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► **To cite this version:**

Jean Godon, Sylvain Argentieri, Bruno Gas. A Formal Account of Structuring Motor Actions With Sensory Prediction for a Naive Agent. *Frontiers in Robotics and AI*, In press, 10.3389/frobt.2020.561660 . hal-02596671v2

HAL Id: hal-02596671

<https://hal.science/hal-02596671v2>

Submitted on 31 Oct 2020

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A Formal Account of Structuring Motor Actions With Sensory Prediction for a Naive Agent

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2 ABSTRACT

3 For naive robots to become truly autonomous, they need a means of developing their perceptive
4 capabilities instead of relying on hand crafted models. The sensorimotor contingency theory
5 asserts that such a way resides in learning invariants of the sensorimotor flow. We propose a
6 formal framework inspired by this theory for the description of sensorimotor experiences of a naive
7 agent, extending previous related works. We then use said formalism to conduct a theoretical
8 study where we isolate sufficient conditions for the determination of a sensory prediction function.
9 Furthermore, we also show that algebraic structure found in this prediction can be taken as a
10 proxy for structure on the motor displacements, allowing for the discovery of the combinatorial
11 structure of said displacements. Both these claims are further illustrated in simulations where a toy
12 naive agent determines the sensory predictions of its spatial displacements from its uninterpreted
13 sensory flow, which it then uses to infer the combinatorics of said displacements.

14 **Keywords:** Sensory prediction, sensorimotor contingencies, interactive perception, bootstrapping, developmental robotics.

15 **textcount statistics:** words in text = app. 12000, words outside text (captions, etc.) = 767, number of
16 floats/tables/figures: 9.

1 INTRODUCTION

17 *Autonomous* robots need possess the cognitive capabilities to face realistic and uncertain environments.
18 Classical approaches deal with this problem by giving them *a priori* models of their interaction with their
19 environment. These rely on carefully crafted models of the agent's body (Mutambara and Litt, 1998), its
20 sensors, the environments it will encounter and the nature of the tasks it is setting to perform (Marconi
21 et al., 2011). But said models are notoriously difficult to obtain (Lee et al., 2017), by definition incom-
22 plete (Nguyen et al., 2017) and often fail to generalize to interactions varying in unknown spatial and
23 temporal scales. As it has been previously studied, models of an agent sensorimotor apparatus (Censi and
24 Murray, 2012) or of a mobile robot interaction with its environment (Jonschkowski and Brock, 2015) can
25 alternatively be learned. In particular, these capabilities crucially depend on the robot correctly learning its
26 *perception* in that it represents the interface layer between the raw readings of its sensors and its higher
27 level cognitive capabilities, e.g. decision making or task solving layers.

28 While there certainly is an established practice of mostly treating perception as processing the sensory
29 signal, multiple cues argue that perception can only emerge from the joint *sensorimotor* experience (Noë,
30 2004). The field of interactive perception, reviewed in (Bohg et al., 2017), indeed displays several
31 approaches which roughly adhere to this principle. One particular theoretical framework is that of *Sensori-*
32 *motor Contingencies Theory* (O’Regan and Noë, 2001) (SMCT for short) which asserts that perception is
33 the mastery of invariant structures in the sensorimotor flow an agent discovers during its interaction with
34 the environment. Their theoretical origin as an abstract, generic cognitive construct lends them desirable
35 properties for robotic applications: namely, they could support bootstrapping the learning of perceptual
36 capabilities in a way that does *not* depend on the implementation of the artificial agent considered as well as
37 on the environments in which said learning is done. In this regard, it differs significantly from the modern
38 developmental approaches supported by Deep Learning (Ruiz-del-Solar et al., 2018) which rely on specific
39 structural details of neural networks, i.e. the numeric forms of inputs, outputs and activation of the neurons.

40 Early works on SMCT have been shown to lead to discovery of the color spectrum (Philipona and
41 O’Regan, 2006) and of the dimensionality of ambient space (Philipona et al., 2003, 2004; Laflaquière
42 et al., 2010; Laflaquière et al., 2012) in “naive” agents, which can be extended to that of an internal, path
43 independent, notion of space (Terekhov and O’Regan, 2016). We already proposed different contributions
44 in this field, successively dealing with peripersonal space characterization (Laflaquière et al., 2015), self-
45 contact and body representation (Marcel et al., 2017) and the emergence of a topological representation
46 of sensors poses (Marcel et al., 2019). These works, as well as (Laflaquière et al., 2018; Laflaquière
47 and Ortiz, 2019), devote a significant effort to providing formalisms suited to make explicit –and, where
48 applicable, formally prove– not only the processing required to capture the contingencies, i.e. invariants in
49 the sensorimotor flow of the agent, but also the mechanisms by which said contingencies should appear.
50 This is an attempt to pinpoint the exact conditions of validity of the proposed processes in order to deliver
51 on the promises of genericity of SMCT.

