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Jérôme Lemoine and Arnaud Münch*

Weak least-squares approaches for the 2D Navier-Stokes system

Abstract: We analyze a least-squares approach in order to approximate weak solutions of the 2D-Navier Stokes system. In a first part, we consider the steady case and introduce a quadratic functional based on a weak norm of the state equation. We construct a minimizing sequence for the functional which converges strongly to a solution of the equation. After a finite number of iterates related to the value of the viscosity constant, the convergence is quadratic, from any initial guess. We then apply iteratively the analysis on the backward Euler scheme associated to the unsteady Navier-Stokes equation and prove the convergence of the iterative process uniformly with respect to the time discretization. In a second part, we reproduce the analysis for the unsteady case by introducing a space-time least-squares functional. The method turns out to be related to the globally convergent damped Newton approach applied to the Navier-Stokes operator, in contrast to standard Newton method used to solve the weak formulation of the Navier-Stokes system. Numerical experiments illustrate our analysis.

Keywords: Navier-Stokes equation, Implicit time scheme, Least-squares approach, Space-time variational formulation, Damped Newton method.

1 Introduction - Motivation

Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set whose boundary $\partial\Omega$ is Lipschitz. We denote by $\mathcal{V} = \{v \in \mathcal{D}(\Omega)^2, \nabla \cdot v = 0\}$, \mathbf{H} the closure of \mathcal{V} in $L^2(\Omega)^2$ and \mathbf{V} the closure of \mathcal{V} in $H^1(\Omega)^2$. Endowed with the norm $\|v\|_{\mathbf{V}} = \|\nabla v\|_2 := \|\nabla v\|_{(L^2(\Omega))^4}$, \mathbf{V} is an Hilbert space. The dual \mathbf{V}' of \mathbf{V} , endowed with the dual norm

$$\|v\|_{\mathbf{V}'} = \sup_{w \in \mathbf{V}, \|w\|_{\mathbf{V}}=1} \langle v, w \rangle_{\mathbf{V}' \times \mathbf{V}}$$

is also a Hilbert space. We denote $\langle \cdot, \cdot \rangle_{\mathbf{V}'}$ the scalar product associated to the norm $\|\cdot\|_{\mathbf{V}'}$.

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Let $T > 0$. We note $Q_T := \Omega \times (0, T)$ and $\Sigma_T := \partial\Omega \times (0, T)$.

The Navier-Stokes system describes a viscous incompressible fluid flow in the bounded domain Ω during the time interval $(0, T)$ submitted to the external force f . It reads as follows :

$$\begin{cases} y_t - \nu \Delta y + (y \cdot \nabla) y + \nabla p = f, & \nabla \cdot y = 0 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = u_0, & \text{in } \Omega, \end{cases} \quad (1.1)$$

where y is the velocity of the fluid, p its pressure and ν is the viscosity constant. We refer to [15, 21, 24].

We recall (see [24]) that for $f \in L^2(0, T; \mathbf{V}')$ and $u_0 \in \mathbf{H}$, there exists a unique weak solution $y \in L^2(0, T; \mathbf{V})$, $\partial_t y \in L^2(0, T; \mathbf{V}')$ of the system

$$\begin{cases} \frac{d}{dt} \int_{\Omega} y \cdot w + \nu \int_{\Omega} \nabla y \cdot \nabla w + \int_{\Omega} y \cdot \nabla y \cdot w = \langle f, w \rangle_{\mathbf{V}' \times \mathbf{V}}, & \forall w \in \mathbf{V} \\ y(\cdot, 0) = u_0, & \text{in } \Omega. \end{cases} \quad (1.2)$$

This work is concerned with the approximation of solution for (1.2), that is, the explicit construction of a sequence $(y_k)_{k \in \mathbb{N}}$ converging to a solution y for a suitable norm. In most of the works devoted to this topic (we refer for instance to [8, 18]), the approximation of (1.2) is addressed through a time marching method. Given $\{t_n\}_{n=0 \dots N}$, $N \in \mathbb{N}$, a uniform discretization of the time interval $(0, T)$ and $\delta t = T/N$ the corresponding time discretization step, we mention for instance the unconditionally stable backward Euler scheme

$$\begin{cases} \int_{\Omega} \frac{y^{n+1} - y^n}{\delta t} \cdot w + \nu \int_{\Omega} \nabla y^{n+1} \cdot \nabla w + \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot w \\ = \langle f^n, w \rangle_{\mathbf{V}' \times \mathbf{V}}, \quad \forall n \geq 0, \quad \forall w \in \mathbf{V} \\ y^0(\cdot, 0) = u_0, & \text{in } \Omega \end{cases} \quad (1.3)$$

with $f^n := \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} f(\cdot, s) ds$. The piecewise linear interpolation (in time) of $\{y^n\}_{n \in [0, N]}$ weakly converges in $L^2(0, T; \mathbf{V})$ toward a solution y of (1.2) as δt goes to zero (we refer to [24, chapter 3, section 4]). Moreover, it achieves a first order convergence with respect to δt . We also refer to [25] for a stability analysis of the scheme in long time and to [22].

For each $n \geq 0$, the determination of y^{n+1} from y^n requires the resolution of a (non-linear) steady Navier-Stokes equation, parametrized by ν and δt . This can be done using Newton type methods (see for instance [19, Section 10.3]) for the

weak formulation of (1.3) which reads as follows: find $y = y^{n+1} \in \mathbf{V}$ solution of

$$\alpha \int_{\Omega} y \cdot w + \nu \int_{\Omega} \nabla y \cdot \nabla w + \int_{\Omega} y \cdot \nabla y \cdot w = \langle f, w \rangle_{H^{-1}(\Omega)^2 \times H_0^1(\Omega)^2} + \alpha \int_{\Omega} g \cdot w, \quad \forall w \in \mathbf{V} \quad (1.4)$$

with

$$\alpha := \frac{1}{\delta t} > 0, \quad f := f^n = \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} f(\cdot, s) ds, \quad g = y^n. \quad (1.5)$$

Introducing the application $F : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} F(y, z) := \int_{\Omega} \alpha y \cdot z + \nu \nabla y \cdot \nabla z + (y \cdot \nabla) y \cdot z \\ - \langle f, z \rangle_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} - \alpha \int_{\Omega} g \cdot z = 0, \quad \forall z \in \mathbf{V} \end{aligned} \quad (1.6)$$

the Newton algorithm formally reads

$$\begin{cases} y_0 \in \mathbf{V}, \\ D_y F(y_k, w) \cdot (y_{k+1} - y_k) = -F(y_k, w), \quad \forall w \in \mathbf{V}, k \geq 0. \end{cases} \quad (1.7)$$

If the initial guess y_0 is close enough to a solution of (1.4) (i.e. a solution satisfying $F(y, w) = 0$ for all $w \in \mathbf{V}$) and if $D_y F(y_k, \cdot)$ is invertible, then the sequence $\{y_k\}_k$ converges. We refer to [19, Section 10.3] and [5, Chapter 6]).

Alternatively, we may also employ least-squares methods which consists in minimizing quadratic functional, which measure how an element y is close to the solution. For instance, we may introduce the extremal problem : $\inf_{y \in \mathbf{V}} E(y)$ with $E : \mathbf{V} \rightarrow \mathbb{R}^+$ defined by

$$E(y) := \frac{1}{2} \int_{\Omega} \alpha |v|^2 + \nu |\nabla v|^2 \quad (1.8)$$

where the corrector v is the unique solution in \mathbf{V} of the formulation

$$\begin{aligned} \alpha \int_{\Omega} v \cdot w + \nu \int_{\Omega} \nabla v \cdot \nabla w = -\alpha \int_{\Omega} y \cdot w - \nu \int_{\Omega} \nabla y \cdot \nabla w - \int_{\Omega} y \cdot \nabla y \cdot w \\ + \langle f, w \rangle_{H^{-1}(\Omega)^2 \times H_0^1(\Omega)^2} + \alpha \int_{\Omega} g \cdot w, \quad \forall w \in \mathbf{V}. \end{aligned} \quad (1.9)$$

Remark that $E(y) = 0$ is zero if and only if $y \in \mathbf{V}$ is a (weak) solution of (1.4), i.e. a zero of $F(y, w) = 0$ for all $w \in \mathbf{V}$. As a matter of fact, the infimum is reached.

Least-squares methods to solve nonlinear boundary value problems have been the subject of intensive developments in the last decades, as they present several advantages, notably on computational and stability viewpoints. We refer to the books [1, 9]. The minimization of the functional E over \mathbf{V} leads to a so-called weak least squares method. Actually, there is a close connection between E and F through the equality $\sqrt{2E(y)} = \sup_{w \in \mathbf{V}, w \neq 0} \frac{F(y,w)}{\|w\|_{\mathbf{V}}}$ from which we deduce that E is equivalent to the \mathbf{V}' -norm of the Navier Stokes equation (see Remark 2.9 below). The terminology “ H^{-1} least-squares method” is employed in [2] where this method has been introduced and numerically implemented to approximate the solutions of (1.2) through the scheme (1.3). We also mention [5, Chapter 4, Section 6] which studied later the use of a least-squares strategy to solve a steady Navier-Stokes equation without incompressibility constraint. In a first part of this work, we analyze rigorously the method introduced in [2, 7] and show that one may construct minimizing sequences in \mathbf{V} for E that converge strongly to a solution of (1.2). We then justify the use of that weak least-squares method to solve iteratively the scheme (1.3), leading to an approximation of the solution of (1.2). This requires to show some convergence properties of the minimizing sequence for E , uniformly with respect to n , related to the time discretization. As we shall see, this requires smallness assumptions on the data u_0 and f . In a second part, we extend this analysis to a full space-time setting. More precisely, following the terminology of [2], we introduce the following $L^2(0, T; \mathbf{V}')$ least-squares functional $\tilde{E} : H^1(0, T; \mathbf{V}') \cap L^2(0, T; \mathbf{V}) \rightarrow \mathbb{R}^+$

$$\tilde{E}(y) := \frac{1}{2} \|y_t + \nu B_1(y) + B(y, y) - f\|_{L^2(0, T; \mathbf{V}')}^2 \quad (1.10)$$

where B_1 and B are defined in Lemmas 3.2 and 3.3. Again, the real quantity $\tilde{E}(y)$ measures how the element y is close to the solution of (1.2). The minimization of this functional leads to a so-called continuous weak least-squares type method.

This paper is organized as follows. In Section 2, we analyze the least-squares method (2.4)-(1.9) associated to weak solutions of (1.4). We first show that E is differentiable over \mathbf{V} and that any critical point for E in the ball $B := \{y \in \mathbf{V}, \tau(y) \leq 1\}$ (see Definition 2.1) is also a zero of E . This is done by introducing a descent direction Y_1 for E at any point $y \in \mathbf{V}$ for which $E'(y) \cdot Y_1$ is proportional to $E(y)$. Then, assuming that there exists a least one solution of (1.4) in the ball B , we show that any minimizing sequence $\{y_k\}_{(k \in \mathbb{N})}$ for E uniformly in B strongly converges to a solution of (1.4). Such limit belongs to B and is actually the unique solution. Eventually, we construct a minimizing sequence (defined in (2.18)) based on the element Y_1 and initialized with g assume in \mathbf{V} . We show that, if α is large enough, then this particular sequence is uniformly in B and converges (quadratically after a finite number of iterates related to the values of ν and α)

strongly to the solution of (1.4). A section of remarks emphasizes that this specific sequence coincides the one obtained from the damped Newton method (a globally convergent generalization of (1.7)) and with (1.7) for α large enough. We also emphasize that a similar convergence result hold true with minimizing sequences based on the gradient of E , as used in [2]. Then, in Section 2.4, as an application, we consider the least-squares approach to solve iteratively the backward Euler scheme (see (2.36)). For each $n > 0$, we define a minimizing sequence $\{y_k^{n+1}\}_{k \geq 0}$ based on Y_1^{n+1} and initialize with y^n , in order to approximate the y^{n+1} . Adapting the global convergence result of Section 2, we then show, assume α large enough (which is achieved by taking a small enough time discretization step δt) and smallness property on $\|u_0\|_2 + \|f\|_{L^2(0,T;H^{-1}(\Omega))}$, the strong convergence of the minimizing sequences, uniformly with respect to the time discretization. The analysis is performed in 2D for weak and regular solutions and in 3D for regular solutions. In particular, we justify the use of Newton type methods to solve implicit time schemes for (1.1), as mentioned in [19, Section 10.3]. To the best of our knowledge, such analysis of convergence is original.

In Section 3, we reproduce the analysis with the weak solution of (1.2) associated to initial data u_0 in \mathbf{H} and source term $f \in L^2(0, T; \mathbf{V}')$. In Section 4, we discuss numerical experiments based on finite element approximations in space for two 2D geometries: the celebrated example of the channel with a backward facing step and the semi-circular driven cavity introduced in [6]. We notably exhibit the robustness of the damped Newton method (compared to the Newton one), including for small values of the viscosity constant. Section 5 concludes with some perspectives.

We emphasize that the 3D case can be addressed as well: we refer to [14, 13].

2 Analysis of a Least-squares method for a steady Navier-Stokes equation

We analyse in this section a least-squares method to solve the steady Navier-Stokes equation (1.4): we follow and improve [11] where the particular case $\alpha = 0$ is addressed.

2.1 Technical preliminary results

We endow the space \mathbf{V} with the norm $\|y\|_{\mathbf{V}} := \|\nabla y\|_2$, for all $y \in \mathbf{V}$. We shall also use the following notations

$$\|y\|_{\mathbf{V}}^2 := \alpha\|y\|_2^2 + \nu\|\nabla y\|_2^2, \quad \forall y \in \mathbf{V}$$

and $\langle y, z \rangle_{\mathbf{V}} := \alpha \int_{\Omega} yz + \nu \int_{\Omega} \nabla y \cdot \nabla z$ so that $\langle y, z \rangle_{\mathbf{V}} \leq \|y\|_{\mathbf{V}} \|z\|_{\mathbf{V}}$ for any $y, z \in \mathbf{V}$.

In the sequel, we repeatedly use the following classical estimates (see [24]).

Lemma 2.1. *Let any $u, v \in \mathbf{V}$. Then*

$$-\int_{\Omega} u \cdot \nabla u \cdot v = \int_{\Omega} u \cdot \nabla v \cdot u \leq \sqrt{2}\|u\|_2 \|\nabla v\|_2 \|\nabla u\|_2. \quad (2.1)$$

Definition 2.1. *For any $y \in \mathbf{V}$, we define*

$$\tau(y) := \frac{\|y\|_{\mathbf{V}}}{\sqrt{2\alpha\nu}}.$$

We shall also repeatedly use the following Young type inequalities.

Lemma 2.2. *For any $u, v \in \mathbf{V}$, the following inequality holds true :*

$$\sqrt{2}\|u\|_2 \|\nabla v\|_2 \|\nabla u\|_2 \leq \tau(v) \|u\|_{\mathbf{V}}^2 \quad (2.2)$$

Let $f \in H^{-1}(\Omega)^2$, $g \in L^2(\Omega)^2$ and $\alpha \in \mathbb{R}_+^*$. The following result holds true:

Proposition 2.3. *Assume $\Omega \subset \mathbb{R}^d$ is bounded and Lipschitz. There exists at least one solution y of (1.4) satisfying*

$$\|y\|_{\mathbf{V}}^2 \leq c_0 \nu \|f\|_{H^{-1}(\Omega)^d}^2 + \alpha \|g\|_2^2 \quad (2.3)$$

where $c_0 > 0$, only connected to the Poincaré constant, depends on Ω . If moreover, Ω is C^2 and $f \in L^2(\Omega)^d$, then any solution $y \in \mathbf{V}$ of (1.4) belongs to $H^2(\Omega)^2$.

Proof. We refer to [15]. □

Lemma 2.4. *Assume that a solution $y \in \mathbf{V}$ of (1.4) satisfies $\tau(y) < 1$. Then, such solution is the unique solution of (1.4).*

Proof. Let $y_1 \in \mathbf{V}$ and $y_2 \in \mathbf{V}$ be two solutions of (1.4). Set $Y = y_1 - y_2$. Then,

$$\alpha \int_{\Omega} Y \cdot w + \nu \int_{\Omega} \nabla Y \cdot \nabla w + \int_{\Omega} y_2 \cdot \nabla Y \cdot w + \int_{\Omega} Y \cdot \nabla y_1 \cdot w = 0 \quad \forall w \in \mathbf{V}.$$

We now take $w = Y$ and use that $\int_{\Omega} y_2 \cdot \nabla Y \cdot Y = 0$. We use (2.1) and (2.2) to get

$$\|Y\|_{\mathbf{V}}^2 = - \int_{\Omega} Y \cdot \nabla y_1 \cdot Y \leq \tau(y_1) \|Y\|_{\mathbf{V}}^2$$

leading to $(1 - \tau(y_1)) \|Y\|_{\mathbf{V}}^2 \leq 0$. Consequently, if $\tau(y_1) < 1$ then $Y = 0$ and the solution of (1.4) is unique. In particular, in view of (2.3), this holds if the data satisfy $\nu \|g\|_2^2 + \frac{c_0}{\alpha} \|f\|_{H^{-1}(\Omega)}^2 < 2\nu^3$. \square

We now introduce our least-squares functional $E : \mathbf{V} \rightarrow \mathbb{R}^+$ as follows

$$E(y) := \frac{1}{2} \int_{\Omega} (\alpha |v|^2 + \nu |\nabla v|^2) = \frac{1}{2} \|v\|_{\mathbf{V}}^2 \tag{2.4}$$

where the corrector $v \in \mathbf{V}$ is the unique solution of the linear formulation (1.9). In particular, the corrector v satisfies the estimate:

$$\|v\|_{\mathbf{V}} \leq \|y\|_{\mathbf{V}} \left(1 + \frac{\|y\|_{\mathbf{V}}}{2\sqrt{\alpha\nu}} \right) + \sqrt{\frac{c_0 \|f\|_{H^{-1}}^2}{\nu} + \alpha \|g\|_2^2}. \tag{2.5}$$

Conversely, we also have

$$\|y\|_{\mathbf{V}} \leq \|v\|_{\mathbf{V}} + \sqrt{\frac{c_0 \|f\|_{H^{-1}}^2}{\nu} + \alpha \|g\|_2^2}. \tag{2.6}$$

The infimum of E is equal to zero and is reached by a solution of (1.4). In this sense, the functional E is a so-called error functional which measures, through the corrector variable v , the deviation of the pair y from being a solution of the underlying equation (1.4).

A practical way of taking a functional to its minimum is through some (clever) use of descent directions, i.e. the use of its derivative. In doing so, the presence of local minima is always something that may dramatically spoil the whole scheme. The unique structural property that discards this possibility is the strict convexity of the functional. However, for non-linear equations like (1.4), one cannot expect this property to hold for the functional E in (2.4). Nevertheless, we insist in that for a descent strategy applied to the extremal problem $\min_{y \in \mathbf{V}} E(y)$ numerical procedures cannot converge except to a global minimizer leading E down to zero.

Indeed, we would like to show that the only critical points for E correspond to solutions of (1.4). In such a case, the search for an element y solution of (1.4) is reduced to the minimization of E .

For any $y \in \mathbf{V}$, we now look for an element $Y_1 \in \mathbf{V}$ solution of the following formulation

$$\alpha \int_{\Omega} Y_1 \cdot w + \nu \int_{\Omega} \nabla Y_1 \cdot \nabla w + \int_{\Omega} (y \cdot \nabla Y_1 + Y_1 \cdot \nabla y) \cdot w = -\alpha \int_{\Omega} v \cdot w - \nu \int_{\Omega} \nabla v \cdot \nabla w, \forall w \in \mathbf{V} \tag{2.7}$$

where $v \in \mathbf{V}$ is the corrector (associated to y) solution of (1.9). Y_1 enjoys the following property.

Proposition 2.5. *For all $y \in \mathbf{V}$ satisfying $\tau(y) < 1$, there exists a unique solution Y_1 of (2.7) associated to y . Moreover, this solution satisfies*

$$(1 - \tau(y)) \|Y_1\|_{\mathbf{V}} \leq \sqrt{2E(y)}. \quad (2.8)$$

Proof. The proof uses the arguments of Lemma 2.4. We define the bilinear and continuous form $a : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$ by

$$a(Y, w) = \alpha \int_{\Omega} Y \cdot w + \nu \int_{\Omega} \nabla Y \cdot \nabla w + \int_{\Omega} (y \cdot \nabla Y + Y \cdot \nabla y) \cdot w. \quad (2.9)$$

so that $a(Y, Y) = \|Y\|_{\mathbf{V}}^2 + \int_{\Omega} Y \cdot \nabla y \cdot Y$. Using (2.2), we obtain $a(Y, Y) \geq (1 - \tau(y)) \|Y\|_{\mathbf{V}}^2$ for all $Y \in \mathbf{V}$. Lax-Milgram lemma leads to the existence and uniqueness of Y_1 provided $\tau(y) < 1$. Then, putting $w = Y_1$ in (2.7) implies

$$a(Y_1, Y_1) \leq -\alpha \int_{\Omega} v \cdot Y_1 - \nu \int_{\Omega} \nabla v \cdot \nabla Y_1 \leq \|Y_1\|_{\mathbf{V}} \|v\|_{\mathbf{V}} = \|Y_1\|_{\mathbf{V}} \sqrt{2E(y)}$$

leading to (2.8). \square

We now check the differentiability of the least-squares functional.

