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HOLOMORPHIC PROJECTIVE CONNECTIONS ON COMPACT COMPLEX THREEFOLDS

INDRANIL BISWAS AND SORIN DUMITRESCU

ABSTRACT. We prove that a holomorphic projective connection on a complex projective threefold is either flat, or it is a translation invariant holomorphic projective connection on an abelian threefold. In the second case, a generic translation invariant holomorphic affine connection on the abelian variety is *not* projectively flat. We also prove that a simply connected compact complex threefold with trivial canonical line bundle does not admit any holomorphic projective connection.

1. INTRODUCTION

An important consequence of the uniformization theorem for Riemann surfaces is that any Riemann surface admits a holomorphic projective structure which is isomorphic either to the one-dimensional model $\mathbb{C}P^1$, or to a quotient of the complex affine line \mathbb{C} by a discrete group of translations, or to a quotient of the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^1$ by a torsion-free discrete subgroup of $SU(1, 1) \simeq SL(2, \mathbb{R})$ [Gu, StG]. In higher dimensions compact complex manifolds do not, in general, admit any holomorphic (flat or non-flat) projective connection.

Kobayashi and Ochiai in [KO1, KO2] classified compact Kähler–Einstein manifolds admitting a holomorphic projective connection. Their result says that the only examples of compact Kähler–Einstein manifolds admitting a holomorphic projective connection are the standard ones; we recall that the n -dimensional standard examples are the following: the complex projective space $\mathbb{C}P^n$, all étale quotients of complex n -tori and all compact quotients of the complex hyperbolic n -space $\mathbb{H}_{\mathbb{C}}^n$ by a torsion-free discrete subgroup of $SU(n, 1)$. All of these three types of manifolds are endowed with a standard flat holomorphic projective connection, i.e., a holomorphic projective structure, which is locally modeled on $\mathbb{C}P^n$ (see Section 2).

Moreover, Kobayashi and Ochiai gave a classification of compact complex surfaces admitting holomorphic projective connections [KO1, KO2]. Their classification shows that all those compact complex surfaces also admit flat holomorphic projective connections. The geometry of flat holomorphic projective structures on compact complex surfaces was studied by Klingler in [Kl2]. Subsequently, it was proved in [Du2] that all holomorphic (normal) projective connections on compact complex surfaces are flat.

Here we study the local geometry of holomorphic projective connections on compact complex manifolds of dimension three and higher. For defining holomorphic projective

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connections we adopt the terminology of [Gu, MM]; these connections are holomorphic normal projective connections in the terminology of [Ka, KO1, JR1] (see Section 2).

For compact Kähler–Einstein manifolds, using the classification in [KO1, KO2], and generalizing, to the non-flat case, some results of Mok–Yeung and Klingler on flat projective connections, [MY, Kl1], we prove the following (see Section 3.1):

Theorem 1. *Let M be a compact Kähler–Einstein manifold of complex dimension $n > 1$ endowed with a holomorphic projective connection. Then the following hold:*

- (1) *either M is the complex projective space $\mathbb{C}P^n$ endowed with its standard flat projective connection;*
- (2) *or M is a quotient of the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$, by a discrete subgroup in $SU(n, 1)$, endowed with its induced standard flat projective connection;*
- (3) *or M is an étale quotient of a compact complex n -torus endowed with the holomorphic projective connection induced by a translation invariant holomorphic torsionfree affine connection on the universal cover \mathbb{C}^n . For $n \geq 3$, the general translation invariant holomorphic torsionfree affine connection on \mathbb{C}^n is not projectively flat.*

In particular, a holomorphic projective connection ϕ on a compact Kähler–Einstein manifold of complex dimension n is either flat, or it is locally isomorphic to the projective connection induced by a translation invariant holomorphic affine connection on \mathbb{C}^n . In both cases, ϕ is locally homogeneous; more precisely, the local projective Killing Lie algebra of the projective connection ϕ (see Section 2 for definition) contains a copy of the abelian Lie algebra \mathbb{C}^n which is transitive on M .

The third case in Theorem 1 covers all compact Kähler manifolds with vanishing first Chern class (see Proposition 12). Indeed, on a compact Kähler manifold M with vanishing first Chern class, any holomorphic projective connection admits a global representative which is a holomorphic torsionfree affine connection (see Lemma 7). In this case it is known that M admits a finite unramified cover which is a compact complex torus [IKO] (the pull-back, to the torus, of such a global representative affine connection is a translation invariant holomorphic torsionfree affine connection). This type of results are also valid in the broader context of holomorphic Cartan geometries [BM2, BM3, Du3, BD4]; see [Sha] for holomorphic Cartan geometries.

Kobayashi and Ochiai proved in [KO4] that holomorphic G -structures modeled on Hermitian symmetric spaces of rank ≥ 2 on compact Kähler–Einstein manifolds are always flat (see also [HM] for a similar result for uniruled projective manifolds). The complex projective space being a Hermitian space of rank one, holomorphic projective connections constitute examples of holomorphic G -structures modeled on a Hermitian symmetric space of rank one. Theorem 1 may be seen as a rank one version of the above mentioned result in [KO4]. However, contrary to the situation in rank ≥ 2 , there exist non-flat holomorphic projective connections on compact complex tori of dimension three or more (see Proposition 12).

Complex projective threefolds admitting a holomorphic projective connection were classified by Jahnke and Radloff in [JR1]. Their result says that any such projective threefold is

- either one among the standard ones (the complex projective space $\mathbb{C}\mathbb{P}^3$, étale quotients of abelian threefolds and compact quotients of the complex hyperbolic 3-space $\mathbb{H}_{\mathbb{C}}^3$ by a torsion-free discrete subgroup in $SU(3, 1)$),
- or an étale quotient of a Kuga–Shimura projective threefold (i.e., a smooth modular family of false elliptic curves; their description is recalled in Section 2).

As noted in [JR1], each of these projective threefolds also admit a flat holomorphic projective connection.

We investigate the space of all holomorphic projective connections on Kuga–Shimura projective threefolds. The main result in this direction is the following (proved in Section 4.1):

Theorem 2. *Let $M \rightarrow \Sigma$ be a Kuga–Shimura projective threefold over a Shimura curve Σ of false elliptic curves. Then the following hold:*

- The projective equivalence classes of holomorphic projective connections on M are parametrized by a complex affine space for the complex vector space $(H^0(\Sigma, K_{\Sigma}^{\frac{3}{2}}))^2$.*
- All holomorphic projective connections on M are flat. The fibers of the Kuga–Shimura fibration are totally geodesic with respect to every holomorphic projective connection on M .*

Theorem 2 implies that the space of projective equivalence classes of flat holomorphic projective connections on Kuga–Shimura projective threefolds can have arbitrarily large dimension (see Remark 15).

Theorem 1 and Theorem 2, combined with the classification in [JR1], give the following (proved in Section 4.1):

Corollary 3. *A holomorphic projective connection ϕ on a complex projective threefold is either flat, or it is an étale quotient of a translation invariant holomorphic projective connection on an abelian threefold. In the second case, a generic translation invariant holomorphic projective connection on an abelian variety of dimension three is not flat.*

Our motivation for Corollary 3 comes from the projective Lichnerowicz conjecture. The projective Lichnerowicz conjecture roughly says that compact manifolds M endowed with a projective connection ϕ admitting a connected (or, more generally, infinite) *essential* group G of automorphisms of (M, ϕ) (meaning, G preserves ϕ , but does not preserve any torsionfree affine connection representing ϕ) are actually flat (i.e., ϕ is a flat projective connection). The literature on this subject is vast: see for instance [Ma], [Ze] and references therein. In [Ma], the projective Lichnerowicz conjecture was solved in the Riemannian context (i.e., for the Levi–Civita connection of a Riemannian metric) and for connected essential groups of projective automorphisms G ; in [Ze], the same was proved for discrete infinite essential groups of projective automorphisms G . For local results in this direction, see, for instance, Theorem 3.1 in [CM] which implies that analytic projective connections admitting an essential local projective Killing field are flat (compare this with [NO]).

We formulate here a version of the projective Lichnerowicz conjecture for holomorphic pseudo-groups; there is no global transformation group in our formulation; instead we replace it with the pseudo-group of local biholomorphisms which are the transition maps of a compact complex manifold endowed with a holomorphic projective connection.

Conjecture 4. Let M be a compact complex manifold bearing a holomorphic projective connection ϕ . Assume that M does not admit any global holomorphic torsionfree affine connection projectively equivalent to ϕ . Then ϕ is flat.

Lemma 7 shows that the assumption in Conjecture 4 is equivalent to the assumption that the canonical line bundle K_M does not admit any holomorphic connection. Moreover, if M is Kähler, this assumption is equivalent to the assumption that $c_1(M) \neq 0$. So, in the Kähler setting, Conjecture 4 simplifies to the following:

Conjecture 5. Holomorphic projective connections on compact Kähler manifolds with nonzero first Chern class are flat.

Theorem 1 gives a positive answer to Conjecture 5 for Kähler–Einstein manifolds, while Corollary 3 gives a positive answer to Conjecture 5 for projective threefolds.

All simply connected Kähler manifolds, and, more generally, all simply connected manifolds in the Fujiki class \mathcal{C} [Fu] (i.e., compact complex manifolds bimeromorphic to a Kähler manifold [Va]), bearing a holomorphic projective connection are actually complex projective manifold [BD3, Theorem 4.3]. In view of this, Corollary 3 also gives a positive answer to Conjecture 4 for simply connected threefolds belonging to the Fujiki class \mathcal{C} . More precisely, a compact simply connected complex threefold in the Fujiki class \mathcal{C} equipped with a holomorphic projective connection is isomorphic to $\mathbb{C}\mathbb{P}^3$ endowed with its standard flat projective connection.

For higher dimensions, a classification of complex projective manifolds admitting a flat holomorphic projective connection was obtained in [JR2]. The classification of complex projective manifolds admitting a holomorphic projective connection (non necessarily flat) is still an open question. Notice that Conjecture 5 implies that compact Kähler manifolds bearing a holomorphic projective connection also admit flat holomorphic projective connections (all holomorphic projective connections are actually expected to be flat, except the étale quotients of generic translation-invariant projective connections on compact complex tori).

An interesting class of compact non-Kähler threefolds with trivial canonical bundle and admitting a flat holomorphic projective connection is provided by parallelizable manifolds $\mathrm{SL}(2, \mathbb{C})/\Gamma$, where Γ is a cocompact lattice in $\mathrm{SL}(2, \mathbb{C})$, along with the deformations of $\mathrm{SL}(2, \mathbb{C})/\Gamma$ constructed in [Gh] (which are, in general, not parallelizable manifolds). The details about the geometry of these projective connections can be found in [Gh] and [BD1, Section 5]. It should be mentioned that compact complex non-Kähler parallelizable manifolds admitting a holomorphic projective connection, but not admitting any flat holomorphic projective connection, were constructed in [BD1, Proposition 5.7].

Section 2 provides an introduction to the geometry of holomorphic projective connections as well as presentations of the standard models and the Kuga–Shimura threefolds. In Section 3 we study holomorphic projective connections on Kähler–Einstein manifolds. Section 4 is about holomorphic projective connections on Kuga–Shimura manifolds, and contains proofs of Theorem 2 and Corollary 3.

Section 5 deals with the compact non-Kähler threefolds, and the following theorem is proved there (see Theorem 17).

Theorem 6. *A simply connected compact complex threefold with trivial canonical line bundle does not admit any holomorphic projective connection*

A key ingredient in the proof of Theorem 6 is the result that the projective Killing Lie algebra of a holomorphic projective connection on a compact complex threefold is nontrivial (see Proposition 18). We think that the statement proved in Theorem 6 is likely to be true in higher dimensions. Our Proposition 19 is a step in that direction.

More generally, we conjecture that a simply connected compact complex manifold bearing a holomorphic projective connection is isomorphic to the complex projective space (endowed with its standard flat structure); this is a version of Conjecture 4 for simply connected manifolds. We have seen above that this conjecture is verified for complex threefolds in Fujiki class \mathcal{C} , while Theorem 6 verifies it for complex threefolds with trivial canonical bundle.

2. HOLOMORPHIC PROJECTIVE CONNECTIONS

Recall that using the standard action of $\mathrm{PGL}(n+1, \mathbb{C})$ on the complex projective space $\mathbb{C}\mathbb{P}^n$, the group of holomorphic automorphisms of $\mathbb{C}\mathbb{P}^n$ is identified with $\mathrm{PGL}(n+1, \mathbb{C})$.

Let M be a complex manifold of complex dimension n . A holomorphic coordinate function on M is a pair of the form (U, ϕ) , where $U \subset M$ is an open subset and $\phi : U \rightarrow \mathbb{C}\mathbb{P}^n$ is a holomorphic embedding. A *flat holomorphic projective connection* (also called a *holomorphic projective structure*) on M is given by a collection of holomorphic coordinate functions $(U_i, \phi_i)_{i \in I}$ such that

- $\bigcup_{i \in I} U_i = M$, and
- for $i, j \in I$, and each connected component $U_{ij,c} \subset U_i \cap U_j$, the transition function

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_{ij,c}) \rightarrow \phi_i(U_{ij,c})$$

coincides with the restriction of some $\phi_{ij,c} \in \mathrm{PGL}(n+1, \mathbb{C})$.

An important consequence of the uniformization theorem for Riemann surfaces is that any Riemann surface admits a holomorphic projective structure [Gu]. In higher dimension the situation is much more stringent. All compact Kähler–Einstein manifolds admitting a holomorphic projective structure actually lie in one of the three standard examples described below [KO1, KO2].

2.1. The standard examples. The complex projective space $\mathbb{C}\mathbb{P}^n$ is endowed with its standard flat holomorphic projective connection. This first of the three standard examples is the model for any holomorphic projective structure in the following sense.

If M is a complex simply connected manifold of complex dimension n , any holomorphic projective structure on M is given by a holomorphic submersion (equivalently, immersion)

$$\mathrm{dev} : M \rightarrow \mathbb{C}\mathbb{P}^n$$

which is known as the developing map. In particular, if M is also compact, this dev is a covering map and hence it is a biholomorphism, because $\mathbb{C}\mathbb{P}^n$ is simply connected. Therefore, the only compact simply connected complex n -manifold endowed with a holomorphic projective structure is $\mathbb{C}\mathbb{P}^n$ equipped with its standard projective structure.

Assume now that M is endowed with a holomorphic projective structure ϕ , but it is not simply connected anymore. Fix a point $x_0 \in M$, denote by $\varpi : \widetilde{M} \rightarrow M$ the corresponding universal covering of M , and pull-back the holomorphic projective structure ϕ to \widetilde{M} . Consider the developing map $\text{dev} : \widetilde{M} \rightarrow \mathbb{C}\mathbb{P}^n$ for $\varpi^*\phi$. We have a unique homomorphism

$$\rho : \pi_1(M, x_0) \longrightarrow \text{PGL}(n+1, \mathbb{C})$$

such that dev is $\pi_1(M, x_0)$ -equivariant for the natural action of $\pi_1(M, x_0)$ on \widetilde{M} and the action of $\pi_1(M, x_0)$ on $\mathbb{C}\mathbb{P}^n$ given by ρ together with the standard action of $\text{PGL}(n+1, \mathbb{C})$ on $\mathbb{C}\mathbb{P}^n$; this ρ is called the *monodromy homomorphism* for the projective structure.