52 Many recent contributions drawing from SMCT revolve around sensorimotor prediction in some way:
53 the ability to discover a sensorimotor prediction is empirically shown to arise from both the temporal
54 structure of the sensorimotor experience (Maye and Engel, 2012) and the spatial coherence of a natural
55 visual environment for a sensor based on a retina (Laflaquière, 2017). Moreover, said ability to predict
56 sensory outcomes has been shown to provide in robots a basis for an egocentric representation of ambient
57 space (Laflaquière and Ortiz, 2019), object perception (Maye and Engel, 2011; Le Hir et al., 2018), action
58 selection (Maye and Engel, 2012), motor control (Schröder-Schetelig et al., 2010) and motor sequence
59 compression (Ortiz and Laflaquière, 2018). Along the “Bayesian brain” approach, predictive processing is
60 even argued to form the mechanistic implementation of sensorimotor contingencies (Seth, 2014). This is
61 very much in line with classical findings in cognitive psychology, both those regarding the physiological
62 implementations of sensorimotor prediction via *efference copies* (von Helmholtz et al., 1925; Sperry,
63 1950; von Holst and Mittelstaedt, 1950) and how it supports, albeit incompletely, a number of perceptual
64 processes (Bridgeman, 1995; Imamizu, 2010; Pynn and DeSouza, 2012; Bhanpuri et al., 2013), as well as
65 those supporting ideomotor theory (Stock and Stock, 2004) according to which actions are equated to their
66 perceptual consequences from a cognitive standpoint.

67 This article follows much of the same approach begun in (Philipona et al., 2003) and subsequently
68 developed in e.g. (Marcel et al., 2017; Laflaquière et al., 2018; Laflaquière and Ortiz, 2019; Marcel et al.,
69 2019). In particular, it sets out to mathematically describe from an *exterior*, “objective”, point of view
70 some properties of the interaction between the robot and its environment which should appear in its
71 sensorimotor flow. To this end we propose a revised formalism building upon the previous instances in

72 these contribution. One notable contribution indeed resides in our proposal remedying their requirement
73 of the agent having a fixed base by transposing the location of sensorimotor contingencies in sets of
74 “displacements” instead of that of motor or sensory configurations. In accordance with the previous remark
75 about genericity, a particular attention is given to the construction of said formalism with assumptions and
76 proofs explicitly detailed. Moreover, the bootstrapping aspect is emphasized throughout the work, much
77 in the spirit of (Marcel et al., 2017; Marcel et al., 2019), highlighting the distinction between the points
78 of view of the agent and of the observer in the description of the problems and an explicit discussion of
79 the degree of *a priori* knowledge given to the agent, in terms of both data and computations available to
80 it. There lie two contributions of this article: while the formalism is used to formally describe why and
81 how spatial coherence lead to the discovery of sensory prediction very much like alluded to in (Laflaquière,
82 2017) and how this sensory prediction encodes spatial structure akin to that of (Terekhov and O’Regan,
83 2016; Laflaquière and Ortiz, 2019), it does so with a greater emphasis put on the precise relations between
84 the algebraic structures at play and with much weakened assumptions about *a priori* capabilities of the
85 agent, much closer to those put forward in (O’Regan and Noë, 2001). We argue that this formalism unifies
86 and extends those found in previous works; that the formal structures its expressive power makes explicit
87 (e.g. group morphisms between action and prediction) give a conceptual explanation of results previously
88 achieved by more complex means in experimental contexts (Ortiz and Laflaquière, 2018; Laflaquière and
89 Ortiz, 2019); and on a somewhat “philosophical” level that it allows for a clearer picture of the applicability
90 and function of SMCT in the process of bootstrapping perception *via* its systematic distinction of points of
91 view.

92 The paper is organized as follows: to begin with, we introduce in Section 2 all the notations and concepts
93 used for describing the sensorimotor experience. On this basis, Section 3 defines the two distinct points of
94 view and enunciates generic properties of the sensorimotor experience that motivate the proposed study of
95 internal sensorimotor prediction. In particular, the equivalence between the combinatorial structures of
96 actions and sensory prediction is proved. Then, some simulations are proposed in Section 4 to assess the
97 mathematical formalism through a careful evaluation of each step of the proposed framework. We establish
98 that the spatial shifts mediating the sensory experience of a naive agent allow it to determine the sensory
99 outcome of particular actions, in particular those corresponding to displacements of the agent. Further,
100 we show that the ability to predict said outcomes can be used as a proxy to the hidden combinatorial
101 structure of its motor actions. We argue that the theoretical focus adopted in this work provides some new
102 valuable insight into the mechanisms supporting these results, as well as several similar findings presented
103 in aforementioned related works.