Proposition 2.6. *For all $y \in \mathbf{V}$, the map $Y \mapsto E(y + Y)$ is a differentiable function on the Hilbert space \mathbf{V} and for any $Y \in \mathbf{V}$, we have*

$$E'(y) \cdot Y = \int_{\Omega} \alpha v \cdot V + \nu \nabla v \cdot \nabla V \quad (2.10)$$

where $V \in \mathbf{V}$ is the unique solution of

$$\alpha \int_{\Omega} V \cdot w + \nu \int_{\Omega} \nabla V \cdot \nabla w = -\alpha \int_{\Omega} Y \cdot w - \nu \int_{\Omega} \nabla Y \cdot \nabla w - \int_{\Omega} (y \cdot \nabla Y + Y \cdot \nabla y) \cdot w, \quad \forall w \in \mathbf{V}. \quad (2.11)$$

Proof. Let $y \in \mathbf{V}$ and $Y \in \mathbf{V}$. We have $E(y + Y) = \frac{1}{2} \| \bar{V} \|_{\mathbf{V}}^2$ where $\bar{V} \in \mathbf{V}$ is the unique solution of

$$\begin{aligned} \alpha \int_{\Omega} \bar{V} \cdot w + \nu \int_{\Omega} \nabla \bar{V} \cdot \nabla w + \alpha \int_{\Omega} (y + Y) \cdot w + \nu \int_{\Omega} \nabla (y + Y) \cdot \nabla w \\ + \int_{\Omega} (y + Y) \cdot \nabla (y + Y) \cdot w - \langle f, w \rangle_{\mathbf{V}' \times \mathbf{V}} - \alpha \int_{\Omega} g w = 0, \quad \forall w \in \mathbf{V}. \end{aligned}$$

If $v \in \mathbf{V}$ is the solution of (1.9) associated to y , $v' \in \mathbf{V}$ is the unique solution of

$$\alpha \int_{\Omega} v' \cdot w + \nu \int_{\Omega} \nabla v' \cdot \nabla w + \int_{\Omega} Y \cdot \nabla Y \cdot w = 0, \quad \forall w \in \mathbf{V} \quad (2.12)$$

and $V \in \mathbf{V}$ is the unique solution of (2.11), then it is straightforward to check that $\bar{V} - v - v' - V \in \mathbf{V}$ is solution of

$$\alpha \int_{\Omega} (\bar{V} - v - v' - V) \cdot w + \nu \int_{\Omega} \nabla (\bar{V} - v - v' - V) \cdot \nabla w = 0, \quad \forall w \in \mathbf{V}$$

and therefore $\bar{V} - v - v' - V = 0$. Thus

$$\begin{aligned} E(y + Y) &= \frac{1}{2} \left\| v + v' + V \right\|_{\mathbf{V}}^2 \\ &= \frac{1}{2} \|v\|_{\mathbf{V}}^2 + \frac{1}{2} \|v'\|_{\mathbf{V}}^2 + \frac{1}{2} \|V\|_{\mathbf{V}}^2 + \langle v, v' \rangle_{\mathbf{V}} + \langle v, v' \rangle_{\mathbf{V}} + \langle v, v' \rangle_{\mathbf{V}}. \end{aligned} \quad (2.13)$$

Then, writing (2.11) with $w = V$ and using (2.1), we obtain

$$\begin{aligned} \|V\|_{\mathbf{V}}^2 &\leq \|V\|_{\mathbf{V}} \|Y\|_{\mathbf{V}} + \sqrt{2} (\|y\|_2 \|\nabla Y\|_2 + \|Y\|_2 \|\nabla y\|_2) \|\nabla V\|_2 \\ &\leq \|V\|_{\mathbf{V}} \|Y\|_{\mathbf{V}} + \frac{\sqrt{2}}{\sqrt{\alpha\nu}} \|y\|_{\mathbf{V}} \|Y\|_{\mathbf{V}} \|\nabla V\|_2 \end{aligned}$$

leading to $\|V\|_{\mathbf{V}} \leq \|Y\|_{\mathbf{V}} (1 + \frac{\sqrt{2}}{\sqrt{\alpha\nu}} \|y\|_{\mathbf{V}})$. Similarly, using (2.12), we obtain $\|v'\|_{\mathbf{V}} \leq \frac{1}{\sqrt{2\alpha\nu}} \|Y\|_{\mathbf{V}}^2$. It follows that $\frac{1}{2} \|v'\|_{\mathbf{V}}^2 + \frac{1}{2} \|V\|_{\mathbf{V}}^2 + \langle v, v' \rangle_{\mathbf{V}} + \langle v, v' \rangle_{\mathbf{V}} = o(\|Y\|_{\mathbf{V}})$ and from (2.13) that

$$E(y + Y) = E(y) + \langle v, V \rangle + o(\|Y\|_{\mathbf{V}}).$$

Eventually, the estimate $|\langle v, V \rangle| \leq \|v\|_{\mathbf{V}} \|V\|_{\mathbf{V}} \leq (1 + \frac{\sqrt{2}}{\sqrt{\alpha\nu}} \|y\|_{\mathbf{V}}) \sqrt{E(y)} \|Y\|_{\mathbf{V}}$ gives the continuity of the linear map $Y \mapsto \langle v, V \rangle$. \square

We are now in position to prove the following result which indicates that, in the ball $\{\tau_d(y) < 1\}$ of \mathbf{V} , any critical point for E is also a zero of E .

Proposition 2.7. *For all $y \in \mathbf{V}$ satisfying $\tau(y) < 1$,*

$$(1 - \tau(y)) \sqrt{2E(y)} \leq \frac{1}{\sqrt{\nu}} \|E'(y)\|_{\mathbf{V}'}$$

Proof. For any $Y \in \mathbf{V}$, $E'(y) \cdot Y = \int_{\Omega} \alpha v \cdot V + \nu \nabla v \cdot \nabla V$ where $V \in \mathbf{V}$ is the unique solution of (2.11). In particular, taking $Y = Y_1$ defined by (2.7), we obtain an element $V_1 \in \mathbf{V}$ solution of

$$\alpha \int_{\Omega} V_1 \cdot w + \nu \int_{\Omega} \nabla V_1 \cdot \nabla w = -\alpha \int_{\Omega} Y_1 \cdot w - \nu \int_{\Omega} \nabla Y_1 \cdot \nabla w - \int_{\Omega} (y \cdot \nabla Y_1 + Y_1 \cdot \nabla y) \cdot w, \quad \forall w \in \mathbf{V}. \quad (2.14)$$

Summing (2.7) and (2.14), we obtain that $v - V_1 \in \mathbf{V}$ solves

$$\alpha \int_{\Omega} (v - V_1)w + \nu \int_{\Omega} (\nabla v - \nabla V_1) \cdot w = 0, \quad \forall w \in \mathbf{V}.$$

This implies that v and V_1 coincide and then that

$$E'(y) \cdot Y_1 = \int_{\Omega} \alpha |v|^2 + \nu |\nabla v|^2 = 2E(y), \quad \forall y \in \mathbf{V}. \quad (2.15)$$

It follows that $2E(y) = E'(y) \cdot Y_1 \leq \|E'(y)\|_{\mathbf{V}'} \|Y_1\|_{\mathbf{V}} \leq \|E'(y)\|_{\mathbf{V}'} \frac{\|Y_1\|_{\mathbf{V}}}{\sqrt{\nu}}$. Proposition 2.5 allows to conclude. \square

Eventually, we prove the following coercivity type inequality for the error functional E .

Proposition 2.8. *Assume that a solution $\bar{y} \in \mathbf{V}$ of (1.4) satisfies $\tau(\bar{y}) < 1$. Then, for all $y \in \mathbf{V}$,*

$$\|y - \bar{y}\|_{\mathbf{V}} \leq (1 - \tau(\bar{y}))^{-1} \sqrt{2E(y)}. \quad (2.16)$$

Proof. For any $y \in \mathbf{V}$, let v be the corresponding corrector and let $Y = y - \bar{y}$. We have

$$\alpha \int_{\Omega} Y \cdot w + \nu \int_{\Omega} \nabla Y \cdot \nabla w + \int_{\Omega} y \cdot \nabla Y \cdot w + \int_{\Omega} Y \cdot \nabla \bar{y} \cdot w = -\alpha \int_{\Omega} v \cdot w - \nu \int_{\Omega} \nabla v \cdot \nabla w, \quad \forall w \in \mathbf{V}. \quad (2.17)$$

For $w = Y$, this equality rewrites

$$\|Y\|_{\mathbf{V}}^2 = - \int_{\Omega} Y \cdot \nabla \bar{y} \cdot Y - \alpha \int_{\Omega} v \cdot Y - \nu \int_{\Omega} \nabla v \cdot \nabla Y.$$

Repeating the arguments of the proof of Proposition 2.5, the results follows. \square

Assuming the existence of a solution of (1.4) in the ball $\{y \in \mathbf{V}, \tau(y) < 1\}$, Proposition 2.7 and Proposition 2.8 imply that any minimizing sequence $\{y_k\}_{(k \in \mathbb{N})}$ for E uniformly in $\{y \in \mathbf{V}, \tau(y) \leq 1\}$ strongly converges to a solution of (1.4). Remark that, from Lemma 2.4, such solution is unique. In the next section, we construct, assuming the parameter α large enough, such sequence $\{y_k\}_{(k \in \mathbb{N})}$.

Remark 2.9. *In order to simplify notations, we have introduced the corrector variable v leading to the functional E . Instead, we may consider the functional $\tilde{E} : \mathbf{V} \rightarrow \mathbb{R}$ defined by*

$$\tilde{E}(y) := \frac{1}{2} \|\alpha y + \nu B_1(y) + B(y, y) - f + \alpha g\|_{\mathbf{V}'}^2,$$

with $B_1 : \mathbf{V} \rightarrow L^2(\Omega)^2$ and $B : \mathbf{V} \times \mathbf{V} \rightarrow L^2(\Omega)^2$ defined by $(B_1(y), w) := (\nabla y, \nabla w)_2$ and $(B(y, z), w) := \int_{\Omega} y \cdot \nabla z \cdot w$ respectively. E and \tilde{E} are equivalent. Precisely, from the definition of v (see (1.9)), we deduce that

$$E(y) = \frac{1}{2} \|v\|_{\mathbf{V}}^2 \leq \frac{c_0^2}{2\nu^2} \|\alpha y + \nu B_1(y) + B(y, y) - f + \alpha g\|_{\mathbf{V}'}^2 = \frac{c_0^2}{\nu^2} \tilde{E}(y), \quad \forall y \in \mathbf{V}.$$

Conversely,

$$\begin{aligned} \|\alpha y + \nu B_1(y) + B(y, y) - f + \alpha g\|_{\mathbf{V}'} &= \sup_{w \in \mathbf{V}, w \neq 0} \frac{\int_{\Omega} (\alpha v w + \nu \nabla v \cdot \nabla w)}{\|w\|_{\mathbf{V}}} \\ &\leq \|v\|_{\mathbf{V}} \sup_{w \in \mathbf{V}, w \neq 0} \frac{\|w\|_{\mathbf{V}}}{\|w\|_{\mathbf{V}}} \leq \sqrt{\alpha c_0^2 + \nu} \|v\|_{\mathbf{V}} \end{aligned}$$

so that $\tilde{E}(y) \leq (\alpha c_0^2 + \nu) E(y)$ for all $y \in \mathbf{V}$.

2.2 A strongly convergent minimizing sequence for E - Link with the damped Newton method

We define in this section a sequence converging strongly to a solution of (1.4) for which E vanishes. According to Proposition 2.7, it suffices to define a minimizing sequence for E included in the ball $\mathbb{B} = \{y \in \mathbf{V}, \tau(y) < 1\}$. In this respect, remark that equality (2.15) shows that $-Y_1$ given by the solution of (2.7) is a descent direction for the functional E . Therefore, we can define at least formally, for any $m \geq 1$, a minimizing sequence $\{y_k\}_{(k \geq 0)}$ as follows:

$$\begin{cases} y_0 \in \mathbf{H} \text{ given,} \\ y_{k+1} = y_k - \lambda_k Y_{1,k}, \quad k \geq 0, \\ \lambda_k = \operatorname{argmin}_{\lambda \in [0, m]} E(y_k - \lambda Y_{1,k}) \end{cases} \quad (2.18)$$

with $Y_{1,k} \in \mathbf{V}$ the solution of the formulation

$$\begin{aligned} \alpha \int_{\Omega} Y_{1,k} \cdot w + \nu \int_{\Omega} \nabla Y_{1,k} \cdot \nabla w + \int_{\Omega} (y_k \cdot \nabla Y_{1,k} + Y_{1,k} \cdot \nabla y_k) \cdot w \\ = -\alpha \int_{\Omega} v_k \cdot w - \nu \int_{\Omega} \nabla v_k \cdot \nabla w, \quad \forall w \in \mathbf{V} \end{aligned} \quad (2.19)$$

and $v_k \in \mathbf{V}$ the corrector (associated to y_k) solution of (1.9) leading (see (2.15)) to $E'(y_k) \cdot Y_{1,k} = 2E(y_k)$.

Remark that from (2.6), the sequence $\{y_k\}_{k > 0}$ is uniformly bounded since y_k satisfies $\|y_k\|_{\mathbf{V}} \leq \sqrt{2E(y_k)} + \sqrt{\frac{c_0}{\nu} \|f\|_{H^{-1}}^2 + \alpha \|g\|_2^2}$. However, we insist that,

in order to justify the existence of the element $Y_{1,k}$, y_k should satisfy $\tau(y_k) < 1$, i.e. $\|\nabla y_k\|_2 < \sqrt{2\alpha\nu}$. We proceed in two steps: first, assuming that the sequence $\{y_k\}_{(k>0)}$ defined by (2.18) satisfies $\tau(y_k) \leq c_1 < 1$ for any k , we show that $E(y_k) \rightarrow 0$ and that $\{y_k\}$ converges strongly in \mathbf{V} to a solution of (1.4). Then, we determine sufficient conditions on the initial guess $y_0 \in \mathbf{V}$ in order that $\tau(y_k) < 1$ for all $k \in \mathbb{N}$.

We start with the following lemma which provides the main property of the sequence $\{E(y_k)\}_{(k \geq 0)}$.

Lemma 2.10. *Assume that the sequence $\{y_k\}_{(k \geq 0)}$ defined by (2.18) satisfy $\tau(y_k) < 1$. Then, for all $\lambda \in \mathbb{R}$, the following estimate holds true*

$$E(y_k - \lambda Y_{1,k}) \leq E(y_k) \left(|1 - \lambda| + \lambda^2 \frac{(1 - \tau(y_k))^{-2}}{\sqrt{\alpha\nu}} \sqrt{E(y_k)} \right)^2. \quad (2.20)$$

Proof. For any real λ and any $y_k, w_k \in \mathbf{V}$ we get the following expansion :

$$\begin{aligned} E(y_k - \lambda w_k) &= E(y_k) - \lambda \int_{\Omega} (\alpha v_k \bar{v}_k + \nu \nabla v_k \cdot \nabla \bar{v}_k) \\ &\quad + \frac{\lambda^2}{2} \int_{\Omega} (\alpha |\bar{v}_k|^2 + \nu |\nabla \bar{v}_k|^2 + 2(\alpha v_k \bar{v}_k + \nu \nabla v_k \cdot \nabla \bar{v}_k)) \\ &\quad - \lambda^3 \int_{\Omega} \alpha \bar{v}_k \bar{v}_k + \nu \nabla \bar{v}_k \cdot \nabla \bar{v}_k + \frac{\lambda^4}{2} \int_{\Omega} \alpha |\bar{v}_k|^2 + \nu |\nabla \bar{v}_k|^2 \end{aligned} \quad (2.21)$$

where $v_k, \bar{v}_k \in \mathbf{V}$ and $\bar{v}_k \in \mathbf{V}$ solves respectively

$$\begin{aligned} \alpha \int_{\Omega} v_k \cdot w + \nu \int_{\Omega} \nabla v_k \cdot \nabla w + \alpha \int_{\Omega} y_k \cdot w + \nu \int_{\Omega} \nabla y_k \cdot \nabla w + \int_{\Omega} y_k \cdot \nabla y_k \cdot w \\ = \langle f, w \rangle_{H^{-1}(\Omega)^2 \times H_0^1(\Omega)^2} + \alpha \int_{\Omega} g \cdot w, \quad \forall w \in \mathbf{V}, \end{aligned} \quad (2.22)$$

$$\begin{aligned} \alpha \int_{\Omega} \bar{v}_k \cdot w + \nu \int_{\Omega} \nabla \bar{v}_k \cdot \nabla w + \alpha \int_{\Omega} w_k \cdot w + \nu \int_{\Omega} \nabla w_k \cdot \nabla w \\ + \int_{\Omega} w_k \cdot \nabla y_k \cdot w + y_k \cdot \nabla w_k \cdot w = 0, \quad \forall w \in \mathbf{V}, \end{aligned} \quad (2.23)$$

and

$$\alpha \int_{\Omega} \bar{v}_k \cdot w + \nu \int_{\Omega} \nabla \bar{v}_k \cdot \nabla w + \int_{\Omega} w_k \cdot \nabla w_k \cdot w = 0, \quad \forall w \in \mathbf{V}. \quad (2.24)$$

Since the corrector \bar{v}_k associated to $Y_{1,k}$ coincides with the corrector v_k associated to y_k , expansion (2.21) reduces to

$$\begin{aligned}
 E(y_k - \lambda Y_{1,k}) &= (1 - \lambda)^2 E(y_k) + \lambda^2 (1 - \lambda) \int_{\Omega} \alpha v_k \bar{v}_k + \nu \nabla v_k \nabla \bar{v}_k \\
 &\quad + \frac{\lambda^4}{2} \int_{\Omega} \alpha |\bar{v}_k|^2 + \nu |\nabla \bar{v}_k|^2 \\
 &\leq (1 - \lambda)^2 E(y_k) + \lambda^2 (1 - \lambda) \|v_k\|_{\mathbf{V}} \|\bar{v}_k\|_{\mathbf{V}} + \frac{\lambda^4}{2} \|\bar{v}_k\|_{\mathbf{V}}^2 \\
 &\leq \left(|1 - \lambda| \sqrt{E(y_k)} + \frac{\lambda^2}{\sqrt{2}} \|\bar{v}_k\|_{\mathbf{V}} \right)^2.
 \end{aligned} \tag{2.25}$$

(2.24) then leads to $\|\bar{v}_k\|_{\mathbf{V}} \leq \frac{\|Y_{1,k}\|_{\mathbf{V}}^2}{\sqrt{2\alpha\nu}} \leq \sqrt{2}(1 - \tau(y_k))^{-2} \frac{E(y_k)}{\sqrt{\alpha\nu}}$ and to (2.20). \square

We are now in position to prove the following convergence result for the sequence $\{E(y_k)\}_{(k \geq 0)}$.

Proposition 2.11. *Let $\{y_k\}_{k \geq 0}$ be the sequence defined by (2.18). Assume that there exists a constant $c_1 \in (0, 1)$ such that $\tau(y_k) \leq c_1$ for all k . Then $E(y_k) \rightarrow 0$ as $k \rightarrow \infty$. Moreover, there exists $k_0 \in \mathbb{N}$ such that the sequence $\{E(y_k)\}_{(k \geq k_0)}$ decays quadratically.*

Proof. The inequality $\tau(y_k) \leq c_1$ and (2.20) implies that

$$E(y_k - \lambda Y_{1,k}) \leq E(y_k) \left(|1 - \lambda| + \lambda^2 c_{\alpha,\nu} \sqrt{E(y_k)} \right)^2, \quad c_{\alpha,\nu} := \frac{(1 - c_1)^{-2}}{\sqrt{\alpha\nu}}. \tag{2.26}$$

Let us denote the polynomial p_k by $p_k(\lambda) = |1 - \lambda| + \lambda^2 c_{\alpha,\nu} \sqrt{E(y_k)}$ for all $\lambda \in [0, m]$. So, we can write

$$\sqrt{E(y_{k+1})} = \min_{\lambda \in [0, m]} \sqrt{E(y_k - \lambda Y_{1,k})} \leq \min_{\lambda \in [0, m]} p_k(\lambda) \sqrt{E(y_k)}.$$

If $c_{\alpha,\nu} \sqrt{E(y_0)} < 1$ (and thus $c_{\alpha,\nu} \sqrt{E(y_k)} < 1$ for all $k \in \mathbb{N}$) then

$$p_k(\tilde{\lambda}_k) := \min_{\lambda \in [0, m]} p_k(\lambda) \leq p_k(1) = c_{\alpha,\nu} \sqrt{E(y_k)}$$

and thus

$$c_{\alpha,\nu} \sqrt{E(y_{k+1})} \leq (c_{\alpha,\nu} \sqrt{E(y_k)})^2 \tag{2.27}$$

implying that $c_{\alpha,\nu} \sqrt{E(y_k)} \rightarrow 0$ as $k \rightarrow \infty$ with a quadratic rate.

Suppose now that $c_{\alpha,\nu} \sqrt{E(y_0)} \geq 1$ and denote $I = \{k \in \mathbb{N}, c_{\alpha,\nu} \sqrt{E(y_k)} \geq 1\}$. Let us prove that I is a finite subset of \mathbb{N} . For all $k \in I$, since $c_{\alpha,\nu} \sqrt{E(y_k)} \geq 1$,

$$\min_{\lambda \in [0, m]} p_k(\lambda) = \min_{\lambda \in [0, 1]} p_k(\lambda) = p_k\left(\frac{1}{2c_{\alpha,\nu} \sqrt{E(y_k)}}\right) = 1 - \frac{1}{4c_{\alpha,\nu} \sqrt{E(y_k)}}$$

and thus, for all $k \in I$,

$$c_{\alpha,\nu}\sqrt{E(y_{k+1})} \leq \left(1 - \frac{1}{4c_{\alpha,\nu}\sqrt{E(y_k)}}\right)c_{\alpha,\nu}\sqrt{E(y_k)} = c_{\alpha,\nu}\sqrt{E(y_k)} - \frac{1}{4}.$$

This inequality implies that the sequence $\{c_{\alpha,\nu}\sqrt{E(y_k)}\}_{k \in \mathbb{N}}$ strictly decreases and thus, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $c_{\alpha,\nu}\sqrt{E(y_k)} < 1$. Thus I is a finite subset of \mathbb{N} . Arguing as in the first case, it follows that $c_{\alpha,\nu}\sqrt{E(y_k)} \rightarrow 0$ as $k \rightarrow \infty$.

In both cases, remark that $p_k(\tilde{\lambda}_k)$ decreases with respect to k . \square

Lemma 2.12. *Assume that the sequence $\{y_k\}_{(k \geq 0)}$ defined by (2.18) satisfies $\tau(y_k) \leq c_1$ for all k and some $c_1 \in (0, 1)$. Then $\lambda_k \rightarrow 1$ as $k \rightarrow \infty$.*

Proof. In view of (2.25), we have, as long as $E(y_k) > 0$,

$$(1 - \lambda_k)^2 = \frac{E(y_{k+1})}{E(y_k)} - \lambda_k^2(1 - \lambda_k) \frac{\langle v_k, \bar{v}_k \rangle_{\mathbf{V}}}{E(y_k)} - \lambda_k^4 \frac{\|\bar{v}_k\|_{\mathbf{V}}^2}{2E(y_k)}.$$

From the proof of lemma 2.10, $\frac{\langle v_k, \bar{v}_k \rangle_{\mathbf{V}}}{E(y_k)} \leq C(\alpha, \nu)(1 - c_1)^{-2}\sqrt{E(y_k)}$ while $\frac{\|\bar{v}_k\|_{\mathbf{V}}^2}{E(y_k)} \leq C(\alpha, \nu)^2(1 - c_1)^{-4}E(y_k)$. Consequently, since $\lambda_k \in [0, m]$ and $\frac{E(y_{k+1})}{E(y_k)} \rightarrow 0$, we deduce that $(1 - \lambda_k)^2 \rightarrow 0$, that is $\lambda_k \rightarrow 1$ as $k \rightarrow \infty$. \square

Proposition 2.13. *Let $\{y_k\}$ be the sequence defined by (2.18). Assume that there exists a constant $c_1 \in (0, 1)$ such that $\tau(y_k) \leq c_1$ for all k . Then, $y_k \rightarrow \bar{y}$ in \mathbf{V} where $\bar{y} \in \mathbf{V}$ is the unique solution of (1.4).*

Proof. Remark that we can not use Proposition 2.8 since we do not know yet that there exists a solution, say z , of (1.4) satisfying $\tau(z) < 1$. In view of $y_{k+1} = y_0 - \sum_{n=0}^k \lambda_n Y_{1,n}$, we write

$$\sum_{n=0}^k |\lambda_n| \|Y_{1,n}\|_{\mathbf{V}} \leq m \sum_{n=0}^k \|Y_{1,n}\|_{\mathbf{V}} \leq m\sqrt{2} \sum_{n=0}^k \frac{\sqrt{E(y_n)}}{1 - \tau_d(y_n)} \leq \frac{m\sqrt{2}}{1 - c_1} \sum_{n=0}^k \sqrt{E(y_n)}.$$

Using that $p_n(\tilde{\lambda}_n) \leq p_0(\tilde{\lambda}_0)$ for all $n \geq 0$, we can write for $n > 0$,

$$\sqrt{E(y_n)} \leq p_{n-1}(\tilde{\lambda}_{n-1})\sqrt{E(y_{n-1})} \leq p_0(\tilde{\lambda}_0)\sqrt{E(y_{n-1})} \leq p_0(\tilde{\lambda}_0)^n \sqrt{E(y_0)},$$

we finally obtain

$$\sum_{n=0}^k |\lambda_n| \|Y_{1,n}\|_{\mathbf{V}} \leq \frac{m\sqrt{2}}{1 - c_1} \frac{\sqrt{E(y_0)}}{1 - p_0(\tilde{\lambda}_0)}$$

for which we deduce that the serie $\sum_{k \geq 0} \lambda_k Y_{1,k}$ converges in \mathbf{V} . Then, y_k converges in \mathbf{V} to $\bar{y} := y_0 + \sum_{k \geq 0} \lambda_k Y_{1,k}$. Eventually, the convergence of $E(y_k)$ to 0 implies

the convergence of the corrector v_k to 0 in \mathbf{V} ; taking the limit in the corrector equation (2.22) shows that \bar{y} solves (1.4). Since $\tau(\bar{y}) \leq c_1 < 1$, lemma 2.4 shows that this solution is unique. \square

As mentioned earlier, the remaining and crucial point is to show that the sequence $\{y_k\}$ may satisfies the uniform property $\tau(y_k) \leq c_1$ for some $c_1 < 1$.

