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# RESCALED ENTROPY OF CELLULAR AUTOMATA

DAVID BURGNET

ABSTRACT. For a  $d$ -dimensional cellular automaton with  $d \geq 1$  we introduce a rescaled entropy which estimates the growth rate of the entropy at small scales by generalizing previous approaches [1, 9]. We also define a notion of Lyapunov exponent and proves a Ruelle inequality as already established for  $d = 1$  in [16, 15]. Finally we generalize the entropy formula for 1-dimensional permutative cellular automata [18] to the rescaled entropy in higher dimensions. This last result extends recent works [17] of Shinoda and Tsukamoto dealing with the metric mean dimensions of two-dimensional symbolic dynamics.

## 1. INTRODUCTION

In this paper we estimate the dynamical complexity of multidimensional cellular automata. In the following the main results will be stated in a more general setting, but let us focus in this introduction on the following algebraic cellular automaton on  $(\mathbb{F}_p)^{\mathbb{Z}^d}$  with  $p$  prime given for some finite family  $(a_i)_{i \in I}$  in  $\mathbb{F}_p^*$  by

$$\forall (x_j)_j \in (\mathbb{F}_p)^{\mathbb{Z}^d}, f((x_j)_j) = \left( \sum_{i \in I} a_i x_{i+j} \right)_j.$$

Let  $I' = I \cup \{0\}$ . For  $d = 1$  the topological entropy of  $f$  is finite and equal to  $\text{diam}(I') \log p$  where  $\text{diam}(I')$  denotes the diameter of  $I'$  for the usual distance on  $\mathbb{R}$  [18]. However in higher dimensions the topological entropy of  $f$  is always infinite unless  $f$  is the identity map [13, 10]. Moreover the topological entropy of the  $\mathbb{Z}^{d+1}$ -action given by  $f$  and the shift vanishes. In this paper we investigate the growth rate of  $(h_{\text{top}}(f, \mathcal{P}_{J_n}))_n$  for nondecreasing sequences  $(J_n)$  of convex subsets of  $\mathbb{R}^d$  where  $(\mathcal{P}_{J_n})_n$  denotes the clopen partitions into  $J_n$ -coordinates with  $J_n := J_n \cap \mathbb{Z}^d$ . This sequence appears to increase as the perimeter  $p(J_n)$  of  $J_n$ . We define the rescaled entropy  $h_{\text{top}}^d(f)$  of  $f$  as  $\limsup_{J_n} \frac{h_{\text{top}}(f, \mathcal{P}_{J_n})}{p(J_n)}$ . In [9] another renormalization is used, whereas in [1] the authors only investigate the case of squares  $J_n = [-n, n]^2$ ,  $n \in \mathbb{N}$ . For  $d = 1$  we get  $h_{\text{top}}^1(f) = \frac{h_{\text{top}}(f)}{2}$ . We generalize the entropy formula for algebraic cellular automata as follows :

**Theorem 1.** *Let  $f$  be an algebraic cellular automaton on  $(\mathbb{F}_p)^{\mathbb{Z}^d}$  as above, then*

$$h_{\text{top}}^d(f) = R_{I'} \log p,$$

where  $R_{I'}$  denotes the radius of the smallest bounding sphere containing  $I'$ .

In fact we establish such a formula for any permutative cellular automaton (see Section 7). In [17] the authors compute, inter alia, the metric mean dimension of the horizontal shift in  $\mathbb{Z}^2$  for some standard distances. These dimensions may be interpreted as the rescaled entropy with respect to some particular sequence of convex sets  $(J_n)_n$ . In particular we extend these results in higher dimensions for general permutative cellular automata.

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We also consider a measure theoretical analogous quantity of the rescaled entropy. In dimension one, a notion of Lyapunov exponent has been defined in [15]. Then Tisseur [16] proved in this case a Ruelle inequality relating this exponent with the Kolmogorov-Sinai entropy. In this paper we also introduce a notion of Lyapunov exponent in higher dimensions, which bounds from above the rescaled entropy of measures.

The paper is organized as follows. In Section 2 we state some measure geometrical properties of convex sets in  $\mathbb{R}^d$ . We recall the dynamical background of cellular automata in Section 4 and we introduce then a Lyapunov exponent for multidimensional cellular automata. In Section 5 we define and study the topological and measure theoretical rescaled entropy. The last section is devoted to the proof of the entropy formula for permutative cellular automata.

## 2. BACKGROUND ON CONVEX GEOMETRY

**2.1. Convex bodies, domains and polytopes.** For a fixed positive integer  $d$  we endow the vector space  $\mathbb{R}^d$  with its usual Euclidean structure. The associated scalar product is simply denoted by  $\cdot$  and we let  $\mathbb{S}^d$  be the unit sphere. For a subset  $F$  of  $\mathbb{R}^d$  we let  $\bar{F}$ ,  $\text{Int}(F)$  and  $\partial F$  be respectively its closure, interior set and boundary. We denote by  $\underline{F} = F \cap \mathbb{Z}^d$  the set of integer points in  $F$ , i.e.  $\underline{F} = F \cap \mathbb{Z}^d$ . We also denote by  $V(F)$  the  $d$ -Lebesgue measure of  $F$  (also called the volume of  $F$ ) when the set  $F$  is Borel.

The extremal set of a convex set  $J$  is denoted by  $\text{ex}(J)$  and the convex hull of  $F \subset \mathbb{R}^d$  by  $\text{cv}(F)$ . A convex body is a compact convex set of  $\mathbb{R}^d$ . A convex body containing the origin  $0 \in \mathbb{R}^d$  in its interior set is said to be a **convex domain**. The set of convex bodies endowed with the Hausdorff topology is a locally compact metrizable space. In the following we denote by  $\mathcal{D}$ , resp.  $\mathcal{D}^1$ , the set of convex domains, resp. with unit perimeter, endowed with the Hausdorff topology. A **convex polytope** (resp.  $k$ -polytope with  $k \leq d$ ) in  $\mathbb{R}^d$  is a convex body given by the convex hull of a finite set (resp. with topological dimension equal to  $k$ ). When this set lies inside the lattice  $\mathbb{Z}^d$ , the convex polytope is said **integral**. We let  $\mathcal{F}(P)$  be the set of faces of a convex polytope  $P$ . For a convex body  $J$  we denote by  $\mathbb{J}$  the integral polytope given by the convex hull of integer points in  $J$ , i.e.  $\mathbb{J} = \text{cv}(\underline{J})$ .

A convex domain  $J$  has Lipschitz boundary and finite perimeter  $p(J)$ . For convex domains the perimeter in the distributional sense of De Giorgi coincides with the  $(d-1)$ -Hausdorff measure  $\mathcal{H}_{d-1}$  of the boundary. For  $J \in \mathcal{D}$  we let  $\partial'J$  be the subset of points  $x \in \partial J$ , where the tangent space  $T_x J$  is well defined. The set  $\partial'J$  has full  $\mathcal{H}_{d-1}$ -measure in  $\partial J$ . We let  $N^J(x) \in \mathbb{S}^d$  be the unit  $J$ -external normal vector at  $x \in \partial'J$ . For any  $x \in \partial'J$  we let  $T_x^+ J$  (resp.  $T_x^- J$ ) be the open external (resp. closed internal) semi-space with boundary  $T_x J$ . With these notations we have  $J = \bigcap_{x \in \partial'J} T_x^- J$ . For  $\epsilon \in \mathbb{R}$  we denote by  $T_x^\pm J(\epsilon)$  the semi-planes  $T_x^\pm J(\epsilon) = T_x^\pm J + \epsilon N^J(x)$ . When  $J$  is a convex polytope and  $F \in \mathcal{F}(J)$ , we write  $T_F$  to denote the tangent affine space supporting  $F$ ,  $T_F^\pm$  for the associated semi-spaces and  $N^F$  for the unit external normal to  $F$ .

The **support function** of a convex body  $I$  is the real continuous function  $h_I$  on  $\mathbb{S}^d$  :

$$\forall x \in \mathbb{S}^d, h_I(x) = \max_{u \in I} u \cdot x.$$

The support function completely characterizes the convex body  $I$ . The **area measure**  $\sigma_J$  of a convex domain  $J$  is the Borel measure on  $\mathbb{S}^d$  given by  $N_*^J \mathcal{H}_{d-1}$  :

$$\forall B \text{ Borel of } \mathbb{S}^d, \sigma_J(B) = \mathcal{H}_{d-1}((N^J)^{-1}B).$$

If a sequence  $(J_n)_n$  in  $\mathcal{D}$  is converging to  $J_\infty \in \mathcal{D}$  (for the Hausdorff topology), then  $\sigma_{J_n}$  is converging weakly to  $\sigma_{J_\infty}$ , in particular the perimeter of  $J_n$  goes to the perimeter of  $J_\infty$  (see Proposition 10.2 in [7]).

**2.2. Convex exhaustions.** We consider sequences  $\mathcal{J} = (J_n)_{n \in \mathbb{N}}$  of convex domains with  $p(J_n) \xrightarrow{n} +\infty$ , such that the sets  $\tilde{J}_n = p(J_n)^{-\frac{1}{d-1}} J_n \in \mathcal{D}^1$  are converging to a limit  $J_\infty \in \mathcal{D}$  in the Hausdorff topology. In particular  $\bigcup_n J_n = \mathbb{R}^d$ . Moreover the limit  $J_\infty$  has unit perimeter.

The sequences  $\mathcal{J} = (J_n)_n$  satisfying the above properties are said to be **convex exhaustions**. For  $O \in \mathcal{D}^1$  we denote by  $\mathcal{E}(O)$  the set of convex exhaustions  $\mathcal{J} = (J_n)_n$  with  $J_\infty = O$ . Moreover for  $O \in \mathcal{D}$  we let  $\mathcal{J}_O \in \mathcal{E}(p(O)^{-1}O)$  be the convex exhaustion given by  $\mathcal{J}_O := (nO)_n$ . A convex exhaustion is said integral when  $J_n$  is an integral polytope for all  $n$ .