2 DEFINING A FORMALISM FOR SENSORIMOTOR INTERACTION

104 This first section aims at expanding several previous results in *Interactive Perception* as obtained for
105 example in (Bohg et al., 2017). These have made use of several classical objects such as the *pose* (or
106 *working*) *space* and the *forward* (either geometrical or sensory) *maps*, at times rearranging their definitions
107 or making them more precise to allow for formal proofs to be derived. Such work is followed upon in this
108 contribution, with a somewhat significant overhaul of the formal definitions. This section is thus devoted to
109 the definitions of the terms we will use to describe a sensorimotor problem, showing during the exposition
110 how they appear in a simple classical example and how they differ from previous theoretical formulations.
111 We then leverage these definitions to propose and prove new perceptive bootstrapping algorithms in the
112 following section.

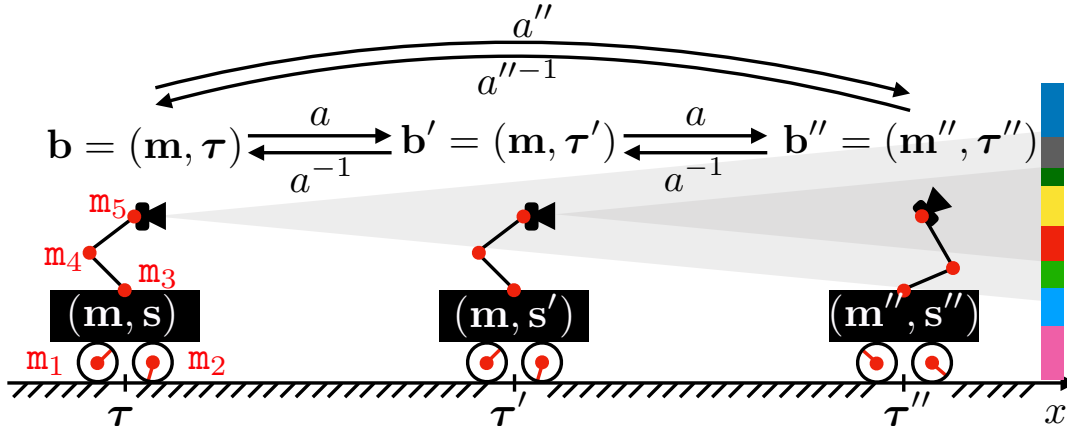


Figure 1. Illustration of the motor actions effects. The agent actuator states m_i are regrouped into its motor configuration \mathbf{m} , while \mathbf{s} denotes its sensor output. Both define the internal agent configuration i.e. the sensorimotor flow the agent has access to. The agent position and orientation in space is captured by τ , which together with \mathbf{m} makes the absolute configuration \mathbf{b} , partially unknown to the agent. Note that one motor configuration \mathbf{m} can be associated to distinct sensory outputs \mathbf{s} and \mathbf{s}' provided a displacement in space from τ to τ' , thus preventing the existence of a mapping $\mathbf{m} \mapsto \mathbf{s}$. The agent may also perform an action a to modify its absolute configuration. One such action a is partially represented, mapping \mathbf{b} to $\mathbf{b}' = (\mathbf{m}' = \mathbf{m}, \tau')$, and \mathbf{b}' to $\mathbf{b}'' = (\mathbf{m}'', \tau'')$. Note that it induces a rigid displacement in the first case, while a change of posture occurs in the second. Also depicted are the inverse action a^{-1} of a and the combination a'' obtained by repeating a twice.

113 2.1 Motor actions

114 As a first step, this subsection is devoted to the introduction of all the notions and definitions of the
 115 *motor* side of the proposed sensorimotor framework. After highlighting the limitations of the previous
 116 approaches, as seen in e.g. (Marcel et al., 2017; Laflaquière et al., 2018), we show how to reparameterize
 117 the sensorimotor interaction by introducing motor actions. Their definition and properties are then carefully
 118 discussed.

119 2.1.1 A look back to previous formalisms

120 Let us consider in all the following an agent endowed with motor and sensing capabilities. In the refer-
 121 enced previous contributions, the sensorimotor interaction is defined by the *internal* motor configuration \mathbf{m}
 122 and the sensory configuration \mathbf{s} of this agent, which lie respectively into some sets \mathcal{M} and \mathcal{S} . Both of them
 123 define the *internal* agent configuration (\mathbf{m}, \mathbf{s}) , i.e. the sensorimotor flow the agent has access to. There is a
 124 clear dependency between the sensory and motor configurations that can be captured by the sensorimotor
 125 maps $\psi : \mathcal{M} \times \mathcal{E} \rightarrow \mathcal{S}$, such that $\psi(\mathbf{m}, \epsilon) = \mathbf{s}$, where $\epsilon \in \mathcal{E}$ represents the state of the environment. As
 126 said in the introduction, other contributions already exploited this kind of parameterization (Philipona
 127 et al., 2003; Laflaquière et al., 2015; Marcel et al., 2017). In all these contributions, only fixed base agents
 128 are considered, since a single internal motor configuration $\mathbf{m} \in \mathcal{M}$ is only mapped to a single sensory
 129 configuration $\mathbf{s} \in \mathcal{S}$ for a fixed environment configuration ϵ .