Lemma 2.14. *Assume that $y_0 = g \in \mathbf{V}$. For all $c_1 \in (0, 1)$ there exists $\alpha_0 > 0$, such that, for any $\alpha \geq \alpha_0$, the unique sequence defined by (2.18) satisfies $\tau(y_k) \leq c_1$ for all $k \geq 0$.*

Proof. Let $c_1 \in (0, 1)$ and assume that y_0 belongs to \mathbf{V} . There exists $\alpha_1 > 0$ such that, for all $\alpha \geq \alpha_1$, $\tau(y_0) \leq \frac{c_1}{2}$.

Moreover, in view of the above computation, for all $\alpha > 0$, since for all $v \in \mathbf{V}$ $\|v\|_{\mathbf{V}} \leq \frac{1}{\nu} \|v\|_{\mathbf{V}}$, for all $k \in \mathbb{N}$

$$\|y_{k+1}\|_{\mathbf{V}} \leq \|y_0\|_{\mathbf{V}} + \frac{m\sqrt{2}}{\nu(1-c_1)} \frac{\sqrt{E(y_0)}}{1-p_0(\tilde{\lambda}_0)}$$

where

$$\frac{\sqrt{E(y_0)}}{1-p_0(\tilde{\lambda}_0)} \leq \begin{cases} \frac{\sqrt{E(y_0)}}{1-c_{\alpha,\nu}\sqrt{E(y_0)}}, & \text{if } c_{\alpha,\nu}\sqrt{E(y_0)} < 1, \\ 4c_{\alpha,\nu}E(y_0), & \text{if } c_{\alpha,\nu}\sqrt{E(y_0)} \geq 1. \end{cases}$$

From (1.9), we obtain that

$$\|v\|_{\mathbf{V}}^2 \leq \alpha \|g - y\|_2^2 + \frac{1}{\nu} \left(\nu \|\nabla y\|_2 + \|y\|_2 \|\nabla y\|_2 + \sqrt{c_0} \|f\|_{H^{-1}(\Omega)^2} \right)^2.$$

In particular, taking $y = y_0 = g$ allows to remove the α term in the right hand side and gives

$$E(g) \leq \frac{1}{2\nu} \left(\|g\|_{\mathbf{V}}(\nu + \|g\|_2) + \sqrt{c_0} \|f\|_{H^{-1}(\Omega)^2} \right)^2 := \frac{1}{2\nu} c_2(f, g), \quad (2.28)$$

and thus, if $c_{\alpha_1,\nu}\sqrt{E(g)} \geq 1$, then for all $\alpha \geq \alpha_1$ such that $c_{\alpha,\nu}\sqrt{E(g)} \geq 1$ and for all $k \in \mathbb{N}$:

$$\|y_{k+1}\|_{\mathbf{V}} \leq \|g\|_{\mathbf{V}} + \frac{m\sqrt{2}}{\nu(1-c_1)} \frac{\sqrt{E(g)}}{1-p_0(\tilde{\lambda}_0)} \leq \|g\|_{\mathbf{V}} + \frac{2m\sqrt{2}}{\nu^3\sqrt{\alpha}(1-c_1)^3} c_2(f, g). \quad (2.29)$$

If now $c_{\alpha_1,\nu}\sqrt{E(g)} < 1$ then there exists $0 < K < 1$ such that for all $\alpha \geq \alpha_1$ we have $c_{\alpha,\nu}\sqrt{E(g)} \leq K$. We therefore have for all $\alpha \geq \alpha_1$

$$\frac{\sqrt{E(g)}}{1-p_0(\tilde{\lambda}_0)} \leq \frac{\sqrt{E(g)}}{1-K}$$

and thus for all $k \in \mathbb{N}$:

$$\|y_{k+1}\|_{\mathbf{V}} \leq \|g\|_{\mathbf{V}} + \frac{m\sqrt{2}}{\nu(1-c_1)} \frac{\sqrt{E(g)}}{1-p_0(\tilde{\lambda}_0)} \leq \|g\|_{\mathbf{V}} + \frac{m}{\nu^{3/2}(1-c_1)(1-K)} \sqrt{c_2(f,g)}. \quad (2.30)$$

On the other hand, there exists $\alpha_0 \geq \alpha_1$ such that, for all $\alpha \geq \alpha_0$ we have

$$\frac{2m\sqrt{2}}{\nu^3\sqrt{\alpha}(1-c_1)^3} c_2(f,g) \leq \frac{c_1}{2} \sqrt{2\alpha\nu}$$

and

$$\frac{m}{\nu^{3/2}(1-c_1)(1-K)} \sqrt{c_2(f,g)} \leq \frac{c_1}{2} \sqrt{2\alpha\nu}.$$

We then deduce from (2.29) and (2.30) that for all $\alpha \geq \alpha_0$ and for all $k \in \mathbb{N}$:

$$\|y_{k+1}\|_{\mathbf{V}} \leq \frac{c_1}{2} \sqrt{2\alpha\nu} + \frac{c_1}{2} \sqrt{2\alpha\nu} = c_1 \sqrt{2\alpha\nu}$$

that is $\tau(y_{k+1}) \leq c_1$. □

Gathering the previous lemmas and propositions, we can now deduce the strong convergence of the sequence $\{y_k\}_{k \geq 0}$ defined by (2.18), initialized by $y_0 = g$.

Theorem 2.15. *Let $c_1 \in (0, 1)$. Assume that $y_0 = g \in \mathbf{V}$ and α is large enough so that*

$$c_2(f,g) \leq \max\left(\frac{1-c_1}{2}, \frac{c_1\sqrt{\nu}(1-K^2)}{m}\right) \frac{c_1}{4m} \nu^{5/2} (1-c_1)^2 2\alpha\nu \quad (2.31)$$

Then, the sequence $\{y_k\}_{(k \in \mathbb{N})}$ defined by (2.18) strongly converges to the unique solution y of (1.4). Moreover, there exists $k_0 \in \mathbb{N}$ such that the sequence $\{y_k\}_{k \geq k_0}$ converges quadratically to y . Moreover, this solution satisfies $\tau(y) < 1$.

2.3 Remarks

The following remarks are in order.

Remark 2.16. *Estimate (2.3) is usually used to obtain a sufficient condition on the data f, g to ensure the uniqueness of the solution of (1.4) (i.e. $\tau(y) < 1$): it leads to*

$$\alpha \|g\|_2^2 + c_0 \nu \|f\|_{(H^{-1}(\Omega))^2}^2 \leq 2\alpha\nu^2, \quad \text{if } d = 2, \quad (2.32)$$

We emphasize that such (sufficient) conditions are more restrictive than (2.31), as they impose smallness properties on g : precisely $\|g\|_2^2 \leq 2\nu^2$. In particular, this latter yields a restrictive condition for α large contrary to (2.31).

Remark 2.17. *It seems surprising that the algorithm (2.18) achieves a quadratic rate for k large. Let us consider the application $\mathcal{F} : \mathbf{V} \rightarrow \mathbf{V}'$ defined as $\mathcal{F}(y) = \alpha y + \nu B_1(y) + B(y, y) - f - \alpha g$. The sequence $\{y_k\}_{k>0}$ associated to the Newton method to find the zero of F is defined as follows:*

$$\begin{cases} y_0 \in \mathbf{V}, \\ \mathcal{F}'(y_k) \cdot (y_{k+1} - y_k) = -\mathcal{F}(y_k), \quad k \geq 0. \end{cases} \quad (2.33)$$

We check that this sequence coincides with the sequence obtained from (2.18) if λ_k is fixed equal to one and if $y_0 \in \mathbf{V}$. The algorithm (2.18) which consists to optimize the parameter $\lambda_k \in [0, m]$, $m \geq 1$, in order to minimize $E(y_k)$, equivalently $\|\mathcal{F}(y_k)\|_{\mathbf{V}'}$, corresponds to the so-called in the literature damped Newton method for the application \mathcal{F} (see [4]). As the iterates increase, the optimal parameter λ_k converges to one (according to Lemma 2.12), this globally convergent method behaves like the standard Newton method (for which λ_k is fixed equal to one): this explains the quadratic rate after a finite number of iterates. To the best of our knowledge, this is the first analysis of the damped Newton method for a partial differential equation. Among the few numerical works devoted to the damped Newton method for partial differential equations, we mention [20] for computing viscoplastic fluid flows.

Remark 2.18. *Section 6, chapter 6 of the book [5] introduces a least-squares method in order to solve an Oseen type equation (without incompressibility constraint). The convergence of any minimizing sequence toward a solution y is proved under the a priori assumption that the operator $DF(y)$ defined as follows*

$$DF(y) \cdot w = \alpha w - \nu \Delta w + [(w \cdot \nabla)y + (y \cdot \nabla)w], \quad \forall w \in \mathbf{V} \quad (2.34)$$

(for some $\alpha > 0$) is an isomorphism from \mathbf{V} onto \mathbf{V}' . y is then said to be a nonsingular point. According to Proposition 2.5, a sufficient condition for y to be a nonsingular point is $\tau(y) < 1$. Recall that τ depends on α . As far as we know, determining a weaker condition ensuring that $DF(y)$ is an isomorphism is an open question. Moreover, according to Lemma 2.4, it turns out that this condition is also a sufficient condition for the uniqueness of (1.4). Theorem 2.15 asserts that, if α is large enough, then the sequence $\{y_k\}_{(k \in \mathbb{N})}$ defined in (2.18), initialized with $y_0 = g$, is a convergent sequence of nonsingular points. Since λ_k is constant equal to one, this shows the convergence of the Newton method to solve the steady Navier-Stokes equation.

Remark 2.19. We may also define a minimizing sequence for E using the gradient E' :

$$\begin{cases} y_0 \in \mathbf{H} \text{ given,} \\ y_{k+1} = y_k - \lambda_k g_k, \quad k \geq 0, \\ \lambda_k = \operatorname{argmin}_{\lambda \in [0, m]} E(y_k - \lambda g_k) \end{cases} \quad (2.35)$$

with $g_k \in \mathbf{V}$ such that $(g_k, w)_{\mathbf{V}} = (E'(y_k), w)_{\mathbf{V}'}$ for all $w \in \mathbf{V}$. In particular, $\|g_k\|_{\mathbf{V}} = \|E'(y_k)\|_{\mathbf{V}'}$. Using the expansion (2.13) with $w_k = g_k$, we can prove the linear decrease of the sequence $\{E(y_k)\}_{k>0}$ to zero assuming however that $E(y_0)$ is small enough, of the order of ν^2 , independently of the value of α .

2.4 Application to the backward Euler scheme

We use the analysis of the previous section to discuss the resolution of the backward Euler scheme (1.3) through a least-squares method. The weak formulation of this scheme reads as follows: given $y^0 = u_0 \in \mathbf{H}$, the sequence $\{y^n\}_{n>0}$ in \mathbf{V} is defined by recurrence as follows:

$$\int_{\Omega} \frac{y^{n+1} - y^n}{\delta t} \cdot w + \nu \int_{\Omega} \nabla y^{n+1} \cdot \nabla w + \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot w = \langle f^n, w \rangle_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} \quad (2.36)$$

with f^n defined by (1.5) in term of the external force of the Navier-Stokes model (1.1). We recall that a piecewise linear interpolation in time of $\{y^n\}_{n \geq 0}$ weakly converges in $L^2(0, T, \mathbf{V})$ toward a solution of (1.2)

As done in [2], one may use the least-squares method (analyzed in Section 2) to solve iteratively (2.36). Precisely, in order to approximate y^{n+1} from y^n , one may consider the following extremal problem

$$\inf_{y \in \mathbf{V}} E_n(y), \quad E_n(y) = \frac{1}{2} \| \|v\| \|_{\mathbf{V}}^2 \quad (2.37)$$

where the corrector $v \in \mathbf{V}$ solves

$$\begin{aligned} \alpha \int_{\Omega} v \cdot w + \nu \int_{\Omega} \nabla v \cdot \nabla w = -\alpha \int_{\Omega} y \cdot w - \nu \int_{\Omega} \nabla y \cdot \nabla w - \int_{\Omega} y \cdot \nabla y \cdot w \\ + \langle f^n, w \rangle_{H^{-1}(\Omega)^2 \times H_0^1(\Omega)^2} + \alpha \int_{\Omega} y^n \cdot w, \quad \forall w \in \mathbf{V} \end{aligned} \quad (2.38)$$

with α and f^n given by (1.5). For any $n \geq 0$, a minimizing sequence $\{y_k^n\}_{(k \geq 0)}$ for E_n is defined as follows :

$$\begin{cases} y_0^{n+1} = y^n, \\ y_{k+1}^{n+1} = y_k^{n+1} - \lambda_k Y_{1,k}^{n+1}, \quad k \geq 0, \\ \lambda_k = \operatorname{argmin}_{\lambda \in [0, m]} E_n(y_k^n - \lambda Y_{1,k}^n) \end{cases} \quad (2.39)$$

where $Y_{1,k}^n \in \mathbf{V}$ solves (2.19). Remark that, in view of Theorem 2.15, the first element of the minimizing sequence is chosen equal to y^n , i.e. the minimizer of E_{n-1} .

The main goal of this section is to prove that for all $n \in \mathbb{N}$, the minimizing sequence $(y_k^{n+1})_{k \in \mathbb{N}}$ do converges to a solution y^{n+1} of (2.36). This allows to justify the use of least-squares method to solve the backward Euler scheme. Arguing as in Lemma 2.14, we have to prove the existence of a constant $c_1 \in (0, 1)$, such that $\tau(y_k^n) \leq c_1$ for all n and k in \mathbb{N} . Remark that the initialization y_0^{n+1} is fixed as the minimizer of the functional E^{n-1} , obtained at the previous iterate. Consequently, the uniform property $\tau(y_k^n) \leq c_1$ is related to the initial guess y_0^0 equal to the initial position u_0 , to the external force f (see (1.2)) and to the value of α . u_0 and f are given a priori, On the other hand, the parameter α , related to the discretization parameter δt , can be chosen as large as necessary. As we shall see, this uniform property, which is essential to set up the least-squares procedure, requires smallness properties on u_0 and f .

We start with the following result analogue to Proposition 2.3.

Proposition 2.20. *Let $(f^n)_{n \in \mathbb{N}}$ be a sequence in $H^{-1}(\Omega)^2$, $\alpha > 0$ and $y^0 = u_0 \in \mathbf{H}$. For any $n \in \mathbb{N}$, there exists a solution $y^{n+1} \in \mathbf{V}$ of*

$$\alpha \int_{\Omega} (y^{n+1} - y^n) \cdot w + \nu \int_{\Omega} \nabla y^{n+1} \cdot \nabla w + \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot w = \langle f^n, w \rangle_{H^{-1}(\Omega)^2 \times H_0^1(\Omega)} \quad (2.40)$$

for all $w \in \mathbf{V}$. Moreover, for all $n \in \mathbb{N}$, y^{n+1} satisfies

$$\| \| y^{n+1} \| \|_{\mathbf{V}}^2 \leq \frac{c_0}{\nu} \| f^n \|_{H^{-1}(\Omega)^2}^2 + \alpha \| y^n \|_2^2 \quad (2.41)$$

where $c_0 > 0$, only connected to the Poincaré constant, depends on Ω . Moreover, for all $n \in \mathbb{N}^*$:

$$\| y^n \|_2^2 + \frac{\nu}{\alpha} \sum_{k=1}^n \| \nabla y^k \|_2^2 \leq \frac{1}{\nu} \left(\frac{c_0}{\alpha} \sum_{k=0}^{n-1} \| f^k \|_{H^{-1}(\Omega)^2}^2 + \nu \| u_0 \|_2^2 \right). \quad (2.42)$$

Proof. The existence of y^{n+1} is given in Proposition 2.3. (2.42) is obtained by summing (2.41). □

Remark 2.21. *Arguing as in Lemma 2.4, if there exists a solution y^{n+1} in \mathbf{V} of (2.38) satisfying $\tau(y^{n+1}) < 1$, then such solution is unique. In view of Proposition 2.20, this holds true if notably the quantity $\mathcal{M}(f, \alpha, \nu)$ defined as follows*

$$\mathcal{M}(f, \alpha, \nu) = \frac{1}{\nu^2} \left(\frac{c_0}{\alpha} \sum_{k=0}^{n-1} \|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu \|u_0\|_2^2 \right) \quad (2.43)$$

is small enough.

2.4.1 Uniform convergence of the least-squares method w.r.t. n

We have the following convergence for weak solutions of (2.40).

Theorem 2.22. *Suppose $f \in L^2(0, T; H^{-1}(\Omega)^2)$, $u_0 \in \mathbf{V}$ and let $c(u_0, f)$ be defined as follows :*

$$c(u_0, f) := \max \left(\frac{1}{\alpha} \|u_0\|_{\mathbf{V}}^2 (\nu + \|u_0\|_2)^2 + c_0 \|f\|_{L^2(0, T; H^{-1}(\Omega)^2)}^2, \right. \\ \left. 2c_0 \|f\|_{L^2(0, T; H^{-1}(\Omega)^2)}^2 + \nu \|u_0\|_2^2 \right).$$

Let α large enough and f^n be given (1.5) by all $n \in \{0, \dots, N-1\}$ and let $\{y^n\}_{n \in \mathbb{N}}$ in \mathbf{V} solution of (2.40). If there exists a constant $c > 0$ such that

$$c(u_0, f) \leq c\nu^4 \quad (2.44)$$

then, for any $n \geq 0$, the minimizing sequence $\{y_k^{n+1}\}_{k \in \mathbb{N}}$ defined by (2.39) strongly converges to the unique of solution of (2.40).

Proof. According to Proposition 2.13, we have to prove the existence of a constant $c_1 \in (0, 1)$ such that, for all $n \in \{0, \dots, N-1\}$ and all $k \in \mathbb{N}$, $\tau(y_k^n) \leq c_1$.

For $n = 0$, as in the previous section, it suffices to takes α large enough to ensure the conditions (2.31) with $g = y_0^0 = u_0$ leading to the property $\tau(y_k^0) < c_1$ for all $k \in \mathbb{N}$ and therefore $\tau(y^1) < c_1$.

For the next minimizing sequences, let us recall (see Lemma 2.14) that for all $n \in \{0, \dots, N-1\}$ and all $k \in \mathbb{N}$

$$\|y_k^{n+1}\|_{\mathbf{V}} \leq \|y^n\|_{\mathbf{V}} + \frac{m\sqrt{2}}{\nu(1-c_1)} \frac{\sqrt{E_n(y^n)}}{1-p_{n,0}(\tilde{\lambda}_{n,0})}$$

where $p_{n,0}(\tilde{\lambda}_{n,0})$ is defined as in the proof of Proposition 2.7.

First, since for all $n \in \{0, \dots, N-1\}$, $\|f^n\|_{H^{-1}(\Omega)^2}^2 \leq \alpha \|f\|_{L^2(0,T;H^{-1}(\Omega)^2)}^2$, we can write

$$\begin{aligned} E_0(y^0) = E_0(u_0) &\leq \frac{1}{2\nu} \left(\|u_0\|_{\mathbf{V}}(\nu + \|u_0\|_2) + \sqrt{c_0} \|f^0\|_{H^{-1}(\Omega)^2} \right)^2 \\ &\leq \frac{1}{\nu} \left(\|u_0\|_{\mathbf{V}}^2(\nu + \|u_0\|_2)^2 + c_0 \|u_0\|_{H^{-1}(\Omega)^2}^2 \right) \\ &\leq \frac{\alpha}{\nu} \left(\frac{1}{\alpha} \|u_0\|_{\mathbf{V}}^2(\nu + \|u_0\|_2)^2 + c_0 \|f\|_{L^2(0,T;H^{-1}(\Omega)^2)}^2 \right). \end{aligned}$$

Since y^n is solution of (2.40), it follows from (2.38), that for all $n \in \{1, \dots, N-1\}$:

$$\begin{aligned} E_n(y^n) &\leq \frac{c_0}{2\nu} \|f^n - f^{n-1}\|_{H^{-1}(\Omega)^2}^2 + \frac{\alpha}{2} \|y^n - y^{n-1}\|_2^2 \\ &\leq \frac{\alpha}{\nu} \left(2c_0 \|f\|_{L^2(0,T;H^{-1}(\Omega)^2)}^2 + \nu \|u_0\|_2^2 \right). \end{aligned}$$

Therefore, for all $n \in \{0, \dots, N-1\}$, $E_n(y^n) \leq \frac{\alpha}{\nu} c(u_0, f)$.

Let $c_1 \in (0, 1)$ and suppose that $c(u_0, f) < (1-c_1)^4 \nu^3$. Then, for any $K \in (0, 1)$, there exists $\alpha_0 > 0$ such that, for all $\alpha \geq \alpha_0$ $c_{\alpha, \nu} \sqrt{E_n(y^n)} \leq K < 1$. We therefore have (see Lemma 2.14), for all $\alpha \geq \alpha_0$, all $n \in \{0, \dots, N-1\}$ and all $k \in \mathbb{N}$:

$$\begin{aligned} \|y_k^{n+1}\|_{\mathbf{V}} &\leq \|y^n\|_{\mathbf{V}} + \frac{m\sqrt{2}}{\nu(1-c_1)} \frac{\sqrt{E_n(y^n)}}{1-c_{\alpha, \nu} \sqrt{E_n(y^n)}} \\ &\leq \|y^n\|_{\mathbf{V}} + \frac{m\sqrt{2}}{\nu(1-c_1)} \frac{\sqrt{E_n(y^n)}}{1-K} \\ &\leq \|y^n\|_{\mathbf{V}} + \frac{m\sqrt{2\alpha}}{\nu^{3/2}(1-c_1)(1-K)} \sqrt{c(u_0, f)}. \end{aligned} \tag{2.45}$$

From (2.42) we then obtain, for all $n \in \{0, \dots, N-1\}$,

$$\|y^n\|_{\mathbf{V}} \leq \frac{\sqrt{\alpha}}{\nu} \sqrt{\frac{c_0}{\alpha} \sum_{k=0}^{n-1} \|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu \|u_0\|_2^2}$$

and since $\frac{c_0}{\alpha} \sum_{k=0}^{n-1} \|f^k\|_{H^{-1}(\Omega)^2}^2 \leq c_0 \|f\|_{L^2(0,T;H^{-1}(\Omega)^2)}^2$, we deduce that if

$$c_0 \|f\|_{L^2(0,T;H^{-1}(\Omega)^2)}^2 + \nu \|u_0\|_2^2 \leq \frac{c_1^2}{2} \nu^3$$

then $\|y^n\|_{\mathbf{V}} \leq \frac{c_1}{2} \sqrt{2\alpha\nu}$. Moreover, assuming $c(u_0, f) \leq \frac{c_1^2(1-c_1)^2(1-K)^2}{4m} \nu^4$, we deduce from (2.45), for all $n \in \{0, \dots, N-1\}$ and for all $k \in \mathbb{N}$:

$$\|y_k^n\|_{\mathbf{V}} \leq \frac{c_1}{2} \sqrt{2\alpha\nu} + \frac{c_1}{2} \sqrt{2\alpha\nu} = c_1 \sqrt{2\alpha\nu}$$

that is $\tau(y_k^n) \leq c_1$. The result follow from Proposition 2.13. \square

We emphasize that, for each $n \in \mathbb{N}$, the limit y^{n+1} of the sequence $\{y_k^{n+1}\}_{k \in \mathbb{N}}$ satisfies $\tau(y^{n+1}) < 1$ and is therefore the unique solution of (2.40). Moreover, for α large enough, the condition (2.44) reads as the following smallness property on the data u_0 and $f : c_0 \|f\|_{L^2(0,T;H^{-1}(\Omega)^2)}^2 + \nu \|u_0\|_2^2 \leq c\nu^4$. In contrast with the static case of Section (2) where the unique condition (2.31) on the data g is fulfilled as soon as α is large, the iterated case requires a condition on the data u_0 and f , whatever be the amplitude of α . Again, this smallness property is introduced in order to guarantee the condition $\tau(y^n) < 1$ for all n . In view of (2.42), this condition implies notably that $\|y^n\|_2 \leq c\nu^{3/2}$ for all $n > 0$.