**Lemma 1.** *When  $\mathcal{J} = (J_n)_n$  is a convex exhaustion, the associated sequence of integral convex polytopes  $(J_n)_n$  defines an integral convex exhaustion with the same limit  $J_\infty = J_\infty$  and  $p(J_n) \sim^n p(J_\infty)$  when  $n$  goes to infinity.*

*Proof.* As  $J_n \subset J_n$  the monotonicity of the perimeter of convex domains implies  $p(J_n) \leq p(J_n)$  (see e.g. Chapter 7 in [3]). Then for any  $x \in \text{Int}(J_\infty)$  there is a ball  $B_x \subset J_\infty$  centered at  $x$  such that  $\tilde{J}_n$  contains  $B_x$  for  $n$  large enough. Moreover the rescaled ball  $p(J_n)^{\frac{1}{d-1}} B_x \subset J_n$  has a non empty intersection with  $\mathbb{Z}^d$  when  $n$  goes to infinity. Therefore there is  $u_n \in J_n$  with  $p(J_n)^{-\frac{1}{d-1}} u_n \in B_x$  for large  $n$ . Let  $(E_p)_{p \in \mathbb{N}}$  be a sequence of finite subsets of  $\text{Int}(J_\infty)$  with  $d_H(E_p, J_\infty) < \frac{1}{p}$ , where  $d_H$  denotes the Hausdorff distance. Fix  $p$ . Arguing as above for any  $x \in E_p$  with  $B_x$  of radius less than  $\frac{1}{p}$ , we get  $d_H(J_\infty \cap p(J_n)^{-\frac{1}{d-1}} J_n, J_\infty) < \frac{2}{p}$  for  $n$  large enough. Therefore  $J_\infty \cap p(J_n)^{-\frac{1}{d-1}} J_n$  goes to  $J_\infty$ , so that we obtain by taking the perimeters in this limit

$$\begin{aligned} \liminf_n \frac{p(J_n)}{p(J_n)} &= \liminf_n p\left(p(J_n)^{-\frac{1}{d-1}} J_n\right), \\ &\geq \lim_n p\left(J_\infty \cap p(J_n)^{-\frac{1}{d-1}} J_n\right), \\ &\geq p(J_\infty) = 1. \end{aligned}$$

Together with  $p(J_n) \leq p(J_n)$ , we get  $p(J_n) \sim^n p(J_n)$ . Now we observe that  $J_\infty \cap \tilde{J}_n \supset J_\infty \cap p(J_n)^{-\frac{1}{d-1}} J_n$  is also converging to  $J_\infty$ . Moreover, it follows from the inclusions  $J_n \subset J_n$  that \*

$$\begin{aligned} \limsup_n \tilde{J}_n &= \limsup_n p(J_n)^{-\frac{1}{d-1}} J_n, \\ &\subset \lim_n \tilde{J}_n = J_\infty. \end{aligned}$$

Consequently  $\tilde{J}_n$  also goes to  $J_\infty$  when  $n$  goes to infinity.  $\square$

**2.3. Internal and external morphological boundary.** We recall some terminology of mathematical morphology used in image processing. For two subsets  $I$  and  $J$  of  $\mathbb{R}^d$ , the **dilation** (also known as the Minkowski sum)  $J \oplus I$  and the **erosion**  $J \ominus I$  of  $J$  by  $I$  are defined as follows

$$\begin{aligned} J \oplus I &= \{i + j \mid i \in I \text{ and } j \in J\}, \\ J \ominus I &= \{j \in \mathbb{R}^d \mid \forall i \in I, i + j \in J\}. \end{aligned}$$

When the origin 0 belongs to  $I$  then we have  $J \subset J \oplus I$  and  $J \ominus I \subset J$ . Assume now moreover that  $I$  and  $J$  are convex bodies. The dilation  $J \oplus I$  is then also a convex body with  $\text{ex}(J \oplus I) \subset \text{ex}(I) \oplus \text{ex}(J)$ . In particular, when  $I$  and  $J$  are moreover convex polytopes, then so is  $J \oplus I$ . We have  $J \ominus I = \bigcap_{x \in \partial' J} T_x^- J(h_I(-N^J(x)))$  (also  $J \oplus I \subset \bigcap_{x \in \partial' J} T_x^- J(h_I(N^J(x)))$ ), but this last inclusion may be strict). Consequently  $J \ominus I$  is a convex body. When  $J$  is a convex polytope, the above intersection is finite, thus  $J \ominus I$  is also a convex polytope. The convex bodies given by the erosion  $J \ominus I$  and the dilation  $J \oplus I$  are also known as the inner and outer parallel bodies of  $J$  relative to  $I$ . We recall that  $h_{J \oplus I} = h_J + h_I$ . In particular when  $I = \{i\}$  is a singleton, we get  $h_{J+i}(x) = h_J(x) + i \cdot x$  for all  $x \in \mathbb{S}^d$ . In general we only have  $h_{J \ominus I} \leq h_J - h_I$ .

\*For a sequence  $(A_n)_n$  of subsets in  $\mathbb{R}^d$ , we let  $\limsup_n A_n = \bigcap_n \overline{\bigcup_{k \geq n} A_k}$ .

The **internal and external (morphological) boundaries** of  $J$  relative to  $I$  denoted respectively by  $\partial_I^- J$  and  $\partial_I^+ J$  are given by

$$\begin{aligned}\partial_I^+ J &= (I \oplus J) \setminus J, \\ \partial_I^- J &= J \setminus (J \ominus I).\end{aligned}$$

Clearly we have  $\partial_I^\pm J = \partial_{I'}^\pm J$  with  $I' = I \cup \{0\}$ . When  $J$  is a convex domain then we have  $\partial_I^- J = \partial_{\text{cv}(I)}^- J$  and  $\partial_I^+ J \subset \partial_{\text{cv}(I)}^+ J$ . In the following the set  $I$  will be fixed so that we omit the index  $I$  in the above definitions when there is no confusion.

### 3. COUNTING INTEGER POINTS IN MORPHOLOGICAL BOUNDARY OF LARGE CONVEX SETS

For a large convex domain  $J$  and a fixed integral polytope  $I$  we estimate the cardinality of the integer points in the morphological boundaries of  $J$  relative to  $I$ . We first compare the cardinality of integer points in the internal and external boundaries of  $J$  and  $\mathbb{J}$ . Recall that  $\underline{F}$  denotes the integer points in a subset set  $F$  of  $\mathbb{R}^d$  and  $\mathbb{J} = \text{cv}(\mathbb{J})$ .

**Lemma 2.** *With the above notations we have*

$$\underline{\partial^- \mathbb{J}} = \underline{\partial^- J}$$

and

$$\underline{\partial^+ \mathbb{J}} \subset \underline{\partial^+ J}.$$

In general the last inclusion is strict.

*Proof.* For any convex domain  $J$ , a point  $u$  of  $J$  belongs to  $\partial^- J$  if and only if there is  $v$  in  $\text{ex}(I)$  such that  $u+v$  does not lie in  $J$ . As  $J \cap \mathbb{Z}^d = \mathbb{J} \cap \mathbb{Z}^d$  and  $\text{ex}(I) \subset \mathbb{Z}^d$ , we get  $\underline{\partial^- J} = \underline{\partial^- \mathbb{J}}$ . Similarly if a point  $u \in \partial^+ J$  is an integer, then  $u \in J \oplus I$  but  $u \notin J$ . Therefore we get  $\underline{\partial^+ \mathbb{J}} \subset \underline{\partial^+ J}$ .  $\square$

**Lemma 3.** *Let  $J$  be a convex polytope.*

$$\#\underline{\partial^- J} \leq \#\underline{\partial^+ J}.$$

*Proof.* We have  $\partial^- J \subset \bigcup_{F \in \mathcal{F}(J)} T_F^+ J(-h_I(N^F))$ . For  $F \in \mathcal{F}(J)$  there exists  $u^F \in \text{ex}(I)$  with  $h_I(N^F) = u^F \cdot N^F$ . Let  $F_1, \dots, F_N$  be an enumeration of  $\mathcal{F}(J)$ . Let  $\phi : \partial^- J \rightarrow \partial^+ J$  be the function defined by  $\phi(x) = x + u^{F_1}$  for  $x \in S_1 := \partial^- J \cap T_{F_1}^+ J(-h_I(N^{F_1}))$  and  $\phi(x) = x + u^{F_l}$  for  $x \in S_l := \partial^- J \cap T_{F_l}^+ J(-h_I(N^{F_l})) \setminus \bigcup_{k < l} T_{F_k}^+ J(-h_I(N^{F_k}))$  by induction on  $l$ .

This map is injective : indeed if  $\phi(x) = \phi(y)$  either  $x$  and  $y$  lie in the same  $S_l$  and then  $\phi(x) = x + u^{F_l} = y + u^{F_l} = \phi(y)$  clearly implies  $x = y$  or  $x \in S_k, y \in S_l$  with  $k \neq l$ . We may assume  $k < l$  without loss of generality. Then  $y + u^{F_l} \in T_{F_k}^- J$  whereas  $x + u^{F_k} \in T_{F_k}^+ J$  and we get thus a contradiction. Finally the map  $\phi$  preserves the integer points since we have  $\text{ex}(I) \subset \mathbb{Z}^d$ .  $\square$

**3.1. First relative quermass integral.** Let  $O$  be a convex domain and let  $I$  be a convex body. For  $\rho \in \mathbb{R}$  we let

$$O_\rho = \begin{cases} O \oplus \rho I & \text{when } \rho \geq 0, \\ O \ominus \rho I & \text{when } \rho < 0. \end{cases}$$

**Proposition 2.**

$$\lim_{\rho \rightarrow 0} \frac{V(O_\rho) - V(O)}{\rho} = \int_{\mathbb{S}^d} h_I d\sigma_O.$$

For  $\rho > 0$  the formula follows from Minkowski's formula on mixed volume (see Theorem 6.5 and Corollary 10.1 in [7]). For  $\rho < 0$  we refer to [12] (see also Lemma 2 in [4] for the 2-dimensional case).

The quantity  $d \int_{\mathbb{S}^d} h_I d\sigma_O$  is known as the **first  $I$ -relative quermass integral** of  $J$ . In the following we denote by  $V_I(O)$  the integral  $\int_{\mathbb{S}^d} h_I d\sigma_O$ . For convex bodies  $I \subset I'$  and  $k \in \mathbb{N}$ , we have  $V_I(O) \leq V_{I'}(O)$  and  $V_{kI}(O) = kV_I(O)$  for any convex domain  $O$ . The support function  $h_I$  being continuous, the first  $I$ -relative quermass integral of  $O$  is continuous with respect to the Hausdorff topology, i.e. if  $(O_n)_n$  is a sequence of convex domains converging to a convex domain  $O_\infty$  in the Hausdorff topology, then we have

$$V_I(O_n) \xrightarrow{n \rightarrow +\infty} V_I(O_\infty).$$

We deduce now from Proposition 2 an estimate on the volume of the morphological boundary for large convex sets.

**Corollary 3.** *Let  $I$  be a convex body containing 0 and let  $O \in \mathcal{D}$ . Then*

$$V(\partial_I^\pm nO) \sim n^{d-1} \int_{\mathbb{S}^d} h_I d\sigma_O.$$

*Proof.* We only consider the case of the external boundary as one may argue similarly for the internal boundary. For all  $n$  we have

$$\begin{aligned} V(\partial_I^+ nO) &= V(nO \oplus I) - V(nO), \\ &= n^d (V(O \oplus n^{-1}I) - V(O)) \end{aligned}$$

According to Proposition 2 we conclude that

$$V(\partial_I^+ nO) \sim n^{d-1} \int_{\mathbb{S}^d} h_I d\sigma_O.$$

□

**3.2. Counting integer points in large convex sets.** Since Gauss circle problem counting lattice points in convex sets has been extensively investigated. Let  $\mathbf{C} = [0, 1]^d$ . Clearly for any Borel subset  $K$  of  $\mathbb{R}^d$  we have always

$$(3.1) \quad \#\underline{K} \leq V(K \oplus \mathbf{C}).$$

In the other hand, Bokowski, Hadwiger and Wills have proved the following general (sharp) inequality for any convex domain  $O$  [2] :

$$(3.2) \quad V(O) - \frac{p(O)}{2} \leq \#\underline{O}.$$

There exists precise asymptotic estimate of  $\#\underline{xO}$  for large  $x > 0$  for convex domains with smooth domain having positive curvature, in particular we have in this case  $\#\underline{xO} = V(xO) + o(x^{d-1})$  [8].

**3.3. First rough estimate for  $\#\partial_I^\pm nO \cap \mathbb{Z}^d$  with  $O \in \mathcal{D}$ .** For a real sequences  $(a_n)_n$  and two numbers  $l$  and  $C > 0$  we write  $a_n \sim^C l$  when the accumulation points of  $(a_n)_n$  lie in  $[l - C, l + C]$ .