The projective structure on $\mathbb{C}\mathbb{P}^n$ induces a projective structure on every open subset of it. Take an open set $\Omega \subset \mathbb{C}\mathbb{P}^n$ and a discrete subgroup $\Gamma \subset \text{PGL}(n+1, \mathbb{C})$ that preserves Ω while acting freely and properly discontinuously on Ω . Then the quotient complex manifold Ω/Γ inherits a holomorphic projective structure induced by that of Ω .

The remaining two standard examples will be described by choosing appropriately the pair (Ω, Γ) .

Complex affine space and its quotients. Take Ω to be the standard open affine subset $\mathbb{C}^n \subset \mathbb{C}\mathbb{P}^n$. Let $\Lambda \simeq \mathbb{Z}^{2n}$ be some lattice in \mathbb{R}^{2n} ; it acts on \mathbb{C}^n by translations. Since all affine transformations of \mathbb{C}^n are restrictions of projective transformations, the complex compact torus \mathbb{C}^n/Λ inherits a holomorphic projective structure.

Complex hyperbolic space and its quotients. Let us now consider the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$ of complex dimension n , seen as the Hermitian symmetric space $\text{SU}(n, 1)/\text{S}(\text{U}(n, 1) \times \text{U}(1))$. The group $\text{SU}(n, 1)$ coincides with the group of holomorphic isometries of $\mathbb{H}_{\mathbb{C}}^n$. The compact dual of $\mathbb{H}_{\mathbb{C}}^n$ is $\mathbb{C}\mathbb{P}^n$ acted on by the holomorphic isometry group $\text{PU}(n+1)$ for its standard Fubini–Study Kähler metric. There is a canonical (Borel) embedding of $\mathbb{H}_{\mathbb{C}}^n$ as an open subset of its compact dual $\mathbb{C}\mathbb{P}^n$. The image of this Borel embedding is the following ball in $\mathbb{C}\mathbb{P}^n$:

$$\mathbb{H}_{\mathbb{C}}^n := \{[Z_0 : Z_1 : \cdots : Z_n] \mid |Z_0|^2 + |Z_1|^2 + \cdots + |Z_{n-1}|^2 < |Z_n|^2\} \subset \mathbb{C}\mathbb{P}^n. \quad (2.1)$$

The action of $\text{SU}(n, 1)$ on $\mathbb{H}_{\mathbb{C}}^n$ evidently extends to an action of $\text{SU}(n, 1)$ on $\mathbb{C}\mathbb{P}^n$ by projective transformations. Therefore any quotient of $\mathbb{H}_{\mathbb{C}}^n$ by a torsion-free discrete subgroup in $\text{SU}(n, 1)$ is a complex manifold endowed with a holomorphic projective structure induced by the natural holomorphic projective structure on the open subset $\mathbb{H}_{\mathbb{C}}^n \subset \mathbb{C}\mathbb{P}^n$ in (2.1).

2.2. False elliptic curves and Shimura curve. The main Theorem in [JR1] asserts that any complex projective threefold bearing a holomorphic projective connection is

- either one of the above (three-dimensional) standard examples,
- or an étale quotient of a smooth modular family of false elliptic curves.

We present here a construction of these compact Shimura curves of false elliptic curves, following the description in [LB] and [JR1].

Let \mathcal{B} be a totally indefinite quaternion algebra over \mathbb{Q} . More precisely, \mathcal{B} is the algebra generated by two elements i, j such that

$$ij = -ji, \quad i^2 = a, \quad j^2 = b$$

for some $a, b \in \mathbb{Q}$. Then \mathcal{B} is a division algebra, and

$$\mathcal{B} \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_{2,2}(\mathbb{R}).$$

Therefore elements of \mathcal{B} may be seen as $(2, 2)$ -matrices with coefficients in some real quadratic number field.

A *false elliptic curve* is an abelian surface T such that $\text{End}(T) \otimes \mathbb{Q} \simeq \mathcal{B}$.

Let $\Lambda \simeq \mathbb{Z}^4$ be some lattice in \mathcal{B} , and choose a nontrivial anti-symmetric matrix

$$M = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \in \mathcal{B}$$

such that $\text{tr}(\Lambda M \Lambda^t) \subset \mathbb{Z}$. Denote by \mathcal{H} the upper-half of complex plane. For any $\tau \in \mathcal{H}$, we construct a complex structure on $\mathcal{B} \otimes_{\mathbb{Q}} \mathbb{R}$ through the \mathbb{R} -vector space isomorphism

$$j_{\tau} : \mathcal{B} \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow \mathbb{C}^2$$

defined as $A \longmapsto A \cdot \begin{pmatrix} \tau \\ \alpha \end{pmatrix}$. There is a free and proper discontinuous action of Λ on $\mathcal{H} \times \mathbb{C}^2$ given by

$$\lambda \cdot (h, (z_1, z_2)) = (h, (z_1, z_2) + j_{\tau}(\lambda))$$

for any $\lambda \in \Lambda$. The quotient for this action is a smooth nontrivial family $\Xi_{\mathcal{B}}$ of abelian surfaces over \mathcal{H} .

Denote by $\bar{\Gamma}$ the stabilizer of the lattice

$$\Lambda \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \subset \text{SL}(2, \mathbb{R}),$$

and choose a torsionfree finite index subgroup $\Gamma \subset \bar{\Gamma}$. Any element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

acts on \mathcal{H} by the conformal map $\tau \longmapsto \frac{a\tau+b}{c\tau+d}$. We note that the fiber of $\Xi_{\mathcal{B}}$ over τ is isomorphic to the fiber of $\Xi_{\mathcal{B}}$ over $\frac{a\tau+b}{c\tau+d}$ through the multiplication by $\frac{1}{c\tau+d}$.

It follows that there is an action of the semi-direct product $\Gamma \ltimes \Lambda$ on $\mathcal{H} \times \mathbb{C}^2$ given by the map

$$(\gamma, \lambda) \cdot (\tau, z_1, z_2) \longmapsto \left(\frac{a\tau+b}{c\tau+d}, \frac{z_1+m\tau+n}{c\tau+d}, \frac{z_2+k\tau+l}{c\tau+d} \right), \quad (2.2)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\lambda = (m, n, k, l) \in \Lambda$. The quotient of $\mathcal{H} \times \mathbb{C}^2$ by this action of $\Gamma \ltimes \Lambda$ is a projective abelian fibration

$$\Xi_{\mathcal{B}} \longrightarrow \Sigma \quad (2.3)$$

with base $\Sigma = \mathcal{H}/\Gamma$ a compact Riemann surface of genus $g \geq 2$; since \mathcal{B} is a division algebra, Γ is a Fuchsian group such that \mathcal{H}/Γ is compact [Shi].

A projective abelian fibration of the above type is called a *Kuga–Shimura projective threefold*.

Considering $\mathcal{H} \times \mathbb{C}^2$ as an open subset of $\mathbb{C}\mathbb{P}^3$, the action of $\Gamma \ltimes \Lambda$ on $\mathcal{H} \times \mathbb{C}^2$ in (2.2) is evidently given by projective transformations. In particular, as it was first observed in

[JR1], a Kuga–Shimura projective threefold is endowed with a flat holomorphic projective connection.

2.3. Holomorphic projective connections and Weyl projective tensor. Let \mathcal{Z} be a complex manifold of complex dimension $n > 1$. A holomorphic connection on the holomorphic tangent bundle $T\mathcal{Z}$ of \mathcal{Z} is called a holomorphic affine connection on \mathcal{Z} (see [At] for holomorphic connection). A holomorphic affine connection ∇ on \mathcal{Z} is called torsionfree if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for all locally defined holomorphic vector fields X and Y on \mathcal{Z} . Two holomorphic torsionfree affine connections ∇^1 and ∇^2 on \mathcal{Z} are called *projectively equivalent* if they have the same non-parametrized holomorphic geodesics. This condition is equivalent to the condition that there is a holomorphic 1–form $\theta \in H^0(\mathcal{Z}, T^*\mathcal{Z})$ such that

$$\nabla^1_X Y = \nabla^2_X Y + \theta(X)Y + \theta(Y)X \quad (2.4)$$

for any locally defined holomorphic vector fields X, Y on \mathcal{Z} (see [MM, p. 3021, Theorem 4.2], [OT, p. 222, Proposition A.3.2]).

Let M be a complex manifold of dimension $n > 1$. A *holomorphic projective connection* on M is given by a collection $(U_i, \nabla_i)_{i \in I}$, where

- $U_i \subset M, i \in I$, are open subsets with $\bigcup_{i \in I} U_i = M$, and
- ∇_i is a torsionfree affine connection on U_i such that for all $i, j \in I$, the two affine connection $\nabla_i|_{U_i \cap U_j}$ and $\nabla_j|_{U_i \cap U_j}$ on $U_i \cap U_j$ are projectively equivalent

(compare this with the equivalent definitions in [KO1] and [MM]). We say that the affine connection (U_i, ∇_i) is a local representative of the holomorphic projective connection.

Two holomorphic projective connections $(U_i, \nabla_i)_{i \in I}$ and $(U'_j, \nabla'_j)_{j \in J}$ are called *projectively equivalent* if their union $\{(U_i, \nabla_i)_{i \in I}, (U'_j, \nabla'_j)_{j \in J}\}$ is again a holomorphic projective connection.

The above definition coincides with Definition 4.4 in [MM] and also with the definition given in [Gu, Chapter 8] (see the proof in [MM] showing that the two definitions are equivalent). It should be mentioned that some authors call those projective connections, which are locally represented by *torsionfree* affine connections, as *normal* [Ka, KO1, JR1]. In their terminology we work, throughout the article, with *holomorphic normal projective connections*.

A holomorphic projective connection is called *flat* if it is projectively equivalent to a holomorphic projective connection $(U_i, \nabla_i)_{i \in I}$, where each ∇_i is flat. This means that a suitable holomorphic coordinate function on U_i takes ∇_i to the standard connection on \mathbb{C}^n . Once we fix holomorphic coordinate functions on every U_i satisfying the above condition that it takes ∇_i to the standard connection on \mathbb{C}^n , the transition functions defined on the intersections $U_i \cap U_j$ are projective transformations between open subsets of $\mathbb{C}\mathbb{P}^n$. Hence manifolds endowed with a flat holomorphic projective connection are locally modeled on the complex projective space. Consequently, flat holomorphic projective connections on M are precisely the holomorphic projective structures on M .

The curvature tensor of a holomorphic affine connection ∇ on M is defined to be

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad (2.5)$$

where X, Y, Z are locally defined holomorphic vector fields on M . So

$$R \in H^0(M, (\bigwedge^2 T^*M) \otimes \text{End}(TM)).$$

The curvature R vanishes identically if and only if ∇ is locally isomorphic to the standard affine connection of \mathbb{C}^n . Let $\text{End}^0(TM) \subset \text{End}(TM)$ be the direct summand given by the endomorphisms of trace zero of the fibers. The trace-free part of R , which is a holomorphic section of $(\bigwedge^2 T^*M) \otimes \text{End}^0(TM)$, is the *Weyl projective curvature* of ∇ . While the curvature R is not a projective invariant, its Weyl projective curvature W is evidently a projective invariant.

The Weyl projective tensor of a holomorphic projective connection on M is the Weyl projective curvature of the local representatives $(U_i, \nabla_i)_{i \in I}$ of the holomorphic projective connection. A holomorphic projective connection is flat if and only if the associated Weyl projective tensor vanishes identically (see [Ei, We] and [Gu, p. 79–83] for the adaptation of the proof to the holomorphic setting).

The expression of the Weyl curvature in dimension three — the case of interest in this article — is the following (see, for example, formula (3.4) on [Ga, p. 114]):

$$\begin{aligned} W(X, Y)Z &= R(X, Y)Z - \frac{1}{4}\text{Tr}R(X, Y)Z - \frac{1}{2}(\text{Ricci}(Y, Z)X \\ &\quad - \text{Ricci}(X, Z)(Y)) - \frac{1}{8}(\text{Tr}R(Y, Z)X - \text{Tr}R(X, Z)(Y)). \end{aligned} \quad (2.6)$$

In the expression in (2.6),

$$\text{Ricci} \in H^0(M, \bigwedge^2 T^*M) \quad (2.7)$$

is the Ricci curvature that sends $\eta \wedge \nu \in \bigwedge^2 T_x M$ to the trace of the endomorphism of $T_x M$ defined by $\xi \mapsto R(x)(\xi, \eta)\nu$. Also, in (2.6),

$$\text{Tr}R \in H^0(M, \bigwedge^2 T^*M)$$

sends $\eta \wedge \nu \in \bigwedge^2 T_x M$ to the trace of the endomorphism of $T_x M$ defined by $\xi \mapsto R(\eta, \nu)\xi$.

We have $\text{Tr}R(X, Y) = \text{Ricci}(Y, X) - \text{Ricci}(X, Y)$, and hence the Weyl tensor in (2.6) can be expressed in terms of just the Ricci tensor (this is exactly the formula (3.4) on [Ga, p. 114]):

$$\begin{aligned} W(X, Y)Z &= R(X, Y)Z + \frac{1}{4}(\text{Ricci}(X, Y)Z - \text{Ricci}(Y, X)Z) \\ &\quad + \frac{1}{8}[(3\text{Ricci}(X, Z) + \text{Ricci}(Z, X))Y - (3\text{Ricci}(Y, Z) + \text{Ricci}(Z, Y))X]. \end{aligned}$$

We note that Ricci is not a projective invariant, while, in contrast, as mentioned before, W is projectively invariant. Moreover, the Weyl tensor W possesses the same tensorial symmetries as R . In particular, $W \in H^0(M, \bigwedge^2 T^*M) \otimes \text{End}^0(TM)$ satisfies the first Bianchi identity which says that

$$W(X, Y)Z + W(Y, Z)X + W(Z, X)Y = 0 \quad (2.8)$$

for all locally defined holomorphic vector fields X, Y, Z on M .

The local symmetries for a holomorphic projective connection are given by the local (*projective*) *Killing field*. For a holomorphic projective connection ϕ on M , a local holomorphic vector field \mathbb{K} on M is a local *projective Killing field* (or briefly Killing field, when there is no ambiguity) if the local flow for \mathbb{K} preserves ϕ . When the local flow for \mathbb{K} preserves a holomorphic affine connection ∇ representing ϕ , then \mathbb{K} is called a local *affine Killing field*.