For regular solutions of (2.40) which we now consider, we may slightly improve the results, notably based on the control of two consecutive elements of the corresponding sequel $\{y^n\}_{n \in \mathbb{N}}$ for the L^2 norm. We first start with the following result of regularity.

Proposition 2.23. *Assume that Ω is C^2 , that $(f^n)_n$ is a sequence in $L^2(\Omega)^2$ and that $u_0 \in \mathbf{V}$. Then, for all $n \in \mathbb{N}$, any solution $y^{n+1} \in \mathbf{V}$ of (2.40) belongs to $H^2(\Omega)^2$.*

If moreover, there exists $C > 0$ such that

$$\frac{c_0}{\alpha} \sum_{k=0}^n \|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu \|y^0\|_2^2 < C\nu^3, \quad (2.46)$$

then y^{n+1} satisfies

$$\int_{\Omega} |\nabla y^{n+1}|^2 + \frac{\nu}{2\alpha} \sum_{k=1}^{n+1} \int_{\Omega} |P\Delta y^k|^2 \leq \frac{1}{\nu} \left(\frac{1}{\alpha} \sum_{k=0}^n \|f^k\|_2^2 + \nu \|\nabla u_0\|_2^2 \right) \quad (2.47)$$

where P is the operator of projection from $L^2(\Omega)^d$ into \mathbf{H} .

Proof. From Proposition 2.3, we know that for all $n \in \mathbb{N}^*$, $y^n \in H^2(\Omega)^2 \cap \mathbf{V}$. Taking $w = P\Delta y^{n+1}$ in (2.40) leads to :

$$\begin{aligned} \alpha \int_{\Omega} |\nabla y^{n+1}|^2 + \nu \int_{\Omega} |P\Delta y^{n+1}|^2 &= - \int_{\Omega} f^n P\Delta y^{n+1} + \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot P\Delta y^{n+1} \\ &\quad + \alpha \int_{\Omega} \nabla y^n \cdot \nabla y^{n+1}. \end{aligned} \quad (2.48)$$

Recall that

$$\int_{\Omega} f^n P\Delta y^{n+1} \leq \frac{1}{2\nu} \|f^n\|_2^2 + \frac{\nu}{2} \|P\Delta y^{n+1}\|_2^2, \quad \alpha \int_{\Omega} \nabla y^n \cdot \nabla y^{n+1} \leq \frac{\alpha}{2} \|y^n\|_{\mathbf{V}}^2 + \frac{\alpha}{2} \|y^{n+1}\|_{\mathbf{V}}^2.$$

We also have

$$\left| \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot P \Delta y^{n+1} \right| \leq \|y^{n+1}\|_{\infty} \|\nabla y^{n+1}\|_2 \|P \Delta y^{n+1}\|_2.$$

We now use (see [23, chapter 5]) that there exist three constants c_1, c_2 and c_3 such that

$$\|\Delta y^{n+1}\|_2 \leq c_1 \|P \Delta y^{n+1}\|_2, \quad \|y^{n+1}\|_{\infty} \leq c_2 \|y^{n+1}\|_2^{\frac{1}{2}} \|\Delta y^{n+1}\|_2^{\frac{1}{2}}$$

and

$$\|\nabla y^{n+1}\|_2 \leq c_3 \|y^{n+1}\|_2^{\frac{1}{2}} \|\Delta y^{n+1}\|_2^{\frac{1}{2}}.$$

This implies that (for $c = c_1 c_2 c_3$)

$$\left| \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot P \Delta y^{n+1} \right| \leq c \|y^{n+1}\|_2 \|P \Delta y^{n+1}\|_2^2.$$

Recalling (2.48), it follows that

$$\frac{\alpha}{2} \int_{\Omega} |\nabla y^{n+1}|^2 + \left(\frac{\nu}{2} - c \|y^{n+1}\|_2 \right) \int_{\Omega} |P \Delta y^{n+1}|^2 \leq \frac{1}{2\nu} \|f^n\|_2^2 + \frac{\alpha}{2} \int_{\Omega} |\nabla y^n|^2.$$

But, from estimate (2.42), the assumption (2.46) implies that $\|y^{n+1}\|_2 \leq \frac{\nu}{4c}$ and

$$\int_{\Omega} |\nabla y^{n+1}|^2 + \frac{\nu}{2\alpha} \int_{\Omega} |P \Delta y^{n+1}|^2 \leq \frac{1}{\nu\alpha} \|f^n\|_2^2 + \int_{\Omega} |\nabla y^n|^2.$$

Summing then implies (2.47) for all $n \in \mathbb{N}$. □

Remark 2.24. *Under the hypothesis of Proposition 2.23, suppose that*

$$B_{\alpha, \nu} := (\alpha \nu^5)^{-1} \left(c_0 \alpha^{-1} \sum_{k=0}^n \|f^k\|_{H^{-1}(\Omega)^2 + \nu \|y^0\|_2^2} \right) \left(\alpha^{-1} \sum_{k=0}^{n-1} \|f^k\|_2^2 + \nu \|\nabla y^0\|_2^2 \right)$$

is small (which is satisfied as soon as α is large enough). Then, the solution of (2.40) is unique.

Indeed, let $n \in \mathbb{N}$ and let $y_1^{n+1}, y_2^{n+1} \in \mathbf{V}$ be two solutions of (2.40). Then $Y := y_1^{n+1} - y_2^{n+1}$ satisfies

$$\alpha \int_{\Omega} Y \cdot w + \nu \int_{\Omega} \nabla Y \cdot \nabla w + \int_{\Omega} y_2^{n+1} \cdot \nabla Y \cdot w + \int_{\Omega} Y \cdot \nabla y_1^{n+1} \cdot w = 0 \quad \forall w \in \mathbf{V}$$

and in particular, for $w = Y$ (since $\int_{\Omega} y_2^{n+1} \cdot \nabla Y \cdot Y = 0$)

$$\begin{aligned} \alpha \int_{\Omega} |Y|^2 + \nu \int_{\Omega} |\nabla Y|^2 &= - \int_{\Omega} Y \cdot \nabla y_1^{n+1} \cdot Y = \int_{\Omega} Y \cdot \nabla Y \cdot y_1^{n+1} \\ &\leq c \|y_1^{n+1}\|_{\infty} \|\nabla Y\|_2 \|Y\|_2 \\ &\leq c \|y_1^{n+1}\|_2^{1/2} \|P\Delta y_1^{n+1}\|_2^{1/2} \|\nabla Y\|_2 \|Y\|_2 \\ &\leq \alpha \|Y\|_2^2 + \frac{c}{\alpha} \|y_1^{n+1}\|_2 \|P\Delta y_1^{n+1}\|_2 \|\nabla Y\|_2^2 \end{aligned}$$

leading to

$$\left(\nu - \frac{c}{\alpha} \|y_1^{n+1}\|_2 \|P\Delta y_1^{n+1}\|_2 \right) \|\nabla Y\|_2^2 \leq 0.$$

If

$$\|y_1^{n+1}\|_2 \|P\Delta y_1^{n+1}\|_2 < \frac{\nu\alpha}{c}, \quad (2.49)$$

then $Y = 0$ and the solution is unique. But, from (2.42) and (2.47),

$$\|y_1^{n+1}\|_2^2 \|P\Delta y_1^{n+1}\|_2^2 \leq \frac{4\alpha}{\nu^3} \left(\frac{c_0}{\alpha} \sum_{k=0}^n \|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu \|y^0\|_2^2 \right) \left(\frac{1}{\alpha} \sum_{k=0}^n \|f^k\|_2^2 + \nu \|\nabla y^0\|_2^2 \right).$$

Therefore, if there exists a constant C such that $B_{\alpha, \nu} < C$, then (2.49) holds true.

Proposition 2.23 then allows to obtain the following estimate of $\|y^{n+1} - y^n\|_2$ in term of the parameter α .

Theorem 2.25. *We assume that Ω is C^2 , that $(f^n)_n$ is a sequence in $L^2(\Omega)^2$ satisfies $\alpha^{-1} \sum_{k=0}^{+\infty} \|f^k\|_2^2 < +\infty$, that $u_0 \in \mathbf{V}$ and that for all $n \in \mathbb{N}$, $y^{n+1} \in H^2(\Omega)^2 \cap \mathbf{V}$ is a solution of (2.40) satisfying $\|y^{n+1}\|_2 \leq \frac{\nu}{4c}$. Then, there exists $C_1 > 0$ such that the sequence $(y^n)_n$ satisfies*

$$\|y^{n+1} - y^n\|_2^2 \leq \frac{C_1}{\alpha\nu^{3/2}}. \quad (2.50)$$

Proof. For all $n \in \mathbb{N}$, $w = y^{n+1} - y^n$ in (2.40) gives :

$$\begin{aligned} \alpha \|y^{n+1} - y^n\|_2^2 + \nu \|\nabla y^{n+1}\|_2^2 &\leq \left| \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot (y^{n+1} - y^n) \right| \\ &\quad + \left| \int_{\Omega} f^n \cdot (y^{n+1} - y^n) \right| + \nu \left| \int_{\Omega} \nabla y^n \cdot \nabla y^{n+1} \right|. \end{aligned}$$

Moreover,

$$\begin{aligned} \left| \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot (y^{n+1} - y^n) \right| &\leq c \|\nabla y^{n+1}\|_2^2 \|\nabla (y^{n+1} - y^n)\|_2 \\ &\leq c \|\nabla y^{n+1}\|_2^2 (\|\nabla y^{n+1}\|_2 + \|\nabla y^n\|_2). \end{aligned}$$

Therefore,

$$\alpha \|y^{n+1} - y^n\|_2^2 + \nu \|\nabla y^{n+1}\|_2^2 \leq c \|\nabla y^{n+1}\|_2^2 (\|\nabla y^{n+1}\|_2 + \|\nabla y^n\|_2) + \frac{1}{\alpha} \|f^n\|_2^2 + \nu \|\nabla y^n\|_2^2.$$

But, (2.47) implies that for all $n \in \mathbb{N}$

$$\int_{\Omega} |\nabla y^{n+1}|^2 \leq \frac{1}{\nu} \left(\frac{1}{\alpha} \sum_{k=0}^{+\infty} \|f^k\|_2^2 + \nu \|\nabla y^0\|_2^2 \right) := \frac{C}{\nu}$$

and thus, since $\nu < 1$

$$\alpha \|y^{n+1} - y^n\|_2^2 + \nu \|\nabla y^{n+1}\|_2^2 \leq \frac{2cC^{3/2}}{\nu^{3/2}} + 2C \leq \frac{C_1}{\nu^{3/2}}$$

leading to $\|y^{n+1} - y^n\|_2^2 = O(\frac{1}{\alpha\nu^{3/2}})$ as announced. □

This result asserts that two consecutive elements of the sequence $\{y^n\}_{n \geq 0}$ defined by recurrence from the scheme (1.3) are close each other as soon as δt , the time step discretization, is small enough. In particular, this justifies the choice of the initial term $y_0^{n+1} = y^n$ of the minimizing sequence in order to approximate y^{n+1} .

We end this section with the following result, analogue of Theorem 2.22, for regular data.

Theorem 2.26. *Suppose $f \in L^2(0, T; L^2(\Omega)^2)$, $u_0 \in \mathbf{V}$, for all $n \in \{0, \dots, N - 1\}$, α and f^n are given by (1.5) and $y^{n+1} \in \mathbf{V}$ solution of (2.40). If $C(u_0, f) := \|f\|_{L^2(0, T; L^2(\Omega)^2)}^2 + \nu \|u_0\|_{\mathbf{V}}^2 \leq C\nu^2$ for some C and α is large enough, then, for any $n \geq 0$, the minimizing sequence $\{y_k^{n+1}\}_{k \in \mathbb{N}}$ defined by (2.39) strongly converges to the unique of solution of (2.40).*

Proof. As for Theorem 2.22, it suffices to prove that there exists $c_1 \in (0, 1)$ such that, for all $n \in \{0, \dots, N - 1\}$ and all $k \in \mathbb{N}$, $\tau(y_k^n) \leq c_1$. Let us recall that for all $n \in \{0, \dots, N - 1\}$ and all $k \in \mathbb{N}$

$$\|y_{k+1}^{n+1}\|_{\mathbf{V}} \leq \|y^n\|_{\mathbf{V}} + \frac{m\sqrt{2}}{\nu(1 - c_1)} \frac{\sqrt{E_n(y^n)}}{1 - p_{n,0}(\tilde{\lambda}_{n,0})}$$

where $p_{n,0}(\tilde{\lambda}_{n,0})$ is defined as in the proof of Proposition 2.7. From (2.38), since for all $n \in \{0, \dots, N - 1\}$, $\|f^n\|_2^2 \leq \alpha \|f\|_{L^2(0, T; L^2(\Omega)^2)}^2$:

$$\begin{aligned} E_0(y^0) = E_0(u_0) &\leq \frac{1}{2\nu} \left(\|u_0\|_{\mathbf{V}} (\nu + \|u_0\|_2) + \sqrt{\frac{\nu}{\alpha}} \|f^1\|_2 \right)^2 \\ &\leq \frac{1}{\nu} \|u_0\|_{\mathbf{V}}^2 (\nu + \|u_0\|_2)^2 + \|f\|_{L^2(0, T; L^2(\Omega)^2)}^2 \end{aligned}$$

and, since y^n is solution of (2.40), then for all $n \in \{1, \dots, N-1\}$:

$$\begin{aligned} E_n(y^n) &\leq \frac{1}{\alpha} \|f^n - f^{n-1}\|_2^2 + \alpha \|y^n - y^{n-1}\|_2^2 \\ &\leq 2\|f\|_{L^2(0,T;L^2(\Omega)^2)}^2 + \alpha \|y^n - y^{n-1}\|_2^2. \end{aligned}$$

From the proof of Theorem 2.25, we deduce that for all $n \in \{0, \dots, N-1\}$:

$$\alpha \|y^{n+1} - y^n\|_2^2 \leq \frac{2cC(u_0, f)^{3/2}}{\nu^{3/2}} + 2C(u_0, f)$$

and thus, for all $n \in \{1, \dots, N-1\}$

$$E_n(y^n) \leq \frac{2cC(u_0, f)^{3/2}}{\nu^{3/2}} + 4C(u_0, f).$$

Moreover, from (2.47), for all $n \in \{0, \dots, N-1\}$:

$$\|y^n\|_{\mathbf{V}}^2 \leq \frac{1}{\nu} \left(\frac{1}{\alpha} \sum_{k=0}^n \|f^k\|_2^2 + \nu \|u_0\|_{\mathbf{V}}^2 \right) \leq \frac{1}{\nu} \left(\|f\|_{L^2(0,T;L^2(\Omega)^2)}^2 + \nu \|u_0\|_{\mathbf{V}}^2 \right) = \frac{1}{\nu} C(u_0, f).$$

Eventually, let $c_1 \in (0, 1)$. Then there exists $\alpha_0 > 0$ such that, for all $\alpha \geq \alpha_0$ $c_{\alpha, \nu} \sqrt{E_n(y^n)} \leq K < 1$. We therefore have (see Theorem 2.22), for all $\alpha \geq \alpha_0$, all $n \in \{0, \dots, N-1\}$ and all $k \in \mathbb{N}$:

$$\|y_{k+1}^{n+1}\|_{\mathbf{V}} \leq \|y^n\|_{\mathbf{V}} + \frac{m\sqrt{2}}{\nu(1-c_1)} \frac{\sqrt{E_n(y^n)}}{1-K}$$

which gives a bound of $\|y_{k+1}^{n+1}\|_{\mathbf{V}}$ independent of $\alpha \geq \alpha_0$.

Taking $\alpha_1 \geq \alpha_0$ large enough, we deduce that, for all $\alpha \geq \alpha_1$, all $n \in \{0, \dots, N-1\}$ and all $k \in \mathbb{N}$, $\|y_k^n\|_{\mathbf{V}} \leq c_1 \sqrt{2\alpha\nu}$, that is $\tau_2(y_k^n) \leq c_1$. The announced convergence follows from Proposition 2.13. \square

3 Space-time least squares method

Adapting the previous section, we introduce and analyze a so-called weak least-squares functional allowing to approximate the solution of the boundary value problem (1.2). Though more technical, the analysis is simpler since (1.2) (in this 2D setting) admits a unique weak solution, independently of the size of the data, contrary to (1.4).

3.1 Preliminary technical results

In the following, we repeatedly use the following classical estimate.

Lemma 3.1. *Let any $u \in \mathbf{H}$, $v, w \in \mathbf{V}$. There exists a constant $c = c(\Omega)$ such that*

$$\int_{\Omega} u \cdot \nabla v \cdot w \leq c \|u\|_{\mathbf{H}} \|v\|_{\mathbf{V}} \|w\|_{\mathbf{V}}. \quad (3.1)$$

Proof. If $u \in \mathbf{H}$, $v, w \in \mathbf{V}$, denoting \tilde{u}, \tilde{v} and \tilde{w} their extension to 0 in \mathbb{R}^2 , we have, see [3] and [23]

$$\begin{aligned} \left| \int_{\Omega} u \cdot \nabla v \cdot w \right| &= \left| \int_{\Omega} \tilde{u} \cdot \nabla \tilde{v} \cdot \tilde{w} \right| \leq \|\tilde{u} \cdot \nabla \tilde{v}\|_{\mathcal{H}^1(\mathbb{R}^2)} \|\tilde{w}\|_{BMO(\mathbb{R}^2)} \leq c \|\tilde{u}\|_2 \|\nabla \tilde{v}\|_2 \|\tilde{w}\|_{H^1(\mathbb{R}^2)} \\ &\leq c \|u\|_2 \|\nabla v\|_2 \|w\|_{H^1(\Omega)^2} \leq c \|u\|_{\mathbf{H}} \|v\|_{\mathbf{V}} \|w\|_{\mathbf{V}}. \end{aligned}$$

□

Lemma 3.2. *Let any $u \in L^\infty(0, T; \mathbf{H})$ and $v \in L^2(0, T; \mathbf{V})$. Then the function $B(u, v)$ defined by*

$$\langle B(u(t), v(t)), w \rangle = \int_{\Omega} u(t) \cdot \nabla v(t) \cdot w \quad \forall w \in \mathbf{V}, \text{ a.e in } t \in [0, T]$$

belongs to $L^2(0, T; \mathbf{V}')$ and

$$\|B(u, v)\|_{L^2(0, T; \mathbf{V}')} \leq c \left(\int_0^T \|u\|_{\mathbf{H}}^2 \|v\|_{\mathbf{V}}^2 \right)^{\frac{1}{2}} \leq c \|u\|_{L^\infty(0, T; \mathbf{H})} \|v\|_{L^2(0, T; \mathbf{V})}. \quad (3.2)$$

Moreover

$$\langle B(u, v), v \rangle_{\mathbf{V}' \times \mathbf{V}} = 0. \quad (3.3)$$

Proof. Indeed, a.e in $t \in [0, T]$ we have (see (3.1)), $\forall w \in \mathbf{V}$

$$|\langle B(u(t), v(t)), w \rangle| \leq c \|u(t)\|_{\mathbf{H}} \|v(t)\|_{\mathbf{V}} \|w\|_{\mathbf{V}}$$

and thus,

$$\int_0^T \|B(u, v)\|_{\mathbf{V}'}^2 \leq c \int_0^T \|u\|_{\mathbf{H}}^2 \|v\|_{\mathbf{V}}^2 \leq c \|u\|_{L^\infty(0, T; \mathbf{H})}^2 \|v\|_{L^2(0, T; \mathbf{V})}^2 < +\infty.$$

We also have a.e in $t \in [0, T]$ (see [24])

$$\langle B(u(t), v(t)), v(t) \rangle_{\mathbf{V}' \times \mathbf{V}} = \int_{\Omega} u(t) \cdot \nabla v(t) \cdot v(t) = 0.$$

□

Lemma 3.3. *Let any $u \in L^2(0, T; \mathbf{V})$. Then the function $B_1(u)$ defined by*

$$\langle B_1(u(t)), w \rangle = \int_{\Omega} \nabla u(t) \cdot \nabla w \quad \forall w \in \mathbf{V}, \text{ a.e in } t \in [0, T]$$

belong to $L^2(0, T; \mathbf{V}')$ and

$$\|B_1(u)\|_{L^2(0, T; \mathbf{V}')} \leq \|u\|_{L^2(0, T; \mathbf{V})}^2 < +\infty. \quad (3.4)$$

Proof. Indeed, a.e in $t \in [0, T]$ we have

$$|\langle B_1(u(t)), w \rangle| \leq \|\nabla u(t)\|_2 \|\nabla w\|_2 = \|u(t)\|_{\mathbf{V}} \|w\|_{\mathbf{V}}$$

and thus, a.e in $t \in [0, T]$

$$\|B_1(u(t))\|_{\mathbf{V}'} \leq \|u(t)\|_{\mathbf{V}}$$

which gives (3.4). □

We also have (see [24, 15]) :

Lemma 3.4. *For all $y \in L^2(0, T, \mathbf{V}) \cap H^1(0, T; \mathbf{V}')$ we have $y \in \mathcal{C}([0, T]; \mathbf{H})$ and in $\mathcal{D}'(0, T)$, for all $w \in \mathbf{V}$:*

$$\langle \partial_t y, w \rangle_{\mathbf{V}' \times \mathbf{V}} = \int_{\Omega} \partial_t y \cdot w = \frac{d}{dt} \int_{\Omega} y \cdot w, \quad \langle \partial_t y, y \rangle_{\mathbf{V}' \times \mathbf{V}} = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |y|^2 \quad (3.5)$$

and

$$\|y\|_{L^\infty(0, T; \mathbf{H})}^2 \leq c \|y\|_{L^2(0, T; \mathbf{V})} \|\partial_t y\|_{L^2(0, T; \mathbf{V}')} . \quad (3.6)$$

We recall that along this section, we suppose that $u_0 \in \mathbf{H}$, $f \in L^2(0, T, \mathbf{V}')$ and Ω is a bounded lipschitz domain of \mathbb{R}^2 . We also denote

$$\mathcal{A} = \{y \in L^2(0, T; \mathbf{V}) \cap H^1(0, T; \mathbf{V}'), y(0) = u_0\}$$

and

$$\mathcal{A}_0 = \{y \in L^2(0, T; \mathbf{V}) \cap H^1(0, T; \mathbf{V}'), y(0) = 0\}.$$

Endowed with the scalar product

$$\langle y, z \rangle_{\mathcal{A}_0} = \int_0^T \langle y, z \rangle_{\mathbf{V}} + \langle \partial_t y, \partial_t z \rangle_{\mathbf{V}'}$$

and the associated norm

$$\|y\|_{\mathcal{A}_0} = \sqrt{\|y\|_{L^2(0, T; \mathbf{V})}^2 + \|\partial_t y\|_{L^2(0, T; \mathbf{V}')}^2}$$

\mathcal{A}_0 is an Hilbert space.

We also recall and introduce several technical results. The first one is well-known (we refer to [15] and [24]).