**Lemma 4.** *There exists a constant  $C$  depending only on  $d$  such that we have for any convex domain  $O$  and any convex body  $I$  of  $\mathbb{R}^d$  with  $0 \in I$  :*

$$\frac{\#\partial_I^\pm nO}{n^{d-1}} \sim^C V_I(O).$$

*Proof.* We only argue for  $\partial_I^+ O$ , the other case being similar. We have  $\#\partial_I^+ nO = \#\underline{nO \oplus I} - \#\underline{nO}$ , and then by combining Equation (3.1) and (3.2) we get :

$$V(nO \oplus I) - \frac{p(nO \oplus I)}{2} - V(nO + \mathbf{C}) \leq \#\partial_I^+ nO \leq V(nO \oplus I \oplus \mathbf{C}) - V(nO) + \frac{p(nO)}{2},$$

After dividing by  $n^{d-1}$ , the right (resp. left) hand side term is going to  $\int_{\mathbb{S}^d}(h_I - h_C - 1/2) d\sigma_O$  (resp.  $\int_{\mathbb{S}^d}(h_I + h_C + 1/2) d\sigma_O$ ) according to Corollary 3.  $\square$

**3.4. Fine estimate of  $\#\partial_I^\pm J_n$  for general convex exhaustion  $(J_n)_n$  in dimension 2.** We compare directly the cardinality of lattice points in the morphological boundary with the first  $I$ -relative quermass integral of  $J_\infty$  for two-dimensional convex exhaustion. This result will not be used directly in the next sections but is potentially of independent interest.

**Proposition 4.** *For any convex exhaustion  $(J_n)_n$  in  $\mathbb{R}^2$ , we have*

$$\lim_n \frac{\#\partial_I^- J_n}{p(J_n)} = V_I(J_\infty).$$

By Lemma 1 and Lemma 2 we only need to consider integral convex exhaustions. In fact in this case we also show the corresponding statement for the external morphological boundary.

**Proposition 5.** *For any integral convex exhaustion  $(J_n)_n$  in  $\mathbb{R}^2$ , we have*

$$\lim_n \frac{\#\partial_I^\pm J_n}{p(J_n)} = V_I(J_\infty).$$

The rest of this subsection is devoted to the proof of Proposition 5. We start by giving some preliminary lemmas.

We denote by  $\angle P$  the minimum of the interior angles at the vertices of a convex polygon  $P \subset \mathbb{R}^2$ .

**Lemma 5.** *For any integral convex exhaustion  $(J_n)_n$  in  $\mathbb{R}^2$ , we have*

$$\liminf_n \angle J_n > 0.$$

*Proof.* We have  $\angle \tilde{J}_n = \angle J_n$ . Moreover the minimal angle is lower semi-continuous for the Hausdorff topology, therefore  $\liminf_n \angle J_n \geq \angle J_\infty$ . Since  $J_\infty$  has non-empty interior, we have  $\angle J_\infty > 0$ .  $\square$

**Lemma 6.** *For any integral convex exhaustion  $(J_n)_n$  in  $\mathbb{R}^2$ , we have*

$$\#\mathcal{F}(J_n) = o(p(J_n)).$$

*Proof.* Two integral polytopes are said equivalent when there is a translation (necessarily by an integer) mapping one to the other. For any  $L$  the number  $a_L$  of equivalence classes of integral 1-polytopes with 1-Hausdorff measure less than  $L$  is finite (these polytopes are just line segments with integral endpoints and their 1-Hausdorff measure is just equal to their length). Moreover for a integral convex polytope there are at most two faces in the same class. Therefore

$$\begin{aligned} \#\mathcal{F}(J_n) &\leq 2a_L + \#\{F \in \mathcal{F}(J_n), \mathcal{H}_1(F) \geq L\}, \\ &\leq 2a_L + \frac{p(J_n)}{L}. \end{aligned}$$

This inequality holds for all  $n$  and  $p(J_n)$  goes to infinity with  $n$  so that we conclude  $\#\mathcal{F}(J_n) = o(p(J_n))$  as  $L$  was arbitrarily fixed.  $\square$

Given two distinct points  $A, B$  in  $\mathbb{R}^2$  and  $h \neq 0$ , the rectangle  $R_{AB}(h)$  of basis  $AB$  and height  $h > 0$  (resp.  $h < 0$ ) is the semi-open rectangle  $[AB[\times]A, D[$  oriented as  $ABCD$  (resp.  $ADCB$ )<sup>†</sup> with  $|AD| = |h|$ . This rectangle is said integral when  $A, B$  belong to  $\mathbb{Z}^2$  and the line  $(CD)$  has a non-empty intersection with  $\mathbb{Z}^2$ .

<sup>†</sup>We denote a convex polytope with its vertices by respecting the usual orientation of the plane.

**Lemma 7.** *For any integral rectangle  $R$ ,*

$$\#R = V(R).$$

*Proof.* After a translation by an integer we may assume that the origin is the vertex  $A$  of the integral rectangle  $R = R_{AB}(h)$ . Let  $(p', q')$  be an integer on the line segment  $[A, B]$  with  $p', q'$  relatively prime. By Bezout theorem there is  $(u, v) \in \mathbb{Z}^2$  with  $up + vq = 1$ . Therefore there is a matrix  $M \in SL_2(\mathbb{Z})$  with  $M(p, q) = (k, 0)$ . As the transformation  $M$  preserve both the volume and the integer points it is enough to consider the semi-open parallelogram  $M(R)$ . But there is a piecewise integral translation, which maps  $M(R)$  to a semi-open integral rectangle with basis  $M([A, B]) \subset \mathbb{R} \times \{0\}$ . For such a rectangle the area is obviously equal to the cardinality of its integer points.  $\square$

For  $A, B \in \mathbb{R}^2$  and  $\epsilon < \frac{|AB|}{2}$  we let  $A^\epsilon$  and  $B^\epsilon$  be the points in the line  $(AB)$  with Euclidean distance  $|\epsilon|$  to  $A$  and  $B$  respectively, which lie inside  $[A, B]$  if  $\epsilon > 0$  and outside if not. As the symmetric difference of  $R_{A^\epsilon B^\epsilon}(h)$  and  $R_{AB}(h)$  is given by the union of two rectangles with sides of length  $|\epsilon|$  and  $|h|$  we have for some constant  $C = C(|\epsilon|, |h|)$

$$(3.3) \quad \left| \#R_{A^\epsilon B^\epsilon}(h) - \#R_{AB}(h) \right| \leq C.$$

This estimate still holds true for  $\epsilon \geq |AB|/2$  when choosing the convention  $R_{A^\epsilon B^\epsilon}(h) = \emptyset$  for such  $\epsilon$ .

**Fact.** *For any convex body  $I$  and for any  $a > 0$ , there exists  $\epsilon^+ = \epsilon^+(I) > 0$  and  $\epsilon^- = \epsilon^-(I, a) > 0$  such that any convex polytope  $J = A_1 \cdots A_n$  with  $\angle J \geq a$  satisfies*

$$\partial^+ J \subset \bigcup_{l < n} R_{A_l^{\epsilon^+} A_{l+1}^{-\epsilon^+}}(-h_I(N^{A_l A_{l+1}}))$$

and

$$\partial^- J \supset \bigcup_{l < n} R_{A_l^{\epsilon^-} A_{l+1}^{-\epsilon^-}}(h_I(N^{A_l A_{l+1}})).$$

This fact is illustrated on Figure 1 and its easy proof is left to the reader. We are now in a position to prove Proposition 5.

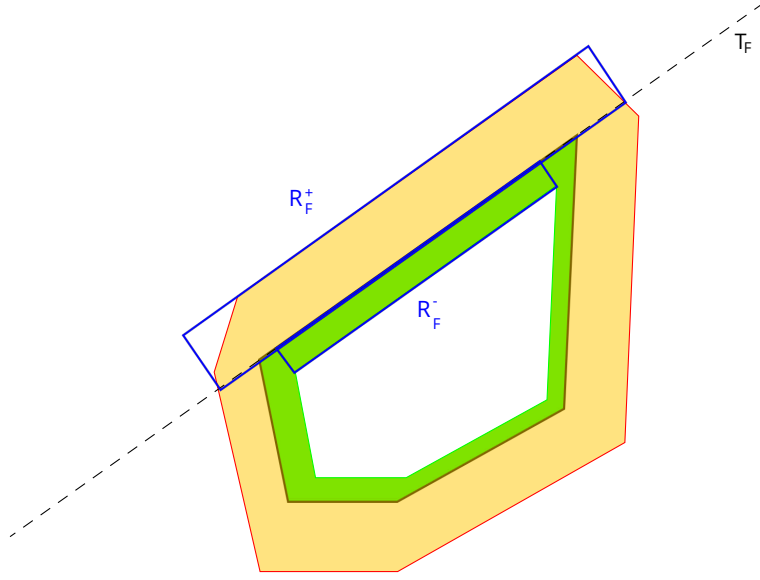
*Proof of Proposition 5.* From the above fact and (3.3) there is  $\epsilon = \epsilon^-(I, \angle J) > 0$  and  $C = C(I, \angle J) > 0$  such that for any convex polytope  $J = A_1 \cdots A_n$

$$\begin{aligned} \#\partial^- J &\geq \sum_{l < n} \#R_{A_l(\epsilon) A_{l+1}(\epsilon)}(-h_I(N^{A_l A_{l+1}})), \\ &\geq \sum_{F \in \mathcal{F}(J)} \left[ \#R_F(-h_I(N^F)) - C \right]. \end{aligned}$$

Then when  $J$  is an integral convex polytope we get by Lemma 7 :

$$\begin{aligned} \#\partial^- J &\geq -C\#\mathcal{F}(J) + \sum_{F \in \mathcal{F}(J)} V(R_F(-h_I(N^F))), \\ &\geq -C\#\mathcal{F}(J) + \int_{\mathbb{S}^d} h_I d\sigma_J. \end{aligned}$$





**Figure 1:** *The external and internal rectangles associated to a face  $F$  of a polygon. The external and internal morphological boundaries are respectively represented by the areas in yellow and green. The rectangles,  $R_F^+$  and  $R_F^-$ , given by Fact 3.4 are drawn in blue.*

For an integral convex exhaustion  $(J_n)_n$  we obtain finally for large  $n$  by using Lemma 6 and Lemma 5

$$\begin{aligned} \# \partial^- J_n &\geq -C(I, \frac{\angle J_\infty}{2}) \cdot \# \mathcal{F}(J_n) + \int_{\mathbb{S}^d} h_I d\sigma_{J_n}, \\ \liminf_n \frac{\# \partial^- J_n}{p(J_n)} &\geq \lim_n \int_{\mathbb{S}^d} h_I d\sigma_{J_n}, \\ &\geq \int_{\mathbb{S}^d} h_I d\sigma_{J_\infty}. \end{aligned}$$

One proves similarly that  $\limsup_n \frac{\# \partial^+ J_n}{p(J_n)} \leq \int_{\mathbb{S}^d} h_I d\sigma_{J_\infty}$  and this concludes the proof as we have  $\# \partial^+ J_n \geq \# \partial^- J_n$  according to Lemma 3.  $\square$

**3.5. Supremum of  $O \mapsto V_I(O)$ .** In this section we investigate the supremum of  $V_I$  on  $\mathcal{D}^1$  for a given convex body  $I$  of  $\mathbb{R}^d$ . We recall that there is a unique sphere  $S_I$  containing  $I$  with minimal radius, usually called the **smallest bounding sphere** of  $I$ . We let  $R_I$  and  $x_I$  be respectively the radius and the center of  $S_I$ . There are at least two distinct points in  $S_I \cap I$ . Moreover we have the following alternative :

- either there is a subset of  $S_I \cap I$  generating an inscribable polytope  $T$  with  $\text{Int}(T) \ni x_I$  (in particular the interior set of  $I$  is non empty),
- or there is a hyperplane  $H$  containing  $x_I$  such that  $I$  lies in an associated semiplane and  $S_I \cap H$  is the smallest bounding sphere of  $I \cap H$ .