130 To illustrate this point, Figure 1 represents a 2D-agent able to translate itself only along one dimension x .
 131 This agent is able to move inside an environment made of colored walls thanks to 5 rotating joints whose
 132 states m_i , $i = 1, \dots, 5$, are captured in its motor configuration $\mathbf{m} = (m_i)_i$ (where the second i subscript
 133 in $(\cdot)_i$ denotes the collection being taken with i for ranging variable). To begin, let us consider the case
 134 where m_1 and m_2 are fixed, so that the agent is only able to move its arm supporting a camera-like sensor
 135 generating a sensation \mathbf{s} , i.e. \mathbf{m} is restricted to (m_3, m_4, m_5) only. In such a scenario, one has a fixed base

136 agent for which each motor configuration \mathbf{m} can be mapped to one corresponding sensor pose, which is
137 itself mapped to a sensation \mathbf{s} . This simple statement allows to build structures in \mathcal{M} by exploiting only the
138 sensorimotor flow (\mathbf{m}, \mathbf{s}) , structures that can be leveraged to build an internal representation of the agent
139 body; they can be further refined into a representation of its peripersonal space (Marcel et al., 2017; Marcel
140 et al., 2019). In these works, \mathbf{m} carries all spatial data, possibly with some redundancy, about the coupling
141 between the agent and its environment: the combination of states \mathbf{m} and \mathbf{e} is sufficient to determine the
142 resulting sensory output \mathbf{s} (as described by the formal sensorimotor map ψ).

143 However, what would happen if the same agent was able to perform translations in its environment? Let
144 us now focus on the case where *all* motor states m_i are actually used, as depicted in Figure 1. Indeed, one
145 can imagine a case where the agent moves in its environment along the x axis from (external) position
146 τ (with internal configuration \mathbf{m}) to τ' (same \mathbf{m}). In this case, the sensor samples two different parts of
147 the color wall so that its generated sensations \mathbf{s} and \mathbf{s}' from these two different positions are different.
148 Then two identical internal configurations \mathbf{m} give two different sensations: there is no more mapping
149 between \mathbf{m} and \mathbf{s} , and all the mathematical developments performed in previous works can no longer apply.
150 Therefore, it seems necessary to generalize these formalisms to cope with agents able to move freely in
151 their environment. In this paper, one proposes a *variational* formulation of motor actions to deal with this
152 issue. Importantly, the term *variational* refers in all the following to the focus given on specific sequences
153 of states (e.g. motor, sensory or external states) rather than any specific one of said states. It is introduced
154 in the next subsections.

155 2.1.2 Dealing with mobile agents: reparameterizing the sensorimotor interaction

156 From previous arguments, the internal motor configuration \mathbf{m} can not be mapped unambiguously to
157 sensations without additional considerations. If one still insists on having a functional relation between
158 motor data and sensations, one then needs to enrich the initial motor set. In this paper, one proposes to
159 introduce some superset \mathcal{B} of \mathcal{M} as initial parameter space. This new set \mathcal{B} can be thought of as the set
160 of all *absolute* configurations \mathbf{b} made of pairs (\mathbf{m}, τ) where \mathbf{m} is the internal motor configuration and τ
161 represents an absolute measure of the pose of the agent in its ambient space (which would most commonly
162 be position and orientation in 3D space). While posture or proprioception give \mathbf{m} a definite meaning,
163 one should instead only think of τ as a choice of reference frame in space. Indeed for spaces much like
164 ours it ought to be somewhat arbitrary since any “displacement” from pose τ to τ' could be instead
165 realized as an opposite motion of the whole ambient space like for compensatory movements as initially
166 introduced in (Poincaré, 1895) and dealt with in (Philippona et al., 2003). This equivalency argument has
167 been mentioned in previous contributions as a possible way to deal with mobile agents as proposed in this
168 contribution. While this could be formalized in a consistent way, either from a quasi static or variational
169 perspective, we argue the proposed point of view offers usability advantages, like for multi-agent situations
170 (where space should emerge as a shared common playground), an easier formulation of compensability, or
171 a clear separation between the internal and external points of view. It is important to understand that the
172 agent itself has no knowledge of the current absolute configuration \mathbf{b} of its interaction with its environment,
173 retaining the same hypotheses about a priori structure. However we may then consider the sensorimotor
174 map as a function $\psi : \mathcal{B} \times \mathcal{E} \rightarrow \mathcal{S}$ instead of $\psi : \mathcal{M} \times \mathcal{E} \rightarrow \mathcal{S}$ to account for possible displacements in the
175 environment. Defining such a new set \mathcal{B} allows then to introduce the notion of *external* agent configuration
176 as the tuple (\mathbf{b}, \mathbf{s}) . As such, two different points of view must be stressed out: (i) the external point of
177 view (i.e. coming from the designer of the system) will allow to characterize some properties of the agent
178 interaction with its environment (through modelization, hypotheses, etc.), and (ii) the internal point of view

179 which represents which data and concepts are available to the agent for its operations. This specific point is
 180 discussed in §3.1.