The local Lie algebra formed by all local projective Killing fields has finite dimension. The dimension of the projective Killing Lie algebra for (M, ϕ) is at most $(n + 1)^2 - 1$, where $n = \dim_{\mathbb{C}} M$. This maximal bound is realized only for projectively flat manifolds: in this case the local projective Killing Lie algebra is isomorphic to the Lie algebra of $\mathrm{PGL}(n + 1, \mathbb{C})$.

3. HOLOMORPHIC PROJECTIVE CONNECTIONS ON KÄHLER-EINSTEIN MANIFOLDS

In this Section we study holomorphic projective connections on compact Kähler–Einstein manifolds and prove Theorem 1.

According to [KO1, KO2], the only compact Kähler–Einstein manifolds of dimension n admitting a holomorphic projective connection are the standard ones: the complex projective space $\mathbb{C}\mathbb{P}^n$, the compact quotients of the complex hyperbolic n -space $\mathbb{H}_{\mathbb{C}}^n$ by a torsion-free discrete subgroup in $\mathrm{SU}(n, 1)$ and the étale quotients of compact complex n -tori.

The case of holomorphic projective connections on quotients of $\mathbb{H}_{\mathbb{C}}^n$ will be settled in Corollary 9. The case of the complex projective space $\mathbb{C}\mathbb{P}^n$ will be settled in Corollary 11 (more general results are known from [Ye, JR1, BM1]; see Proposition 13 below and the discussion there). Both of these results are direct consequences of Lemma 8 which parametrizes the space of projective classes of holomorphic projective connections on a complex manifold (compare this with [K11, Proposition 5.7] and [MY, Proposition 2.1] for the flat case).

Holomorphic projective connections on compact complex tori are studied in Proposition 12.

Let us first state a technical result which will be useful in the sequel (compare it with [BD1, p. 7449, Lemma 5.6] where the sufficient condition was proved for compact manifolds with trivial canonical bundle).

Lemma 7. *Let ϕ be a holomorphic projective connection on a compact complex manifold M . Then M admits a holomorphic torsionfree affine connection ∇ which is projectively equivalent to ϕ if and only if the canonical line bundle K_M admits a holomorphic connection. If M is compact and Kähler, this condition is equivalent with the condition that $c_1(M) = 0$. In particular, the above condition is automatically satisfied if K_M is trivial.*

Proof. The proof is obtained as a direct consequence of the results in [KO1] (see also [Gu]). There exists a holomorphic affine connection representing the projective connection ϕ if and only if the cocycle (3.2) in [KO1] defined as $d \log \Delta_{ij}$, where Δ_{ij} is the 1-cocycle of the canonical bundle K_M , vanishes in the cohomology group $H^1(M, \Omega_M)$ of the sheaf of holomorphic one-forms (see the explicit formula (3.6) on [KO1, p. 78–79]). This vanishing condition is satisfied if and only if the canonical line bundle K_M admits a holomorphic

connection (it coincides with the condition that the Atiyah class for K_M vanishes [At, Theorem 5, p. 195]; see also [Gu, p. 96–97] for an alternative approach). For compact Kähler manifolds, this condition is equivalent to the condition that $c_1(M) = 0$ [At, Proposition 12, p. 196].

If $K_M = \mathcal{O}_M$, the existence of a (global) holomorphic torsionfree affine connection representing ϕ was proved in [BD1, p. 7449, Lemma 5.6]. For the convenience of the reader, we include here a short proof which will be needed in the proof of Proposition 19.

Let $M = \bigcup_{i \in I} U_i$ be an open cover of M such that on each U_i there is a holomorphic torsionfree affine connection ∇_i projectively equivalent to the given projective connection ϕ . In particular, ∇_i and ∇_j are projectively equivalent on $U_i \cap U_j$. Let ω be a holomorphic volume form on M (i.e., ω is a trivializing holomorphic section of K_M). On each U_i , there exists a unique holomorphic torsionfree affine connection $\tilde{\nabla}_i$ projectively equivalent to ∇_i such that ω is parallel with respect to $\tilde{\nabla}_i$ [OT, Appendix A.3]. By uniqueness, $\tilde{\nabla}_i$ and $\tilde{\nabla}_j$ agree on $U_i \cap U_j$ for all $i, j \in I$. Consequently, the connections $\{\tilde{\nabla}_i\}$ together define a global holomorphic torsionfree affine connection on M which is projectively equivalent to ϕ . \square

The i -fold symmetric product of a vector bundle V would be denoted by $S^i(V)$. For a complex manifold M , let

$$\text{div} : S^2(T^*M) \otimes TM \longrightarrow T^*M \quad (3.1)$$

be the map constructed by combining the natural homomorphism

$$S^2(T^*\tilde{M}) \otimes T\tilde{M} \longrightarrow T^*\tilde{M} \otimes \text{End}(T\tilde{M})$$

with the trace map $\text{Tr} : \text{End}(T\tilde{M}) \longrightarrow \mathcal{O}_M$. The resulting map in (3.1) is denoted by div (not to be mixed with the earlier map dev) because it can be seen as a divergence operator defined on the space of quadratic vector fields (see [OT, p. 180]). Now define

$$(S^2(T^*M) \otimes TM)_0 := \text{kernel}(\text{div}) \subset S^2(T^*M) \otimes TM. \quad (3.2)$$

The next lemma generalizes to the non-flat case a known result for flat projective connections; compare it with [Kl1, Proposition 5.7] and [MY, Proposition 2.1], and notice that the bundle $\pi_* \text{Hom}(L, S)$ in [MY, Proposition 2.1] is isomorphic to $(S^2(T^*M) \otimes TM)_0$ defined in (3.2).

Lemma 8. *Let M be a complex manifold of complex dimension $n > 1$ endowed with a holomorphic projective connection. Then the space of projective equivalence classes of holomorphic projective connections on M is identified with $H^0(M, (S^2(T^*M) \otimes TM)_0)$ (see (3.2)).*

Proof. Fix a point $x_0 \in M$, and let

$$\varpi : \tilde{M} \longrightarrow M$$

be the corresponding universal cover of M . Let ϕ be holomorphic projective connection on M . Let $\varpi^*\phi$ be the holomorphic projective connection on \tilde{M} obtained by pulling back ϕ .

First assume that the canonical bundle of \widetilde{M} is holomorphically trivial. Now Lemma 7 implies that $\varpi^*\phi$ is represented by a (globally defined) torsionfree holomorphic affine connection. Let ∇^0 be such a global representative.

Now consider another holomorphic projective connections ϕ' on M . Let ∇ be a holomorphic affine connection on \widetilde{M} that represents $\varpi^*\phi'$. Then

$$\Theta := \nabla - \nabla^0 \in H^0(\widetilde{M}, S^2(T^*\widetilde{M}) \otimes T\widetilde{M}). \quad (3.3)$$

It should be mentioned that Θ lies in the subspace

$$H^0(\widetilde{M}, S^2(T^*\widetilde{M}) \otimes T\widetilde{M}) \subset H^0(\widetilde{M}, (T^*\widetilde{M})^{\otimes 2} \otimes T\widetilde{M})$$

because both ∇ and ∇^0 are torsionfree.

The natural action of $\pi_1(M, x_0)$ on \widetilde{M} is evidently by ∇^0 -projective transformations. Although this action does not preserve the connection ∇^0 , the action of any element $\gamma \in \pi_1(M)$ does send ∇^0 to an affine connection

$$\nabla_\gamma \quad (3.4)$$

which is projectively equivalent to ∇^0 , meaning there is a holomorphic one-form $\phi_\gamma \in H^0(\widetilde{M}, T^*\widetilde{M})$ such that

$$(\nabla_\gamma)_X Y = \nabla_X^0 Y + \phi_\gamma(X)Y + \phi_\gamma(Y)X$$

(see (2.4)). If Θ in (3.3) is invariant under the action of $\pi_1(M, x_0)$ on \widetilde{M} , then $\gamma \in \pi_1(M, x_0)$ sends $\nabla = \nabla^0 + \Theta$ to the projectively equivalent connection $\nabla_\gamma + \Theta$, where ∇_γ is the connection in (3.4). This immediately implies that the action of $\pi_1(M, x_0)$ on \widetilde{M} does factor through the ∇ -projective transformations. Consequently, ∇ descends to M as a holomorphic projective connection.

Therefore, the $\pi_1(M, x_0)$ -invariance of Θ is a sufficient condition for $\nabla^0 + \Theta$ to descend to M as a holomorphic projective connection.

We shall prove that the trace-free part of Θ is $\pi_1(M, x_0)$ -invariant if and only if ∇ descends to M as a holomorphic projective connection.

There is a natural injection

$$\mathcal{J} : T^*\widetilde{M} \longrightarrow S^2(T^*\widetilde{M}) \otimes T\widetilde{M} \quad (3.5)$$

that sends any $l \in T_y^*\widetilde{M}$ to the homomorphism $\mathcal{J}(l) : S^2(T_y\widetilde{M}) \longrightarrow T_y\widetilde{M}$ defined by $u \otimes v \longmapsto l(u)v + l(v)u$. It is straightforward to check that for div in (3.1),

$$\text{div} \circ \mathcal{J} = (n+1)\text{Id},$$

and hence the decomposition into a direct sum

$$S^2(T^*\widetilde{M}) \otimes T\widetilde{M} = (S^2(T^*\widetilde{M}) \otimes T\widetilde{M})_0 \oplus \text{Im}(\mathcal{J})$$

is obtained, where $(S^2(T^*\widetilde{M}) \otimes T\widetilde{M})_0 = \text{kernel}(\text{div})$ (as in (3.2)). The projection

$$\mathbb{F} : S^2(T^*\widetilde{M}) \otimes T\widetilde{M} \longrightarrow (S^2(T^*\widetilde{M}) \otimes T\widetilde{M})_0 \quad (3.6)$$

for the above decomposition coincides with the map defined by $\Theta \longmapsto \Theta - \frac{1}{n+1}(\mathcal{J} \circ \text{div})$, where \mathcal{J} is constructed in (3.5) (see [OT, p. 180]).

Next from the definition of the projectively equivalent connections (compare with the expression of \mathcal{J}) it follows that the action of $\pi_1(M, x_0)$ on \widetilde{M} is via projective equivalent maps with respect to $\nabla^0 + \Theta$ if and only if $\pi_1(M, x_0)$ preserves the trace-free part of Θ . Therefore, the holomorphic affine connection $\nabla^0 + \Theta$ on \widetilde{M} descends to a well-defined holomorphic projective connection on M if and only if

$$\mathbb{F}(\Theta) \in H^0(M, (S^2(T^*\widetilde{M}) \otimes T\widetilde{M})_0)$$

is $\pi_1(M, x_0)$ -invariant, where \mathbb{F} is the projection in (3.6). Consequently, the space of projective equivalence classes of holomorphic projective connections on M is identified with the space of holomorphic sections of $(S^2(T^*M) \otimes TM)_0$.

Now consider the general case where M is any complex manifold endowed with a holomorphic projective connection ϕ_0 . Let $(U_i, \nabla_i^0)_{i \in I}$ be a covering of M by local representatives of ϕ_0 , where ∇_i^0 is a holomorphic torsionfree affine connection on U_i that represents $\phi_0|_{U_i}$. Of course on each intersection $U_i \cap U_j$ the connections ∇_i^0 and ∇_j^0 are projectively equivalent. On each open subset U_i , the canonical line bundle $K_{U_i} = K_M|_{U_i}$ admits a holomorphic affine connection induced by ∇_i^0 .

Next take another holomorphic projective connection ϕ on M . By Lemma 7, on each open subset U_i there exists a holomorphic torsionfree affine connection ∇_i representing $\phi|_{U_i}$. Define

$$\Theta_i := \nabla_i - \nabla_i^0$$

on each U_i ; it is a holomorphic section of $S^2(T^*M) \otimes TM$ over U_i . By the above considerations, the trace zero-part $\mathbb{F}(\Theta_i)$, where \mathbb{F} is constructed in (3.6), does not depend on the choices of the local representatives ∇_i^0 and ∇_i . This implies that the local sections $\mathbb{F}(\Theta_i)$ and $\mathbb{F}(\Theta_j)$ coincide on $U_i \cap U_j$. Consequently, the local sections $\mathbb{F}(\Theta_i)$ glue together compatibly to produce a global holomorphic section of $(S^2(T^*M) \otimes TM)_0$ (defined in (3.2)) over M .

Conversely, take any $\Theta \in H^0(M, (S^2(T^*M) \otimes TM)_0)$. On each open subset U_i , consider the holomorphic affine connection $\nabla_i^0 + \Theta_i$, where Θ_i is the restriction of Θ to U_i . Over $U_i \cap U_j$, the difference $(\nabla_i^0 + \Theta_i) - (\nabla_j^0 + \Theta_j)$ between the two holomorphic affine connections is a holomorphic section of $S^2(T^*M) \otimes TM$ over $U_i \cap U_j$ that lies in the image of the homomorphism \mathcal{J} in (3.5). This implies that the restrictions of $(\nabla_i^0 + \Theta_i)|_{U_i \cap U_j}$ and $(\nabla_j^0 + \Theta_j)|_{U_i \cap U_j}$ are projectively equivalent. Therefore, the collection $(U_i, \nabla_i^0 + \Theta_i)_{i \in I}$ defines a holomorphic projective connection on M , which will be denoted by ϕ .

By construction, the holomorphic projective connection ϕ constructed above is projectively equivalent to ϕ_0 if and only if the corresponding section Θ vanishes identically. \square

Corollary 9. *Let M be a compact quotient of the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$ by a torsion-free discrete subgroup of $SU(n, 1)$, with $n > 1$. Then there is a unique holomorphic projective connection on M , namely the standard flat one.*

Proof. It was proved in [Kl1, Proposition 4.10], and earlier in [MY, Section 3], that $H^0(M, (S^2(T^*M) \otimes TM)_0) = 0$. Therefore, M has at most one holomorphic projective connection by Lemma 8. \square

Remark 10. Note that Lemma 8 does not hold for $n = 1$. Indeed, it is classically known that the space of holomorphic projective structures on a Riemann surface Σ is

an affine space for the vector space of holomorphic quadratic differentials on Σ (see [Gu] or Chapter 8 in [StG]). Lemma 8 is a higher dimensional version of this classical result. Also, Corollary 9 does not hold for $n = 1$ for the same reason.

Corollary 11. *The complex projective space $\mathbb{C}\mathbb{P}^n$ admits a unique holomorphic projective connection, namely the standard flat one.*

Proof. If $n = 1$, a holomorphic projective connection is automatically flat. Since the complex projective line is simply connected, the developing map for a projective structure produces an isomorphism of the projective structure with the standard projective structure of $\mathbb{C}\mathbb{P}^1$.