The smallest bounding sphere  $S_I$  will be said **nondegenerated** (resp. **degenerated**) and an associated polytope  $T$  (resp. hyperplane  $H$ ) is said **generating**. For an inscribable polytope  $T$  in  $\mathbb{R}^d$  we may define its dual  $T'$  as the polytope given by the intersection of the inner semi-spaces tangent to the circumsphere of  $T$  at the vertices of  $T$ . In the following  $T'$  always denote the dual polytope of a generating polytope  $T$  with respect to  $I$ .

When  $S_I$  is degenerated and  $H$  is an associated generating hyperplane, we let  $T'_R$ ,  $R > 0$ , be the polytope given by the intersection of the square  $[-R, R]^d$  with the inner semiplanes tangent to  $S_I$  at  $H \cap S_I \cap I$ . We also denote by  $\mathcal{F}_{\parallel}(T'_R)$  the subspace of  $\mathcal{F}(T'_R)$  given by subfaces of the square  $[-R, R]^d$ .

**Proposition 6.**

$$\sup_{O \in \mathcal{D}^1} V_I(O) = R_I.$$

The supremum of  $V_I$  is achieved if and only if  $S_I$  is nondegenerated. The supremum is then achieved for  $O \in \mathcal{D}^1$  homothetic to the dual polytope  $T'$  of a generating polytope  $T$ .

*Proof.* For any  $v \in \mathbb{R}^d$  we have

$$\begin{aligned} V_{I+v}(O) &= \int h_{I+v} d\sigma_O, \\ &= \int h_I d\sigma_O + \int_{\mathbb{S}^d} v \cdot u d\sigma_O(u), \\ &= \int h_I d\sigma_O + \int_{\partial O} v \cdot N^O d\mathcal{H}_{d-1}. \end{aligned}$$

By the divergence formula we have  $\int_{\partial O} v \cdot N^O d\mathcal{H}_{d-1} = 0$  for any  $v \in \mathbb{R}^d$  and  $O \in \mathcal{D}^1$ . Therefore we may assume  $x_I = 0$ . With the above notations we have  $\max_{i \in I} i \cdot v \leq R_I$  for all  $v \in \mathbb{R}^d$  with  $\|v\| = 1$  with equality iff  $v$  belongs to  $R_I^{-1}I$ . Therefore  $V_I(O) \leq R_I$  for any  $O \in \mathcal{D}^1$ . Moreover if the equality occurs then for  $x$  in a subset  $E$  of  $\partial O$  with full  $\mathcal{H}_{d-1}$ -measure,  $h_I(N^O(x)) = \max_{i \in I} i \cdot x = R_I$  and therefore the normal unit vector  $N^O(x)$  belongs to  $R_I^{-1}I$ . But as  $O$  is a convex domain, we may find  $d+1$  points  $x_1, \dots, x_{d+1}$  in  $E$  in such a way the origin belongs to the interior of the simplex  $T = R_I \text{cv}(N^O(x_1), \dots, N^O(x_{d+1}))$ . Thus  $S_I$  is nondegenerated and the polytope  $T$  is a generating polytope with respect to  $I$ . Moreover we have with the above notations

$$\int h_I d\sigma_{T'} = R_I p(T').$$

Therefore the homothetic tetrahedron of  $T'$  with unit perimeter achieves the supremum of  $V_I$ . We consider now the degenerated case. With the above notations we have  $\mathcal{H}_{d-1} \left( \bigcup_{F \in \mathcal{F}_{\parallel}(T'_R)} F \right) = o(p(T'_R))$  when  $R$  goes to infinity. Therefore the renormalization  $O_R \in \mathcal{D}^1$  of  $T'_R$  satisfies

$$V_I(O_R) \xrightarrow{R \rightarrow +\infty} R_I.$$

□

## 4. CELLULAR AUTOMATA

**4.1. Definitions.** We consider a finite set  $\mathcal{A}$ . We endow the set  $\mathcal{A}$  with the discrete topology and  $X_d = \mathcal{A}^{\mathbb{Z}^d}$  with the product topology. We consider the  $\mathbb{Z}^d$ -shift  $\sigma$  on  $\mathcal{A}^{\mathbb{Z}^d}$  defined for  $l \in \mathbb{Z}^d$  and  $u = (u_k)_k \in X_d$  by  $\sigma^l(u) = (u_{k+l})_k$ . Any closed subset  $X$  of  $X_d$  invariant under the action of  $\sigma$  is called a  $\mathbb{Z}^d$ -**subshift**. We fix such a subshift  $X$  in the remaining of the paper.

For a bounded subset  $J$  of  $\mathbb{R}^d$  we consider the partition  $\mathbb{P}_J$  into  $J$ -**cylinders**, i.e. the element  $\mathbb{P}_J^x$  of  $\mathbb{P}_J$  containing  $x = (x_i)_{i \in \mathbb{Z}^d} \in X$  is given by  $\mathbb{P}_J^x := \{y = (y_i)_{i \in \mathbb{Z}^d} \in X, \forall i \in J y_i = x_i\}$ . In other terms we may define  $\mathbb{P}_J$  as the joined partition  $\bigvee_{j \in J} \sigma^{-j} \mathbb{P}_0$  with  $\mathbb{P}_0$  being the zero-coordinate partition.

A **cellular automaton** (CA for short) defined on a  $\mathbb{Z}^d$ -subshift  $X$  is a continuous map  $f : X \rightarrow X$  which commutes with the shift action  $\sigma$ . By a famous theorem of Hedlund [14] the cellular automaton  $f$  is given by a local rule, i.e. there exists a finite subset  $I$  of  $\mathbb{Z}^d$  and a map  $F : \mathcal{A}^I \rightarrow \mathcal{A}$  such that

$$\forall j \in \mathbb{Z}^d \quad (fx)_j = F((x_{j+i})_{i \in I}).$$

The (minimal) subset  $I$  is called the **domain** of the CA. Recall  $I' = I \cup \{0\}$ .

**4.2. Lyapunov exponents for higher dimensional cellular automata.** Lyapunov exponent of one-dimensional cellular automata have been defined in [15, 16]. We develop a similar theory in higher dimensions. Let  $f$  be a CA on a  $\mathbb{Z}^d$ -subshift  $X$  with domain  $I$ .

Given a convex body  $J$  of  $\mathbb{R}^d$  and  $x \in X$ , we let

$$\mathcal{E}_f^+(x, J) := \{K \text{ convex body, } f\mathbb{P}_K^x \subset \mathbb{P}_J^{fx}\}$$

$$\mathcal{E}_f^-(x, J) := \{K \text{ convex body, } f\mathbb{P}_J^x \subset \mathbb{P}_K^{fx}\}$$

The family  $\mathcal{E}_f^+(x, J)$  has a minimal element for the inclusion given by  $J \oplus f(x) := \bigcap \{K, K \in \mathcal{E}_f^+(x, J)\}$ , but a priori this is not the case of  $\mathcal{E}_f^-(x, J)$ . Observe that  $J \oplus I$  and  $J \ominus I$  belongs respectively to  $\mathcal{E}_f^+(x, J)$  and  $\mathcal{E}_f^-(x, J)$ . Then we let for all  $x$  :

$$\text{gr}_J^+ f(x) := \min\{\#K \setminus J, K \in \mathcal{E}_f^+(x, J)\}$$

$$\text{gr}_J^- f(x) := \min\{\#J \setminus K, K \in \mathcal{E}_f^-(x, J)\}.$$

Observe that the family  $\mathcal{E}_f^-(x, J)$  and the function  $\text{gr}_J^- f(x)$  are constant on each atom  $A$  of  $\mathbb{P}_J$ , then we let  $\mathcal{E}_f^-(A, J)$  and  $\text{gr}_J^- f(A)$  be these quantities. For a convex exhaustion  $\mathcal{J} = (J_n)_n$  and a convex domain  $O \in \mathcal{D}^1$ , we define **the growth**  $\text{gr}_{\mathcal{J}}^\pm f$  with respect to  $\mathcal{J}$  and  $O$  as the following real functions on  $X$  :

$$\begin{aligned} \text{gr}_{\mathcal{J}}^\pm f &:= \limsup_n \frac{\text{gr}_{J_n}^\pm f}{p(J_n)}, \\ \text{gr}_O^\pm f &= \sup_{\mathcal{J} \in \mathcal{E}(O)} \text{gr}_{\mathcal{J}}^\pm f. \end{aligned}$$

**Lemma 8.**

$$\forall O \in \mathcal{D}^1, \text{gr}_O^+ f = \text{gr}_O^- f.$$

*Proof.* It follows from the definitions that we have for any convex bodies  $K, J$  :

$$(4.1) \quad [K \in \mathcal{E}^+(x, J)] \Leftrightarrow [J \in \mathcal{E}^-(x, K)].$$

Then if  $\mathcal{J} = (J_n)_n$  is a convex exhaustion and  $\mathcal{K}^\pm = (K_n^\pm)_n$  is a sequence of convex bodies  $K_n$  which realizes the maximum in the definition of  $\text{gr}_{J_n}^\pm f$ . As  $J \oplus I$  and  $J \ominus I$  belong respectively to  $\mathcal{E}^\pm(x, J)$  the sequence  $\mathcal{K}^\pm$  is a convex exhaustion with  $p(K_n^\pm) \sim^n p(J_n)$ . By the equivalence (4.1) we get

$$\begin{aligned} \text{gr}_{\mathcal{J}}^+ f &\leq \limsup_n \frac{\#K_n^+ \setminus J_n}{p(J_n)}, \\ &\leq \limsup_n \frac{\#K_n^+ \setminus J_n}{p(K_n^+)}, \\ &\leq \text{gr}_{\mathcal{K}^+}^- f, \end{aligned}$$

therefore we get  $\text{gr}_O^+ f \leq \text{gr}_O^- f$  and then  $\text{gr}_O^+ f \geq \text{gr}_O^- f$  by following the same lines.  $\square$

In the following we let  $\text{gr}_O f = \text{gr}_O^\pm f$ .

**Lemma 9.** *The sequence of functions  $(\text{gr}_O f^k)_k$  is subadditive, i.e.*

$$\forall k, l \in \mathbb{N} \quad \forall x \in X, \text{gr}_O f^{k+l}(x) \leq \text{gr}_O f^l(f^k x) + \text{gr}_O f^k(x).$$

*Proof.* Fix  $x \in X$  and  $k, l \in \mathbb{N}$ . Let  $\mathcal{J} = (J_n)_n \in \mathcal{E}(O)$ . As already observed, the sequences  $\mathcal{J}' = (J'_n)_n := (J_n \oplus f^l(f^k x))_n$  and  $\mathcal{J}'' = (J''_n)_n := (J'_n \oplus f^l(x))_n$  both belong to  $\mathcal{E}(O)$  and their perimeters are equivalent to  $p(J_n)$ , when  $n$  goes to infinity. Moreover for all positive integers  $n$  we have

$$\begin{aligned} f^{k+l} \mathbf{P}_{J''_n}^x &= f^l(f^k \mathbf{P}_{J''_n}^x), \\ &\subset f^l \left( \mathbf{P}_{J'_n}^{f^k x} \right), \\ &\subset \mathbf{P}_{J_n}^{f^{k+l} x}. \end{aligned}$$

Therefore we conclude

$$\begin{aligned} \text{gr}_{J_n}^+ f^{k+l}(x) &\leq \# \underline{J''_n} \setminus J_n, \\ &\leq \# \underline{J''_n} \setminus J'_n + \# \underline{J'_n} \setminus J_n, \\ \text{gr}_{\mathcal{J}}^+ f^{k+l}(x) &\leq \text{gr}_{\mathcal{J}'}^+ f^k(x) + \text{gr}_{\mathcal{J}'}^+ f^l(f^k x), \\ \text{gr}_O f^{k+l}(x) &\leq \text{gr}_O f^k(x) + \text{gr}_O f^l(f^k x). \end{aligned}$$

□

For  $d = 2$  the nonnegative function  $\text{gr}_O f$  is bounded from above by  $V_I(O)$  according to Proposition 4. Therefore the subadditive ergodic theorem applies : for any  $\mu \in \mathcal{M}(X, f)$  the sequence  $(\frac{1}{n} \text{gr}_O f^n(x))_k$  converge almost everywhere to a  $f$ -invariant function  $\chi_O$  with  $\int \chi_O d\mu = \lim / \inf_n \frac{1}{n} \int \text{gr}_O f^n d\mu$ . We call the function  $\chi_O$  the Lyapunov exponent of  $f$  with respect to  $O$ .