181 Coming back to Figure 1, the agent moves to three successive absolute configurations \mathbf{b} , \mathbf{b}' and \mathbf{b}'' . All of
 182 them are now different, which was not the case of the internal motor configurations: introducing $\mathbf{b} \in \mathcal{B}$
 183 apparently solves the issue raised at the end of §2.1.1. Let us now explain how the agent actually reaches
 184 some given absolute configuration \mathbf{b} .

185 2.1.3 Going variational: introducing motor actions

186 As explained previously, the agent has no direct access to the configuration data \mathbf{b} : it cannot know where
 187 it *is* in \mathcal{B} . Instead we suppose it starts with some (very limited) knowledge of how it *moves* in this set,
 188 i.e. it is capable of performing some moves in \mathcal{B} and of comparing any two moves for equality. To this
 189 end, we propose to introduce some new set \mathcal{A} behaving in the following manner: an element $a \in \mathcal{A}$ can be
 190 applied to any absolute configuration $\mathbf{b} \in \mathcal{B}$ to give a new configuration $\mathbf{b}' = a\mathbf{b} = a(\mathbf{b})$. Therefore, a can
 191 be seen as a function $\mathcal{B} \rightarrow \mathcal{B}$. We will usually denote $\mathbf{b} \xrightarrow{a} \mathbf{b}'$ this situation, and call a a *motor action*.
 192 Such a definition for “actions” differs from many intuitions since it is restricted to quasistatic differences in
 193 posture and position; it does not account for a notion of dynamical effort exerted by actuators. In particular,
 194 no dynamical effects are considered at this level and no specification is made of the precise motor path
 195 taken from \mathbf{b} to $a\mathbf{b}$. Instead, only these pairs of related $(\mathbf{b}, a\mathbf{b})$ endpoints are relevant to characterizing any
 196 action a . This constitutes a present limitation pervading much of similar works to which a future –and
 197 assuredly significant– contribution shall be devoted.

198 Now as we intend to represent the way in which the agent can move in its environment, one can take for
 199 granted the existence of a special action $e \in \mathcal{A}$ that verifies $\forall \mathbf{b} \in \mathcal{B}, \mathbf{b} \xrightarrow{e} \mathbf{b}$: the agent may decide to stay
 200 still. Note that for certain systems, e.g. drones or bipedal walkers, this is distinctly different from doing
 201 nothing since constant posture and position must still be maintained. Moreover, considering it is able to do
 202 any moves a and a' , it may then chain them in one single move $a'' = a'a \in \mathcal{A}$ which satisfies

$$\forall \mathbf{b} \in \mathcal{B}, \quad \begin{array}{ccc} \mathbf{b} & \xrightarrow{a''} & \mathbf{b}'' \\ & \searrow a & \nearrow a' \\ & \mathbf{b}' & \end{array} \quad (1)$$

203 so that \mathcal{A} naturally carries the structure of a *monoid*. Remark that this composition operation is necessarily
 204 associative since motor actions are assumed to behave as functions $\mathcal{B} \rightarrow \mathcal{B}$. In the following, we will
 205 further restrict ourselves to the case where individual actions are *reversible*, that is for any action a there
 206 exists an action a^{-1} such that

$$\forall \mathbf{b} \in \mathcal{B}, \quad \mathbf{b} \xrightleftharpoons[a^{-1}]{a} \mathbf{b}', \quad (2)$$

207 making \mathcal{A} into a *group*. Seeing as how actions can be thought of as mappings $\mathcal{B} \rightarrow \mathcal{B}$, a necessary (and
 208 sufficient) condition is for all mappings in \mathcal{A} to be bijective. It is clear that this assumption of invertibility
 209 may not apply in some experimental contexts, e.g. an agent may jump down a height which it cannot jump
 210 up. This constitutes a current limitation of the proposed framework, although several factors may limit its
 211 severity. In this example, even if the agent can not jump up the height directly, it may still find a sequence
 212 of actions allowing it to climb back up to its original position. Corresponding to the definition of Eq. (2),
 213 this would result in an inverse action in the formal sense, as illustrated later in Section 4.3.3