Now assume that $n > 1$. We will prove that

$$H^0(\mathbb{C}\mathbb{P}^n, S^2(T^*\mathbb{C}\mathbb{P}^n) \otimes T\mathbb{C}\mathbb{P}^n) = 0. \quad (3.7)$$

The Fubini–Study metric on $\mathbb{C}\mathbb{P}^n$ is Kähler–Einstein. So the Hermitian structure on $S^2(T^*\mathbb{C}\mathbb{P}^n) \otimes T\mathbb{C}\mathbb{P}^n$ induced by the Fubini–Study metric is Hermitian–Einstein. On the other hand,

$$\text{degree}(S^2(T^*\mathbb{C}\mathbb{P}^n) \otimes T\mathbb{C}\mathbb{P}^n) < 0$$

[LT, p. 50, Theorem 2.2.1]. Hence (3.7) holds. Therefore, Lemma 8 implies that $\mathbb{C}\mathbb{P}^n$ has a unique holomorphic projective connection: it is the standard one. \square

The following proposition (statement (ii)) studies holomorphic projective connections on compact complex tori. Statement (i), which was already known in the broader context of holomorphic Cartan geometries (see for example, [BM2, BM3, BD4, Du3]), shows that compact complex tori cover the case of Kähler manifolds with trivial first Chern class.

Proposition 12.

- (i) *Let M be a compact Kähler manifold with $c_1(M) = 0$ bearing a holomorphic projective connection ϕ . Then M admits a finite unramified cover T which is a compact complex torus; the pull-back of ϕ on T is projectively equivalent to a (translation invariant) holomorphic torsionfree affine connection.*
- (ii) *A generic holomorphic projective connection on a compact complex torus of complex dimension $n > 2$ is not projectively flat.*

Proof. (i) By Calabi’s conjecture proved by Yau, [Ya], M admits a Ricci flat Kähler metric. Using this, Bogomolov–Beauville decomposition theorem, [Be, Bo], shows that M admits a finite unramified cover $\psi : M' \rightarrow M$ such that the canonical line bundle $K_{M'}$ is trivial. Now Lemma 7 says that the holomorphic projective connection $\psi^*\phi$ is represented by a holomorphic affine connection on M' . Since M' is Kähler, it is known that M' admits a finite unramified cover T which is a complex torus [IKO]. Any holomorphic affine connection on T is known to be translation invariant [IKO] (see also the proof below).

(ii) Let \mathbb{T}^n be a compact complex torus of complex dimension $n > 2$. Since the canonical bundle $K_{\mathbb{T}^n}$ of the torus is trivial, Lemma 7 says that every holomorphic projective connection on \mathbb{T}^n is represented by some globally defined holomorphic affine connection.

Denote by (z_1, \dots, z_n) a holomorphic linear coordinate function on \mathbb{T}^n and by ∇_0 the standard flat holomorphic affine connection of \mathbb{T}^n (induced by that of \mathbb{C}^n using this coordinate function). Any holomorphic affine connection on \mathbb{T}^n is of the form $\nabla_0 + \Theta$,

where $\Theta \in H^0(\mathbb{T}^n, S^2(T^*\mathbb{T}^n) \otimes T\mathbb{T}^n)$. Since the holomorphic tangent bundle $T\mathbb{T}^n$ is holomorphically trivial, such a section Θ is a sum of terms of the form $f_{ij}^k dz_i dz_j \frac{\partial}{\partial z_k}$, where f_{ij}^k are constant functions on \mathbb{T}^n with values in \mathbb{C} . The coefficients f_{ij}^k are classically called the Christoffel symbols of the affine connection. In particular, the holomorphic affine connection $\nabla_0 + \Theta$ is translation-invariant.

First assume $n = 3$. For the convenience of computations, let us denote by (z_1, z_2, τ) the holomorphic linear coordinate function on \mathbb{T}^3 .

Denote by $\Theta_{A,B,C,D,E}$ the holomorphic section of $S^2(T^*\mathbb{T}^3) \otimes T\mathbb{T}^3$ corresponding to the following Christoffel symbols:

$$\begin{aligned} f_{\tau,\tau}^{z_1} &= A, & f_{\tau,\tau}^{z_2} &= B, \\ f_{z_1,z_1}^{z_1} &= 2f_{\tau,z_1}^\tau = 2f_{z_1,z_2}^{z_1} = C, \\ f_{z_2,z_2}^{z_2} &= 2f_{\tau,z_2}^\tau = 2f_{z_1,z_2}^{z_2} = D, \\ f_{\tau,\tau}^\tau &= 2f_{z_1,\tau}^{z_1} = 2f_{z_2,\tau}^{z_2} = E, \end{aligned}$$

with $A, B, C, D, E \in \mathbb{C}$; all other remaining coefficients are set to zero.

Consider the associated holomorphic affine connection

$$\nabla^{A,B,C,D,E} = \nabla_0 + \Theta_{A,B,C,D,E}.$$

We will show that $\nabla^{A,B,C,D,E}$ is projectively flat if and only if $C = D$.

For that we shall compute the projective Weyl curvature tensor of $\nabla^{A,B,C,D,E}$.

We start by computing the affine curvature tensor. To simplify the notation in the computation, $\nabla^{A,B,C,D,E}$ is denoted simply by ∇ . Recall from (2.5) that the affine curvature tensor of ∇ is given by the formula

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Substituting for X, Y and Z we get the following explicit expressions:

$$\begin{aligned} R\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial z_1}\right) \frac{\partial}{\partial z_1} &= \nabla_{\frac{\partial}{\partial \tau}} \nabla_{\frac{\partial}{\partial z_1}} \frac{\partial}{\partial z_1} - \nabla_{\frac{\partial}{\partial z_1}} \nabla_{\frac{\partial}{\partial \tau}} \frac{\partial}{\partial z_1} = \nabla_{\frac{\partial}{\partial \tau}} \left(f_{z_1, z_1}^{z_1} \frac{\partial}{\partial z_1} \right) \\ &\quad - \nabla_{\frac{\partial}{\partial z_1}} \left(f_{\tau, z_1}^\tau \frac{\partial}{\partial \tau} + f_{\tau, z_1}^{z_1} \frac{\partial}{\partial z_1} \right) \\ &= f_{z_1, z_1}^{z_1} \left(f_{\tau, z_1}^\tau \frac{\partial}{\partial \tau} + f_{\tau, z_1}^{z_1} \frac{\partial}{\partial z_1} \right) - f_{\tau, z_1}^\tau \nabla_{\frac{\partial}{\partial z_1}} \frac{\partial}{\partial \tau} - f_{\tau, z_1}^{z_1} \nabla_{\frac{\partial}{\partial z_1}} \frac{\partial}{\partial z_1} \\ &= f_{z_1, z_1}^{z_1} \left(f_{\tau, z_1}^\tau \frac{\partial}{\partial \tau} + f_{\tau, z_1}^{z_1} \frac{\partial}{\partial z_1} \right) - f_{\tau, z_1}^\tau \left(f_{z_1, \tau}^\tau \frac{\partial}{\partial \tau} + f_{z_1, \tau}^{z_1} \frac{\partial}{\partial z_1} \right) - f_{\tau, z_1}^{z_1} f_{z_1, z_1}^{z_1} \frac{\partial}{\partial z_1} \\ &= \frac{C^2}{4} \frac{\partial}{\partial \tau} - \frac{CE}{4} \frac{\partial}{\partial z_1}. \end{aligned}$$

By symmetry we get

$$R\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial z_2}\right) \frac{\partial}{\partial z_2} = \frac{D^2}{4} \frac{\partial}{\partial \tau} - \frac{DE}{4} \frac{\partial}{\partial z_2},$$

and

$$\begin{aligned} R\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) \frac{\partial}{\partial z_1} &= \nabla_{\frac{\partial}{\partial z_1}} \nabla_{\frac{\partial}{\partial z_2}} \frac{\partial}{\partial z_1} - \nabla_{\frac{\partial}{\partial z_2}} \nabla_{\frac{\partial}{\partial z_1}} \frac{\partial}{\partial z_1} = \nabla_{\frac{\partial}{\partial z_1}} \left(f_{z_1, z_2}^{z_1} \frac{\partial}{\partial z_1} + f_{z_1, z_2}^{z_2} \frac{\partial}{\partial z_2} \right) \\ &\quad - \nabla_{\frac{\partial}{\partial z_2}} \left(f_{z_1, z_1}^{z_1} \frac{\partial}{\partial z_1} \right) = f_{z_1, z_2}^{z_1} \cdot f_{z_1, z_1}^{z_1} \frac{\partial}{\partial z_1} + f_{z_1, z_2}^{z_2} \nabla_{\frac{\partial}{\partial z_1}} \frac{\partial}{\partial z_2} - f_{z_1, z_1}^{z_1} \nabla_{\frac{\partial}{\partial z_2}} \frac{\partial}{\partial z_1} \end{aligned}$$

$$\begin{aligned}
&= \frac{C^2}{2} \frac{\partial}{\partial z_1} + \left(\frac{1}{2}D - C\right) \nabla_{\frac{\partial}{\partial z_1}} \frac{\partial}{\partial z_2} = \frac{C^2}{2} \frac{\partial}{\partial z_1} + \left(\frac{1}{2}D - C\right) \left(f_{z_1, z_2}^{z_1} \frac{\partial}{\partial z_1} + f_{z_1, z_2}^{z_2} \frac{\partial}{\partial z_2}\right) \\
&= \frac{C^2}{2} \frac{\partial}{\partial z_1} + \left(\frac{1}{2}D - C\right) \left(\frac{C}{2} \frac{\partial}{\partial z_1} + \frac{D}{2} \frac{\partial}{\partial z_2}\right) = \frac{CD}{4} \frac{\partial}{\partial z_1} + \frac{D^2 - 2CD}{4} \frac{\partial}{\partial z_2}.
\end{aligned}$$

By symmetry we get

$$R\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) \frac{\partial}{\partial z_2} = -R\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_1}\right) \frac{\partial}{\partial z_2} = -\frac{C^2 - 2CD}{4} \frac{\partial}{\partial z_1} - \frac{CD}{4} \frac{\partial}{\partial z_2}.$$

We have

$$\begin{aligned}
R\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial z_1}\right) \frac{\partial}{\partial z_2} &= \nabla_{\frac{\partial}{\partial \tau}} \nabla_{\frac{\partial}{\partial z_1}} \frac{\partial}{\partial z_2} - \nabla_{\frac{\partial}{\partial z_1}} \nabla_{\frac{\partial}{\partial \tau}} \frac{\partial}{\partial z_2} \\
&= \nabla_{\frac{\partial}{\partial \tau}} \left(f_{z_1, z_2}^{z_1} \frac{\partial}{\partial z_1} + f_{z_1, z_2}^{z_2} \frac{\partial}{\partial z_2}\right) - \nabla_{\frac{\partial}{\partial z_1}} \left(f_{\tau, z_2}^\tau \frac{\partial}{\partial \tau} + f_{\tau, z_2}^{z_2} \frac{\partial}{\partial z_2}\right) \\
&= \frac{1}{2}C \left(f_{\tau, z_1}^\tau \frac{\partial}{\partial \tau} + f_{\tau, z_1}^{z_1} \frac{\partial}{\partial z_1}\right) + \frac{1}{2}D \left(f_{\tau, z_2}^\tau \frac{\partial}{\partial \tau} + f_{\tau, z_2}^{z_2} \frac{\partial}{\partial z_2}\right) - f_{z_2, \tau}^\tau \nabla_{\frac{\partial}{\partial z_1}} \frac{\partial}{\partial \tau} - f_{z_2, \tau}^{z_2} \nabla_{\frac{\partial}{\partial z_1}} \frac{\partial}{\partial z_2} \\
&= \frac{1}{4}C^2 \frac{\partial}{\partial \tau} + \frac{1}{4}CE \frac{\partial}{\partial z_1} + \frac{1}{4}D^2 \frac{\partial}{\partial \tau} + \frac{1}{4}DE \frac{\partial}{\partial z_2} - \frac{1}{2}D \left(f_{z_1, \tau}^\tau \frac{\partial}{\partial \tau} + f_{z_1, \tau}^{z_1} \frac{\partial}{\partial z_1}\right) \\
&\quad - \frac{1}{2}E \left(f_{z_1, z_2}^{z_1} \frac{\partial}{\partial z_1} + f_{z_1, z_2}^{z_2} \frac{\partial}{\partial z_2}\right) = \frac{1}{4}(C^2 + D^2 - CD) \frac{\partial}{\partial \tau} - \frac{1}{4}DE \frac{\partial}{\partial z_1}.
\end{aligned}$$

By symmetry,

$$R\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial z_2}\right) \frac{\partial}{\partial z_1} = \frac{1}{4}(C^2 + D^2 - CD) \frac{\partial}{\partial \tau} - \frac{1}{4}CE \frac{\partial}{\partial z_2}.$$

We also compute that

$$\begin{aligned}
R\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) \frac{\partial}{\partial \tau} &= \nabla_{\frac{\partial}{\partial z_1}} \nabla_{\frac{\partial}{\partial z_2}} \frac{\partial}{\partial \tau} - \nabla_{\frac{\partial}{\partial z_2}} \nabla_{\frac{\partial}{\partial z_1}} \frac{\partial}{\partial \tau} \\
&= \nabla_{\frac{\partial}{\partial z_1}} \left(f_{\tau, z_2}^\tau \frac{\partial}{\partial \tau} + f_{\tau, z_2}^{z_2} \frac{\partial}{\partial z_2}\right) - \nabla_{\frac{\partial}{\partial z_2}} \left(f_{\tau, z_1}^\tau \frac{\partial}{\partial \tau} + f_{\tau, z_1}^{z_1} \frac{\partial}{\partial z_1}\right) \\
&= f_{\tau, z_2}^\tau \left(f_{\tau, z_1}^{z_1} \frac{\partial}{\partial z_1} + f_{\tau, z_1}^\tau \frac{\partial}{\partial \tau}\right) + f_{\tau, z_2}^{z_2} \left(f_{z_1, z_2}^{z_1} \frac{\partial}{\partial z_1} + f_{z_1, z_2}^{z_2} \frac{\partial}{\partial z_2}\right) \\
&\quad - f_{\tau, z_1}^\tau \left(f_{\tau, z_2}^{z_2} \frac{\partial}{\partial z_2} + f_{\tau, z_2}^\tau \frac{\partial}{\partial \tau}\right) - f_{\tau, z_1}^{z_1} \left(f_{z_1, z_2}^{z_1} \frac{\partial}{\partial z_1} + f_{z_1, z_2}^{z_2} \frac{\partial}{\partial z_2}\right) \\
&= \frac{DE}{4} \frac{\partial}{\partial z_1} - \frac{CE}{4} \frac{\partial}{\partial z_2},
\end{aligned}$$

and

$$\begin{aligned}
R\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial z_1}\right) \frac{\partial}{\partial \tau} &= \nabla_{\frac{\partial}{\partial \tau}} \nabla_{\frac{\partial}{\partial z_1}} \frac{\partial}{\partial \tau} - \nabla_{\frac{\partial}{\partial z_1}} \nabla_{\frac{\partial}{\partial \tau}} \frac{\partial}{\partial \tau} \\
&= \nabla_{\frac{\partial}{\partial \tau}} \left(f_{\tau, z_1}^{z_1} \frac{\partial}{\partial z_1} + f_{\tau, z_1}^\tau \frac{\partial}{\partial \tau}\right) - \nabla_{\frac{\partial}{\partial z_1}} \left(f_{\tau, \tau}^\tau \frac{\partial}{\partial \tau} + f_{\tau, \tau}^{z_1} \frac{\partial}{\partial z_1} + f_{\tau, \tau}^{z_2} \frac{\partial}{\partial z_2}\right) \\
&= f_{\tau, z_1}^{z_1} \left(f_{\tau, z_1}^\tau \frac{\partial}{\partial \tau} + f_{\tau, z_1}^{z_1} \frac{\partial}{\partial z_1}\right) + f_{\tau, z_1}^\tau \left(f_{\tau, \tau}^\tau \frac{\partial}{\partial \tau} + f_{\tau, \tau}^{z_1} \frac{\partial}{\partial z_1} + f_{\tau, \tau}^{z_2} \frac{\partial}{\partial z_2}\right) \\
&\quad - f_{\tau, \tau}^\tau \left(f_{\tau, z_1}^{z_1} \frac{\partial}{\partial z_1} + f_{\tau, z_1}^\tau \frac{\partial}{\partial \tau}\right) - f_{\tau, \tau}^{z_1} f_{z_1, z_1}^{z_1} \frac{\partial}{\partial z_1} - f_{\tau, \tau}^{z_2} \left(f_{z_1, z_2}^{z_1} \frac{\partial}{\partial z_1} + f_{z_1, z_2}^{z_2} \frac{\partial}{\partial z_2}\right) \\
&= \frac{EC}{4} \frac{\partial}{\partial \tau} + \left(-\frac{E^2}{4} - \frac{1}{2}C(A + B)\right) \frac{\partial}{\partial z_1} + \frac{1}{2}B(C - D) \frac{\partial}{\partial z_2}.
\end{aligned}$$