**Remark 7.** *The exponent  $\chi_O$  for  $O \in \mathcal{D}$  plays some how the role of the sum of the positive Lyapunov exponents in smooth dynamical systems.*

## 5. RESCALED ENTROPY OF CELLULAR AUTOMATA

**5.1. Definition.** We let  $\mathcal{M}(f)$  (resp.  $\mathcal{M}(f, \sigma)$ ) be the set of invariant Borel probability measures on  $X$  which are  $f$ -invariant (resp.  $f$ - and  $\sigma$ -invariant). For a finite clopen partition  $\mathbf{P}$  of  $X$  we let  $H_{\text{top}}(\mathbf{P}) = \log \# \mathbf{P}$  and  $H_\mu(\mathbf{P}) = - \sum_{A \in \mathbf{P}} \mu(A) \log \mu(A)$  with  $\mu \in \mathcal{M}(f)$ . In the following the symbol  $\ast$  denotes either  $\ast = \text{top}$  or  $\ast = \mu \in \mathcal{M}(f)$ . We let  $h_\ast(f, \mathbf{P})$  be the entropy with respect to the clopen partition  $\mathbf{P}$  :

$$h_\ast(f, \mathbf{P}) := \lim_n \frac{1}{n} H_\ast \left( \bigvee_{k=0}^{n-1} f^{-k} \mathbf{P} \right).$$

For two partitions  $\mathbf{P}, \mathbf{Q}$  of  $X$ , we say  $\mathbf{P}$  is finer than  $\mathbf{Q}$  and we write  $\mathbf{P} > \mathbf{Q}$ , when any atom of  $\mathbf{P}$  is contained in an atom of  $\mathbf{Q}$ . The functions  $H_\ast(\cdot)$  and  $h_\ast(f, \cdot)$  are nondecreasing with respect to this order.

The rescaled entropy with respect to a convex exhaustion  $\mathcal{J} = (J_n)_n$  is defined as follows

$$h_\ast^d(f, \mathcal{J}) = \limsup_n \frac{h_\ast(f, \mathbf{P}_{J_n})}{p(J_n)}.$$

In [9] the authors defines a similar notion for the rescaled topological entropy with the renormalization factor  $\# \partial_I^- J_n$  (which depends on the domain  $I$  of  $f$ ) rather than  $p(J_n)$ .

**Remark 8.** When  $J = \bigcup_{i \in I} J_i$  is a finite disjoint union of Jordan domains  $J_i$  with Lipschitz boundary, we have

$$\begin{aligned} \frac{h_{top}(f, P_J)}{p(J)} &\leq \frac{\sum_{i \in I} h_{top}(f, P_{J_i})}{\sum_{i \in I} p(J_i)}, \\ &\leq \sup_{i \in I} \frac{h_{top}(f, P_{J_i})}{p(J_i)}. \end{aligned}$$

Moreover for each  $i$ , we have  $p(J_i) \geq p(\text{cv}(J_i))$  and  $P_{\text{cv}(J_i)}$  is finer than  $P_{J_i}$ . Therefore

$$\begin{aligned} \frac{h_{top}(f, P_J)}{p(J)} &\leq \frac{\sum_{i \in I} h_{top}(f, P_{J_i})}{\sum_{i \in I} p(J_i)}, \\ &\leq \sup_{i \in I} \frac{h_{top}(f, P_{\text{cv}(J_i)})}{p(\text{cv}(J_i))}. \end{aligned}$$

This inequality justifies somehow that we focus on convex bodies  $J$  of  $\mathbb{R}^d$ .

We let also

$$h_{top}^d(f) = \sup_{\mathcal{J}} h_*^d(f, \mathcal{J}).$$

For  $d = 1$  we have  $p(J) = 2$  for any convex subset  $J$ . Therefore up to a factor 2 we recover the usual definition of entropy,  $2h_*^1(f) = h_*(f)$ .

As the CA  $f$  commutes with the shift action  $\sigma$  we have for all  $k \in \mathbb{Z}^d$  and any subset  $J$  of  $\mathbb{Z}^d$   $h_{top}(f, P_{J+k}) = h_{top}(f, \sigma^{-k}P_J) = h_{top}(f, P_J)$  and the same holds for the measure theoretical entropy with respect to measures in  $\mathcal{M}(f, \sigma)$ . In particular we have  $h_{top}^d(f, O) = h_{top}^d(f, O + \alpha)$  for any  $\alpha \in \mathbb{R}^d$ . Also for any  $\mu$  in  $\mathcal{M}(f)$  and  $k \in \mathbb{Z}^d$  we have  $h_\mu^d(f, O) = h_{\sigma^k \mu}^d(f, O)$  because for  $\mathcal{J} = (J_n)_n \in \mathcal{E}(O)$ , the sequence  $\sigma^k(\mathcal{J}) = (k + J_n)_n$  belongs to  $\mathcal{E}(O)$ .

**Remark 9.** (1) The partition  $P_{J_n}$  may be written as  $\bigvee_{k \in J_n} \sigma^{-k}P_0$  with  $P_0$  being the zero-coordinate partition. Instead of  $P_0$  we could choose another clopen generating partition  $P$ , i.e. a partition of  $X$  into clopen sets with  $\bigvee_{k \in \mathbb{Z}^d} \sigma^{-k}P$  equal to the partition of  $X$  into points. But for a finite subset  $J$  of  $\mathbb{Z}^d$  we have  $\bigvee_{k \in J} \sigma^{-k}P > P_0$  and  $\bigvee_{k \in J} \sigma^{-k}P_0 > P$  so that in the definition of the rescaled entropy we may replace  $P_0$  by any other generator  $P$  of  $X$ , i.e.  $P_{J_n}$  by  $\bigvee_{k \in J_n} \sigma^{-k}P$ .

(2) Let  $X$  be a zerodimensional compact metrizable space endowed with a expansive  $\mathbb{Z}^d$ -action  $\tau$ . We consider a map  $f$  preserving  $(X, \tau)$  i.e.  $f$  is an homeomorphism of  $X$  commuting with  $\tau$ . The triple  $(X, \tau, f)$  is called a topological  $\mathbb{Z}^d$ -expansive preserving system (t.e.p.s. for short). Two t.e.p.s.  $(Y, \phi, g)$  are conjugated when there is a homeomorphism  $h : X \rightarrow Y$  such that  $h \circ f \circ h^{-1} = g$  and  $h \circ \tau \circ h^{-1} = \phi$ . We may define the rescaled entropy as we did for a CA and all the previous results hold in this more general setting. Moreover two conjugated t.e.p.s. have the same rescaled entropy. Any t.e.p.s. is conjugated to a CA.

**5.2. Link with the metric mean dimension.** In a compact metric space  $(X, d)$ , the ball of radius  $\epsilon \geq 0$  centered at  $x \in X$  will be denoted by  $B_d(x, \epsilon)$ . For a continuous map  $f : X \rightarrow X$  we denote by  $d_n$  the dynamical distance defined for all  $n \in \mathbb{N}$  by

$$\forall x, y \in X, d_n(x, y) = \max\{d(f^k x, f^k y), 0 \leq k < n\}.$$

The metric mean dimension of  $f$  is defined as  $\text{mdim}(f, d) = \limsup_{\epsilon \rightarrow 0} \frac{h_{top}(f, \epsilon)}{|\log \epsilon|}$  where  $h_{top}(f, \epsilon)$  denotes the topological entropy at the scale  $\epsilon > 0$  :

$$h_{top}(f, \epsilon) := \limsup_n \frac{1}{n} \log \min\{\#C, \bigcup_{x \in C} B_{d_n}(x, \epsilon) = X\}.$$

The topological mean dimension is the infimum of  $\text{mdim}(f, \mathbf{d})$  over all distances on  $X$ . We refer to [11] for alternative definitions and further properties of mean dimension. The topological mean dimension of a finite dimensional topological system is null.

Here  $f$  is a CA on a subshift of  $\mathbb{Z}^d$ . In particular it has zero topological mean dimension. For a norm  $\|\cdot\|$  of  $\mathbb{R}^d$  we may associate a metric  $\mathbf{d}_{\|\cdot\|}$  on  $X_d$  by letting  $\mathbf{d}_{\|\cdot\|}(u, v) = \alpha^{-\min\{\|k\|, k \in \mathbb{Z}^d, u_k \neq v_k\}}$  for all  $u = (u_k)_k, v = (v_k)_k \in X_d$ . Then for  $l \in \mathbb{N}$  the (open) ball  $B_{\mathbf{d}_{\|\cdot\|}}(x, 2^{-l})$  with respect to  $\mathbf{d}_{\|\cdot\|}$  coincides with the cylinder  $\mathbb{P}_{J_l}^x$  with  $J_l = B_{\|\cdot\|}(0, l)$ .

As there is a correspondence between convex symmetric bodies and unit balls of norms on  $\mathbb{R}^d$ , the mean dimension with respect to such distances  $\mathbf{d}_{\|\cdot\|}$  are given by  $h_{top}^d(f, \mathcal{J}_O)$  for convex symmetric bodies  $O$ .

**Remark 10.** In [17] the authors work with a measure theoretical quantity, called the measure distortion rate dimension and show a variational principle with the metric mean dimension of  $\mathbf{d}_{\|\cdot\|}$ . Does this quantity coincides with  $\mu \mapsto h_\mu^d(f, O)$  with  $O$  being the symmetric convex domain associated to the norm  $\|\cdot\|$ ?