Proposition 3.5. *There exists a unique $\bar{y} \in \mathcal{A}$ solution in $\mathcal{D}'(0, T)$ of (1.2). This solution satisfies the following estimates :*

$$\begin{aligned} \|\bar{y}\|_{L^\infty(0, T; \mathbf{H})}^2 + \nu \|\bar{y}\|_{L^2(0, T; \mathbf{V})}^2 &\leq \|u_0\|_{\mathbf{H}}^2 + \frac{1}{\nu} \|f\|_{L^2(0, T; \mathbf{V}')}^2, \\ \|\partial_t \bar{y}\|_{L^2(0, T; \mathbf{V}')} &\leq \sqrt{\nu} \|u_0\|_{\mathbf{H}} + 2\|f\|_{L^2(0, T; \mathbf{V}')} + \frac{c}{\nu^{\frac{3}{2}}} (\nu \|u_0\|_{\mathbf{H}}^2 + \|f\|_{L^2(0, T; \mathbf{V}')}^2). \end{aligned}$$

We also introduce the following result :

Proposition 3.6. *For all $y \in L^2(0, T, \mathbf{V}) \cap H^1(0, T; \mathbf{V}')$, there exists a unique $v \in \mathcal{A}_0$ solution in $\mathcal{D}'(0, T)$ of*

$$\left\{ \begin{aligned} \frac{d}{dt} \int_{\Omega} v \cdot w + \int_{\Omega} \nabla v \cdot \nabla w + \frac{d}{dt} \int_{\Omega} y \cdot w + \nu \int_{\Omega} \nabla y \cdot \nabla w \\ + \int_{\Omega} y \cdot \nabla y \cdot w = \langle f, w \rangle_{\mathbf{V}' \times \mathbf{V}}, \quad \forall w \in \mathbf{V} \quad (3.7) \\ v(0) = 0. \end{aligned} \right.$$

Moreover, for all $t \in [0, T]$,

$$\|v(t)\|_{\mathbf{H}}^2 + \|v\|_{L^2(0, t; \mathbf{V})}^2 \leq \|f - B(y, y) - \nu B_1(y) - \partial_t y\|_{L^2(0, t; \mathbf{V}')}^2$$

and

$$\begin{aligned} \|\partial_t v\|_{L^2(0, T; \mathbf{V}')} &\leq \|v\|_{L^2(0, T; \mathbf{V})} + \|f - B(y, y) - \nu B_1(y) - \partial_t y\|_{L^2(0, T; \mathbf{V}')} \\ &\leq 2\|f - B(y, y) - \nu B_1(y) - \partial_t y\|_{L^2(0, T; \mathbf{V}')} \end{aligned}$$

The proof of this proposition is a consequence of the following standard result (see [23, 15]).

Proposition 3.7. *For all $z_0 \in \mathbf{H}$ and all $F \in L^2(0, T; \mathbf{V}')$, there exists a unique $z \in L^2(0, T, \mathbf{V}) \cap H^1(0, T; \mathbf{V}')$ solution in $\mathcal{D}'(0, T)$ of*

$$\left\{ \begin{aligned} \frac{d}{dt} \int_{\Omega} z \cdot w + \int_{\Omega} \nabla z \cdot \nabla w = \langle F, w \rangle_{\mathbf{V}' \times \mathbf{V}}, \quad \forall w \in \mathbf{V} \\ z(0) = z_0. \end{aligned} \right. \quad (3.8)$$

Moreover, for all $t \in [0, T]$,

$$\|z(t)\|_{\mathbf{H}}^2 + \|z\|_{L^2(0, t; \mathbf{V})}^2 \leq \|F\|_{L^2(0, t; \mathbf{V}')}^2 + \|z_0\|_{\mathbf{H}}^2 \quad (3.9)$$

and

$$\|\partial_t z\|_{L^2(0,T;\mathbf{V}')} \leq \|z\|_{L^2(0,T,\mathbf{V})} + \|F\|_{L^2(0,T;\mathbf{V}')} \leq 2\|F\|_{L^2(0,T;\mathbf{V}')} + \|z_0\|_{\mathbf{H}}. \quad (3.10)$$

Proof. (of Proposition 3.6) Let $y \in L^2(0, T, \mathbf{V}) \cap H^1(0, T; \mathbf{V}')$. Then the functions $B(y, y)$ and $B_1(y)$ defined in $\mathcal{D}'(0, T)$ by

$$\langle B(y, y), w \rangle = \int_{\Omega} y \cdot \nabla y \cdot w \quad \text{and} \quad \langle B_1(y), w \rangle = \int_{\Omega} \nabla y \cdot \nabla w, \quad \forall w \in \mathbf{V}$$

belong to $L^2(0, T; \mathbf{V}')$ (see Lemma 3.2 and 3.3).

Moreover, since $y \in L^2(0, T, \mathbf{V}) \cap H^1(0, T, \mathbf{V}')$ then, in view of (3.5), in $\mathcal{D}'(0, T)$, for all $w \in \mathbf{V}$ we have :

$$\frac{d}{dt} \int_{\Omega} y \cdot w = \langle \partial_t y, w \rangle_{\mathbf{V}' \times \mathbf{V}}.$$

Then (1.9) may be rewritten as

$$\begin{cases} \frac{d}{dt} \int_{\Omega} v \cdot w + \int_{\Omega} \nabla v \cdot \nabla w = \langle F, w \rangle_{\mathbf{V}' \times \mathbf{V}}, & \forall w \in \mathbf{V} \\ v(0) = 0, \end{cases}$$

where $F = f - B(y, y) - \nu B_1(y) - \partial_t y \in L^2(0, T, \mathbf{V}')$; Proposition 3.6 is therefore a consequence of Proposition 3.7. \square

3.2 The least-squares functional

We now introduce our least-squares functional $E : H^1(0, T, \mathbf{V}') \cap L^2(0, T, \mathbf{V}) \rightarrow \mathbb{R}^+$ by putting

$$E(y) = \frac{1}{2} \int_0^T \|v\|_{\mathbf{V}}^2 + \frac{1}{2} \int_0^T \|\partial_t v\|_{\mathbf{V}'}^2 = \frac{1}{2} \|v\|_{\mathcal{A}_0}^2 \quad (3.11)$$

where the corrector v is the unique solution of (3.7). The infimum of E is equal to zero and is reached by a solution of (1.2). In this sense, the functional E is a so-called error functional which measures, through the corrector variable v , the deviation of y from being a solution of the underlying equation (1.2).

Beyond this statement, we would like to argue why we believe it is a good idea to use a (minimization) least-squares approach to approximate the solution of (1.2) by minimizing the functional E . Our main result of this section is a follows:

Theorem 3.8. *Let $\{y_k\}_{k \in \mathbb{N}}$ be a sequence of \mathcal{A} bounded in $L^2(0, T; \mathbf{V}) \cap H^1(0, T; \mathbf{V}')$. If $E'(y_k) \rightarrow 0$ as $k \rightarrow \infty$, then the whole sequence $\{y_k\}_{k \in \mathbb{N}}$ converges strongly as $k \rightarrow \infty$ in $L^2(0, T; \mathbf{V}) \cap H^1(0, T; \mathbf{V}')$ to the solution \bar{y} of (1.2).*

As in the previous section, we divide the proof in two main steps.

1. First, we use a typical *a priori* bound to show that leading the error functional E down to zero implies strong convergence to the unique solution of (1.2).
2. Next, we show that taking the derivative E' to zero actually suffices to take E to zero.

Before to prove this result, we mention the following equivalence which justifies the least-squares terminology we have used in the following sense: the minimization of the functional E is equivalent to the minimization of the $L^2(0, T; \mathbf{V}')$ -norm of the main equation of the Navier-Stokes system.

Lemma 3.9. *There exists $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 E(y) \leq \|y_t + \nu B_1(y) + B(y, y) - f\|_{L^2(0, T; \mathbf{V}')}^2 \leq c_2 E(y)$$

for all $y \in L^2(0, T; \mathbf{V}) \cap H^1(0, T; \mathbf{V}')$.

Proof. From Proposition 3.6 we deduce that

$$2E(y) = \|v\|_{\mathcal{A}_0}^2 \leq 5\|y_t + \nu B_1(y) + B(y, y) - f\|_{L^2(0, T; \mathbf{V}')}^2.$$

On the other hand, from the definition of v ,

$$\begin{aligned} \|y_t + \nu B_1(y) + B(y, y) - f\|_{L^2(0, T; \mathbf{V}')} &= \|v_t + B_1(v)\|_{L^2(0, T; \mathbf{V}')} \\ &\leq \|v_t\|_{L^2(0, T; \mathbf{V}')} + \|B_1(v)\|_{L^2(0, T; \mathbf{V}')} \leq \sqrt{2}\|v\|_{\mathcal{A}_0} = 2\sqrt{E(y)}. \end{aligned}$$

□

We start with the following proposition which establishes that as we take down the error E to zero, we get closer, in the norm $L^2(0, T; \mathbf{V})$ and $H^1(0, T; \mathbf{V}')$, to the solution \bar{y} of the problem (1.2), and so, it justifies why a promising strategy to find good approximations of the solution of problem (1.2) is to look for global minimizers of (3.11).

Proposition 3.10. *Let $\bar{y} \in \mathcal{A}$ be the solution of (1.2), $M \in \mathbb{R}$ such that $\|\partial_t \bar{y}\|_{L^2(0, T; \mathbf{V}')} \leq M$ and $\sqrt{\nu}\|\nabla \bar{y}\|_{L^2(Q_T)^4} \leq M$ and let $y \in \mathcal{A}$. If $\|\partial_t y\|_{L^2(0, T; \mathbf{V}')} \leq M$ and $\sqrt{\nu}\|\nabla y\|_{L^2(Q_T)^4} \leq M$, then there exists a constant $c(M)$ such that*

$$\|y - \bar{y}\|_{L^\infty(0, T; \mathbf{H})} + \sqrt{\nu}\|y - \bar{y}\|_{L^2(0, T; \mathbf{V})} + \|\partial_t y - \partial_t \bar{y}\|_{L^2(0, T; \mathbf{V}')} \leq c(M)\sqrt{E(y)}. \quad (3.12)$$

Proof. Let $Y = y - \bar{y}$. The functions $B(Y, y)$, $B(\bar{y}, Y)$ and $B_1(v)$ defined in $\mathcal{D}'(0, T)$ by

$$\begin{aligned} \langle B(Y, y), w \rangle &= \int_{\Omega} Y \cdot \nabla y \cdot w, \quad \langle B(\bar{y}, Y), w \rangle = \int_{\Omega} \bar{y} \cdot \nabla Y \cdot w, \\ \langle B_1(v), w \rangle &= \int_{\Omega} \nabla v \cdot \nabla w, \quad \forall w \in \mathbf{V} \end{aligned}$$

belong to $L^2(0, T; \mathbf{V}')$ (see Lemma 3.2 and 3.3), and from (1.2), (3.7) and (3.5) we deduce that

$$\begin{cases} \frac{d}{dt} \int_{\Omega} Y \cdot w + \nu \int_{\Omega} \nabla Y \cdot \nabla w = -\langle \partial_t v + B_1(v) + B(Y, y) + B(\bar{y}, Y), w \rangle_{\mathbf{V}' \times \mathbf{V}}, & \forall w \in \mathbf{V} \\ Y(0) = 0, \end{cases}$$

and from (3.9), (3.10), (3.2), (3.3) and (3.4) we deduce that for all $t \in [0, T]$

$$\begin{aligned} \int_{\Omega} |Y(t)|^2 + \nu \int_{Q_t} |\nabla Y|^2 &\leq \frac{1}{\nu} \int_0^t \|\partial_t v + B_1(v) + B(Y, y)\|_{\mathbf{V}'}^2 \\ &\leq \frac{4}{\nu} (\|\partial_t v\|_{L^2(0, T, \mathbf{V}')}^2 + \|v\|_{L^2(0, T, \mathbf{V})}^2 + c \int_0^t \|Y\|_2^2 \|y\|_{\mathbf{V}}^2) \\ &\leq \frac{4}{\nu} (2E(y) + c \int_0^t \|Y\|_2^2 \|y\|_{\mathbf{V}}^2). \end{aligned}$$

Gronwall's lemma then implies that for all $t \in [0, T]$

$$\int_{\Omega} |Y(t)|^2 + \nu \int_{Q_t} |\nabla Y|^2 \leq \frac{8}{\nu} E(y) \exp\left(\frac{c}{\nu} \int_0^t \|y\|_{\mathbf{V}}^2\right) \leq \frac{8}{\nu} E(y) \exp\left(\frac{c}{\nu^2} M^2\right)$$

which gives

$$\|Y\|_{L^\infty(0, T; \mathbf{H})} + \sqrt{\nu} \|Y\|_{L^2(0, T; \mathbf{V})} \leq \frac{4\sqrt{2}}{\sqrt{\nu}} \sqrt{E(y)} \exp\left(\frac{c}{\nu^2} M^2\right) \leq C(M) \sqrt{E(y)}.$$

Now

$$\begin{aligned} \|\partial_t Y\|_{L^2(0, T, \mathbf{V}')} &\leq \|\partial_t v + B_1(v) + \nu B_1(Y) + B(Y, y) + B(\bar{y}, Y)\|_{L^2(0, T, \mathbf{V}')} \\ &\leq \nu \|Y\|_{L^2(0, T, \mathbf{V})} + \|\partial_t v\|_{L^2(0, T, \mathbf{V}')} + \|v\|_{L^2(0, T, \mathbf{V})} \\ &\quad + c \|Y\|_{L^\infty(0, T, \mathbf{H})} \|y\|_{L^2(0, T, \mathbf{V})} + c \|\bar{y}\|_{L^\infty(0, T, \mathbf{H})} \|Y\|_{L^2(0, T, \mathbf{V})} \\ &\leq \sqrt{E(y)} \left(2\sqrt{2} \exp\left(\frac{c}{\nu^2} M^2\right) + 2\sqrt{2} + cM \frac{4\sqrt{2}}{\nu} \exp\left(\frac{c}{\nu^2} M^2\right) \right) \end{aligned}$$

and thus

$$\|\partial_t Y\|_{L^2(0,T,\mathbf{V}')} \leq c(M)\sqrt{E(y)}.$$

□

We now proceed with the second part of the proof and would like to show that the only critical points for E correspond to solutions of (1.2). In such a case, the search for an element y solution of (1.2) is reduced to the minimization of E .

For any $y \in \mathcal{A}$, we now look for an element $Y_1 \in \mathcal{A}_0$ solution of the following formulation

$$\left\{ \begin{array}{l} \frac{d}{dt} \int_{\Omega} Y_1 \cdot w + \nu \int_{\Omega} \nabla Y_1 \cdot \nabla w + \int_{\Omega} y \cdot \nabla Y_1 \cdot w \\ \quad + \int_{\Omega} Y_1 \cdot \nabla y \cdot w = -\frac{d}{dt} \int_{\Omega} v \cdot w - \int_{\Omega} \nabla v \cdot \nabla w, \quad \forall w \in \mathbf{V} \\ Y_1(0) = 0, \end{array} \right. \quad (3.13)$$

where $v \in \mathcal{A}_0$ is the corrector (associated to y) solution of (3.7). Y_1 enjoys the following property:

Proposition 3.11. *For all $y \in \mathcal{A}$, there exists a unique $Y_1 \in \mathcal{A}_0$ solution of (3.13). Moreover if for some $M \in \mathbb{R}$, $\|\partial_t y\|_{L^2(0,T,\mathbf{V}')} \leq M$ and $\sqrt{\nu}\|\nabla y\|_{L^2(Q_T)^4} \leq M$, then this solution satisfies*

$$\|\partial_t Y_1\|_{L^2(0,T,\mathbf{V}')} + \sqrt{\nu}\|\nabla Y_1\|_{L^2(Q_T)^4} \leq c(M)\sqrt{E(y)}$$

for some constant $c(M) > 0$.

Proof. As in Proposition 3.10, (3.13) can be written as

$$\left\{ \begin{array}{l} \frac{d}{dt} \int_{\Omega} Y_1 \cdot w + \nu \int_{\Omega} \nabla Y_1 \cdot \nabla w + \int_{\Omega} y \cdot \nabla Y_1 \cdot w + \int_{\Omega} Y_1 \cdot \nabla y \cdot w \\ \quad = -\langle \partial_t v + B_1(v), w \rangle_{\mathbf{V}' \times \mathbf{V}}, \quad \forall w \in \mathbf{V} \\ Y_1(0) = 0. \end{array} \right. \quad (3.14)$$

(3.14) admits a unique solution $Y_1 \in \mathcal{A}_0$. Indeed, let $y_1 \in L^2(0,T;\mathbf{V}) \cap \mathcal{C}([0,T];\mathbf{H})$. Moreover, there exists (see [24]) a unique $z_1 \in \mathcal{A}_0$ solution of

$$\left\{ \begin{array}{l} \frac{d}{dt} \int_{\Omega} z_1 \cdot w + \nu \int_{\Omega} \nabla z_1 \cdot \nabla w + \int_{\Omega} y \cdot \nabla z_1 \cdot w + \int_{\Omega} y_1 \cdot \nabla y \cdot w \\ \quad = -\langle \partial_t v + B_1(v), w \rangle_{\mathbf{V}' \times \mathbf{V}}, \quad \forall w \in \mathbf{V} \\ z_1(0) = 0. \end{array} \right. \quad (3.15)$$

Let $\mathcal{T} : y_1 \mapsto z_1$. Then if $z_2 = \mathcal{T}(y_2)$, $z_1 - z_2$ is solution of

$$\left\{ \begin{array}{l} \frac{d}{dt} \int_{\Omega} (z_1 - z_2) \cdot w + \nu \int_{\Omega} \nabla(z_1 - z_2) \cdot \nabla w + \int_{\Omega} y \cdot \nabla(z_1 - z_2) \cdot w \\ \quad + \int_{\Omega} (y_1 - y_2) \cdot \nabla y \cdot w = 0, \quad \forall w \in \mathbf{V} \\ (z_1 - z_2)(0) = 0, \end{array} \right.$$

and thus, for $w = z_1 - z_2$

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |z_1 - z_2|^2 + \nu \int_{\Omega} |\nabla(z_1 - z_2)|^2 = - \int_{\Omega} (y_1 - y_2) \cdot \nabla y \cdot (z_1 - z_2).$$

But

$$\begin{aligned} \left| \int_{\Omega} (y_1 - y_2) \cdot \nabla y \cdot (z_1 - z_2) \right| &\leq c \|y_1 - y_2\|_2 \|y\|_{\mathbf{V}} \|\nabla(z_1 - z_2)\|_2 \\ &\leq c \|y_1 - y_2\|_2^2 \|y\|_{\mathbf{V}}^2 + \frac{\nu}{2} \|\nabla(z_1 - z_2)\|_2^2 \end{aligned}$$

so that

$$\frac{d}{dt} \int_{\Omega} |z_1 - z_2|^2 + \nu \int_{\Omega} |\nabla(z_1 - z_2)|^2 \leq c \|y_1 - y_2\|_2^2 \|y\|_{\mathbf{V}}^2,$$

and for all $t \in [0, T]$

$$\|z_1 - z_2\|_{L^\infty(0,t;\mathbf{H})}^2 + \nu \int_0^t \int_{\Omega} |\nabla(z_1 - z_2)|^2 \leq c \|y_1 - y_2\|_{L^\infty(0,t;\mathbf{H})}^2 \int_0^t \|y\|_{\mathbf{V}}^2.$$

Since $y \in L^2(0, T; \mathbf{V})$, there exists $t' \in]0, T]$ such that $\int_0^{t'} \|y\|_{\mathbf{V}}^2 \leq \frac{1}{2c}$. We then have

$$\|z_1 - z_2\|_{L^\infty(0,t';\mathbf{H})}^2 + \nu \iint_{0\Omega}^{t'} |\nabla(z_1 - z_2)|^2 \leq \frac{1}{2} \|y_1 - y_2\|_{L^\infty(0,t';\mathbf{H})}^2$$

and the map \mathcal{T} is a contraction mapping on $X = \mathcal{C}([0, t']; \mathbf{H}) \cap L^2(0, t'; \mathbf{V})$. So \mathcal{T} admits a unique fixed point $Y_1 \in X$. Moreover, from (3.15) we deduce that $\partial_t Y_1 \in L^2(0, t', \mathbf{V}')$. Since the map $t \mapsto \int_0^t \|\nabla y\|_2^2$ is a uniformly continuous function, we can take $t' = T$.

For this solution we have, for all $t \in [0, T]$, since $\int_{Q_t} y \cdot \nabla Y_1 \cdot Y_1 = 0$

$$\frac{1}{2} \int_{\Omega} |Y_1(t)|^2 + \nu \int_{Q_t} |\nabla Y_1|^2 = - \int_0^t \langle B(Y_1, y) + \partial_t v + B_1(v), Y_1 \rangle_{\mathbf{V}' \times \mathbf{V}}.$$

Moreover, as in the proof of Proposition 3.10, we have

$$\int_{\Omega} |Y_1(t)|^2 + \nu \int_{Q_t} |\nabla Y_1|^2 \leq \frac{8}{\nu} E(y) \exp\left(\frac{c}{\nu} \int_0^t \|y\|_{\mathbf{V}}^2\right) \quad (3.16)$$

and thus

$$\begin{aligned} \sqrt{\nu} \|Y_1\|_{L^2(0,T;\mathbf{V})} &\leq \frac{2\sqrt{2}}{\sqrt{\nu}} \sqrt{E(y)} \exp\left(\frac{c}{\nu} \int_0^T \|y\|_{\mathbf{V}}^2\right) \\ &\leq \frac{2\sqrt{2}}{\sqrt{\nu}} \sqrt{E(y)} \exp\left(\frac{c}{\nu^2} M^2\right) \leq c(M) \sqrt{E(y)} \end{aligned}$$

and

$$\begin{aligned} &\|\partial_t Y_1\|_{L^2(0,T;\mathbf{V}')} \\ &\leq \sqrt{E(y)} \left(2\sqrt{2} \exp\left(\frac{c}{\nu} \int_0^T \|y\|_{\mathbf{V}}^2\right) + 2\sqrt{2} + c \|y\|_{L^2(0,T;\mathbf{V})} \frac{2\sqrt{2}}{\sqrt{\nu}} \exp\left(\frac{c}{\nu} \int_0^T \|y\|_{\mathbf{V}}^2\right) \right. \\ &\quad \left. + c \|y\|_{L^\infty(0,T;\mathbf{H})} \frac{2\sqrt{2}}{\nu} \exp\left(\frac{c}{\nu} \int_0^T \|y\|_{\mathbf{V}}^2\right) \right) \\ &\leq \sqrt{E(y)} \left(2\sqrt{2} \exp\left(\frac{c}{\nu^2} M^2\right) + 2\sqrt{2} + cM \frac{4\sqrt{2}}{\nu} \exp\left(\frac{c}{\nu^2} M^2\right) \right) \leq c(M) \sqrt{E(y)}. \end{aligned} \quad (3.17)$$

□

Proposition 3.12. *For all $y \in \mathcal{A}$, the map $Y \mapsto E(y + Y)$ is a differentiable function on the Hilbert space \mathcal{A}_0 and for any $Y \in \mathcal{A}_0$, we have*

$$E'(y) \cdot Y = \langle v, V \rangle_{\mathcal{A}_0} = \int_0^T \langle v, V \rangle_{\mathbf{V}} + \int_0^T \langle \partial_t v, \partial_t V \rangle_{\mathbf{V}'}$$

where $V \in \mathcal{A}_0$ is the unique solution in $\mathcal{D}'(0, T)$ of

$$\begin{cases} \frac{d}{dt} \int_{\Omega} V \cdot w + \int_{\Omega} \nabla V \cdot \nabla w + \frac{d}{dt} \int_{\Omega} Y \cdot w + \nu \int_{\Omega} \nabla Y \cdot \nabla w \\ \quad + \int_{\Omega} y \cdot \nabla Y \cdot w + \int_{\Omega} Y \cdot \nabla y \cdot w = 0, \quad \forall w \in \mathbf{V} \\ V(0) = 0. \end{cases} \quad (3.18)$$

Proof. Let $y \in \mathcal{A}$ and $Y \in \mathcal{A}_0$. We have $E(y+Y) = \frac{1}{2} \|\bar{V}\|_{\mathcal{A}_0}^2$ where $\bar{V} \in \mathcal{A}_0$ is the unique solution of

$$\left\{ \begin{array}{l} \frac{d}{dt} \int_{\Omega} \bar{V} \cdot w + \int_{\Omega} \nabla \bar{V} \cdot \nabla w + \frac{d}{dt} \int_{\Omega} (y+Y) \cdot w + \nu \int_{\Omega} \nabla (y+Y) \cdot \nabla w \\ \quad + \int_{\Omega} (y+Y) \cdot \nabla (y+Y) \cdot w - \langle f, w \rangle_{\mathbf{V}' \times \mathbf{V}} = 0, \quad \forall w \in \mathbf{V} \\ \bar{V}(0) = 0. \end{array} \right.$$