By symmetry,

$$R\left(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial\tau} = \frac{ED}{4}\frac{\partial}{\partial\tau} + \left(-\frac{E^2}{4} - \frac{1}{2}D(A+B)\right)\frac{\partial}{\partial z_2} + \frac{1}{2}A(D-C)\frac{\partial}{\partial z_1}.$$

Now we compute the Ricci curvature defined in (2.7):

$$\begin{aligned} \text{Ricci}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) &= \text{Ricci}\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_1}\right) = \frac{1}{4}(C^2 + D^2); \\ \text{Ricci}\left(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial z_1}\right) &= \text{Ricci}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial\tau}\right) = \frac{1}{2}CE; \\ \text{Ricci}\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial\tau}\right) &= \text{Ricci}\left(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial z_2}\right) = \frac{1}{2}DE; \\ \text{Ricci}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_1}\right) &= \frac{1}{4}(C^2 + 2CD - D^2); \\ \text{Ricci}\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_2}\right) &= \frac{1}{4}(D^2 + 2CD - C^2); \\ \text{Ricci}\left(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial\tau}\right) &= \frac{1}{2}E^2 + \frac{1}{2}(A+B)(C+D). \end{aligned}$$

Recall from (2.6) the formula for the Weyl projective tensor W in dimension three:

$$\begin{aligned} W(X, Y)Z &= R(X, Y)Z - \frac{1}{4}\text{Tr}R(X, Y)Z - \frac{1}{2}[\text{Ricci}(Y, Z)X - \text{Ricci}(X, Z)(Y)] \\ &\quad - \frac{1}{8}[\text{Tr}R(Y, Z)X - \text{Tr}R(X, Z)(Y)]. \end{aligned}$$

Also, recall that the connection ∇ is projectively flat if and only if the tensor W vanishes identically. The Weyl projective tensor W is anti-symmetric in (X, Y) and satisfies the first Bianchi identity in (2.8).

Since Ricci for ∇ is symmetric, it follows that $\text{Tr}R$ vanishes identically. Connections with symmetric Ricci tensor are called equiaffine. The geometrical meaning of it is that there is a parallel holomorphic volume form [OT, p. 222, Appendix A.3]. The above formula for Weyl projective tensor for ∇ reduces to

$$W(X, Y)Z = R(X, Y)Z - \frac{1}{2}[\text{Ricci}(Y, Z)X - \text{Ricci}(X, Z)(Y)].$$

The computation for $W\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial z_2}$ is as follows:

$$\begin{aligned} W\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial z_2} &= R\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial z_2} - \frac{1}{2}[\text{Ricci}\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial z_1} - \text{Ricci}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial z_2}] \\ &= \frac{1}{4}(2CD - C^2)\frac{\partial}{\partial z_1} - \frac{1}{4}CD\frac{\partial}{\partial z_2} - \frac{1}{8}(D^2 + 2CD - C^2)\frac{\partial}{\partial z_1} + \frac{1}{8}(C^2 + D^2)\frac{\partial}{\partial z_2} \\ &= -\frac{1}{8}(C - D)^2\frac{\partial}{\partial z_1} + \frac{1}{8}(C - D)^2\frac{\partial}{\partial z_2}. \end{aligned}$$

Hence

$$W\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial z_2} = -\frac{1}{8}(C - D)^2\frac{\partial}{\partial z_1} + \frac{1}{8}(C - D)^2\frac{\partial}{\partial z_2}.$$

Also,

$$\begin{aligned} W\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial z_1} &= R\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial z_1} - \frac{1}{2}[\text{Ricci}\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_1}\right)\frac{\partial}{\partial z_1} - \text{Ricci}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_1}\right)\frac{\partial}{\partial z_2}] \\ &= \frac{1}{4}CD\frac{\partial}{\partial z_1} + \frac{1}{4}(D^2 - 2CD)\frac{\partial}{\partial z_2} - \frac{1}{8}(C^2 + D^2)\frac{\partial}{\partial z_1} + \frac{1}{8}(C^2 + 2CD - D^2)\frac{\partial}{\partial z_2} \\ &= -\frac{1}{8}(C - D)^2\frac{\partial}{\partial z_1} + \frac{1}{8}(C - D)^2\frac{\partial}{\partial z_2}. \end{aligned}$$

In conclusion,

$$W\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial z_1} = -\frac{1}{8}(C - D)^2\frac{\partial}{\partial z_1} + \frac{1}{8}(C - D)^2\frac{\partial}{\partial z_2}.$$

We get that

$$W\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial \tau} = R\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial \tau} - \frac{1}{2}[\text{Ricci}\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial \tau}\right)\frac{\partial}{\partial z_1} - \text{Ricci}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \tau}\right)\frac{\partial}{\partial z_2}] = 0.$$

Hence we have

$$W\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial \tau} = 0.$$

By similar direct computations we get that

$$W\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial z_1}\right)\frac{\partial}{\partial z_2} = \frac{1}{8}(C - D)^2\frac{\partial}{\partial \tau}$$

and

$$W\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial z_1}\right)\frac{\partial}{\partial \tau} = \frac{1}{4}(A + B)(D - C)\frac{\partial}{\partial z_1} + \frac{1}{2}B(C - D)\frac{\partial}{\partial z_2}.$$

Also by direct computation:

$$W\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial z_1}\right)\frac{\partial}{\partial z_1} = \frac{1}{8}(C - D)^2\frac{\partial}{\partial \tau}$$

and

$$W\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial z_2} = \frac{1}{8}(C - D)^2\frac{\partial}{\partial \tau}.$$

Again by a direct computation,

$$W\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial \tau}\right)\frac{\partial}{\partial \tau} = \frac{1}{2}A(C - D)\frac{\partial}{\partial z_1} + \frac{1}{4}(A + B)(D - C)\frac{\partial}{\partial z_2}.$$

The other components of the Weyl tensor can be obtained using the first Bianchi identity in (2.8). Indeed, from

$$W\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial \tau} + W\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial \tau}\right)\frac{\partial}{\partial z_1} + W\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial z_1}\right)\frac{\partial}{\partial z_2} = 0$$

we infer that

$$W\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial \tau}\right)\frac{\partial}{\partial z_1} = -\frac{1}{8}(C - D)^2\frac{\partial}{\partial \tau}. \quad (3.8)$$

Notice that the Weyl projective tensor W does not depend on the parameter E . This is due to the facts that W is a projective invariant and $\nabla^{A,B,C,D,E}$ is projectively equivalent with $\nabla^{A,B,C,D,0}$. Indeed, let ϕ_τ be the holomorphic one-form on \mathbb{T}^3 defined by

$$\phi_\tau\left(\frac{\partial}{\partial\tau}\right) = \frac{1}{2}E \quad \text{and} \quad \phi_\tau\left(\frac{\partial}{\partial z_i}\right) = 0$$

for $i = 1, 2$. Then

$$\nabla_X^{A,B,C,D,E}Y - \nabla_X^{A,B,C,D,0}Y = \phi_\tau(X)(Y) + \phi_\tau(Y)X \quad (3.9)$$

for all holomorphic vector fields X, Y ; the identity in (3.9) being tensorial it can be easily verified for any pair of vectors chosen from the basis $(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$. From (3.9) it follows immediately that $\nabla^{A,B,C,D,E}$ and $\nabla^{A,B,C,D,0}$ are projectively equivalent.

From (3.8) and the expression of all components of the Weyl projective tensor, it follows that W vanishes identically if and only if $C = D$. Therefore, for generic A, B, C, D, E , the Weyl projective tensor W does not vanish identically. Consequently, the associated projective connection for generic A, B, C, D, E is not projectively flat.

Now consider the general connection on \mathbb{T}^3 given by

$$\nabla = \nabla_0 + \Theta,$$

where $\Theta \in H^0(\mathbb{T}^3, S^2(T^*\mathbb{T}^3) \otimes T\mathbb{T}^3)$. The vanishing of the Weyl tensor for ∇ is an algebraic (quadratic) equation in the Christoffel symbols f_{ij}^k . Since all connections $\nabla^{A,B,C,D,E}$ with $C \neq D$ have nonzero Weyl tensor, the space of flat projective connections has positive codimension in the space of all connections. Consequently, the general connection is not flat.

Now consider the case where the complex dimension of the torus is $n > 3$. Denote by $(z_1, z_2, \tau = z_3, z_4, \dots, z_n)$ a global linear holomorphic coordinate function on \mathbb{T}^n . Consider the holomorphic projective connection represented by $\nabla_n^{A,B,C,D,E} = \nabla_0 + \Theta_n^{A,B,C,D,E}$, where

$$\Theta_n^{A,B,C,D,E} \in H^0(\mathbb{T}^n, S^2(T^*\mathbb{T}^n) \otimes T\mathbb{T}^n)$$

is defined below by the Christoffel symbols:

$$\begin{aligned} f_{\tau,\tau}^{z_1} &= A, f_{\tau,\tau}^{z_2} = B, \\ f_{z_1,z_1}^{z_1} &= 2f_{\tau,z_1}^\tau = 2f_{z_1,z_2}^{z_1} = C, \\ f_{z_2,z_2}^{z_2} &= 2f_{\tau,z_2}^\tau = 2f_{z_1,z_2}^{z_2} = D, \\ f_{\tau,\tau}^\tau &= 2f_{z_1,\tau}^{z_1} = 2f_{z_2,\tau}^{z_2} = E, \end{aligned}$$

where $A, B, C, D, E \in \mathbb{C}$; the remaining symbols are trivial.

Identify \mathbb{C}^n with the universal cover of \mathbb{T}^n , and equip \mathbb{C}^n with the connection given by the connection $\nabla_n^{A,B,C,D,E}$ on \mathbb{T}^n using this identification. By construction, the three dimensional linear subspace

$$\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_4 = \dots = z_n = 0\} \subset \mathbb{C}^n$$

is totally geodesic, and the induced connection on this subspace is the connection $\nabla^{A,B,C,D,E}$ studied earlier. For $C \neq D$, the connection $\nabla^{A,B,C,D,E}$ is not projectively flat, and hence $\nabla_n^{A,B,C,D,E}$ is not projectively flat either.

The same argument as in the dimension three case proves that the generic connection ∇ on \mathbb{T}^n (meaning ∇ lying in a Zariski dense open set, whose complement of a proper algebraic subvariety defined by quadratic equations) is not flat. \square

Ye proved in [Ye] that Fano manifolds bearing a holomorphic projective connection are flat, isomorphic to the standard complex projective space. This was generalized in [JR1] to compact Kähler manifolds admitting nontrivial rational curves. More recently this result was extended to the more general context of holomorphic Cartan geometries (see Theorem 2 in [BM1]). We produce below a simple proof specific to the case of projective connections (this proof uses the same circle of ideas as in [JR1]).

Proposition 13. *Let M be a complex projective manifold of complex dimension n equipped with a holomorphic projective connection ϕ . Assume that there is a nonconstant holomorphic map*

$$\beta : \mathbb{C}\mathbb{P}^1 \longrightarrow M.$$

Then (M, ϕ) is the complex projective space $\mathbb{C}\mathbb{P}^n$ endowed with its standard (flat) projective connection.

Proof. Take any nonconstant holomorphic map $\beta : \mathbb{C}\mathbb{P}^1 \longrightarrow M$. There are integers a_1, \dots, a_n such that

$$\beta^*TM = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{C}\mathbb{P}^1}(a_i)$$

[Grot, p. 122, Théorème 1]. The vector bundle β^*TM admits a holomorphic projective connection, because TM does so, and a holomorphic projective connection on β^*TM induces a holomorphic connection on

$$\text{End}(\beta^*TM) = \bigoplus_{i,j=1}^n \mathcal{O}_{\mathbb{C}\mathbb{P}^1}(a_i - a_j).$$

Hence we have $a_i = a_j$ for all $1 \leq i, j \leq n$ [At, p. 202, Proposition 17]. On the other hand, we have the nonzero homomorphism

$$d\beta : T\mathbb{C}\mathbb{P}^1 = \mathcal{O}_{\mathbb{C}\mathbb{P}^1}(2) \longrightarrow \beta^*TM.$$

Combining these it follows that $a_1 = a_2 = \dots = a_n \geq 2$. Therefore,

$$H^0(\mathbb{C}\mathbb{P}^1, \beta^*((\bigwedge^2 T^*M) \otimes \text{End}(TM))) = 0, \quad (3.10)$$

because $\beta^*((\bigwedge^2 T^*M) \otimes \text{End}(TM)) = \mathcal{O}_{\mathbb{C}\mathbb{P}^1}(-2a_1)^{\oplus n^3(n-1)/2}$.