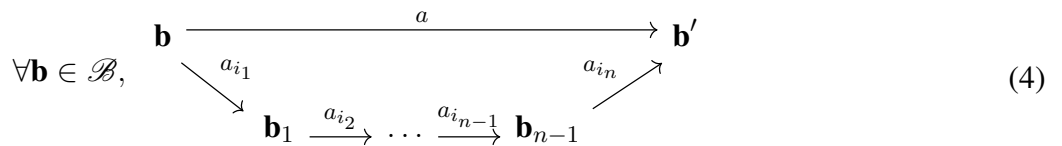
214 Figure 1 illustrates these notions, with the agent moving from external configuration \mathbf{b} to \mathbf{b}' through
 215 an action a . This action, applied at $\mathbf{b} = (\mathbf{m}, \tau)$, happens to produce a translation of the agent so that its
 216 internal motor configuration finishes at the same \mathbf{m} . Note that the agent would be able to return back to
 217 its initial absolute configuration by applying the inverse action a^{-1} of a . Moreover, since a is a function
 218 defined on the whole \mathcal{B} set, the same action can be applied at $\mathbf{b}' = (\mathbf{m}, \tau')$ to reach a third configuration
 219 $\mathbf{b}'' = (\mathbf{m}'', \tau'')$. This time, the same action a has conducted to a global displacement of the agent in the
 220 environment, combined with a change in its internal motor configuration. Indeed, while it represents cases
 221 which are mostly avoided for practical reasons, it is not required for a to only depend on \mathbf{m} in the general
 222 case: the outcome of the same action a may depend on the position τ of the agent in the environment.
 223 Finally, the agent would have been able to move from \mathbf{b} to \mathbf{b}'' by applying the action $a'' = a^2$, as per
 224 Equation (1).

225

226 With these structure assumptions, for a given subset of *motor primitives* $\mathcal{A}' \subset \mathcal{A}$ available to the agent,
 227 we can search for the set of composed moves the agent can actually reach by iteration of its known ones.
 228 We shall say an action $a \in \mathcal{A}$ *decomposes over* $\mathcal{A}' = \{a_i\}_{i \in I}$ if it can be written in the form

$$a = a_{i_n} \dots a_{i_1} = \prod_{1 \leq k \leq n} a_{i_k}, i_k \in I \tag{3}$$

229 This represents a formal property functionally similar to that of compositionality of motor trajectories,
 230 with \mathcal{A}' filling for actions the role of primitives (Flash and Hochner, 2006). Indeed, the interest of these
 231 decompositions appears because the effect of composed moves on motor configurations boils down to the
 232 effects of its components as per the following diagram:



233 In the example in Figure 1, it may well be that the agent can move to any configuration \mathbf{b} i.e. that its
 234 action set is $\mathcal{A} = \mathbb{R}^5$ (for the 5 possible angular increments of its 5 joints). But it may also be restricted
 235 to a limited set of moves, for example if it only can send discrete commands to its joints. For instance, if
 236 each actuator is a stepper motor, then its action set turns into $\mathcal{A} = \mathbb{Z}^5$. In this case, a would be written
 237 as the tuple $(\Delta q_i)_i, i = 1, \dots, 5$, where $\Delta q_i \in \mathbb{Z}$ is the i^{th} motor increment expressed in step increments.
 238 Consequently, any action a would decompose over $\mathcal{A}' = \{a_i\}_i$ where action a_i corresponds to adding
 239 one step to the i^{th} actuator. In this specific case, while \mathcal{A} is infinite, it is sufficient for the agent to know
 240 the 5 motor primitives a_i to generate any action $a \in \mathcal{A}$. This is very similar to the notion of reducing a
 241 (finite dimensional) vector space, which is usually infinite, to the very finite subset of a generating set or if
 242 possible a base. However it can be proven that any finite subset of \mathbb{R} will *not* generate it as a group, and that
 243 it will often only generate a discrete $k\mathbb{Z}$ subgroup. This occurs in the proposed simulations in §4, where all
 244 combinations of a finite subset of starting actions lead to the discovery of a discrete generated action group.

245 2.2 Grounding sensations in space

246 The previous subsection was devoted to the introduction of actions on the motor side of the proposed
 247 sensorimotor framework. This subsection accordingly deals with the *sensory* side of it, and more particularly

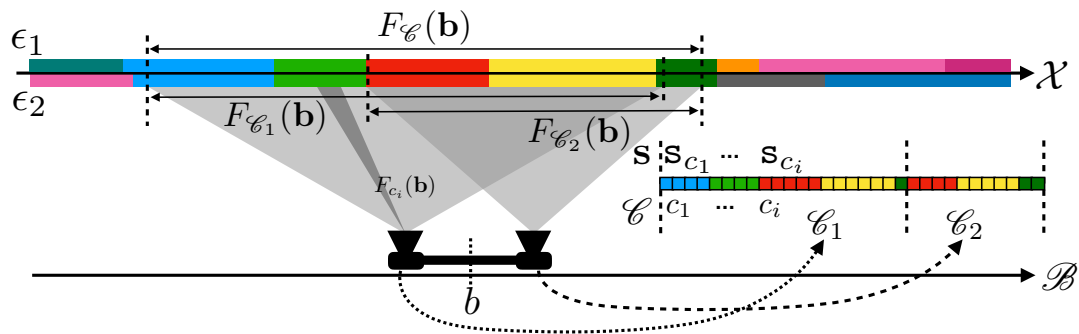


Figure 2. Illustration of the receptive fields for a sensor made of two rigidly linked cameras at configuration **b**. Each pixel c_i of either camera produces a sensory value s_{c_i} in the overall sensory array \mathbf{s} explained only by a small subset of space $F_{c_i}(\mathbf{b})$. The same applies for both cameras, thus explaining how a sensation for the agent can be explained by the perception of a subset of space.