If $v \in \mathcal{A}_0$ is the solution of (3.7) associated to y , $v' \in \mathcal{A}_0$ is the unique solution of

$$\left\{ \begin{array}{l} \frac{d}{dt} \int_{\Omega} v' \cdot w + \int_{\Omega} \nabla v' \cdot \nabla w + \int_{\Omega} Y \cdot \nabla Y \cdot w = 0, \quad \forall w \in \mathbf{V} \\ v'(0) = 0 \end{array} \right.$$

and $V \in \mathcal{A}_0$ is the unique solution of (3.18), then it is straightforward to check that $\bar{V} - v - v' - V \in \mathcal{A}_0$ is solution of

$$\left\{ \begin{array}{l} \frac{d}{dt} \int_{\Omega} (\bar{V} - v - v' - V) \cdot w + \int_{\Omega} \nabla (\bar{V} - v - v' - V) \cdot \nabla w = 0, \quad \forall w \in \mathbf{V} \\ (\bar{V} - v - v' - V)(0) = 0 \end{array} \right.$$

and therefore $\bar{V} - v - v' - V = 0$. Thus

$$\begin{aligned} E(y+Y) &= \frac{1}{2} \|v + v' + V\|_{\mathcal{A}_0}^2 \\ &= \frac{1}{2} \|v\|_{\mathcal{A}_0}^2 + \frac{1}{2} \|v'\|_{\mathcal{A}_0}^2 + \frac{1}{2} \|V\|_{\mathcal{A}_0}^2 + \langle V, v' \rangle_{\mathcal{A}_0} + \langle V, v \rangle_{\mathcal{A}_0} + \langle v, v' \rangle_{\mathcal{A}_0}. \end{aligned}$$

We deduce from (3.18) and (3.9) that

$$\begin{aligned} \|V\|_{L^2(0,T,\mathbf{V})}^2 &\leq c \left(\|\partial_t Y\|_{L^2(0,T,\mathbf{V}')}^2 + \nu^2 \|B_1(Y)\|_{L^2(0,T,\mathbf{V}')}^2 \right. \\ &\quad \left. + \|B(y,Y)\|_{L^2(0,T,\mathbf{V}')}^2 + \|B(Y,y)\|_{L^2(0,T,\mathbf{V}')}^2 \right) \end{aligned}$$

and from (3.4), (3.2) and (3.6) that

$$\|V\|_{L^2(0,T,\mathbf{V})}^2 \leq c \|Y\|_{\mathcal{A}_0}^2.$$

Similarly, we deduce from (3.10) that

$$\|\partial_t V\|_{L^2(0,T,\mathbf{V}')}^2 \leq c \|Y\|_{\mathcal{A}_0}^2.$$

Thus

$$\|V\|_{\mathcal{A}_0}^2 \leq c\|Y\|_{\mathcal{A}_0}^2 = o(\|Y\|_{\mathcal{A}_0}).$$

From (3.9), (3.10) and (3.2), we also deduce that

$$\|v'\|_{L^2(0,T;\mathbf{V})}^2 \leq \|B(Y, Y)\|_{L^2(0,T;\mathbf{V}')}^2 \leq c\|Y\|_{L^\infty(0,T;\mathbf{H})}^2 \|Y\|_{L^2(0,T;\mathbf{V})}^2 \leq c\|Y\|_{\mathcal{A}_0}^4$$

and

$$\|\partial_t v'\|_{L^2(0,T;\mathbf{V}')}^2 \leq c\|Y\|_{L^\infty(0,T;\mathbf{H})}^2 \|Y\|_{L^2(0,T;\mathbf{V})}^2 \leq c\|Y\|_{\mathcal{A}_0}^4,$$

thus we also have

$$\|v'\|_{\mathcal{A}_0}^2 \leq c\|Y\|_{\mathcal{A}_0}^4 = o(\|Y\|_{\mathcal{A}_0}).$$

From the previous estimates, we then obtain

$$|\langle V, v' \rangle_{\mathcal{A}_0}| \leq \|V\|_{\mathcal{A}_0} \|v'\|_{\mathcal{A}_0} \leq c\|Y\|_{\mathcal{A}_0}^3 = o(\|Y\|_{\mathcal{A}_0})$$

and

$$|\langle v, v' \rangle_{\mathcal{A}_0}| \leq \|v\|_{\mathcal{A}_0} \|v'\|_{\mathcal{A}_0} \leq c\sqrt{E(y)} \|Y\|_{\mathcal{A}_0}^2 = o(\|Y\|_{\mathcal{A}_0}),$$

thus

$$E(y + Y) = E(y) + \langle v, V \rangle_{\mathcal{A}_0} + o(\|Y\|_{\mathcal{A}_0}).$$

Eventually, the estimate

$$|\langle v, V \rangle_{\mathcal{A}_0}| \leq \|v\|_{\mathcal{A}_0} \|V\|_{\mathcal{A}_0} \leq c\sqrt{E(y)} \|Y\|_{\mathcal{A}_0}$$

gives the continuity of the linear map $Y \mapsto \langle v, V \rangle_{\mathcal{A}_0}$. □

We are now in position to prove the following result.

Proposition 3.13. *If $\{y_k\}_{k \in \mathbb{N}}$ is a sequence of \mathcal{A} bounded in $L^2(0, T; \mathbf{V}) \cap H^1(0, T; \mathbf{V}')$ satisfying $E'(y_k) \rightarrow 0$ as $k \rightarrow \infty$, then $E(y_k) \rightarrow 0$ as $k \rightarrow \infty$.*

Proof. For any $y \in \mathcal{A}$ and $Y \in \mathcal{A}_0$, we have

$$E'(y) \cdot Y = \langle v, V \rangle_{\mathcal{A}_0} = \int_0^T \langle v, V \rangle_{\mathbf{V}} + \int_0^T \langle \partial_t v, \partial_t V \rangle_{\mathbf{V}'}$$

where $V \in \mathcal{A}_0$ is the unique solution in $\mathcal{D}'(0, T)$ of (3.18). In particular, taking $Y = Y_1$ defined by (3.13), we define an element V_1 solution of

$$\left\{ \begin{array}{l} \frac{d}{dt} \int_{\Omega} V_1 \cdot w + \int_{\Omega} \nabla V_1 \cdot \nabla w + \frac{d}{dt} \int_{\Omega} Y_1 \cdot w + \nu \int_{\Omega} \nabla Y_1 \cdot \nabla w + \int_{\Omega} y \cdot \nabla Y_1 \cdot w \\ \quad + \int_{\Omega} Y_1 \cdot \nabla y \cdot w = 0, \quad \forall w \in \mathbf{V} \\ V_1(0) = 0. \end{array} \right. \tag{3.19}$$

Summing (3.19) and the (3.13), we obtain that $V_1 - v$ solves (3.8) with $F \equiv 0$ and $z_0 = 0$. This implies that V_1 and v coincide, and then

$$E'(y) \cdot Y_1 = \int_0^T \|v\|_{\mathbf{V}}^2 + \int_0^T \|\partial_t v\|_{\mathbf{V}'}^2 = 2E(y), \quad \forall y \in \mathcal{A}. \quad (3.20)$$

Let now, for any $k \in \mathbb{N}$, $Y_{1,k}$ be the solution of (3.13) associated to y_k . The previous equality writes $E'(y_k) \cdot Y_{1,k} = 2E(y_k)$ and implies our statement, since from Proposition 3.11, $Y_{1,k}$ is uniformly bounded in \mathcal{A}_0 . \square

3.3 Minimizing sequence for E

As in the previous section, equality (3.20) shows that $-Y_1$ given by the solution of (3.13) is a descent direction for the functional E . Remark also, in view of (3.13), that the corrector V associated to Y_1 , given by (3.18) with $Y = Y_1$, is nothing else than the corrector v itself. Therefore, we can define, for any $m \geq 1$, a minimizing sequence $\{y_k\}_{(k \in \mathbb{N})}$ for E as follows:

$$\begin{cases} y_0 \in \mathcal{A}, \\ y_{k+1} = y_k - \lambda_k Y_{1,k}, \quad k \geq 0, \\ E(y_k - \lambda_k Y_{1,k}) = \min_{\lambda \in [0, m]} E(y_k - \lambda Y_{1,k}) \end{cases} \quad (3.21)$$

with $Y_{1,k} \in \mathcal{A}_0$ the solution of the formulation

$$\begin{cases} \frac{d}{dt} \int_{\Omega} Y_{1,k} \cdot w + \nu \int_{\Omega} \nabla Y_{1,k} \cdot \nabla w + \int_{\Omega} y_k \cdot \nabla Y_{1,k} \cdot w \\ \quad + \int_{\Omega} Y_{1,k} \cdot \nabla y_k \cdot w = -\frac{d}{dt} \int_{\Omega} v_k \cdot w - \int_{\Omega} \nabla v_k \cdot \nabla w, \quad \forall w \in \mathbf{V} \\ Y_{1,k}(0) = 0, \end{cases} \quad (3.22)$$

where $v_k \in \mathcal{A}_0$ is the corrector (associated to y_k) solution of (3.7) leading (see (3.20)) to $E'(y_k) \cdot Y_{1,k} = 2E(y_k)$. For any $k > 0$, the direction $Y_{1,k}$ vanishes when $E(y_k)$ vanishes.

Lemma 3.14. *Let $\{y_k\}_{k \in \mathbb{N}}$ the sequence of \mathcal{A} defined by (3.21). Then $\{y_k\}_{k \in \mathbb{N}}$ is a bounded sequence of $H^1(0, T; \mathbf{V}') \cap L^2(0, T; \mathbf{V})$ and $\{E(y_k)\}_{k \in \mathbb{N}}$ is a decreasing sequence.*

Proof. From (3.21) we deduce that, for all $k \in \mathbb{N}$:

$$E(y_{k+1}) = E(y_k - \lambda_k Y_{1,k}) = \min_{\lambda \in [0, m]} E(y_k - \lambda Y_{1,k}) \leq E(y_k)$$

and thus the sequence $\{E(y_k)\}_{k \in \mathbb{N}}$ decreases and, for all $k \in \mathbb{N}$: $E(y_k) \leq E(y_0)$. Moreover, from the construction of the corrector $v_k \in \mathcal{A}_0$ associated to $y_k \in \mathcal{A}$ given by (3.7), we deduce from Proposition 3.5 that $y_k \in \mathcal{A}$ is the unique solution of

$$\left\{ \begin{array}{l} \frac{d}{dt} \int_{\Omega} y_k \cdot w + \nu \int_{\Omega} \nabla y_k \cdot \nabla w + \int_{\Omega} y_k \cdot \nabla y_k \cdot w = \langle f, w \rangle_{\mathbf{V}' \times \mathbf{V}} \\ - \frac{d}{dt} \int_{\Omega} v_k \cdot w - \int_{\Omega} \nabla v_k \cdot \nabla w, \quad \forall w \in \mathbf{V} \\ y_k(0) = u_0, \end{array} \right.$$

and, using (3.2) and (3.4)

$$\begin{aligned} \|y_k\|_{L^\infty(0, T; \mathbf{H})}^2 &\leq \|u_0\|_{\mathbf{H}}^2 + \frac{1}{\nu} \|f - \partial_t v_k - B_1(v_k)\|_{L^2(0, T; \mathbf{V}')}^2 \\ &\leq \|u_0\|_{\mathbf{H}}^2 + \frac{2}{\nu} \|f\|_{L^2(0, T; \mathbf{V}')}^2 + \frac{2}{\nu} \|\partial_t v_k\|_{L^2(0, T; \mathbf{V}')}^2 + \frac{2}{\nu} \|v_k\|_{L^2(0, T; \mathbf{V})}^2 \\ &\leq \|u_0\|_{\mathbf{H}}^2 + \frac{2}{\nu} \|f\|_{L^2(0, T; \mathbf{V}')}^2 + \frac{4}{\nu} E(y_k) \\ &\leq \|u_0\|_{\mathbf{H}}^2 + \frac{2}{\nu} \|f\|_{L^2(0, T; \mathbf{V}')}^2 + \frac{4}{\nu} E(y_0), \end{aligned} \tag{3.23}$$

$$\begin{aligned} \nu \|y_k\|_{L^2(0, T; \mathbf{V})}^2 &\leq \|u_0\|_{\mathbf{H}}^2 + \frac{1}{\nu} \|f - \partial_t v_k - B_1(v_k)\|_{L^2(0, T; \mathbf{V}')}^2 \\ &\leq \|u_0\|_{\mathbf{H}}^2 + \frac{2}{\nu} \|f\|_{L^2(0, T; \mathbf{V}')}^2 + \frac{4}{\nu} E(y_0) \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \|\partial_t y_k\|_{L^2(0, T; \mathbf{V}')} &\leq \|f - \partial_t v_k - B_1(v_k) - B(y_k, y_k) - \nu B_1(y_k)\|_{L^2(0, T; \mathbf{V}')} \\ &\leq \|f\|_{L^2(0, T; \mathbf{V}')} + \|\partial_t v_k\|_{L^2(0, T; \mathbf{V}')} + \|v_k\|_{L^2(0, T; \mathbf{V})} \\ &\quad + c \|y_k\|_{L^\infty(0, T; \mathbf{H})} \|y_k\|_{L^2(0, T; \mathbf{V})} + \nu \|y_k\|_{L^2(0, T; \mathbf{V})} \\ &\leq \|f\|_{L^2(0, T; \mathbf{V}')} + 2\sqrt{E(y_k)} + \sqrt{\nu} \|u_0\|_{\mathbf{H}} + \sqrt{2} \|f\|_{L^2(0, T; \mathbf{V}')} \\ &\quad + 2\sqrt{E(y_0)} + c \left(\|u_0\|_{\mathbf{H}}^2 + \frac{2}{\nu} \|f\|_{L^2(0, T; \mathbf{V}')}^2 + \frac{4}{\nu} E(y_0) \right) \\ &\leq 3\|f\|_{L^2(0, T; \mathbf{V}')} + 4\sqrt{E(y_0)} + \sqrt{\nu} \|u_0\|_{\mathbf{H}} \\ &\quad + c \left(\|u_0\|_{\mathbf{H}}^2 + \frac{2}{\nu} \|f\|_{L^2(0, T; \mathbf{V}')}^2 + \frac{4}{\nu} E(y_0) \right). \end{aligned}$$

□

Lemma 3.15. *Let $\{y_k\}_{k \in \mathbb{N}}$ the sequence of \mathcal{A} defined by (3.21). Then for all $\lambda \in [0, m]$, the following estimate holds*

$$E(y_k - \lambda Y_{1,k}) \leq E(y_k) \left(|1 - \lambda| + \lambda^2 \frac{c}{\nu\sqrt{\nu}} \sqrt{E(y_k)} \exp\left(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2\right) \right)^2. \quad (3.25)$$

Proof. Let V_k be the corrector associated to $y_k - \lambda Y_{1,k}$. It is easy to check that V_k is given by $(1 - \lambda)v_k + \lambda^2 \bar{v}_k$ where $\bar{v}_k \in \mathcal{A}_0$ solves

$$\begin{cases} \frac{d}{dt} \int_{\Omega} \bar{v}_k \cdot w + \int_{\Omega} \nabla \bar{v}_k \cdot \nabla w + \int_{\Omega} Y_{1,k} \cdot \nabla Y_{1,k} \cdot w = 0, & \forall w \in \mathbf{V} \\ \bar{v}_k(0) = 0, \end{cases} \quad (3.26)$$

and thus

$$\begin{aligned} 2E(y_k - \lambda Y_{1,k}) &= \|V_k\|_{\mathcal{A}_0}^2 = \|(1 - \lambda)v_k + \lambda^2 \bar{v}_k\|_{\mathcal{A}_0}^2 \\ &\leq (|1 - \lambda| \|v_k\|_{\mathcal{A}_0} + \lambda^2 \|\bar{v}_k\|_{\mathcal{A}_0})^2 \\ &\leq (\sqrt{2}|1 - \lambda| \sqrt{E(y_k)} + \lambda^2 \|\bar{v}_k\|_{\mathcal{A}_0})^2, \end{aligned} \quad (3.27)$$

which gives

$$E(y_k - \lambda Y_{1,k}) \leq \left(|1 - \lambda| \sqrt{E(y_k)} + \frac{\lambda^2}{\sqrt{2}} \|\bar{v}_k\|_{\mathcal{A}_0} \right)^2 := g(\lambda, y_k). \quad (3.28)$$

From (3.26), (3.9), (3.10) and (3.2) we deduce that

$$\|\bar{v}_k\|_{L^2(0,T;\mathbf{V})} \leq \|B(Y_{1,k}, Y_{1,k})\|_{L^2(0,T;\mathbf{V}')} \leq c \|Y_{1,k}\|_{L^\infty(0,T;\mathbf{H})} \|Y_{1,k}\|_{L^2(0,T;\mathbf{V})}$$

and

$$\begin{aligned} \|\partial_t \bar{v}_k\|_{L^2(0,T;\mathbf{V}')} &\leq \| -B_1(\bar{v}_k) - B(Y_{1,k}, Y_{1,k}) \|_{L^2(0,T;\mathbf{V}')} \\ &\leq \|\bar{v}_k\|_{L^2(0,T;\mathbf{V})} + c \|Y_{1,k}\|_{L^\infty(0,T;\mathbf{H})} \|Y_{1,k}\|_{L^2(0,T;\mathbf{V})} \\ &\leq c \|Y_{1,k}\|_{L^\infty(0,T;\mathbf{H})} \|Y_{1,k}\|_{L^2(0,T;\mathbf{V})}. \end{aligned}$$

On the other hand, from (3.16) we deduce that

$$\|Y_{1,k}\|_{L^\infty(0,T;\mathbf{H})}^2 + \nu \|Y_{1,k}\|_{L^2(0,T;\mathbf{V})}^2 \leq \frac{16}{\nu} E(y_k) \exp\left(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2\right). \quad (3.29)$$

Thus

$$\|\bar{v}_k\|_{L^2(0,T;\mathbf{V})} \leq \frac{c}{\nu\sqrt{\nu}} E(y_k) \exp\left(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2\right)$$

and

$$\|\partial_t \bar{v}_k\|_{L^2(0,T;\mathbf{V}')} \leq \frac{c}{\nu\sqrt{\nu}} E(y_k) \exp\left(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2\right)$$

which gives

$$\|\bar{v}_k\|_{\mathcal{A}_0} = \sqrt{\|\bar{v}_k\|_{L^2(0,T;\mathbf{V})}^2 + \|\partial_t \bar{v}_k\|_{L^2(0,T;\mathbf{V}')}^2} \leq \frac{c}{\nu\sqrt{\nu}} E(y_k) \exp\left(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2\right).$$

From (3.28) we then deduce (3.25). \square

Lemma 3.16. *Let $\{y_k\}_{k \in \mathbb{N}}$ the sequence of \mathcal{A} defined by (3.21). Then $E(y_k) \rightarrow 0$ as $k \rightarrow \infty$. Moreover, there exists a $k_0 \in \mathbb{N}$ such that the sequence $\{E(y_k)\}_{k \geq k_0}$ decays quadratically.*

Proof. We deduce from (3.25), using (3.24) that, for all $\lambda \in [0, m]$ and $k \in \mathbb{N}^*$:

$$\sqrt{E(y_{k+1})} \leq \sqrt{E(y_k)} \left(|1 - \lambda| + \lambda^2 C_1 \sqrt{E(y_k)} \right) \quad (3.30)$$

where $C_1 = \frac{c}{\nu\sqrt{\nu}} \exp\left(\frac{c}{\nu^2} \|u_0\|_{\mathbf{H}}^2 + \frac{c}{\nu^3} \|f\|_{L^2(0,T;\mathbf{V}')}^2 + \frac{c}{\nu^3} E(y_0)\right)$ does not depend on y_k .

Let us denote the polynomial p_k by $p_k(\lambda) = |1 - \lambda| + \lambda^2 C_1 \sqrt{E(y_k)}$ for all $\lambda \in [0, m]$. If $C_1 \sqrt{E(y_0)} < 1$ (and thus $C_1 \sqrt{E(y_k)} < 1$ for all $k \in \mathbb{N}$) then

$$\min_{\lambda \in [0, m]} p_k(\lambda) \leq p_k(1) = C_1 \sqrt{E(y_k)}$$

and thus

$$C_1 \sqrt{E(y_{k+1})} \leq (C_1 \sqrt{E(y_k)})^2 \quad (3.31)$$

implying that $C_1 \sqrt{E(y_k)} \rightarrow 0$ as $k \rightarrow \infty$ with a quadratic rate.

Suppose now that $C_1 \sqrt{E(y_0)} \geq 1$ and denote $I = \{k \in \mathbb{N}, C_1 \sqrt{E(y_k)} \geq 1\}$. Let us prove that I is a finite subset of \mathbb{N} . For all $k \in I$, since $C_1 \sqrt{E(y_k)} \geq 1$,

$$\begin{aligned} \min_{\lambda \in [0, m]} p_k(\lambda) &= \min_{\lambda \in [0, 1]} p_k(\lambda) = p_k\left(\frac{1}{2C_1 \sqrt{E(y_k)}}\right) \\ &= 1 - \frac{1}{4C_1 \sqrt{E(y_k)}} \leq 1 - \frac{1}{4C_1 \sqrt{E(y_0)}} < 1 \end{aligned}$$

and thus, for all $k \in I$,

$$\sqrt{E(y_{k+1})} \leq \left(1 - \frac{1}{4C_1 \sqrt{E(y_0)}}\right) \sqrt{E(y_k)} \leq \left(1 - \frac{1}{4C_1 \sqrt{E(y_0)}}\right)^{k+1} \sqrt{E(y_0)}.$$

Since $\left(1 - \frac{1}{4C_1\sqrt{E(y_0)}}\right)^{k+1} \rightarrow 0$ as $k \rightarrow +\infty$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $C_1\sqrt{E(y_{k+1})} < 1$. Thus I is a finite subset of \mathbb{N} . Arguing as in the first case, it follows that $C_1\sqrt{E(y_k)} \rightarrow 0$ as $k \rightarrow \infty$. \square

Remark 3.17. *In view of the estimate above of the constant C_1 , the number of iterates k_1 necessary to achieve the quadratic regime (from which the convergence is very fast) is of the order $\nu^{-3/2} \exp\left(\frac{c}{\nu^2}\|u_0\|_{\mathbf{H}}^2 + \frac{c}{\nu^3}\|f\|_{L^2(0,T;\mathbf{V}')}^2 + \frac{c}{\nu^3}E(y_0)\right)$, and therefore increases rapidly as $\nu \rightarrow 0$. In order to reduce the effect of the term $\nu^{-3}E(y_0)$, it is thus relevant, for small values of ν , to couple the algorithm with a continuation approach with respect to ν (i.e. start the sequence $\{y_k\}_{(k \geq 0)}$ with an element y_0 solution of (1.2) associated to $\bar{\nu} > \nu$ so that $\nu^{-3}E(y_0)$ be at most of the order $\mathcal{O}(\nu^{-2})$).*

Lemma 3.18. *Let $\{y_k\}_{k \in \mathbb{N}}$ the sequence of \mathcal{A} defined by (3.21). Then $\lambda_k \rightarrow 1$ as $k \rightarrow \infty$.*

Proof. We have

$$\begin{aligned} 2E(y_{k+1}) &= 2E(y_k - \lambda_k Y_{1,k}) \\ &= (1 - \lambda_k)^2 \|v_k\|_{\mathcal{A}_0}^2 + 2\lambda_k^2 (1 - \lambda_k) \langle v_k, \bar{v}_k \rangle_{\mathcal{A}_0} + \lambda_k^4 \|\bar{v}_k\|_{\mathcal{A}_0}^2 \\ &= 2(1 - \lambda_k)^2 E(y_k) + 2\lambda_k^2 (1 - \lambda_k) \langle v_k, \bar{v}_k \rangle_{\mathcal{A}_0} + \lambda_k^4 \|\bar{v}_k\|_{\mathcal{A}_0}^2, \end{aligned}$$

and thus, as long as $E(y_k) \neq 0$,

$$(1 - \lambda_k)^2 = \frac{E(y_{k+1})}{E(y_k)} - \lambda_k^2 (1 - \lambda_k) \frac{\langle v_k, \bar{v}_k \rangle_{\mathcal{A}_0}}{E(y_k)} - \lambda_k^4 \frac{\|\bar{v}_k\|_{\mathcal{A}_0}^2}{2E(y_k)}.$$

Since

$$\|\bar{v}_k\|_{\mathcal{A}_0} \leq \frac{c}{\nu\sqrt{\nu}} E(y_k) \exp\left(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2\right),$$

we then have

$$\left| \frac{\langle v_k, \bar{v}_k \rangle_{\mathcal{A}_0}}{E(y_k)} \right| \leq \frac{\|v_k\|_{\mathcal{A}_0} \|\bar{v}_k\|_{\mathcal{A}_0}}{E(y_k)} \leq \frac{c}{\nu\sqrt{\nu}} \sqrt{E(y_k)} \exp\left(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2\right) \rightarrow 0$$

and

$$0 \leq \frac{\|\bar{v}_k\|_{\mathcal{A}_0}^2}{E(y_k)} \leq \frac{c}{\nu^3} E(y_k) \exp\left(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2\right) \rightarrow 0$$

as $k \rightarrow +\infty$. Consequently, since $\lambda_k \in [0, m]$ and $\frac{E(y_{k+1})}{E(y_k)} \rightarrow 0$, we deduce that $(1 - \lambda_k)^2 \rightarrow 0$, that is $\lambda_k \rightarrow 1$ as $k \rightarrow \infty$. \square

From Lemmas 3.14, 3.16 and Proposition 3.10 we can deduce that :

Proposition 3.19. *Let $\{y_k\}_{k \in \mathbb{N}}$ the sequence of \mathcal{A} defined by (3.21). Then $y_k \rightarrow \bar{y}$ in $H^1(0, T; \mathbf{V}') \cap L^2(0, T; \mathbf{V})$ where $\bar{y} \in \mathcal{A}$ is the unique solution of (1.2) given in Proposition 3.5.*

3.4 Remarks

The following remarks are in order.

