
- [42] Benchohra, M., S. Litimein, and J. J. Nieto. "Semilinear fractional differential equations with infinite delay and non-instantaneous impulses." *J. Fixed Point Theory Appl.* 21.1 (2019): 21.
- [43] Boudjerida, A., D. Seba, and G. M. N'Guérékata. "Controllability of coupled systems for impulsive φ -Hilfer fractional integro-differential inclusions." *Applicable Anal.* (2020): 1-18.
- [44] Sakthivel, R., N. I. Mahmudov, and J. J. Nieto. "Controllability for a class of fractional-order neutral evolution control systems." *Appl. Math. Comput.* 218.20 (2012): 10334-10340.
- [45] Shi, Y. "A study on the mild solution of impulsive fractional evolution equations." *Appl. Math. Comput.* 273 (2016): 465-476.
- [46] Liu, Z., J. Lv, and R. Sakthivel. "Approximate controllability of fractional functional evolution inclusions with delay in Hilbert spaces." *IMA J. Math. Control Infor.* 31.3 (2014): 363-383.
- [47] Chalishajar, D. N., K. Malar, and R. Ilavarasi. "Existence and controllability results of impulsive fractional neutral integro-differential equation with sectorial operator and infinite delay." *AIP Conference Proceedings*. Vol. 2159. No. 1. AIP Publishing LLC, 2019.
- [48] Dabas, J., and A. Chauhan. "Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay." *Math. Computer Model.* 57.3-4 (2013): 754-763.
- [49] Nirmala, R. J., K. Balachandran, and J. J. Trujillo. "Null controllability of fractional dynamical systems with constrained control." *Frac. Calc. Appl. Anal.* 20.2 (2017): 553-565.

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