**5.3. Monotonicity and Power.** We investigate now basic properties of the rescaled entropy.

**Lemma 10.** For any  $O \in \mathcal{D}$  and any  $\alpha > 0$ , we have

$$h_*^d(f, \mathcal{J}_O) = h_*^d(f, \mathcal{J}_{\alpha O}).$$

*Proof.* For  $n \in \mathbb{N}$ , we let  $k_n = \lceil \frac{n}{\alpha} \rceil$ , thus  $nO \subset k_n \alpha O$  and  $p(nO) \sim^n p(k_n \alpha O)$ . Therefore

$$\begin{aligned} h_*^d(f, \mathcal{J}_O) &= \limsup_n \frac{h_*(f, \mathbb{P}_{nO})}{p(nO)}, \\ &\leq \limsup_n \frac{h_*(f, \mathbb{P}_{k_n \alpha O})}{p(nO)}, \\ &\leq \limsup_n \frac{h_*(f, \mathbb{P}_{k_n \alpha O})}{p(k_n \alpha O)}, \\ &\leq h_*^d(f, \mathcal{J}_{\alpha O}). \end{aligned}$$

The other inequality is obtained by considering  $\alpha O$  and  $\alpha^{-1}$  in place of  $O$  and  $\alpha$ .  $\square$

**Lemma 11.** For any  $O \in \mathcal{D}^1$  and  $O' \in \mathcal{D}$  with  $O \subset \text{Int}(O')$ , we have

$$h_*^d(f, \mathcal{J}_O) \leq h_*^d(f, O) \leq p(O') h_*^d(f, \mathcal{J}_{O'}).$$

*Proof.* As  $\mathcal{J}_O \in \mathcal{E}(O)$  the inequality  $h_*^d(f, \mathcal{J}_O) \leq h_*^d(f, O)$  follows from the definitions. Let now  $\mathcal{J} \in \mathcal{E}(O)$ . For  $n$  large enough we have  $\tilde{J}_n \subset \text{Int}(O')$ , therefore  $J_n \subset k_n O'$  with  $k_n = ([p(J_n)] + 1)^{\frac{1}{d-1}}$ . Therefore we conclude that

$$\begin{aligned} h_*^d(f, \mathcal{J}) &\leq \limsup_n \frac{p(k_n O')}{p(J_n)} h_*^d(f, \mathcal{J}_{O'}), \\ &\leq p(O') h_*^d(f, \mathcal{J}_{O'}). \end{aligned}$$

$\square$

For  $O \in \mathcal{D}^1$  the origin belongs to  $\text{Int}(O)$  so that  $\alpha O \in \mathcal{D}$  and  $O \subset \text{Int}(\alpha O)$  for any  $\alpha > 1$ . Moreover we have  $h_*^d(f, \mathcal{J}_{\alpha O}) = h_*^d(f, \mathcal{J}_O)$  by Lemma 10. Together with Lemma 11 we get immediately :

**Corollary 11.**

$$\forall O \in \mathcal{D}^1, \quad h_*^d(f, O) = h_*^d(f, \mathcal{J}_O).$$

**Corollary 12.**

$$O \mapsto h_*^d(f, O) \text{ is continuous on } \mathcal{D}^1.$$

Convex polytopes are dense in  $\mathcal{D}$ . Therefore we get with  $\mathcal{P}$  being the collections of convex  $d$ -polytopes with the origin in their interior set :

**Corollary 13.**

$$\sup_{O \in \mathcal{D}^1} h_*^d(f, O) = \sup_{P \in \mathcal{P}} h_*^d(f, \mathcal{J}_P).$$

However we will see that the supremum is not always achieved. We prove now a formula for the rescaled entropy of a power.

**Lemma 12.**

$$\forall O \in \mathcal{D}^1 \quad \forall k \in \mathbb{N}, \quad h_*^d(f^k, O) = kh_*^d(f, O).$$

*Proof.* Let  $O \in \mathcal{D}^1$  and  $\mathcal{J} = (J_n)_n \in \mathcal{E}(O)$ . Let  $J_n^k = J_n \oplus \underbrace{I \oplus \cdots \oplus I}_{k \text{ times}}$  for all  $n$ . The sequence  $\mathcal{J}^k = (J_n^k)_n$  belongs also to  $\mathcal{E}(O)$ . Moreover the partition  $\mathbb{P}_{J_n^k}$  is finer than  $\bigvee_{l=0}^{k-1} f^{-l} \mathbb{P}_{J_n}$ . Therefore

$$h_*(f^k, \mathbb{P}_{J_n}) \leq kh_*(f, \mathbb{P}_{J_n}) = h_* \left( f^k, \bigvee_{l=0}^{k-1} f^{-l} \mathbb{P}_{J_n} \right) \leq h_*(f^k, \mathbb{P}_{J_n^k})$$

and we then obtain

$$h_*^d(f^k, \mathcal{J}) \leq kh_*^d(f, \mathcal{J}) \leq h_*^d(f^k, \mathcal{J}^k).$$

We conclude by taking the supremum in  $\mathcal{J} \in \mathcal{E}(O)$ . □

**5.4. Asymptotic properties of the rescaled entropy.** Let  $(X, f)$  be a cellular automaton with domain  $I$ . We will show that in dimension two, the limsup defining  $h_{top}^d(f, \mathcal{J}_O)$  with  $O \in \mathcal{D}^1$  is in fact a limit. We first relate the entropy of  $\mathbb{P}_J$  with the entropy of  $\mathbb{P}_{\partial^\pm J}$  and we prove an upperbound for the rescaled entropy  $h_{top}^d(f, O)$  in term of the first relative quermass integral  $V_{\mathbb{I}}(O)$  with  $\mathbb{I}$  being the convex hull of  $I'$ .

**Lemma 13.** *For any bounded subset  $J$  of  $\mathbb{R}^d$ , we have*

$$h_*(f, \mathbb{P}_J) = h_*(f, \mathbb{P}_{\partial_I^- J}) \text{ and } h_*(f, \mathbb{P}_J) \leq h_*(f, \mathbb{P}_{\partial_I^+ J}).$$

*Proof.* The inequality  $h_*(f, \mathbb{P}_J) \geq h_*(f, \mathbb{P}_{\partial_I^- J})$  follows directly from the inclusion  $\partial^- J \subset J$ . By definition of the domain  $I$  and the erosion  $J \ominus I$ , we have  $\mathbb{P}_J > f^{-1} \mathbb{P}_{J \ominus I}$ . Therefore we get  $f^{-1} \mathbb{P}_J \vee \mathbb{P}_J = f^{-1} \mathbb{P}_{\partial^- J} \vee \mathbb{P}_J$  and then by induction  $\mathbb{P}_J \vee \bigvee_{l=0}^{k-1} f^{-l} \mathbb{P}_{\partial^- J} = \bigvee_{l=0}^{k-1} f^{-l} \mathbb{P}_J$  for all  $k$ . We conclude that :

$$\begin{aligned} h_*(f, \mathbb{P}_J) &= \lim_k \frac{1}{k} H_*(f, \bigvee_{l=0}^{k-1} f^{-l} \mathbb{P}_J), \\ &\leq \lim_k \frac{1}{k} \left( H_*(\mathbb{P}_J) + H_* \left( \bigvee_{l=0}^{k-1} f^{-l} \mathbb{P}_{\partial^- J} \right) \right), \\ &\leq h_*(f, \mathbb{P}_{\partial^- J}). \end{aligned}$$

We also have

$$\mathbb{P}_J \vee \mathbb{P}_{\partial^+ J} > \mathbb{P}_{J \oplus I} > f^{-1} \mathbb{P}_J.$$

Therefore we get now by induction on  $k$

$$\mathbb{P}_J \vee \bigvee_{l=0}^{k-2} f^{-l} \mathbb{P}_{\partial^+ J} > \bigvee_{l=0}^{k-1} f^{-l} \mathbb{P}_J.$$

This implies  $h_*(f, \mathbb{P}_{\partial^+ J}) \leq h_*(f, \mathbb{P}_J)$ . □

**Proposition 14.** *For any  $O \in \mathcal{D}^1$ ,*

$$h_{top}^d(f, O) \leq V_{\mathbb{I}}(O) \log \#\mathcal{A}.$$

*Proof.* Recall that

$$\begin{aligned} h_{top}^d(f, O) &= h_{top}^d(f, \mathcal{J}_O), \\ &= \limsup_n \frac{h_{top}(f, \mathbf{P}_{nO})}{p(nO)}. \end{aligned}$$

Then by applying Lemma 13 we obtain

$$\begin{aligned} h_{top}^d(f, O) &\leq \limsup_n \frac{h_{top}(f, \mathbf{P}_{\partial^\pm nO})}{p(nO)}, \\ &\leq \limsup_n \frac{\#\partial^\pm nO \log \#\mathcal{A}}{p(nO)}. \end{aligned}$$

For all  $k \in \mathbb{N} \setminus \{0\}$  we let  $I_k$  be the domain of  $f^k$  and we denote by  $\mathbb{I}_k$  the convex hull of  $I'_k = I_k \cup \{0\}$ . Clearly we have  $I_k \subset \underbrace{I \oplus \cdots \oplus I}_{k \text{ times}}$ , therefore  $\mathbb{I}_k \subset k\mathbb{I}$ . By Lemma 4, we get for

some constant  $C = C(d)$  :

$$\begin{aligned} h_{top}^d(f^k, O) &\leq (V_{\mathbb{I}_k}(O) + C) \log \#\mathcal{A}, \\ &\leq (V_{k\mathbb{I}}(O) + C) \log \#\mathcal{A}, \\ &\leq (kV_{\mathbb{I}}(O) + C) \log \#\mathcal{A}. \end{aligned}$$

But by Lemma 17 we have  $h_{top}^d(f^k, O) = kh_{top}^d(f, O)$ , so that we finally conclude when  $k$  goes to infinity

$$h_{top}^d(f, O) \leq V_{\mathbb{I}}(O) \log \#\mathcal{A}.$$

□

**Lemma 14.** *Assume from now  $d = 2$ . For any integral polygon  $O$  there is a sequence  $(\epsilon_m)_m$  going to zero such that for all  $m \in \mathbb{N}$*

$$\exists N \forall n > N \forall \mu \in \mathcal{M}(f) \exists k \in \mathbb{Z}^d, \frac{h_\mu(f, \mathbf{P}_{nO})}{n} < \frac{h_{\sigma^k \mu}(f, \mathbf{P}_{mO})}{m} + \epsilon_m.$$

*Proof.* For a fixed face  $F$  of  $O$ , we let  $E_F$  be the band given by  $E_F := T_F^+ O(-h_I(N^F)) \cap T_F^- O$ , thus  $\partial^- nO \subset \bigcup_{F \in \mathcal{F}(O)} E_F$ . There is  $M > 0$  depending only on  $O$  and  $I$  such that for  $m > M$  we may cover  $E_F$  by a collection  $\mathcal{C}_F$  of polygons (a priori not pairwise disjoint) such that

- for any  $C \in \mathcal{C}_F$ , there is  $\alpha_C \in \mathbb{Z}^d$  with  $C = \alpha_C + mO$ ,
- for any  $C \in \mathcal{C}_F$  we have  $\alpha_C + mF \subset T_F$ ,
- the segments  $\alpha_C + mF$  for  $C \in \mathcal{C}_F$  are tiling the line  $T_F$ .



Then for  $1 \ll m \ll n$  the set  $\partial^- nO$  may be covered by a subfamily  $\mathcal{D}$  of  $\bigcup_F \mathcal{C}_F$  with  $\#\mathcal{D} \lesssim n/m$ . Therefore we get

$$\begin{aligned} h_\mu(\mathbf{P}_n O) &= h_\mu(\mathbf{P}_{\partial^- nO}), \\ &\leq h_\mu\left(\bigvee_{C \in \mathcal{D}} \mathbf{P}_{\alpha_C + mO}\right), \\ &\leq \sum_{C \in \mathcal{D}} h_\mu(\mathbf{P}_{\alpha_C + mO}), \\ &\leq \sum_{C \in \mathcal{D}} h_{\sigma^{\alpha_C} \mu}(\mathbf{P}_m O), \\ h_\mu(\mathbf{P}_n O) &\lesssim \frac{n}{m} \sup_{k \in \mathbb{Z}^d} h_{\sigma^k \mu}(\mathbf{P}_m O). \end{aligned}$$

□

**Remark 15.** We ignore if the above Lemma also holds in higher dimensions. In the proof we use the fact that any face  $F$  tiles the hyperplane  $H_F$ . Therefore we may reproduce the proof in higher dimension for polygons satisfying this property, e.g. parrallepipeds or hexagonal prisms.