Remark 3.20. *The strong convergence of the sequence $\{y_k\}_{k > 0}$ is a consequence of the coercivity inequality (3.12), which is itself a consequence of the uniqueness of the solution y of (1.2). Actually, we can directly prove that $\{y_k\}_{k > 0}$ is a convergent sequence in the Hilbert space \mathcal{A} as follows; from (3.21) and the previous proposition, we deduce that the serie $\sum_{k \geq 0} \lambda_k Y_{1k}$ converges in $H^1(0, T; \mathbf{V}') \cap L^2(0, T; \mathbf{V})$ and $\bar{y} = y_0 + \sum_{k=0}^{+\infty} \lambda_k Y_{1k}$. Moreover $\sum \lambda_k \|Y_{1k}\|_{L^2(0, T; \mathbf{V})}$ converges and, if we denote by k_0 one $k \in \mathbb{N}$ such that $C_1 \sqrt{E(y_k)} < 1$ (see Lemma 3.16), then for all $k \geq k_0$, using (3.29), (3.24) and (3.31) (since we can choose $C_1 > 1$), we have*

$$\begin{aligned}
 \|\bar{y} - y_k\|_{L^2(0, T; \mathbf{V})} &= \left\| \sum_{i=k+1}^{+\infty} \lambda_i Y_{1i} \right\|_{L^2(0, T; \mathbf{V})} \leq \sum_{i=k+1}^{+\infty} \lambda_i \|Y_{1i}\|_{L^2(0, T; \mathbf{V})} \\
 &\leq m \sum_{i=k+1}^{+\infty} \sqrt{C_1 E(y_i)} \leq m \sum_{i=k+1}^{+\infty} C_1 \sqrt{E(y_i)} \\
 &\leq m \sum_{i=k+1}^{+\infty} (C_1 \sqrt{E(y_{k_0})})^{2^{i-k_0}} \leq m \sum_{i=0}^{+\infty} (C_1 \sqrt{E(y_{k_0})})^{2^{i+k+1-k_0}} \\
 &\leq m (C_1 \sqrt{E(y_{k_0})})^{2^{k+1-k_0}} \sum_{i=0}^{+\infty} (C_1 \sqrt{E(y_{k_0})})^{2^i} \\
 &\leq cm (C_1 \sqrt{E(y_{k_0})})^{2^{k+1-k_0}}.
 \end{aligned} \tag{3.32}$$

From (3.17) we deduce that

$$\begin{aligned} & \|\partial_t Y_{1,k}\|_{L^2(0,T;\mathbf{V}')} \\ & \leq \sqrt{E(y_k)}(2\sqrt{2} \exp(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2) + 2\sqrt{2} + c\|y_k\|_{L^2(0,T;\mathbf{V})}) \frac{2\sqrt{2}}{\sqrt{\nu}} \exp(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2) \\ & + c\|y_k\|_{L^\infty(0,T;\mathbf{H})} \frac{2\sqrt{2}}{\nu} \exp(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2) \end{aligned}$$

which gives, using (3.23) and (3.24)

$$\begin{aligned} \|\partial_t Y_{1,k}\|_{L^2(0,T;\mathbf{V}')} & \leq \sqrt{E(y_k)}(2\sqrt{2} \exp(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2) + 2\sqrt{2} + \\ & + \frac{c}{\nu\sqrt{\nu}}(\sqrt{\nu}\|u_0\|_{\mathbf{H}} + \sqrt{2}\|f\|_{L^2(0,T;\mathbf{V}')} + 2\sqrt{E(y_0)}) \exp(\frac{c}{\nu} \int_0^T \|y_k\|_{\mathbf{V}}^2) \\ & \leq C_2\sqrt{E(y_k)} \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} C_2 & = c \exp\left(\frac{c}{\nu^2}\|u_0\|_{\mathbf{H}}^2 + \frac{c}{\nu^3}\|f\|_{L^2(0,T;\mathbf{V}')}^2 + \frac{c}{\nu^3}E(y_0)\right) \\ & \times \left(1 + \frac{1}{\nu\sqrt{\nu}}(\sqrt{\nu}\|u_0\|_{\mathbf{H}} + \sqrt{2}\|f\|_{L^2(0,T;\mathbf{V}')} + 2\sqrt{E(y_0)})\right). \end{aligned}$$

Arguing as in the proof of Lemma 3.16, there exists $k_1 \in \mathbb{N}$ such that, for all $k \geq k_1$

$$C_2\sqrt{E(y_{k+1})} \leq (C_2\sqrt{E(y_k)})^2$$

thus

$$\begin{aligned} \|\partial_t \bar{y} - \partial_t y_k\|_{L^2(0,T;\mathbf{V}')} & = \left\| \sum_{i=k+1}^{+\infty} \lambda_i \partial_t Y_{1i} \right\|_{L^2(0,T;\mathbf{V}')} \leq \sum_{i=k+1}^{+\infty} \lambda_i \|\partial_t Y_{1i}\|_{L^2(0,T;\mathbf{V}')} \\ & \leq m \sum_{i=k+1}^{+\infty} \sqrt{C_2 E(y_i)} \\ & \leq m(C_2\sqrt{E(y_{k_1})})^{2^{k+1-k_1}} \sum_{i=0}^{+\infty} (C_2\sqrt{E(y_{k_1})})^{2^i} \\ & \leq mc(C_2\sqrt{E(y_{k_1})})^{2^{k+1-k_1}}. \end{aligned} \tag{3.34}$$

Remark 3.21. *Lemmas 3.14, 3.15, 3.16 and Proposition 3.19 remain true if we replace the minimization of λ over $[0, m]$ by the minimization over \mathbb{R}_+ . However, the sequence $\{\lambda_k\}_{k>0}$ may not be bounded in \mathbb{R}_+ (the fourth order polynomial $\lambda \rightarrow E(y_k - \lambda Y_k)$ may admit a critical point far from 1. In that case, (3.32) and (3.34) may not hold for some $m > 0$.*

Similarly, Lemmas 3.16, 3.18 and Proposition 3.19 remain true for the sequence $\{y_k\}_{k \geq 0}$ defined as follows

$$\begin{cases} y_0 \in \mathcal{A}, \\ y_{k+1} = y_k - \lambda_k Y_{1,k}, \quad k \geq 0, \\ g(\lambda_k, y_k) = \min_{\lambda \in \mathbb{R}^+} g(\lambda, y_k) \end{cases} \quad (3.35)$$

leading to $\lambda_k \in]0, 1]$ for all $k \geq 0$ and $\lim_{k \rightarrow \infty} \lambda_k \rightarrow 1^-$. The fourth order polynomial g is defined in (3.28).

Remark 3.22. *Let us consider the application $F : \mathcal{A} \rightarrow L^2(0, T; \mathbf{V}')$ defined as $F(y) = y_t + \nu B_1(y) + B(y, y) - f$. The sequence $\{y_k\}_{k>0}$ associated to the Newton method to find the zero of F is defined as follows:*

$$\begin{cases} y_0 \in \mathcal{A}, \\ F'(y_k) \cdot (y_{k+1} - y_k) = -F(y_k), \quad k \geq 0. \end{cases}$$

As in the previous section, this sequence coincides with the sequence obtained from (3.21) if λ_k is fixed equal to one. The algorithm (3.21) which consists to optimize the parameter $\lambda_k \in [0, m]$, $m \geq 1$, in order to minimize $E(y_k)$, equivalently $\|F(y_k)\|_{L^2(0, T; \mathbf{V}')}$ corresponds therefore to the so-called damped Newton method for the application F (see [4]). As the iterates increase, the optimal parameter λ_k converges to one (according to Lemma 2.12), this globally convergent method behaves like the standard Newton method (for which λ_k is fixed equal to one): this explains the quadratic rate after a finite number of iterates.

Remark 3.23. *In a different functional framework, a similar approach is considered in [17]; more precisely, the author introduces the functional $E : \mathcal{V} \rightarrow \mathbb{R}$ defined $E(y) = \frac{1}{2} \|\nabla v\|_{L^2(Q_T)}^2$ with $\mathcal{V} := y_0 + \mathcal{V}_0$, $y_0 \in H^1(Q_T)$ and $\mathcal{V}_0 := \{u \in H^1(Q_T; \mathbb{R}^d), u(t, \cdot) \in \mathbf{V} \forall t \in (0, T), u(0, \cdot) = 0\}$ where $v(t, \cdot)$ solves for all $t \in (0, T)$, the steady Navier-Stokes equation with source term equal to $y_t(t, \cdot) - \nu \Delta y(t, \cdot) + (y(t, \cdot) \cdot \nabla) y(t, \cdot) - f(t, \cdot)$. Strong solutions are therefore considered assuming $u_0 \in \mathbf{V}$ and $f \in (L^2(Q_T))^2$. Bound of $E(y)$ implies bound of y in $L^2(0, T; \mathbf{V})$ but not in $H^1(0, T; L^2(\Omega)^2)$. This prevents to get the convergence of minimizing sequences in \mathcal{V} .*

Remark 3.24. This approach which consist in minimizing an appropriate norm of the solution is refereed to in the literature as *variational approach*. We mention notably the work [17] where strong solution of (1.1) are characterized in the two dimensional case in term of the critical points of a quadratic functional, close to \tilde{E} . Similarly, the authors in [16] show that the functional

$$I^\epsilon(y) = \int\int_{\Omega} e^{-t/\epsilon} \left\{ |\partial_t y + y \cdot \nabla y|^2 + |y \cdot \nabla y|^2 + \frac{\nu}{\epsilon} |\nabla y|^2 \right\}$$

admits minimizers u^ϵ for all $\epsilon > 0$ and, up to subsequences, such minimizers converge weakly to a Leray-Hopf solution of (1.1) as $\epsilon \rightarrow 0$.

4 Numerical illustrations

4.1 Algorithm - Approximation

We detail the main steps of the iterative algorithm (3.21). First, we define the initial term y_0 of the sequence $\{y_k\}_{(k \geq 0)}$ as the solution of the Stokes problem, solved by the backward Euler scheme:

$$\begin{cases} \int_{\Omega} \frac{y_0^{n+1} - y_0^n}{\delta t} \cdot w + \bar{\nu} \int_{\Omega} \nabla y_0^{n+1} \cdot \nabla w = \langle f^n, w \rangle_{\mathbf{V}' \times \mathbf{V}}, \quad \forall w \in \mathbf{V}, \quad \forall n \geq 0, \\ y_0^0(\cdot, 0) = u_0, \quad \text{in } \Omega. \end{cases} \quad (4.1)$$

for some $\bar{\nu} > 0$. The incompressibility constraint is taken into account through a lagrange multiplier $\lambda \in L^2(\Omega)$ leading to the mixed formulation

$$\begin{cases} \int_{\Omega} \frac{y_0^{n+1} - y_0^n}{\delta t} \cdot w + \nu \int_{\Omega} \nabla y_0^{n+1} \cdot \nabla w + \int_{\Omega} \lambda^{n+1} \nabla \cdot w = \langle f^n, w \rangle_{\mathbf{V}' \times \mathbf{V}}, \\ \forall w \in (H_0^1(Q_T))^2, \quad \forall n \geq 0, \\ \int_{\Omega} \mu \nabla \cdot y_0^{n+1} = 0, \quad \forall \mu \in L^2(\Omega), \quad \forall n \geq 0, \\ y_0^0 = u_0, \quad \text{in } \Omega. \end{cases} \quad (4.2)$$

A conformal approximation in space is used for $(H_0^1(\Omega))^2 \times L^2(\Omega)$ based on the inf-sup stable $\mathbb{P}_2/\mathbb{P}_1$ Taylor-Hood finite element. Then, assuming that (an approximation $\{y_{h,k}^n\}_{\{n,h\}}$ of) y_k has been obtained for some $k \geq 0$, y_{k+1} is obtained as follows.

(i) From y_k , computation of (an approximation of) the corrector v_k through the backward Euler scheme

$$\left\{ \begin{array}{l} \int_{\Omega} \frac{v_k^{n+1} - v_k^n}{\delta t} \cdot w + \int_{\Omega} \nabla v_k^{n+1} \cdot \nabla w + \int_{\Omega} \frac{y_k^{n+1} - y_k^n}{\delta t} \cdot w + \nu \int_{\Omega} \nabla y_k^{n+1} \cdot \nabla w \\ \quad + \int_{\Omega} y_k^{n+1} \cdot \nabla y_k^{n+1} \cdot w = \langle f^n, w \rangle_{\mathbf{V}' \times \mathbf{V}}, \\ \forall w \in \mathbf{V}, \forall n \geq 0, \\ v_k^0 = 0. \end{array} \right. \quad (4.3)$$

(ii) Then, in order to compute the term $\|v_{k,t}\|_{L^2(0,T;\mathbf{V}')}$ of $E(y_k)$, introduction of the function $w_k \in L^2(\mathbf{V})$ solution of

$$\int_0^T \int_{\Omega} \nabla w_k \cdot \nabla w + v_{k,t} \cdot w = 0, \quad \forall w \in L^2(\mathbf{V}) \quad (4.4)$$

so that $\|v_{k,t}\|_{L^2(\mathbf{V}')} = \|\nabla w_k\|_{L^2(Q_T)}$. An approximation of w_k is obtained through the scheme

$$\int_{\Omega} \nabla w_k^n \cdot \nabla w + \frac{v_k^{n+1} - v_k^n}{\delta t} \cdot w = 0, \quad \forall w \in \mathbf{V}, \forall n \in \mathbb{N}. \quad (4.5)$$

(iii) Computation of an approximation of $Y_{1,k}$ solution of (3.22) through the scheme

$$\left\{ \begin{array}{l} \int_{\Omega} \frac{Y_{1,k}^{n+1} - Y_{1,k}^n}{\delta t} \cdot w + \nu \int_{\Omega} \nabla Y_{1,k}^{n+1} \cdot \nabla w + \int_{\Omega} y_k^{n+1} \cdot \nabla Y_{1,k}^{n+1} \cdot w \\ \quad + \int_{\Omega} Y_{1,k}^{n+1} \cdot \nabla y_k^{n+1} \cdot w = - \int_{\Omega} \frac{v_k^{n+1} - v_k^n}{\delta t} \cdot w - \int_{\Omega} \nabla v_k^{n+1} \cdot \nabla w, \\ \forall w \in \mathbf{V}, \forall n \geq 0. \\ Y_{1,k}^0 = 0. \end{array} \right. \quad (4.6)$$

(iv) Computation of the corrector function \bar{v}_k solution of (3.26) through the scheme

$$\left\{ \begin{array}{l} \int_{\Omega} \frac{\bar{v}_k^{n+1} - \bar{v}_k^n}{\delta t} \cdot w + \int_{\Omega} \nabla \bar{v}_k^{n+1} \cdot \nabla w + \int_{\Omega} Y_{1,k}^{n+1} \cdot \nabla Y_{1,k}^{n+1} \cdot w = 0, \quad \forall w \in \mathbf{V}, \quad n \geq 0, \\ \bar{v}_k(0) = 0. \end{array} \right. \quad (4.7)$$

(v) Computation of $\|v_k\|_{\mathcal{A}_0}^2$, $\langle v_k, \bar{v}_k \rangle_{\mathcal{A}_0}$ and $\|\bar{v}_k\|_{\mathcal{A}_0}^2$ appearing in $E(y_k - \lambda Y_{1,k})$ (see (3.27)). The computation of $\|\bar{v}_k\|_{\mathcal{A}_0}$ requires the computation of $\|\bar{w}_k\|_{L^2(\mathbf{V}^\prime)}$, i.e. the introduction of \bar{w}_k solution of

$$\iint_0^T \int_{\Omega} \nabla \bar{w}_k \cdot \nabla w + v_{k,t} \cdot w = 0, \quad \forall w \in L^2(\mathbf{V})$$

so that $\|\bar{v}_{k,t}\|_{L^2(\mathbf{V}^\prime)} = \|\nabla \bar{w}_k\|_{L^2(Q_T)}$ through the scheme

$$\int_{\Omega} \nabla \bar{w}_k^n \cdot \nabla w + \frac{\bar{v}_k^{n+1} - \bar{v}_k^n}{\delta t} \cdot w = 0, \quad \forall w \in \mathbf{V}, \quad \forall n \in \mathbb{N}. \quad (4.8)$$

(vi) Determination of the minimum $\lambda_k \in (0, m]$ of

$$\lambda \rightarrow E(y_k - \lambda Y_{1,k}) = (1 - \lambda)^2 \|v_k\|_{\mathcal{A}_0}^2 + 2\lambda^2 (1 - \lambda) \langle v_k, \bar{v}_k \rangle_{\mathcal{A}_0} + \lambda^4 \|\bar{v}_k\|_{\mathcal{A}_0}^2$$

through a Newton-Raphson method starting from 0 and finally update of the sequence $y_{k+1} = y_k - \lambda_k Y_{1,k}$.

As a summary, the determination of y_{k+1} from y_k involves the resolution of four Stokes types formulations, namely (4.3),(4.5),(4.7) and (4.8) plus the resolution of the linearized Navier-Stokes formulation (4.6). This latter concentrates most of the computational times ressources since the operator (to be inverted) varies with the index n .

Instead of minimizing exactly the fourth order polynomial $\lambda \rightarrow E(y_k - \lambda Y_{1,k})$ in step (vi), we may simpler minimize w.r.t. $\lambda \in (0, 1]$ the right hand side of the estimate

$$E(y_k - \lambda Y_{1,k}) \leq \left(|1 - \lambda| \sqrt{E(y_k)} + \frac{\lambda^2}{\sqrt{2}} \|\bar{v}_k\|_{\mathcal{A}_0} \right)^2$$

(appearing in the proof of Lemma 3.15) leading to $\hat{\lambda}_k = \min \left(1, \frac{\sqrt{E(y_k)}}{\sqrt{2} \|\bar{v}_k\|_{\mathcal{A}_0}} \right)$ (see remark 3.21). This avoids the computation of the scalar product $\langle v_k, \bar{v}_k \rangle_{\mathcal{A}_0}$ and one resolution of Stokes type formulations.

Remark 4.1. *Similarly, we may also consider the equivalent functional \tilde{E} defined in (1.10). This avoids the introduction of the auxillary corrector function v and reduces to three (instead of four) the number of Stokes type formulations to be solved. Precisely, using the initialization defined in (4.1), the algorithm is as follows:*

(i) Computation of $\tilde{E}(y_k) = \|h_k\|_{L^2(\mathbf{V})} = \|\nabla h_k\|_{L^2(Q_T)}$ where h_k solves

$$\iint_0^T \int_{\Omega} \nabla h_k \cdot \nabla w + (y_{k,t} - \nu \Delta y_k + y_k \cdot \nabla y_k - f) \cdot w = 0, \quad \forall w \in L^2(\mathbf{V})$$

through the scheme

$$\int_{\Omega} \nabla h_k^n \cdot \nabla w + \frac{y_k^{n+1} - y_k^n}{\delta t} \cdot w + \nu \nabla y_k^{n+1} \cdot \nabla w + y_k^{n+1} \cdot \nabla y_k^{n+1} \quad (4.9)$$

$$= \langle f^n, w \rangle_{\mathbf{V}', \mathbf{V}}, \forall w \in \mathbf{V}, \forall n \in \mathbb{N}.$$

(ii) Computation of an approximation of $Y_{1,k}$ from y_k through the scheme

$$\left\{ \begin{array}{l} \int_{\Omega} \frac{Y_{1,k}^{n+1} - Y_{1,k}^n}{\delta t} \cdot w + \nu \int_{\Omega} \nabla Y_{1,k}^{n+1} \cdot \nabla w + \int_{\Omega} y_k^{n+1} \cdot \nabla Y_{1,k}^{n+1} \cdot w \\ + \int_{\Omega} Y_k^{n+1} \cdot \nabla y_{1,k}^{n+1} \cdot w = \int_{\Omega} \frac{y_k^{n+1} - y_k^n}{\delta t} \cdot w + \nu \int_{\Omega} \nabla y_k^{n+1} \cdot \nabla w \\ + \int_{\Omega} y_k^{n+1} \cdot \nabla y_k^{n+1} \cdot w - \langle f^n, w \rangle_{\mathbf{V}' \times \mathbf{V}}, \forall w \in \mathbf{V}, \forall n \geq 0, \\ Y_{1,k}^0 = 0. \end{array} \right. \quad (4.10)$$

(iii) Computation of $\|B(Y_{1,k}, Y_{1,k})\|_{L^2(0,T;\mathbf{V}')} = \|h_k\|_{L^2(\mathbf{V})} = \|\nabla h_k\|_{L^2(Q_T)}$ where h_k solves

$$\int_0^T \int_{\Omega} \nabla h_k \cdot \nabla w + Y_{1,k} \cdot \nabla Y_{1,k} \cdot w = 0, \forall w \in L^2(\mathbf{V})$$

and similarly of the term $\langle y_{k,t} + \nu B_1(y_k) + B(y_k, y_k), B(Y_{1,k}, Y_{1,k}) \rangle_{L^2(0,T;\mathbf{V}')}.$

(iv) Determination of the minimum $\lambda_k \in (0, m]$ of

$$\begin{aligned} \lambda \rightarrow \tilde{E}(y_k - \lambda Y_{1,k}) &= (1 - \lambda)^2 \tilde{E}(y_k) \\ &+ \lambda^2 (1 - \lambda) \langle y_{k,t} + \nu B_1(y_k) + B(y_k, y_k) - f, B(Y_{1,k}, Y_{1,k}) \rangle_{L^2(0,T;\mathbf{V}')} \\ &+ \frac{\lambda^4}{2} \|B(Y_{1,k}, Y_{1,k})\|_{L^2(0,T;\mathbf{V}')}^2 \end{aligned}$$

through a Newton-Raphson method starting from 0 and finally update of the sequence $y_{k+1} = y_k - \lambda_k Y_{1,k}$ until $\tilde{E}(y_k)$ is small enough.