Since β^*TM is ample for any rational curve, a theorem of Mori says that $M = \mathbb{C}\mathbb{P}^n$ [Mor, p. 594] (see also [MP, p. 41, Theorem 4.2]). From (3.10) it follows that the Weyl tensor W for ϕ vanishes identically. Therefore ϕ is flat, and its developing map realizes an isomorphism between (M, ϕ) and $\mathbb{C}\mathbb{P}^n$ endowed with its standard projective structure. \square

Hwang and Mok proved in [HM] (Theorem 2 and Proposition 8) that uniruled projective manifolds bearing a holomorphic G -structure modeled on a Hermitian symmetric spaces of rank ≥ 2 are flat, and globally isomorphic to the corresponding Hermitian symmetric space endowed with its standard G -structure. Proposition 13 could be seen as a rank one version of Theorem 2 in [HM].

3.1. Proof of Theorem 1. Let M be a compact Kähler–Einstein manifold of complex dimension $n > 1$ endowed with a holomorphic projective connection. By the classification result of Kobayashi and Ochiai, [KO1, KO2], M is biholomorphic to one of the standard models: either to $\mathbb{C}\mathbb{P}^n$, or to a compact quotient of $\mathbb{H}_{\mathbb{C}}^n$ by a torsion-free discrete subgroup of $\mathrm{SU}(n, 1)$ or to an étale quotient of a compact complex n -torus.

Corollary 11 proves that the only holomorphic projective connection on $\mathbb{C}\mathbb{P}^n$ is the flat standard one (this was already known; see [Ye, JR1, BM1]).

Corollary 9 proves that for $n > 1$, the only holomorphic projective connection on compact quotients of $\mathbb{H}_{\mathbb{C}}^n$ by a torsion-free discrete subgroup of $\mathrm{SU}(n, 1)$ is the flat standard one. The result was already known for flat holomorphic projective connections [MY] (see also [K11]); Corollary 9 uses arguments from their proof.

Proposition 12 shows that a holomorphic projective connection on a complex compact n -torus is represented by a global translation-invariant torsionfree holomorphic affine connection. Moreover, Proposition 12 proves that, for $n > 2$, the generic translation invariant holomorphic affine connection on a compact complex n -torus is not projectively flat. This completes the proof of Theorem 1.

4. PROJECTIVE CONNECTIONS ON KUGA-SHIMURA THREEFOLDS

As mentioned in the introduction, complex projective threefolds admitting holomorphic projective connections have been classified by Jahnke and Radloff in [JR1]. Their result says that the only examples are either the standard ones or an étale quotient of a Kuga–Shimura projective threefold (see Section 2). The holomorphic projective connections on the standard examples were studied in Section 3.

In this Section we study holomorphic projective connections on Kuga–Shimura threefolds and prove Theorem 2 and Corollary 3.

Proposition 14. *The projective equivalence classes of holomorphic projective connections on a Kuga–Shimura projective threefold $M \rightarrow \Sigma$ are parametrized by a complex affine space on the vector space $H^0(\Sigma, K_{\Sigma}^{\frac{3}{2}})^{\oplus 2}$, where K_{Σ} is the canonical bundle of the base Riemann surface Σ (see (2.3)).*

Proof. Let

$$\varpi : \widetilde{M} = \mathcal{H} \times \mathbb{C}^2 \longrightarrow M \tag{4.1}$$

be the universal cover of M . Let ϕ be a holomorphic projective connection on M . By Lemma 7, the holomorphic projective connection $\varpi^*\phi$ on \widetilde{M} is represented by a torsionfree holomorphic affine connection ∇ on \widetilde{M} .

Let ∇_0 be the standard flat affine connection of $\mathcal{H} \times \mathbb{C}^2$ (seen as an open subset in \mathbb{C}^3). Then

$$\Theta = \nabla - \nabla_0 \in H^0(\widetilde{M}, S^2(T^*\widetilde{M}) \otimes T\widetilde{M}),$$

because both ∇ and ∇_0 are torsionfree.

We saw in the proof of Lemma 8 that $\mathbb{F}(\Theta) \in H^0(\widetilde{M}, (S^2(T^*\widetilde{M}) \otimes T\widetilde{M})_0)^{\pi_1(M)}$, where \mathbb{F} is the projection in (3.6). Conversely, every element of $H^0(\widetilde{M}, (S^2(T^*\widetilde{M}) \otimes T\widetilde{M})_0)^{\pi_1(M)}$

defines a holomorphic projective connection on M . Consequently, the space of projective equivalence classes of holomorphic projective connections on M is identified with $H^0(\widetilde{M}, (S^2(T^*\widetilde{M}) \otimes T\widetilde{M})_0)^{\pi_1(M)}$ (see the proof of Lemma 8). We will first compute

$$H^0(\widetilde{M}, S^2(T^*\widetilde{M}) \otimes T\widetilde{M})^{\pi_1(M)},$$

and then compute the locus in it of the trace-free ones.

As seen in Section 2.2, $\pi_1(M)$ is a semi-direct product $\Gamma \ltimes \Lambda$, where Γ is a Fuchsian group and $\Lambda \simeq \mathbb{Z}^{\oplus 4}$. We recall from (2.2) that the action of $\pi_1(M)$ on \widetilde{M} is given by:

$$(\gamma, \lambda) \cdot (\tau, z_1, z_2) \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{z_1 + m\tau + n}{c\tau + d}, \frac{z_2 + k\tau + l}{c\tau + d} \right) \quad (4.2)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\lambda = (m, n, k, l) \in \Lambda$. It follows that the action of the same element $(\gamma, \lambda) \in \pi_1(M)$ on the standard basis of holomorphic one-forms on \widetilde{M} is given by:

$$\begin{aligned} & (d\tau, dz_1, dz_2) \\ \mapsto & \left(\frac{1}{(c\tau + d)^2} d\tau, \frac{1}{c\tau + d} dz_1 - \frac{cz_1 - md + nc}{(c\tau + d)^2} d\tau, \frac{1}{c\tau + d} dz_2 - \frac{cz_2 - kd + lc}{(c\tau + d)^2} d\tau \right). \end{aligned}$$

The action of the element $(\gamma, \lambda) \in \pi_1(M)$ on the dual basis is computed to be the following:

$$\begin{aligned} \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \right) \mapsto & \left((c\tau + d)^2 \frac{\partial}{\partial \tau} + (c\tau + d)(cz_1 - md + nc) \frac{\partial}{\partial z_1} \right. \\ & \left. + (c\tau + d)(cz_2 - kd + lc) \frac{\partial}{\partial z_2}, (c\tau + d) \frac{\partial}{\partial z_1}, (c\tau + d) \frac{\partial}{\partial z_2} \right). \end{aligned}$$

The expression for the general section of $S^2(T^*\widetilde{M}) \otimes T\widetilde{M}$ is a sum of type

$$\sum_{i,j,k=1}^3 f_{ij}^k dx_i dx_j \frac{\partial}{\partial x_k},$$

where x_1, x_2, x_3 are chosen among the global coordinates $(\tau, z_1, z_2) \in \mathcal{H} \otimes \mathbb{C}^2$ and f_{ij}^k are holomorphic functions defined on $\mathcal{H} \times \mathbb{C}^2$ such that $f_{ij}^k = f_{ji}^k$. Therefore, in global coordinates (τ, z_1, z_2) of the universal cover $\widetilde{M} = \mathcal{H} \times \mathbb{C}^2$, the expression for the general section of $S^2(T^*\widetilde{M}) \otimes T\widetilde{M}$ is a sum of 18 terms:

$$\begin{aligned} & f_{\tau,\tau}^{z_1} d\tau \otimes d\tau \otimes \frac{\partial}{\partial z_1} + f_{\tau,\tau}^{z_2} d\tau \otimes d\tau \otimes \frac{\partial}{\partial z_2} \\ & + f_{\tau,\tau}^{\tau} d\tau \otimes d\tau \otimes \frac{\partial}{\partial \tau} + f_{z_1,\tau}^{z_1} dz_1 \otimes d\tau \otimes \frac{\partial}{\partial z_1} + f_{z_2,\tau}^{z_2} dz_2 \otimes d\tau \otimes \frac{\partial}{\partial z_2} + \dots \end{aligned}$$

This tensor is trace-free if and only if the following three conditions hold:

$$\begin{aligned} f_{z_1,z_1}^{z_1} + f_{z_1,\tau}^{\tau} + f_{z_1,z_2}^{z_2} &= 0, \\ f_{z_2,z_1}^{z_1} + f_{z_2,\tau}^{\tau} + f_{z_2,z_2}^{z_2} &= 0, \\ f_{\tau,z_1}^{z_1} + f_{\tau,\tau}^{\tau} + f_{\tau,z_2}^{z_2} &= 0. \end{aligned}$$

The action of any $\lambda = (m, n, k, l) \in \mathbb{Z}^{\oplus 4} = \Lambda$ on \widetilde{M} is given by

$$\lambda \cdot (\tau, z_1, z_2) \longmapsto (\tau, z_1 + m\tau + n, z_2 + k\tau + l)$$

(see (4.2)). In particular, the action of the normal subgroup $\Lambda \subset \pi_1(M)$ is trivial on the τ -coordinate and it preserves the fibration defined by the projection

$$\mathcal{H} \times \mathbb{C}^2 \longrightarrow \mathcal{H}. \quad (4.3)$$

The symbol functions f_{ij}^k are evidently invariant under the action of Λ . In particular, the functions f_{ij}^k are constants on the fibers of the projection in (4.3), i.e., every f_{ij}^k depends only of the parameter τ .

We now compute the components of the tensor Θ enforcing the invariance condition under the action $\pi_1(M)$.

Take any $(\gamma, \lambda) \in \pi_1(M)$. Identifying the coefficient of $dz_1 \otimes dz_2 \otimes \frac{\partial}{\partial z_1}$ in the expressions of Θ and $(\gamma, \lambda)^* \Theta$ we get:

$$f_{z_1, z_2}^{z_1}(\tau) = \frac{1}{c\tau + d} f_{z_1, z_2}^{z_1} \left(\frac{a\tau + b}{c\tau + d} \right).$$

This implies that $f_{z_1, z_2}^{z_1}$ is a holomorphic section of $K_{\Sigma}^{\frac{1}{2}}$, meaning the holomorphic weighted-form $f_{z_1, z_2}^{z_1}(\tau) d\tau$ descends to M and the descended section coincides with the pull-back of a holomorphic section of $K_{\Sigma}^{\frac{1}{2}}$ through the Kuga–Shimura fibration in (2.3).

Identifying the coefficient of $d\tau \otimes d\tau \otimes \frac{\partial}{\partial z_1}$ in the expressions of Θ and $(\gamma, \lambda)^* \Theta$ we get:

$$\begin{aligned} f_{\tau, \tau}^{z_1}(\tau) &= \frac{1}{(c\tau + d)^3} f_{\tau, \tau}^{z_1} \left(\frac{a\tau + b}{c\tau + d} \right) + \frac{1}{(c\tau + d)^3} f_{z_1, z_1}^{z_1} \left(\frac{a\tau + b}{c\tau + d} \right) (cz_1 - md + nc)^2 + \\ &\frac{1}{(c\tau + d)^3} f_{\tau, \tau}^{\tau} \left(\frac{a\tau + b}{c\tau + d} \right) (cz_1 - md + nc) + \frac{1}{(c\tau + d)^3} f_{z_1, z_1}^{\tau} \left(\frac{a\tau + b}{c\tau + d} \right) (cz_1 - md + nc)^3 \\ &\quad + \frac{1}{(c\tau + d)^3} f_{z_2, z_2}^{z_1} \left(\frac{a\tau + b}{c\tau + d} \right) (cz_2 - kd + lc)^2 \\ &\quad + 2 \frac{1}{(c\tau + d)^3} f_{z_1, z_2}^{\tau} \left(\frac{a\tau + b}{c\tau + d} \right) (cz_1 - md + nc)^2 (cz_2 - kd + lc) \\ &\quad + 2 \frac{1}{(c\tau + d)^3} f_{z_1, \tau}^{z_1} \left(\frac{a\tau + b}{c\tau + d} \right) (-cz_1 + md - nc) + 2 \frac{1}{(c\tau + d)^3} f_{z_2, \tau}^{z_1} \left(\frac{a\tau + b}{c\tau + d} \right) (-cz_2 + kd - lc) \\ &\quad - 2 \frac{1}{(c\tau + d)^3} f_{z_1, \tau}^{\tau} \left(\frac{a\tau + b}{c\tau + d} \right) (cz_1 - md + nc)^2 - 2 \frac{1}{(c\tau + d)^3} f_{z_2, \tau}^{\tau} \left(\frac{a\tau + b}{c\tau + d} \right) (cz_1 - md + nc) (cz_2 - kd + lc) \\ &\quad + 2 \frac{1}{(c\tau + d)^3} f_{z_1, z_2}^{z_1} \left(\frac{a\tau + b}{c\tau + d} \right) (cz_1 - md + nc) (cz_2 - kd + lc). \end{aligned}$$

This yields

$$\begin{aligned} f_{z_1, z_1}^{\tau} &= f_{z_1, z_2}^{\tau} = f_{z_2, z_1}^{\tau} = f_{z_2, \tau}^{z_1} = f_{\tau, z_2}^{z_1} = f_{z_1, \tau}^{z_2} = f_{\tau, z_1}^{z_2} = 0, \\ f_{z_1, z_2}^{z_1} &= f_{z_2, z_1}^{z_1} = f_{\tau, z_2}^{\tau} = f_{z_2, \tau}^{\tau}, \\ f_{z_1, z_1}^{z_1} &= 2f_{z_1, \tau}^{\tau} = 2f_{\tau, z_1}^{\tau}, \\ f_{\tau, \tau}^{\tau} &= 2f_{z_1, \tau}^{z_1} = 2f_{\tau, z_1}^{z_1}. \end{aligned}$$

Also, we get that $f_{\tau,\tau}^{z_1}$ is a holomorphic section of $K_\Sigma^{\frac{3}{2}}$, meaning the holomorphic weighted-form $f_{\tau,\tau}^{z_1}$ descends to M and the descended section coincides with the pull-back of a holomorphic section of $K_\Sigma^{\frac{3}{2}}$ through the Kuga–Shimura fibration in (2.3).

Performing the same computation for the coefficient of $d\tau \otimes d\tau \otimes \frac{\partial}{\partial z_2}$ we get that

$$\begin{aligned} f_{z_2,z_2}^\tau &= 0, \\ f_{z_1,z_2}^{z_2} &= f_{z_2,z_1}^{z_2} = f_{\tau,z_1}^\tau = f_{z_1,\tau}^\tau, \\ f_{z_2,z_2}^{z_2} &= 2f_{z_2,\tau}^\tau = 2f_{\tau,z_2}^\tau, \\ f_{\tau,\tau}^\tau &= 2f_{z_2,\tau}^{z_2} = 2f_{\tau,z_2}^{z_2}. \end{aligned}$$

We also get that $f_{\tau,\tau}^{z_2}$ is a holomorphic section of $K_\Sigma^{\frac{3}{2}}$ in the sense explained above.