**Corollary 16.** For any  $O \in \mathcal{D}$ ,

$$h_{top}^d(f, \mathcal{J}_O) = \lim_n \frac{h_{top}(f, \mathbf{P}_n O)}{p(nO)}.$$

*Proof.* When  $O$  is an integral convex polygon, the statement follows directly from Lemma 14 and the variational principle  $h_{top}(f, \mathbf{P}) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f, \mathbf{P})$  for clopen partitions  $P$ . Consider now the general case  $O \in \mathcal{D}$ . Let  $O'$  be a convex polygon with rational vertices satisfying  $O \subset \text{Int}(O') \subset O' \subset \alpha O$  for  $\alpha \ll 1$ . For some  $m \in \mathbb{N}$ , the polygon  $mO'$  is integral. Then we get :

$$\begin{aligned} h_*^d(f, \mathcal{J}_O) &\leq p(O') h_{top}^d(f, \mathcal{J}_{O'}), \text{ according to Lemma 11,} \\ &\lesssim h_{top}^d(f, \mathcal{J}_{O'}), \text{ as } p(O') \sim p(O) \text{ by monotonicity of the perimeter,} \\ &\lesssim h_{top}^d(f, \mathcal{J}_{mO'}), \text{ as } p(O') \sim p(O) \text{ by Lemma 10,} \\ &\lesssim \lim_n \frac{h_{top}(f, \mathbf{P}_{nmO'})}{p(nmO')}, \text{ by the formula for integral convex polygons,} \\ &\lesssim \liminf_n \frac{h_{top}(f, \mathbf{P}_{nm\alpha O})}{p(nm\alpha O)}, \\ &\lesssim \liminf_n \frac{h_{top}(f, \mathbf{P}_n O)}{p(nO)} \text{ by applying again Lemma 10.} \end{aligned}$$

□

Similarly we get :

**Corollary 17.** For any  $\mu \in \mathcal{M}(f, \sigma)$  and any  $O \in \mathcal{D}$ ,

$$h_\mu^d(f, \mathcal{J}_O) = \lim_n \frac{h_\mu(f, \mathbf{P}_n O)}{p(nO)}.$$

**Remark 18.** Clearly we have  $h_\mu^d \leq h_{top}^d$  for any  $\mu \in \mathcal{M}(f)$  but we ignore if a general variational principle holds true.

## 6. RUELLE INEQUALITY

Recall  $(X, \sigma)$  denotes a  $\mathbb{Z}^d$ -subshift. The topological entropy of  $\sigma$  is defined for any Følner sequence  $\mathcal{L} = (L_n)_n$  (see e.g. [19]) as

$$h_{top}(\sigma) = \limsup_n \frac{H_{top}(\mathbf{P}_{L_n})}{\#L_n}.$$

**Lemma 15.** *Then for all  $\epsilon > 0$  there exists  $c > 0$  such that we have for any  $K \subset J$  convex bodies :*

$$H_{top}(\mathbf{P}_{J \setminus K}) \leq \left( \#J \setminus K + cp(J) \right) \cdot (h_{top}(\sigma) + \epsilon).$$

*Proof.* Let  $\epsilon > 0$ . As the sequence of cubes  $\mathcal{C} = (C_n)_n$  defined by  $C_n = [-n, n]^d \cap \mathbb{Z}^d$  is a Følner sequence, there is a positive integer  $m$  such that  $\frac{H_{top}(\mathbf{P}_{C_m})}{\#C_m} < h_{top}(\sigma) + \epsilon$ . Then for some  $c = c(m) > 0$  we may cover  $J \setminus K$  by a family  $\mathcal{F}$  at most  $\frac{\#J \setminus K + cp(J)}{\#C_m}$  disjoint translated copies of  $C_m$ . Therefore

$$\begin{aligned} H_{top}(\mathbf{P}_{J \setminus K}) &\leq \left( \#J \setminus K + cp(J) \right) \frac{H_{top}(\mathbf{P}_{C_m})}{\#C_m}, \\ &\leq \left( \#J \setminus K + cp(J) \right) \cdot (h_{top}(\sigma) + \epsilon). \end{aligned}$$

□

We refine now the inequality obtained in Lemma 14 at the level of invariant measures :

**Lemma 16.**

$$\forall \mu \in \mathcal{M}(f), \quad h_\mu(f, O) \leq h_{top}(\sigma) \int \chi_O d\mu.$$

*Proof.* For any convex domain  $J$  and any  $\mu \in \mathcal{M}(f)$  we have

$$\begin{aligned} h_\mu(f, \mathbf{P}_J) &\leq H_\mu(f^{-1}\mathbf{P}_J | \mathbf{P}_J), \\ &\leq \sum_{A \in \mathbf{P}_J} \mu(A) H_{\mu_A}(f^{-1}\mathbf{P}_J). \end{aligned}$$

Fix  $\epsilon > 0$  and let  $c$  be as in Lemma 15. Then if  $(K_A)_{A \in \mathbf{P}_J}$  is a family of convex bodies in  $\prod_{A \in \mathbf{P}_J} \mathcal{E}_f^-(A, J)$  we obtain

$$\begin{aligned} h_\mu(f, \mathbf{P}_J) &\leq \sum_{A \in \mathbf{P}_J} \mu(A) H_{\mu_A}(f^{-1}\mathbf{P}_{J \setminus K_A}), \\ &\leq \sum_{A \in \mathbf{P}_J} \mu(A) H_{top}(\mathbf{P}_{J \setminus K_A}), \\ &\leq \sum_{A \in \mathbf{P}_J} \mu(A) \left( \#J \setminus K_A + cp(J) \right) \cdot (h_{top}(\sigma) + \epsilon). \end{aligned}$$

By choosing  $K_A$  with  $\#J \setminus K_A$  minimal we obtain

$$h_\mu(f, \mathbf{P}_J) \leq (h_{top}(\sigma) + \epsilon) \cdot \left( \int \text{gr}_J^- f d\mu + cp(J) \right).$$

Therefore we have for any convex exhaustion  $\mathcal{J} = (J_n)_n \in \mathcal{E}(O)$  :

$$\begin{aligned} h_\mu^d(f, \mathcal{J}) &= \limsup_n \frac{h_\mu(f, P_J)}{p(J_n)}, \\ &\leq (h_{top}(\sigma) + \epsilon) \cdot \left( \int \limsup_n \frac{\text{gr}_{J_n}^- f}{p(J_n)} d\mu + c \right), \\ &\leq (h_{top}(\sigma) + \epsilon) \left( \int \text{gr}_{\mathcal{J}}^- f d\mu + c \right). \end{aligned}$$

By taking the supremum over  $\mathcal{J} \in \mathcal{E}(O)$  we get

$$h_\mu^d(f, O) \leq (h_{top}(\sigma) + \epsilon) \left( \int \text{gr}_O f d\mu + c \right).$$

By Lemma 12 we have  $\frac{h_\mu^d(f^k, O)}{k} = h_\mu^d(f, O)$  for any  $k$ . Apply the above inequality to  $f^k$  :

$$h_\mu^d(f, O) \leq (h_{top}(\sigma) + \epsilon) \left( \int \frac{\text{gr}_O f^k}{k} d\mu + \frac{c}{k} \right).$$

When  $k$  goes to infinity and then  $\epsilon$  goes to zero, we conclude  $h_\mu^d(f, O) \leq h_{top}(\sigma) \int \chi_O d\mu$ .  $\square$

## 7. ENTROPY FORMULA FOR PERMUTATIVE CA

The cellular automaton  $f$  is said **permutative** at  $i \in \mathbb{Z}^d$  if for all pattern  $P$  on  $I \setminus \{i\}$  and for all  $a \in \mathcal{A}$  there is  $b \in \mathcal{A}$  such that the pattern  $P_b^i$  on  $I \cup \{i\}$  given by the completion of  $P$  at  $i$  by  $b$  satisfies  $F(P_b^i) = a$ , in particular  $i$  belongs to the domain  $I$  of  $f$ . The CA is said permutative when it is permutative at the nonzero extreme points of the convex hull  $\mathbb{I}$  of  $I' = I \cup \{0\}$  (these points lie in  $I$ ). The algebraic CA as described in the introduction are permutative. The sets  $I'$  and  $\mathbb{I}$  have the same smallest bounding sphere, thus  $R_{I'} = R_{\mathbb{I}}$ .

**Proposition 19.** *The topological rescaled entropy of a permutative CA  $f$  on  $X_d$  is given by*

$$h_{top}^d(f) = R_{I'} \log \#\mathcal{A}.$$

Theorem 1, stated in the introduction, follows from Proposition 19.

**Question.** For a permutative CA, the uniform measure  $\lambda^{\mathbb{Z}^d}$  with  $\lambda$  being the uniform measure on  $\mathcal{A}$  is known to be invariant [20]. Does the uniform measure maximize the entropy ?

Recall that for any  $k \in \mathbb{N} \setminus \{0\}$  we denote by  $I_k$  the domain of  $f^k$  and  $\mathbb{I}_k$  the convex hull of  $I'_k = I_k \cup \{0\}$ . In the following we also let  $C(P, L)$  be the cylinder associated to the pattern  $P$  on  $L \subset \mathbb{Z}^d$ . We also write  $C(P)$  for this cylinder when there is no confusion on  $J$ .

**Lemma 17.** *For any permutative CA  $f$  and any  $k \in \mathbb{N} \setminus \{0\}$ , the CA  $f^k$  is also permutative and*

$$\mathbb{I}_k = k\mathbb{I}.$$

*Proof.* As already observe, the inclusion  $\mathbb{I}_k \subset k\mathbb{I}$  holds for any CA (not necessarily permutative). We will show  $k \text{ex}(\mathbb{I}) \subset I'_k$ , which implies together with  $\mathbb{I}_k \subset k\mathbb{I}$  the equality  $\mathbb{I}_k \subset k\mathbb{I}$ . Let  $i \in \text{ex}(\mathbb{I}) \setminus \{0\} \subset I$ . For a fixed  $k$  we prove by induction on  $k$  that  $f^k$  is permutative at  $ki$ , in particular  $ki \in I'_k$ . Let  $P$  be a pattern on  $I_k \setminus \{ki\}$  and let  $a \in \mathcal{A}$ . Since we have  $I_k \subset I_{k-1} \oplus I$ , we may complete  $P$  by a pattern  $Q$  on  $(I_{k-1} \oplus I) \setminus \{ki\}$ . By induction hypothesis,  $(k-1)i$  lies in  $\text{ex}(\mathbb{I}_{k-1})$  and  $i$  lies in  $\text{ex}(\mathbb{I})$ , therefore  $ki$  does not belong to  $I_{k-1} \oplus (I \setminus \{i\})$ . Therefore there is a pattern  $R$  on  $I \setminus \{i\}$  such that  $f^{k-1}C(Q, (I_{k-1} \oplus I) \setminus \{ki\})$  is contained in the cylinder  $C(R, I \setminus \{i\})$ . As  $f$  is permutative at  $i$  there is  $b \in \mathcal{A}$  with  $F(R_b^i) = a$  or in other terms

$f(C(R_b^i, I)) \subset C(a, \{0\})$ . Since  $f^{k-1}$  is permutative at  $(k-1)i$ , we may find  $c \in \mathcal{A}$  with  $f^{k-1}(C(Q_c^{ki}, I_{k-1} \oplus I)) \subset C(b, \{i\})$ . Therefore we get

$$f^k(C(Q_c^{ki}, I_{k-1} \oplus I)) \subset f(C(R_b^i, I)) \subset C(a, \{0\}).$$