248 with its relation to a persistent “space” which was entirely absent from previous considerations. After a
 249 more precise definition of the meaning of “environment configuration”, the link between local perception
 250 and spatial considerations is formalized. This will constitute the root of the theoretical developments
 251 proposed in the next section.

252 2.2.1 Decoupling space and environment : the *where* and the *what*

253 In previous works, a traditional way for parameterizing the environment was to introduce the environment
 254 configuration ϵ . The meaning of such a variable was often left unspecified, almost without any formal
 255 semantics linking it to the sensorimotor experience of the agent (Laflaquière et al., 2015). In this paper
 256 it is proposed to stress the difference between the ambient geometrical *space*—in which sensorimotor
 257 experience occurs—and the *environment* itself—that is the state of “things” lying in this space. The former
 258 takes the form of some set \mathcal{X} endowed with a spatial structure as encoded by a group $\mathcal{G}(\mathcal{X})$ of *admissible*
 259 *transformations*. These spatial transformations are mappings $\mathcal{X} \rightarrow \mathcal{X}$ preserving some “geometry” of \mathcal{X} .
 260 The most common illustration is the usual affine geometry of \mathbb{R}^3 given by the group $\text{SE}_3(\mathbb{R}) = \text{SO}_3(\mathbb{R}) \rtimes \mathbb{R}^3$
 261 of its rigid transformations, made of 3D rotations, translations and their compositions. On this basis, one
 262 chooses to particularize a “state of the environment” as a valuation that maps each point of \mathcal{X} to its
 263 corresponding physical properties such as temperature, color, luminance, etc. These states are therefore
 264 best represented as functions $\epsilon : \mathcal{X} \rightarrow \mathcal{P}$ where \mathcal{P} is a set describing the different physical properties the
 265 agent can observe. Consequently, $\epsilon(\mathbf{x})$ represents the observable physical properties at point $\mathbf{x} \in \mathcal{X}$. We
 266 will henceforth denote \mathcal{E} the set of *environment states*, i.e. a set of such functions ϵ .

267 Figure 2 illustrates these considerations. In this simple case, the geometrical space \mathcal{X} is monodimensional,
 268 represented as an axis where each point \mathbf{x} is assigned a color through a function ϵ_1 or ϵ_2 . Interestingly,
 269 one can now distinguish points in \mathcal{X} on which the physics described by different environment states ϵ_1
 270 and ϵ_2 locally coincide, as far as the agent is able to observe this coincidence. Particularizing the former
 271 unspecified ϵ state to a function ϵ of the spatial variable will allow to express new properties of the
 272 sensorimotor experience, as in the next subsection.

273 2.2.2 Local perception and receptive fields

274 Now that we have formally defined what is “out there” from an external point of view, let us now focus on
 275 the sensory capabilities of the agent. On this specific point, most previous contributions were considering
 276 the full sensory output as atomic data: although it is implemented as a possibly high dimensional vector,

277 elements and subarrays were generally kept from scrutiny. On the contrary, we now take interest at the
278 subarray level and accordingly adapt the formalism. Therefore,

279 in all the following the sensorimotor map is written as $\psi_c : \mathcal{B} \times \mathcal{E} \rightarrow \mathcal{S}_c$ where the c subscript outlines
280 that the sensory map is explicitly written for a sensory element c (or *sensel*, i.e. one pixel for a camera, the
281 cochlea cell coding for one sound frequency, etc.). Thus, the sensorimotor map $\psi_{\mathcal{C}}$ for the entire sensory
282 apparatus is made of the aggregate of all sensels along $\psi_{\mathcal{C}} : \mathcal{B} \times \mathcal{E} \rightarrow \mathcal{S} = \prod_{c \in \mathcal{C}} \mathcal{S}_c$ with \mathcal{C} the set of
283 all sensels¹. An illustration of these points is proposed in Figure 2 for a similar agent endowed this time
284 with two cameras so as to better show the descriptive capabilities of the formalism. In this case, the sensels
285 c_i —each depicted as elements in a color array— represent the pixels of either camera. Separate sensors in
286 the apparatus thus appear as sub-arrays in \mathcal{C} : the first (resp. second) camera is figured by \mathcal{C}_1 (resp. \mathcal{C}_2).
287 Note that this decomposition of \mathcal{C} as $\mathcal{C}_1 \cup \mathcal{C}_2$ directly comes from our external understanding of the agent
288 structure (i.e. with one camera corresponding to one set of sensels, i.e. one sensor). One could have selected
289 others sub-arrays to form a distinct set of (virtual) sensors not necessarily corresponding to their (physical)
290 implementation on the agent.