We now identify the coefficient of $d\tau \otimes d\tau \otimes \frac{\partial}{\partial \tau}$. Since we already know that $f_{z_1,z_1}^\tau = f_{z_2,z_2}^\tau = f_{z_1,z_2}^\tau = 0$, the equation is:

$$\begin{aligned} f_{\tau,\tau}^\tau(\tau) &= f_{\tau,\tau}^\tau \left(\frac{a\tau + b}{c\tau + d} \right) \frac{1}{(c\tau + d)^2} + 2f_{z_1,\tau}^\tau \left(\frac{a\tau + b}{c\tau + d} \right) \frac{1}{(c\tau + d)^2} (-cz_1 + md - nc) \\ &\quad + 2f_{z_2,\tau}^\tau \left(\frac{a\tau + b}{c\tau + d} \right) \frac{1}{(c\tau + d)^2} (-cz_2 + kd - lc). \end{aligned}$$

This implies that $f_{z_1,\tau}^\tau = f_{z_2,\tau}^\tau = 0$, and $f_{\tau,\tau}^\tau$ is a holomorphic section of K_Σ in the sense explained above.

Let us now identify the coefficient of $dz_1 \otimes dz_1 \otimes \frac{\partial}{\partial z_1}$. Since $f_{z_1,z_2}^\tau = 0$, we get that

$$f_{z_1,z_1}^{z_1}(\tau) = f_{z_1,z_1}^{z_1} \left(\frac{a\tau + b}{c\tau + d} \right) \frac{1}{(c\tau + d)}.$$

It follows that $f_{z_1,z_1}^{z_1}$ is a holomorphic section of $K_\Sigma^{\frac{1}{2}}$ in the sense explained above.

Consider the coefficient of $dz_2 \otimes dz_2 \otimes \frac{\partial}{\partial z_2}$; we obtain the same result for $f_{z_2,z_2}^{z_2}$.

The conclusion is that the only non-vanishing coefficients are:

$$\begin{aligned} f_{\tau,\tau}^\tau &= 2f_{z_1,\tau}^{z_1} = 2f_{z_2,\tau}^{z_2} \in H^0(\Sigma, K_\Sigma); \\ f_{\tau,\tau}^{z_1}, f_{\tau,\tau}^{z_2} &\in H^0(\Sigma, K_\Sigma^{\frac{3}{2}}). \end{aligned}$$

Therefore, we have a canonical identification

$$H^0(\Sigma, K_\Sigma^{\frac{3}{2}})^{\oplus 2} \oplus H^0(\Sigma, K_\Sigma) \xrightarrow{\sim} H^0(\widetilde{M}, S^2(T^*\widetilde{M}) \otimes T\widetilde{M})^{\pi_1(M)}. \quad (4.4)$$

For $A, B \in H^0(\Sigma, K_\Sigma^{\frac{3}{2}})$ and $C \in H^0(\Sigma, K_\Sigma)$, the corresponding element

$$\Theta_{A,B,C} \in H^0(\widetilde{M}, S^2(T^*\widetilde{M}) \otimes T\widetilde{M})^{\pi_1(M)} \quad (4.5)$$

by the isomorphism in (4.4) is given by

$$f_{\tau,\tau}^{z_1} = A, \quad f_{\tau,\tau}^{z_2} = B, \quad f_{\tau,\tau}^\tau = 2f_{z_1,\tau}^{z_1} = 2f_{z_2,\tau}^{z_2} = C.$$

It can be shown that $\nabla^{A,B,C}$ and $\nabla^{A,B,0}$ are projectively equivalent. Indeed, define ϕ_τ to be the holomorphic one-form on $\mathcal{H} \times \mathbb{C}^2$ such that

$$\phi_\tau \left(\frac{\partial}{\partial \tau} \right) = \frac{1}{2}C \quad \text{and} \quad \phi_\tau \left(\frac{\partial}{\partial z_i} \right) = 0$$

for $i = 1, 2$. It follows that

$$\nabla_X^{A,B,C} Y - \nabla_X^{A,B,0} Y = \phi_\tau(X)(Y) + \phi_\tau(Y)X \quad (4.6)$$

for all locally defined holomorphic vector fields X, Y ; see (3.9). From (4.6) it follows that $\nabla^{A,B,C}$ and $\nabla^{A,B,0}$ are projectively equivalent.

For the section $\Theta_{A,B,C}$ in (4.5), we have $\operatorname{div}(\Theta_{A,B,C}) = 2Cd\tau$, where div is constructed as in (3.1). Consequently, the trace-free condition $\operatorname{div}(\Theta_{A,B,C}) = 0$ holds if and only if

$$C = 0.$$

In fact, the trace-free part of $\Theta^{A,B,C}$ is

$$\mathbb{F}(\Theta_{A,B,C}) = \Theta_{A,B,0} \in H^0(\widetilde{M}, (S^2(T^*\widetilde{M}) \otimes T^*\widetilde{M})_0),$$

where \mathbb{F} is the projection in (3.6).

Therefore, from the isomorphism in (4.4) we have a canonical identification

$$H^0(\Sigma, K_\Sigma^{\frac{3}{2}})^{\oplus 2} \xrightarrow{\sim} H^0(\widetilde{M}, (S^2(T^*\widetilde{M}) \otimes T^*\widetilde{M})_0)^{\pi_1(M)} \quad (4.7)$$

that sends any $(A, B) \in H^0(K_\Sigma^{\frac{3}{2}})^{\oplus 2}$ to $\Theta_{A,B,0}$ in (4.5).

Recall that $\nabla^{A,B,0} = \nabla_0 + \Theta_{A,B,0}$ is a $\pi_1(M)$ -invariant holomorphic projective connection on \widetilde{M} . Therefore, it descends to a holomorphic projective connection $\nabla^{A,B}$ on M . Moreover, we have seen that the space of projective equivalence classes of holomorphic projective connections on M is identified with $H^0(\widetilde{M}, (S^2(T^*\widetilde{M}) \otimes T^*\widetilde{M})_0)^{\pi_1(M)}$. Therefore the projective connection $\nabla^{A,B}$ is projectively equivalent to $\nabla^{A',B'}$ if and only if $(A, B) = (A', B')$; we take the base projective connection $\nabla^{0,0}$ to be the standard flat projective connection induced by the open embedding of \widetilde{M} in $\mathbb{C}\mathbb{P}^3$. \square

Remark 15. There are Kuga–Shimura projective threefolds over Shimura (compact) curves of arbitrarily large genus (see, for instance, [KV, Section 5]). Using Riemann-Roch theorem we deduce that $\dim H^0(\Sigma, K_\Sigma^{\frac{3}{2}})^{\oplus 2} = 4(\operatorname{genus}(\Sigma) - 1)$, so it can be arbitrarily large. This implies that the space of projective classes of holomorphic projective connections on a Kuga–Shimura projective threefold can have dimension arbitrarily large. The next Proposition 16 shows that all these holomorphic projective connections are flat.

Proposition 16. *Let M be a projective Kuga–Shimura threefold which fibers over a Shimura curve Σ . Then all holomorphic projective connections ϕ on M are flat. The fibers of the Kuga–Shimura fibration are (flat) totally geodesic for ϕ .*

Proof. Take a holomorphic projective connection ϕ on M . Let

$$(A, B) \in (H^0(K_\Sigma^{\frac{3}{2}})^{\oplus 2})$$

be the pair associated to ϕ by Proposition 14. Recall that $\varpi^*\phi$ (see (4.1)) is represented by a holomorphic affine connection

$$\nabla^{A,B} := \nabla^{A,B,0} = \nabla_0 + \Theta_{A,B},$$

where $\Theta_{A,B} := \Theta_{A,B,0} \in H^0(\widetilde{M}, (S^2(T^*\widetilde{M}) \otimes T^*\widetilde{M})_0)^{\pi_1(M)}$ corresponds to the Christoffel symbols

$$(f_{\tau,\tau}^{z_1}, f_{\tau,\tau}^{z_2}) = (A, B) \in H^0(\Sigma, K_\Sigma^{\frac{3}{2}})^{\oplus 2}$$

(see (4.5)). As noted in the proof of Proposition 14, the holomorphic weighted-forms $f_{\tau,\tau}^{z_1}(\tau)d\tau$ (respectively, $f_{\tau,\tau}^{z_2}(\tau)d\tau$) defined on $\widetilde{M} = \mathcal{H} \times \mathbb{C}^2$ descends to M as the holomorphic section A (respectively, B) of $K_{\Sigma}^{\frac{3}{2}}$.

We now compute the curvature of the associated affine connection $\nabla^{A,B} = \nabla_0 + \Theta_{A,B}$.

This computation is formally the same as the computation of the curvature of the connection $\nabla^{A,B,0,0,0}$ in the proof of Proposition 12. The only difference is that here A and B depend on the variable τ , while in the proof of Proposition 12 the parameters A and B are two constants. Nevertheless, the curvature tensor $R(X, Y)Z$ being anti-symmetric in variables (X, Y) we have $R(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial\tau}) = 0$; note that $R(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial\tau})$ is an endomorphism of $T^*\widetilde{M}$. Therefore, the components of the affine curvature tensor R do not depend on the derivatives of the functions $\tau \mapsto A(\tau)$ and $\tau \mapsto B(\tau)$.

It follows that the computation of the curvature tensor R is the same as for $\nabla^{A,B,0,0,0}$ in the proof of Proposition 12. In the case where $C = D = E = 0$, the conclusion of the computations in the proof of Proposition 12 is that the tensor R , and hence W , vanishes identically. This proves that $\nabla^{A,B}$ is projectively flat.

Moreover, $\nabla^{A,B}$ preserves the holomorphic two-dimensional foliation \mathcal{F} of $\widetilde{M} = \mathcal{H} \times \mathbb{C}^2$ defined, in global coordinates (τ, z_1, z_2) , by $d\tau = 0$. The connection $\nabla^{A,B}$ coincide with ∇_0 when restriction to \mathcal{F} . More precisely, each leaf of \mathcal{F} is a fiber $\{h\} \times \mathbb{C}^2 \subset \mathcal{H} \times \mathbb{C}^2$, and the restrictions of $\nabla_{A,B}$ and ∇_0 to each fiber coincide. Observe that the projection on M of the \mathcal{F} -leaves are exactly the fibers of the Kuga–Shimura fibration in (2.3). \square

4.1. Proofs of Theorem 2 and Corollary 3.

Proof of Theorem 2. Let M be a Kuga–Shimura projective threefold. Proposition 14 provides a proof of statement (i) of the theorem, showing that the space of the projective classes of holomorphic projective connections on M is identified with a complex affine space for the $H^0(\Sigma, K_{\Sigma}^{\frac{3}{2}})^{\oplus 2}$.

Proposition 16 proves statement (ii). \square

Now we can deduce Corollary 3.

Proof of Corollary 3. Let M be a projective threefold endowed with a holomorphic projective connection ϕ . The main result in [JR1] proves that M is either a Kähler–Einstein threefold (and hence one of the standard examples [KO1, KO2]), or it is an étale quotient of a Kuga–Shimura projective threefold. Theorem 2 shows that on Kuga–Shimura projective threefolds (and hence also on their étale quotients), all holomorphic projective connections are flat. Theorem 1 implies that on Kähler–Einstein projective threefolds ϕ is flat, except for abelian varieties (and their étale quotients) on which ϕ is translation invariant (but generically not flat). \square

5. SIMPLY CONNECTED COMPLEX THREEFOLDS WITH TRIVIAL CANONICAL BUNDLE

In this final section we deal with compact complex manifolds of dimension three. For any cocompact lattice $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$, the quotient $\mathrm{SL}(2, \mathbb{C})/\Gamma$ is compact non-Kähler threefold with trivial tangent bundle admitting a flat holomorphic projective connection.

Ghys constructed their deformations that have canonical bundle trivial and admit a flat holomorphic projective connection [Gh] (see also [BD1, Section 5]).

A simply connected manifold M with trivial canonical bundle does not admit a flat holomorphic projective connection. Indeed, the developing map of such a holomorphic projective structure would realize a biholomorphism between M and the complex projective space (which has nontrivial first Chern class). For dimension three, the following stronger result holds.

Theorem 17. *A simply connected compact complex threefold with trivial canonical bundle does not admit any holomorphic projective connection.*

To prove Theorem 17, we will make use of the theory of *rigid geometric structure* as developed in [Grom, DG]. A holomorphic torsionfree affine connection is known to be rigid (of order one) in the sense of [Grom, DG] because a local automorphism of the connection is completely determined by its underlying one-jet at any given point (i.e., the differential sends parametrized holomorphic geodesic curves to parametrized holomorphic geodesic curves). A holomorphic projective connection is known to be rigid (of order two) in the sense of [Grom, DG] because a local automorphism of the connection is completely determined by its underlying two-jet at any given point.

The algebraic dimension $a(N)$ of a compact complex manifold N is the transcendence degree of the field of meromorphic functions $\mathbb{C}(N)$ over \mathbb{C} (see [Ue, p. 24, Chapter 3]). We have $a(N) \leq \dim N$, and $a(N) = \dim N$ if and only if N is bimeromorphic to a complex projective variety [Moi]. The manifolds with maximal algebraic dimension are called *Moishezon manifolds*.

The Killing Lie algebra of a holomorphic rigid geometric structure (here the geometric structure is a holomorphic projective or affine connection) on a compact complex manifold N of complex dimension n has generic orbits of complex dimension $\geq n - a(N)$ [Du1, p. 568, Theorem 3]. This result will be useful in the proof of the following propositions.

Proposition 18. *Any holomorphic projective connection ϕ on a compact complex threefold M admits a nontrivial Killing Lie algebra.*

Proof. Assume, by contradiction, that the Killing Lie algebra for ϕ is trivial. Now Theorem 3 in [Du1] says that the manifold M is Moishezon. But Moishezon manifolds bearing a holomorphic Cartan geometry (in particular, a holomorphic affine connection [Sha]) are known to be projective (see Corollary 2 in [BM1]).

On the other hand, Corollary 3 implies that the Killing Lie algebra of a holomorphic projective connection on a projective threefold is transitive, hence it has dimension at least three: a contradiction. \square

Proposition 19. *Let M be a compact complex manifold with trivial canonical bundle bearing a holomorphic projective connection ϕ . Then the following five hold:*

- (i) *The Killing Lie algebra of ϕ is nontrivial.*

In the following four, assume that M is simply connected.

- (ii) *The automorphism group $\text{Aut}(M, \phi)$ of (M, ϕ) is a complex Lie group of positive dimension.*

- (iii) *A maximal connected abelian complex subgroup A in $\text{Aut}(M, \phi)$ has positive dimension.*
- (iv) *The A -orbits in M coincide with those of the maximal (real) compact subgroup $K(A) \subset A$; they all are compact complex tori.*
- (v) *The A -action on M does not admit any fixed point.*

Proof. Denote by n the complex dimension of M . Consider a holomorphic volume form ω on M ; this means that ω is a holomorphic trivializing section of the canonical bundle K_M . Lemma 7, and its proof, imply that there is a unique torsionfree affine connection ∇ on M such that

- ∇ is projectively equivalent to ϕ , and
- ω is parallel with respect to ∇ .