But  $I_k$  is the domain of  $f^k$  and  $P$  is the restriction of  $Q$  to  $I_k \setminus \{ki\}$ , so that we also have  $f^k(C(P_c^{ki}, I_k)) \subset C(a, \{0\})$ , i.e.  $f^k$  is permutative at  $ki$ .  $\square$

For a convex polytope  $J$  and a face  $F$  of  $J$  we consider the subset of  $\partial_{\mathbb{I}}^- J$  given by  $\partial_{\mathbb{I}}^- F := \partial_{\mathbb{I}}^- J \cap T_F^+ J(-h_{\mathbb{I}}(N^F))$ . The sets  $\partial_{\mathbb{I}}^- F$  for  $F \in \mathcal{F}(J)$  are covering  $\partial_{\mathbb{I}}^- J$  but do not define a partition in general. For any  $F \in \mathcal{F}(J)$  we let  $u^F \in \text{ex}(\mathbb{I}) \subset I'$  with  $u^F \cdot N^F = h_{\mathbb{I}}(N^F)$  and we also let  $d_F$  be the the Euclidean distance to  $T_F$ . Then for  $j \in \partial_{\mathbb{I}}^- J$  we let  $F_j$  be a face of  $J$  such that  $d_{F_j}(j + u^{F_j}) = -d_{F_j}(j) + u^{F_j} \cdot N^{F_j}$  is maximal among faces  $F$  with  $j \in \partial_{\mathbb{I}}^- F$ . We consider then a total order  $\prec$  on  $\partial_{\mathbb{I}}^- J$  such that  $i \prec j$  if  $d_{F_i}(j + u^{F_i}) < d_{F_j}(j + u^{F_j})$ . We also let  $\mathcal{F}_{\mathbb{I}}(J)$  be the subset of  $\mathcal{F}(J)$  given by faces  $F$  for which  $u_F$  is uniquely defined. We denote by  $\partial_{\mathbb{I}}^\perp J$  the subset of  $\partial_{\mathbb{I}}^- J$  given by

$$\partial_{\mathbb{I}}^\perp J := \bigcup_{F \in \mathcal{F}_{\mathbb{I}}(J)} \partial_{\mathbb{I}}^- F.$$

**Lemma 18.** *Let  $j \in \partial_{\mathbb{I}}^\perp J$ . Then*

$$\forall k \in \mathbb{N}, j + ku^{F_j} \notin \{j' \prec j\} \oplus k\mathbb{I}.$$

*Proof.* We argue by contradiction : there are  $j' \prec j$  and  $u \in \mathbb{I}$  with  $j + ku^{F_j} = j' + ku$ . Observe that

$$\begin{aligned} d_{F_j}(j + ku^{F_j}) &= d_{F_j}(j + u^{F_j}) + (k-1)u^{F_j} \cdot N^{F_j}, \\ d_{F_j}(j' + ku) &= d_{F_j}(j' + u) + (k-1)u \cdot N^{F_j}. \end{aligned}$$

We will show that the equality between these two distances implies  $u = u^{F_j}$ , therefore  $j = j'$ . Indeed we have

$$\begin{aligned} d_{F_j}(j' + u) &\leq \sup_{v \in \text{ex}(\mathbb{I})} d_{F_j}(j' + v), & u \cdot N^{F_j} &\leq \sup_{v \in \text{ex}(\mathbb{I})} v \cdot N^{F_j}, \\ &\leq d_{F_{j'}}(j' + u^{F_{j'}}), & &\leq h_{\mathbb{I}}(N^{F_{j'}}), \\ d_{F_j}(j' + u) &\leq d_{F_j}(j + u^{F_j}) & u \cdot N^{F_j} &\leq u^{F_j} \cdot N^{F_j}, \end{aligned}$$

therefore  $u \cdot N^{F_j} = u^{F_j} \cdot N^{F_j}$ , and finally  $u = u^{F_j}$  as  $j$  belongs to  $\partial_{\mathbb{I}}^\perp J$ .  $\square$

For a partition  $\mathbf{P}$  of  $X$  and a positive integer  $k$ , we write  $\mathbf{P}^k$  to denote the iterated partition  $\bigvee_{l=0}^{k-1} f^{-l}\mathbf{P}$  in order to simplify the notations.

**Lemma 19.** *Let  $O \in \mathcal{P}$  and let  $k, n$  be positive integers. For any  $A^k \in \mathbf{P}_{nO}^k$  and any pattern  $P$  on  $\partial_{\mathbb{I}}^\perp nO$ , there is  $w \in A^k$  such that  $f^k w$  belongs to  $C(P, \partial_{\mathbb{I}}^\perp nO)$ .*

*Proof.* For any  $j \in \partial_{\mathbb{I}}^\perp nO$  we let  $P_j$  be the restriction of  $P = (p_l)_{l \in \partial_{\mathbb{I}}^\perp nO}$  to  $\{j' \prec j\}$ . We show now by induction on  $j \in \partial_{\mathbb{I}}^\perp nO$  that there is  $w \in A^k$  with  $f^k w \in C(P_j)$ . By Lemma 17 the CA  $f^k$  is permutative at  $ku^{F_j}$  so that we may change the  $(j + ku^{F_j})^{\text{tearith}}$ -coordinate of  $w$  to get  $w' \in X$  with  $(f^k w')_j = p_j$ . Moreover the  $j'$ -coordinates of  $f^k w$  for  $j' \prec j$  only depends on the coordinates of  $w$  on  $\{j' \prec j\} \oplus k\mathbb{I}$  so that by Lemma 18 we still have  $f^k w' \in C(P_j, \{j' \prec j\})$ , thus  $f^k w' \in C(P_{j''})$  with  $j''$  being the successor of  $j$  for  $\prec$  in  $\partial_{\mathbb{I}}^\perp nO$ .  $\square$

**Lemma 20.** *With the notations of Subsection 3.5, we have*

$$\mathcal{F}(T') = \mathcal{F}_{\mathbb{I}}(T')$$

and

$$\forall R > 0, \mathcal{F}(T'_R) \setminus \mathcal{F}_{\parallel}(T'_R) \subset \mathcal{F}_{\mathbb{I}}(T'_R).$$

*Proof.* Let  $F \in \mathcal{F}(T')$  or  $F \in \mathcal{F}(T'_R) \setminus \mathcal{F}_{\parallel}(T'_R)$ . Such a face  $F$  is tangent to  $S_{I'}$  at some  $u \in \text{ex}(\mathbb{I})$  with  $u \cdot N^F = h_{\mathbb{I}}(N^F)$ . Then any  $v$  with  $v \cdot N^F = h_{\mathbb{I}}(N^F)$  belongs to  $T_F$ . But  $T_F \cap \mathbb{I} \subset T_F \cap S_{I'} = \{u\}$ , therefore we have necessarily  $u_F = u$ .  $\square$

We are now in a position to prove Proposition 19.

*Proof of Proposition 19.* The inequality  $h_{top}^d(f) \leq R_{I'} \log \#\mathcal{A}$  follows immediately from Proposition 14 and Proposition 6. By Lemma 19 we have for any  $O \in \mathcal{P}$

$$\forall A^k \in \mathbf{P}_{nO}^k, \#\{A^{k+1} \in \mathbf{P}_{nO}^{k+1}, A^{k+1} \subset A^k\} \geq \#\underline{\partial^{\perp} nO}.$$

Consequently we have

$$\begin{aligned} h_{top}(f, \mathbf{P}_{nO}) &\geq \#\underline{\partial^{\perp} nO} \log \#\mathcal{A}, \\ h_{top}^d(f, \mathcal{J}_O) &\geq \limsup_n \frac{\#\underline{\partial^{\perp} nO}}{n^{d-1}p(O)} \log \#\mathcal{A}. \end{aligned}$$

We first assume that  $S_{\mathbb{I}} = S_{I'}$  is nondegenerated. Let  $T'$  be the dual polytope of a generating polytope  $T$ . By Lemma 20 we have  $\mathcal{F}(T') = \mathcal{F}_{\mathbb{I}}(T')$ , therefore  $\mathcal{F}(nT') = \mathcal{F}_{\mathbb{I}}(nT')$  and  $\partial^{\perp} nT' = \partial^{-} nT'$  for all  $n$ . Applying then Lemma 4 we get for some constant  $C = C(d)$  :

$$\begin{aligned} h_{top}^d(f, \mathcal{J}_{T'}) &\geq \limsup_n \frac{\#\underline{\partial^{-} nT'}}{n^{d-1}p(T')} \log \#\mathcal{A}, \\ &\geq \frac{V_{\mathbb{I}}(T')}{p(T')} \log \#\mathcal{A} - C. \end{aligned}$$

Then it follows from Proposition 6 that :

$$h_{top}^d(f, \mathcal{J}_{T'}) \geq R_{\mathbb{I}} \log \#\mathcal{A} - C.$$

For any positive integer  $k$ , we have  $\mathbb{I}_k = k\mathbb{I}$  according to Lemma 17, so that we get together with the power formula of Lemma 12 :

$$\begin{aligned} h_{top}^d(f, \mathcal{J}_{T'}) &= \frac{h_{top}^d(f^k, \mathcal{J}_{T'})}{k}, \\ &\geq \frac{R_{\mathbb{I}_k}}{k} \log \#\mathcal{A} - \frac{C}{k}, \\ &\geq \frac{R_{k\mathbb{I}}}{k} \log \#\mathcal{A} - \frac{C}{k}, \\ &\geq R_{\mathbb{I}} \log \#\mathcal{A} - \frac{C}{k}, \\ h_{top}^d(f, T') &\geq R_{I'} \log \#\mathcal{A}. \end{aligned}$$

Let us deal now with the degenerated case. Then we have for all  $R > 0$  by Lemma 20

$$h_{top}^d(f, \mathcal{J}_{T'_R}) \geq \limsup_n \frac{\#\underline{\partial^{-} nT'_R} - \sum_{F \in \mathcal{F}_{\parallel}(nT'_R)} \#\underline{\partial^{-} F}}{p(nT'_R)} \log \#\mathcal{A}.$$

As the  $(d-1)$ -Hausdorff measure of the faces in  $\mathcal{F}_{\parallel}(T'_R)$  is small compared to  $p(T'_R)$  for large  $R$ , we also have

$$\limsup_n \frac{\sum_{F \in \mathcal{F}_{\parallel}(nT'_R)} \#\underline{\partial^{-} F}}{p(nT'_R)} \xrightarrow{R \rightarrow +\infty} 0.$$

We conclude as in the degenerated case. Fix  $\epsilon > 0$  and let  $k > C\epsilon^{-1}$ . We obtain finally

$$\begin{aligned} h_{top}^d(f, \mathcal{J}_{T'_R}) &= \frac{h_{top}^d(f^k, \mathcal{J}_{T'_R})}{k}, \\ &\geq \left( \frac{V_{\mathbb{I}_k}(T'_R)}{kp(T'_R)} - \epsilon - \limsup_n \frac{\sum_{F \in \mathcal{F}_{\parallel}(nT'_R)} \#\partial^- F}{p(nT'_R)} \right) \log \#\mathcal{A}, \\ &\geq \left( \frac{V_{\mathbb{I}}(T'_R)}{p(T'_R)} - \epsilon - \limsup_n \frac{\sum_{F \in \mathcal{F}_{\parallel}(nT'_R)} \#\partial^- F}{p(nT'_R)} \right) \log \#\mathcal{A}, \\ &\xrightarrow{R \rightarrow +\infty} (R_{I'} - \epsilon) \log \#\mathcal{A}. \end{aligned}$$

□

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