We emphasize one more time that the case $\lambda_k = 1$ coincides with the standard Newton algorithm to find zeros of the functional $F : \mathcal{A} \rightarrow L^2(0, T; \mathbf{V}')$ defined by $F(y) = y_t + \nu B_1(y) + B(y, y) - f$. In term of computational time resources, the determination of the optimal descent step λ_k is negligible with respect to the resolution in the step (ii).

4.2 2D semi-circular driven cavity

We illustrate our theoretical results for the 2D semi-circular cavity discussed in [6]. The geometry is a semi-disk $\Omega = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 < 1/4, x_2 \leq 0\}$ depicted on Figure 1. The velocity is imposed to $y = (g, 0)$ on $\Gamma_0 = \{(x_1, 0) \in \mathbb{R}^2, |x_1| < 1/2\}$ with g vanishing at $x_1 = \pm 1/2$ and close to one elsewhere: we take $g(x_1) = (1 - e^{100(x_1-1/2)})(1 - e^{-100(x_1+1/2)})$. On the complementary $\Gamma_1 = \{(x_1, x_2) \in \mathbb{R}^2, x_2 < 0, x_1^2 + x_2^2 = 1/4\}$ of the boundary the velocity is fixed to zero.

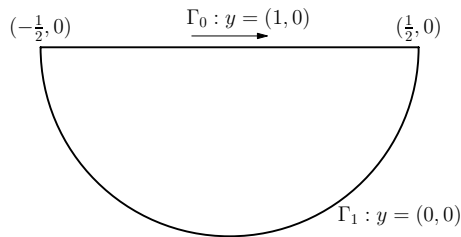


Fig. 1: Semi-disk geometry.

This example has been used in [14] to solve the corresponding steady problem (for which the weak solution is not unique), using again an iterative least-squares strategy. There, the method proved to be robust enough for small values of ν of the order 10^{-4} , while standard Newton method failed. Figure 2 depicts the streamlines of steady state solutions corresponding to $\nu^{-1} = 500$ and to $\nu^{-1} = i \times 10^3$ for $i = 1, \dots, 9$. The figures are in very good agreement with those depicted in [6]. When the Reynolds number (here equal to ν^{-1}) is small, the final steady state consists of one vortex. As the Reynolds number increases, first a secondary vortex then a tertiary vortex arises, whose size depends on the Reynolds number too. Moreover, according to [6], when the Reynolds number exceeds approximately 6 650, an Hopf bifurcation phenomenon occurs in the sense that the unsteady solution does not reach a steady state anymore (at time evolves) but shows an oscillatory behavior. We mention that the Navier-Stokes system is solved in [6] using an operator-splitting/finite elements based methodology. In particular, concerning the time discretization, an explicit scheme is employed.

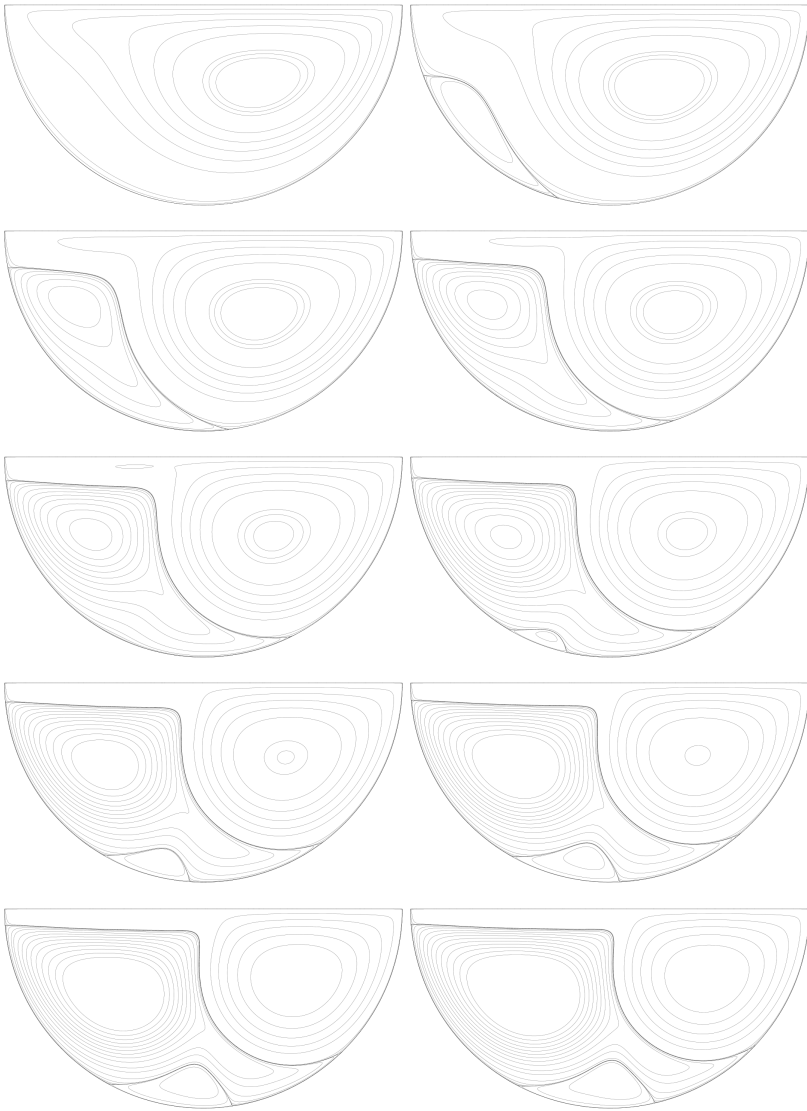


Fig. 2: Streamlines of the steady state solution for $\nu^{-1} = 500, 1000, 2000, 3000, 4000, 5000, 6000, 7000, 8000$ and $\nu^{-1} = 9000$.

4.3 Experiments

We report some numerical results performed with the FreeFem++ package developed at Sorbonne university (see [10]). Regular triangular meshes are used together with the $\mathbb{P}_2/\mathbb{P}_1$ Taylor-Hood finite element, satisfying the Ladyzenskaia-Babushka-Brezzi condition of stability. An example of mesh composed of 9 063 triangles is displayed in Figure 3.

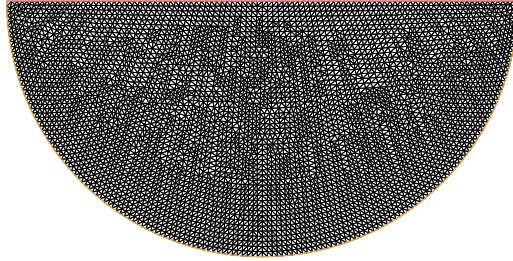


Fig. 3: A regular triangulation of the semi-disk geometry; $\#\text{triangles} = 9\,064$; $\#\text{vertices} = 4\,663$; size $h \approx 1.62 \times 10^{-2}$.

In order to deeply emphasize the influence of the value of ν on the behavior of the algorithm described in Section 4.1, we consider an initial guess y_0 of the sequence $\{y_k\}_{(k>0)}$ independent of ν . Precisely, we define y_0 as the solution of the unsteady Stokes system with viscosity equal to one (i.e. $\bar{\nu} = 1$ in (4.1)) and source term $f \equiv 0$. The initial condition $u_0 \in \mathbf{H}$ is defined as the solution of $-\Delta u_0 + \nabla p = 0$, $\nabla \cdot u_0 = 0$ in Ω and boundary conditions $u_0 = g$ on Γ_0 and $u_0 = 0$ on Γ_1 . u_0 belongs actually to \mathbf{V} .

Table 1 and 2 report numerical values of the sequences $\{\sqrt{2E(y_k)}\}_{(k>0)}$, $\{\lambda_k\}_{(k>0)}$ and $\{\|y_k - y_{k-1}\|_{L^2(\mathbf{V})}/\|y_k\|_{L^2(\mathbf{V})}\}_{(k>0)}$ associated to $\nu = 1/500$ and $\nu = 1/1000$ respectively and $T = 10$, $f = 0$. The tables also display (on the right part) the values obtained when the parameter λ_k is fixed constant equal to one, corresponding to the standard Newton method. The algorithms are stopped when $\sqrt{2E(y_k)} \leq 10^{-8}$. The triangular mesh of Figure 3 for which the discretization parameter h is equal to 1.62×10^{-2} is employed. The number of degrees of freedom is 23 315. Moreover, the time discretization parameter in δt is taken equal to 10^{-2} .

For $\nu = 1/500$, the optimal λ_k are close to one ($\max_k |1 - \lambda_k| \leq 1/5$), so that the two algorithms produce very similar behaviors. The convergence is observed after 6 iterations. For $\nu = 1/1000$, we observe that the optimal λ_k are far from one during the first iterates. The optimization of the parameter allows a faster

convergence (after 9 iterates) than the usual Newton method. For instance, after 8 iterates, $\sqrt{2E(y_k)} \approx 9.931 \times 10^{-11}$ in the first case and $\sqrt{2E(y_k)} \approx 5.669 \times 10^{-5}$ in the second one. In agreement with the theoretical results, we also check that λ_k goes to one. Moreover, the decrease of $\sqrt{2E(y_k)}$ is first linear, then (when λ_k becomes close to one) quadratic.

#iterate k	$\frac{\ y_k - y_{k-1}\ _{L^2(\mathbf{V})}}{\ y_{k-1}\ _{L^2(\mathbf{V})}}$	$\sqrt{2E(y_k)}$	λ_k	$\frac{\ y_k - y_{k-1}\ _{L^2(\mathbf{V})}}{\ y_{k-1}\ _{L^2(\mathbf{V})}}$ ($\lambda_k = 1$)	$\sqrt{2E(y_k)}$ ($\lambda_k = 1$)
0	—	2.690×10^{-2}	0.8112	—	2.690×10^{-2}
1	4.540×10^{-1}	1.077×10^{-2}	0.7758	5.597×10^{-1}	1.254×10^{-2}
2	1.836×10^{-1}	3.653×10^{-3}	0.8749	2.236×10^{-1}	5.174×10^{-3}
3	7.503×10^{-2}	7.794×10^{-4}	0.9919	7.830×10^{-2}	6.133×10^{-4}
4	1.437×10^{-2}	2.564×10^{-5}	1.0006	9.403×10^{-3}	1.253×10^{-5}
5	4.296×10^{-4}	3.180×10^{-8}	1.	1.681×10^{-4}	4.424×10^{-9}
6	5.630×10^{-7}	6.384×10^{-11}	—	—	—

Tab. 1: $\nu = 1/500$; Results for the algorithm (2.18).

#iterate k	$\frac{\ y_k - y_{k-1}\ _{L^2(\mathbf{V})}}{\ y_{k-1}\ _{L^2(\mathbf{V})}}$	$\sqrt{2E(y_k)}$	λ_k	$\frac{\ y_k - y_{k-1}\ _{L^2(\mathbf{V})}}{\ y_{k-1}\ _{L^2(\mathbf{V})}}$ ($\lambda_k = 1$)	$\sqrt{2E(y_k)}$ ($\lambda_k = 1$)
0	—	2.69×10^{-2}	0.634	—	2.69×10^{-2}
1	5.13×10^{-1}	1.49×10^{-2}	0.580	8.10×10^{-1}	2.23×10^{-2}
2	2.53×10^{-1}	7.60×10^{-3}	0.349	4.45×10^{-1}	2.91×10^{-2}
3	1.34×10^{-1}	5.47×10^{-3}	0.402	5.71×10^{-1}	5.68×10^{-2}
4	1.10×10^{-1}	3.81×10^{-3}	0.561	3.68×10^{-1}	2.62×10^{-2}
5	8.95×10^{-2}	2.29×10^{-3}	0.868	2.86×10^{-1}	1.82×10^{-2}
6	6.39×10^{-2}	8.67×10^{-4}	1.036	1.42×10^{-1}	4.30×10^{-3}
7	1.78×10^{-2}	4.15×10^{-5}	0.999	6.05×10^{-2}	9.60×10^{-4}
8	7.98×10^{-4}	9.93×10^{-8}	0.999	1.48×10^{-2}	5.66×10^{-5}
9	2.25×10^{-6}	4.00×10^{-11}	—	9.74×10^{-4}	3.02×10^{-7}
10	—	—	—	4.26×10^{-6}	3.84×10^{-11}

Tab. 2: $\nu = 1/1000$; Results for the algorithm (3.21).

At time $T = 10$, the unsteady state solution is close to the solution of the steady Navier-Stokes equation: the last element $y_{k=9}$ of the converged sequence satisfies $\|y_{k=9}(T, \cdot) - y_{k=9}(T - \delta t, \cdot)\|_{L^2(\Omega)} / \|y_{k=9}(T, \cdot)\|_{L^2(\Omega)} \approx 1.19 \times 10^{-5}$. Figures 4 display the streamlines of the unsteady state solution corresponding to $\nu = 1/1000$

at time 0, 1, 2, 3, 4, 5, 6 and 7 seconds to be compared with the streamlines of the steady solution depicted in Figure 2.

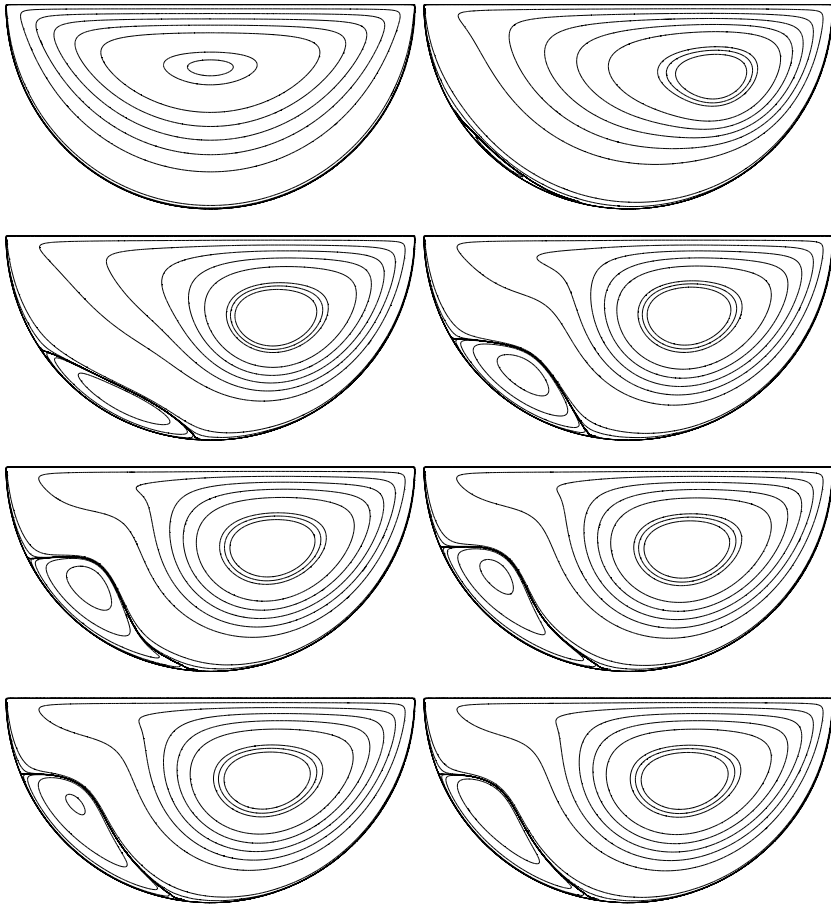


Fig. 4: Streamlines of the unsteady state solution for $\nu^{-1} = 1000$ at time $t = i$, $i = 0, \dots, 7$ s.

For lower values of the viscosity constant, precisely $\nu \leq 1/1100$ approximatively, the initial guess y_0 is too far from the zero of E so that we observe the divergence after few iterates of the Newton method (case $\lambda_k = 1$ for all $k > 0$) but still the convergence of the algorithm described in section 4.1 (see Table 3). The divergence in the case $\lambda_k = 1$ is still observed with a refined discretization both in time and

space, corresponding to $\delta t = 0.5 \times 10^{-3}$ and $h \approx 0.0110$ (19 810 triangles and 10 101 vertices). The divergence of the Newton method suggests that the functional E is not convex far away from the zero of E and that the derivative $E'(y)$ takes small values there. We recall that, in view of the theoretical part, the functional E is coercive and its derivative vanishes only at the zero of E . However, the equality $E'(y_k) \cdot Y_{1,k} = 2E(y_k)$ shows that $E'(y_k)$ can be “small” for “large” $Y_{1,k}$, i.e. “large” y_k . On the other hand, we observe the convergence (after 3 iterates) of the Newton method, when initialized with the approximation corresponding to $\nu = 1/1000$.

iterate k	$\frac{\ y_k - y_{k-1}\ _{L^2(\mathbf{V})}}{\ y_{k-1}\ _{L^2(\mathbf{V})}}$	$\sqrt{2E(y_k)}$	λ_k	$\frac{\ y_k - y_{k-1}\ _{L^2(\mathbf{V})}}{\ y_{k-1}\ _{L^2(\mathbf{V})}} (\lambda_k = 1)$	$\sqrt{2E(y_k)} (\lambda_k = 1)$
0	—	2.69×10^{-2}	0.614	—	2.69×10^{-2}
1	5.24×10^{-1}	1.53×10^{-2}	0.566	8.52×10^{-1}	2.38×10^{-2}
2	2.64×10^{-1}	8.02×10^{-3}	0.323	4.89×10^{-1}	3.55×10^{-2}
3	1.38×10^{-1}	5.98×10^{-3}	0.330	7.17×10^{-1}	8.70×10^{-2}
4	1.11×10^{-1}	4.54×10^{-3}	0.420	4.84×10^{-1}	3.53×10^{-2}
5	9.42×10^{-2}	3.22×10^{-3}	0.587	1.12×10^0	3.90×10^{-1}
6	7.66×10^{-2}	1.94×10^{-3}	0.972	—	1.33×10^4
7	5.68×10^{-2}	5.93×10^{-4}	1.021	—	8.09×10^{27}
8	1.00×10^{-2}	1.08×10^{-5}	0.999	—	—
9	2.83×10^{-4}	1.33×10^{-8}	1.	—	—
10	2.89×10^{-7}	4.61×10^{-11}	—	—	—

Tab. 3: $\nu = 1/1100$; Results for the algorithm (3.21).

Table 4 gives numerical values associated to $\nu = 1/2000$ and $T = 10$. We used a refined discretization: precisely, $\delta t = 1/150$ and a mesh composed of 15 190 triangles, 7 765 vertices ($h \approx 1.343 \times 10^{-2}$). The convergence of the algorithm of section 4.1 is observed after 19 iterates. In agreement with the theoretical results, the sequence $\{\lambda_k\}_{(k>0)}$ goes to one. Moreover, the variation of the error functional $E(y_k)$ is first quite slow, then increases to be very fast after 15 iterates. This behavior is illustrated on Figure 5. For lower values of ν , we still observed the convergence (provided a fine enough discretization so as to capture the third vortex) with an increasing number of iterates. For instance, 28 iterates are necessary to achieve $\sqrt{2E(y_k)} \leq 10^{-8}$ for $\nu = 1/3000$ and 49 iterates for $\nu = 1/4000$. This illustrates the global convergence of the algorithm. In view of the estimate (3.30), a quadratic rate is achieved as soon as $\sqrt{E(y_k)} \leq C_1^{-1}$ with here (since $f \equiv 0$)

$$C_1 = \frac{c}{\nu\sqrt{\nu}} \exp\left(\frac{c}{\nu^2} \|u_0\|_{\mathbf{H}}^2 + \frac{c}{\nu^3} E(y_0)\right)$$

so that $C_1^{-1} \rightarrow 0$ as $\nu \rightarrow 0$. Consequently, for small ν , it is very likely more efficient (in term of computational resources) to couple the algorithm with a continuation method w.r.t. ν , in order to reach faster the quadratic regime. This aspect is not addressed in this work and we refer to [14] where this is illustrated in the steady case.

#iterate k	$\frac{\ y_k - y_{k-1}\ _{L^2(\mathbf{V})}}{\ y_{k-1}\ _{L^2(\mathbf{V})}}$	$\sqrt{2E(y_k)}$	λ_k
0	—	2.691×10^{-2}	0.5215
1	6.003×10^{-1}	1.666×10^{-2}	0.4919
2	3.292×10^{-1}	9.800×10^{-3}	0.1566
3	1.375×10^{-1}	8.753×10^{-3}	0.1467
4	1.346×10^{-1}	7.851×10^{-3}	0.0337
5	5.851×10^{-2}	7.688×10^{-3}	0.0591
6	7.006×10^{-2}	7.417×10^{-3}	0.1196
7	9.691×10^{-2}	6.864×10^{-3}	0.0977
8	8.093×10^{-2}	6.465×10^{-3}	0.0759
9	6.400×10^{-2}	6.182×10^{-3}	0.0968
10	6.723×10^{-2}	5.805×10^{-3}	0.1184
11	6.919×10^{-2}	5.371×10^{-3}	0.1630
12	7.414×10^{-2}	4.825×10^{-3}	0.2479
13	8.228×10^{-2}	4.083×10^{-3}	0.3517
14	8.146×10^{-2}	3.164×10^{-3}	0.4746
15	7.349×10^{-2}	2.207×10^{-3}	0.7294
16	6.683×10^{-2}	1.174×10^{-3}	1.0674
17	3.846×10^{-2}	2.191×10^{-4}	1.0039
18	5.850×10^{-3}	4.674×10^{-5}	0.9998
19	1.573×10^{-4}	5.843×10^{-9}	—

Tab. 4: $\nu = 1/2000$; Results for the algorithm (3.21).

5 Conclusions and perspectives

In order to get an approximation of the solutions of the steady and unsteady 2D Navier-Stokes equations, we have introduced and analyzed a least-squares method based on a minimization of an appropriate norm of the equation. For instance, in the unsteady case, considering the weak solution associated to an initial condition in $\mathbf{H} \subset L^2(\Omega)^2$ and a source $f \in L^2(0, T, \mathbf{V}')$, the least-square functional is based on the $L^2(0, T, \mathbf{V}')$ -norm of the state equation. Using a particular descent direction, we

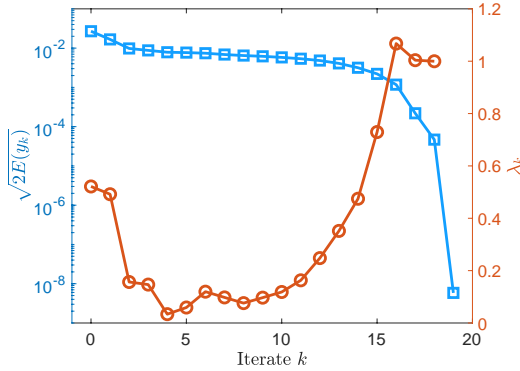


Fig. 5: Evolution of $\sqrt{2E(y_k)}$ and λ_k w.r.t. k ; $\nu = 1/2000$ (see Table 4).

construct explicitly a minimizing sequence for the functional converging strongly, for any initial guess, to the solution of the Navier-Stokes. Moreover, except for the first iterates, the convergence is quadratic. Actually, it turns out that this minimizing sequence coincides with the sequence obtained from the damped Newton method when used to solves the weak formulation associated to the Navier-Stokes equation. The numerical experiments illustrate the global convergence of the method and its robustness including for small values of the viscosity constant.

Moreover, the strong convergence of the whole minimizing sequence has been proved using a coercivity type property of the functional, consequence of the uniqueness of the solution. Actually, it is interesting to remark that this property is not necessary, since such minimizing sequence (which is completely determined by the initial term) is a Cauchy sequence. The approach can therefore be adapted to partial differential equations with multiple solutions or to optimization problem involving various solutions. We mention notably the approximation of null controls for (controllable) nonlinear partial differential equation: the source term f , possibly distributed over a non-empty set of Ω is now, together with the corresponding state, an argument of the least-squares functional. The controllability constraint is incorporated in the set \mathcal{A} of admissible pair (y, f) . In spite of the non uniqueness of the minimizers, the approach introduced in this work still produces a strongly convergent approximation. We refer to [12] for the analysis of this approach for sub linear (null controllable) heat equation.

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