Let us first prove that that M does not admit any nontrivial rational curve.

Take a holomorphic map $f : \mathbb{C}\mathbb{P}^1 \rightarrow M$ and consider the pull-back f^*TM equipped with the holomorphic connection $f^*\nabla$. The connection $f^*\nabla$ is flat (because its curvature is a (bundle valued) holomorphic two-form). This implies that f^*TM is holomorphically trivial, because $\mathbb{C}\mathbb{P}^1$ is simply connected. Since $\text{degree}(T\mathbb{C}\mathbb{P}^1) > 0$, there is no nonzero homomorphism from $T\mathbb{C}\mathbb{P}^1$ to the trivial bundle f^*TM . In particular, the differential df of f vanishes identically. This implies that f is a constant map.

(i) To prove by contradiction, assume that the Killing Lie algebra of ϕ is trivial (in particular the Killing Lie algebra of ∇ is also trivial). Then Theorem 3 in [Du1] implies that M is a Moishezon manifold. But Moishezon manifolds with no rational curves are known to be projective [Cas, p. 307, Theorem 3.1]. A complex projective manifold bearing a holomorphic affine connection ∇ is covered by an abelian variety [IKO]. Moreover, the pull-back of ∇ to the covering abelian variety is translation invariant (see Proposition 12). In particular, ∇ is locally homogeneous, and hence ϕ is also locally homogeneous; so the Killing Lie algebra of ϕ is transitive on M : a contradiction.

Now assume that M is simply connected.

(ii) Each element in the Killing Lie algebra for ϕ extends to a global holomorphic Killing vector field for ϕ (defined on entire M): this was first proved by Nomizu in [No] for Killing vector fields of analytic Riemannian metrics, and subsequently it was generalized to G -structures [Am], to rigid geometric structures [DG, Grom] and also to Cartan geometries [Me, Pe]. This implies that there is a nontrivial connected complex Lie group G acting by biholomorphisms on M that preserves the holomorphic projective connection ϕ . This group G coincides with the connected component, containing the identity element, of the automorphism group $\text{Aut}(M, \phi)$. The Lie algebra $\text{Lie}(G)$ of G is the Lie algebra of global holomorphic Killing vector fields for ϕ .

iii) The nonzero holomorphic section ω of the canonical bundle K_M defines a smooth real volume form on M given by $(\sqrt{-1})^n \cdot \omega \wedge \bar{\omega}$.

We will prove that the action of the group G preserves the smooth measure $(\sqrt{-1})^n \cdot \omega \wedge \bar{\omega}$ on M . To prove this, consider a holomorphic Killing vector field $X \in \text{Lie}(G)$. The Lie derivative $L_X\omega$ of ω is a holomorphic section of K_M . So there is a constant $c \in \mathbb{C}$ such that $L_X\omega = c \cdot \omega$. Hence, if Ψ^t is the one-parameter subgroup of G generated by X , we

get that

$$(\Psi^t)^*\omega = \exp(ct) \cdot \omega$$

for all $t \in \mathbb{C}$; recall that any holomorphic vector field on a compact manifold is complete, and therefore its flow is defined on all of \mathbb{C} . Since the total volume $\int_M (\sqrt{-1})^n \cdot \omega \wedge \bar{\omega}$ of the manifold M is invariant by any automorphism, it follows that $|\exp(ct)| = 1$ for all $t \in \mathbb{C}$. By Liouville Theorem, the entire function $t \mapsto \exp(ct)$ must be constant and equal to 1 (the value of the function at $t = 0$ being 1). This implies that $c = 0$, and ω is X -invariant. Since the complex Lie group G is connected, it is generated by the flows of its fundamental vector fields. It follows that every element of G preserves the volume form.

Moreover since G preserves the holomorphic volume form ω and the holomorphic projective connection ϕ , the action of G also preserves the associated torsionfree holomorphic affine connection ∇ representing ϕ ; it was observed in the proof of Lemma 7 that ∇ is canonically associated to ϕ and ω .

Let us now apply the Gromov abelianization trick (see [DG, Section 3.2.A] or [Grom]) and consider the rigid geometric structures which is a juxtaposition of the holomorphic projective connection ϕ with a family of global holomorphic vector fields $\{X_1, \dots, X_k\} \in H^0(M, TM)$ forming a basis of the Lie algebra of G , seen as a subalgebra of TM .

Denote by A the connected component of the identity element in the automorphism group of the holomorphic rigid geometric structure

$$\phi' = (\phi, X_1, \dots, X_k).$$

Then A is a maximal connected abelian complex Lie subgroup in $\text{Aut}(M, \phi)$ (see [DA, Section 3.1 Lemma], for more details).

Applying [Du1, Theorem 3] to ϕ' it is deduced that A acts with generic orbits of complex dimension at least $n - a(M) > 0$ (as above, M is not Moishezon, so $a(M) < n$). Indeed, we have seen in the proof of point (i) that M Moishezon implies M is covered by an abelian variety: a contradiction (since M is simply connected).

(iv) Since A preserves a smooth measure on M , its orbits are compact and coincide with the orbits of its maximal (real) compact subgroup $K(A) \subset A$ (see [Grom, Section 3.7] and [DG, Section 3.5.4]).

Choose a point $m_0 \in M$, and consider its A -orbit Am_0 . Then Am_0 is biholomorphic to the homogeneous space A/A_{m_0} , where A_{m_0} is the complex subgroup of A that fixes m_0 . Any basis of the quotient space $\text{Lie}(A)/\text{Lie}(A_{m_0})$ is invariant by the adjoint representation of A_{m_0} (because A is abelian) and provides a holomorphic trivialization of the holomorphic tangent bundle of the homogeneous space A/A_{m_0} . So A/A_{m_0} is a compact parallelizable manifold [Wa]. Moreover, the holomorphic tangent bundle is trivialized by commuting vector fields. Therefore A/A_{m_0} is a compact complex torus. Consequently, all A -orbits are compact complex tori (notice that some orbits could be of dimension zero: these are fixed points of the A -action).

(v) To prove this by contradiction, assume that $m_0 \in M$ is fixed by the action of A on M . To any $g \in A$ associate its differential $dg(m_0) \in \text{GL}(T_{m_0}M)$; this gives the isotropy homomorphism $i : A \rightarrow \text{GL}(T_{m_0}M)$ at m_0 . Moreover, since A preserves the holomorphic torsionfree affine connection ∇ , the A action on M is linearizable in local holomorphic ∇ -exponential coordinates in the neighborhood of m_0 . More precisely, there

exists an open neighborhood U of $0 \in T_{m_0}M$ and an open neighborhood U' of $m_0 \in M$ and a biholomorphism $\beta : U \rightarrow U'$, such that β intertwines the actions of $i(A)$ on U and of A on U' . In particular, the homomorphism i is injective (i.e., the isotropy representation of A is faithful).

It is known that $i(A) \subset \mathrm{GL}(T_{m_0}M)$ is a complex algebraic subgroup; this is because $i(A)$ coincides with the stabilizer of a k -jet of the rigid geometric structure ϕ' (for $k \in \mathbb{N}$ large enough) and the $\mathrm{GL}(T_{m_0}M)$ -action on the space of k -jets of ϕ' at m_0 is algebraic (see [Grom, Sections 3.5 and 3.7] and [DG, Sections 3.2A and 3.5] or [Me, Theorem 3.11]).

As before, $K(A) \subset A$ is the maximal compact subgroup. Let $K(A)^0$ be the connected component of $K(A)$ containing the identity element. It is isomorphic to $(S^1)^\ell = \mathrm{U}(1)^\ell$ for some $\ell \geq 1$. Let $i(K(A))_{\mathbb{C}} \subset i(A)$ be the complex Zariski closure of $i(K(A)^0)$. This group $i(K(A))_{\mathbb{C}}$ is isomorphic to $(\mathbb{C}^*)^\ell$. We deduce that $T_{m_0}M$ splits as a direct sum

$$T_{m_0}M = L_1 \oplus L_2 \oplus \dots \oplus L_n$$

of complex $i(K(A))_{\mathbb{C}}$ -invariant lines, such that $i(K(A))_{\mathbb{C}}$ acts on each L_j , $1 \leq j \leq n$, through a multiplicative character

$$\chi_j : (\mathbb{C}^*)^\ell \rightarrow \mathbb{C}^*, \quad (t_1, \dots, t_\ell) \mapsto t_1^{n_1^j} \cdot t_2^{n_2^j} \cdot \dots \cdot t_\ell^{n_\ell^j}$$

defined by a given $(n_1^j, \dots, n_\ell^j) \in \mathbb{Z}^\ell$.

Since the isotropy representation i is faithful, it follows that at least one of the characters $\{\chi_1, \chi_2, \dots, \chi_n\}$ is nontrivial. Recall that β intertwines the actions of $i(K(A))_{\mathbb{C}}$ on U and A on U' . Then there exists points in $U' \setminus \{m_0\}$ such that the A -orbit of any of them accumulates at the fixed point m_0 . This contradicts the fact that the A -orbits in M are compact. Therefore, the A -action on M does not have any fixed points. \square

Let us now give a proof of Theorem 17.

Proof of Theorem 17. To prove by contradiction, assume that there is a complex compact threefold M with trivial canonical bundle K_M and bearing a holomorphic projective connection ϕ . We are exactly in the situation described by Proposition 19. We keep the same notations as in Proposition 19. In particular, we have a holomorphic torsionfree affine connection ∇ on M representing ϕ , which is preserved by the action of the nontrivial connected abelian group A of automorphisms.

It was proved in Proposition 19(iv) that all A -orbits are biholomorphic to compact complex tori. Since M is not homeomorphic to a compact complex torus (it is simply connected), we deduce that the generic A -orbits are either of complex dimension two, or of complex dimension one.

These two cases will be dealt separately.

Case of generic A -orbits being of complex dimension two. Take holomorphic Killing vector fields (X_1, X_2) which span the tangent space to the A -orbit at a generic point. As before, let ω be a nonzero holomorphic section of K_M . We have the holomorphic one-form θ on M defined by

$$\theta(x)(v) = \omega(x)(v, X_1(x), X_2(x))$$

for all $x \in M$ and $v \in T_xM$. This one-form θ is A -invariant and it vanishes on the A -orbits. Since the kernel of θ coincides, at the generic point, with the holomorphic tangent

space of the foliation defined by the A -action, the one-form θ satisfies the Frobenius integrability condition $\theta \wedge d\theta = 0$. Moreover, it can be proved that this nontrivial one-form θ is closed. See the proof of Theorem 4.4 in [BD2] where it is shown that θ is closed because it is projectable on a compact curve; this can also be deduced from the description of non-closed integrable one-forms on threefolds given in [Br, Proposition 3]. This implies that $H^1(M, \mathbb{C}) \neq 0$, and hence the abelianization of the fundamental group of M is infinite: a contradiction.

Case of generic A -orbits being of complex dimension one. Proposition 19(v) proves that the A -action on M does not have any fixed points. It follows from Proposition 19(iv) that all A -orbits are elliptic curves, on which $K(A)$ acts transitively.

We will prove that $K(A)$ acts freely on M .

To prove this, take any $m_0 \in M$, and let $I(m_0) \subset K(A)$ be the stabilizer of m_0 for the action of $K(A)$ on M . Then $I(m_0)$ is a compact abelian group fixing m_0 . Its action linearizes in local holomorphic coordinates at m_0 . For any $k \in I(m_0)$, since K is abelian, the differential $dk(m_0)$ acts trivially on $T_{m_0}(K(A)(m_0)) = T_{m_0}(Am_0)$; recall that any A -orbit is also a $K(A)$ -orbit. On the other hand, the differential $dk(m_0)$ acts trivially on the quotient space $T_{m_0}M/(T_{m_0}(Am_0))$, because any element of A (in particular k) fixes (globally) each A -orbit (i.e., it acts trivially on the space of A -orbits). Since compact groups are reductive, it follows that the differential of k is trivial. The isotropy representation at m_0 being faithful (see the proof of Proposition 19(v)), this implies that k is the identity element. Consequently, $I(m_0)$ is trivial, and the $K(A)$ -action on M is free.

It follows that M is the total space of a real principal $K(A)$ -bundle over a smooth real manifold $B = M/K(A)$. The $K(A)$ -orbits are complex manifolds, because they are A -orbits. This implies that B is also a complex manifold and the projection

$$\delta : M \longrightarrow B = M/K(A) \tag{5.1}$$

is a holomorphic submersion whose fibers are elliptic curves.

We will now prove that the fibration δ in (5.1) is a holomorphic principal elliptic curve bundle over B .

The space of elliptic curves C together with a symplectic basis of $H^1(C, \mathbb{Z})$ is parametrized by Poincaré upper half-plane \mathcal{H} . The base B in (5.1) is simply connected, because M is so (and the fibers of δ are connected). Therefore, fixing a point $b_0 \in B$ and a symplectic basis of $H^1(\delta^{-1}(b_0), \mathbb{Z})$, we get a holomorphic map

$$\Phi : B \longrightarrow \mathcal{H}$$

for the family of elliptic curves in (5.1). Since B is compact, this Φ is a constant function.

Therefore, the fibration δ in (5.1) is isotrivial. By the fundamental result of Fischer and Grauert δ is a holomorphic bundle. Moreover since the fibers of δ are isomorphic elliptic curves on which $K(A)$ acts freely and transitively (by biholomorphisms), for any point $m_0 \in M$, the orbital map $K(A) \longrightarrow K(A)m_0$ induces on $K(A)$ the same complex structure (that of the fiber type of δ). Hence $K(A)$ gets the complex structure of an elliptic curve for which the $K(A)$ -action on M is holomorphic; this elliptic curve will be denoted by $\mathcal{K}(A)$. Therefore δ is a holomorphic principal $\mathcal{K}(A)$ -bundle.

Since δ is a holomorphic principal $\mathcal{K}(A)$ -bundle, and K_M is holomorphically trivial, it follows that K_B is holomorphically trivial. It was noted above that B is simply connected. So B is a K3 surface.

Theorem 1.1 of [BD3] implies that the holomorphic affine connection on M is locally homogeneous, and hence the fundamental group of M is infinite [BD3, Corollary 1.1]: a contradiction. Hence M does not admit any holomorphic projective connection. \square

We conjecture that a simply connected compact complex manifold bearing a holomorphic projective connection is isomorphic to the complex projective space (endowed with its standard flat projective connection). In particular, we conjecture that simply connected compact complex manifolds with trivial canonical bundle do not admit any holomorphic projective connection. The second part of Proposition 19 which led to the proof of Theorem 17 should be seen as a step in this direction. Some other evidence in this direction was provided by the main result in [BD2] which says that simply connected compact complex manifolds do not admit holomorphic Riemannian metrics; in this case the canonical bundle is automatically trivialized by the volume form associated to the holomorphic Riemannian metric.

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