291 With space and sensors made formally precise we can now proceed with the (spatial) *receptive field* of
292 sensor $\mathcal{C}' \subset \mathcal{C}$, that is a region of space which environment state suffices to determine the output of \mathcal{C}' .
293 This region as a subset of \mathcal{X} should naturally depend on the current configuration \mathbf{b} since moving causes
294 one's sensors to sample new parts of space, so that it takes the form of a map $\mathbf{b} \in \mathcal{B} \mapsto F_{\mathcal{C}'}(\mathbf{b}) \subset \mathcal{X}$.
295 Then, its characteristic property is

$$\begin{aligned} \forall \epsilon_1, \epsilon_2 \in \mathcal{E}, \forall \mathbf{b} \in \mathcal{B}, \\ \epsilon_1|_{F_{\mathcal{C}'}(\mathbf{b})} = \epsilon_2|_{F_{\mathcal{C}'}(\mathbf{b})} \Rightarrow \psi_{\mathcal{C}'}(\mathbf{b}, \epsilon_1) = \psi_{\mathcal{C}'}(\mathbf{b}, \epsilon_2). \end{aligned} \quad (5)$$

296 Figure 2 represents some of the receptive fields for the two cameras agent. The first one, $F_{c_i}(\mathbf{b})$, is the
297 receptive field of a single sensel/pixel $c_i \in \mathcal{C}$. The receptive fields $F_{\mathcal{C}_1}(\mathbf{b})$ and $F_{\mathcal{C}_2}(\mathbf{b})$ of each camera can
298 be obtained as the union of the receptive field $F_{c_j}(\mathbf{b})$ of their respective pixels. In the same vein, the overall
299 receptive field of the agent $F_{\mathcal{C}}(\mathbf{b})$ is also given by $F_{\mathcal{C}_1}(\mathbf{b}) \cup F_{\mathcal{C}_2}(\mathbf{b})$. From the same figure, it is clear that
300 even if $\epsilon_1 \neq \epsilon_2$ (since there are areas of different colors on the \mathcal{X} axis), the sensation captured by the agent
301 is the same since the aforementioned differences are restricted to areas of space unseen to the agent.

302 It is important to notice that this is the formal step where the notion of receptive field formalizes an
303 implicit relation between the sensations of the agent and spatial structure. This constitutes one fundamental
304 property sufficient to leverage spatial knowledge from the agent interaction with its environment. The
305 application of these theoretical elements is proposed in the next section.

3 A ZERO-TH LAYER OF SENSORIMOTOR CONTINGENCIES: SPATIAL REGULARITIES THROUGH VARIATIONS

306 In this section, we proceed by describing how the formal elements from Section 2 can be arranged
307 to enunciate some interesting properties of the sensorimotor interaction. First, to keep in line with
308 considerations of minimalist bootstrapping, the assumptions we use relative to the model of knowledge of
309 the agent are discussed, and compared to that of previous contributions. Then, the definitions provided in
310 the previous sections are used to isolate conditions where the spatial structure of the receptive fields can be

¹ In all the following, the map $\psi_{\mathcal{C}}$ will be shortened to ψ when there is no ambiguity, consistently with the initial definition of the sensorimotor map of the agent recalled in §2.1.1.

311 leveraged, in particular *via* a certain class of “conservative” actions which are themselves defined. We prove
 312 that under these conditions a naive agent may achieve the determination of sensory prediction functions
 313 for said conservative actions, and that the algebraic structure of these prediction functions matches that
 314 of their actions. The corresponding results are of two distinct but equally important sorts: some, taking
 315 the viewpoint of an external observer, assert that certain particular objects of interest (such as a *sensory*
 316 *prediction function*) *exist*; others guarantee these objects to be *computable* in the boundaries set by our
 317 model of knowledge. This endeavor is made in an effort to keep *a priori* knowledge to a minimum, and
 318 these proofs are generally of a constructive nature.

319 3.1 Model of knowledge of the agent

320 In the authors’ previous works (Marcel et al., 2017; Marcel et al., 2019), sensorimotor interaction
 321 occurred as a sequence of (generally discrete) steps where at each point, the agent could access both
 322 its proprioception $\mathbf{m} \in \mathcal{M}$ (seen as an array of current joint configuration states) and its corresponding
 323 exteroceptive array $\mathbf{s} = \psi(\mathbf{m}, \epsilon)$. These sensory arrays were then compared for equality (and for equality
 324 *only*) *as total vectors*, that is the agent may not access the vectors component by component. It is crucial
 325 to note that, much like in the referenced articles —and following the argument that “there is no *a priori*
 326 reason why similar neural processes should generate similar percepts” as found in (O’Regan and Noë,
 327 2001)— we will assume here that the sensory signals are *uninterpreted* in the very strong sense that they
 328 retain no other structure than equality. This presents an *a priori* significant hurdle since this includes e.g.
 329 order comparisons, substractions, metric structures and precludes us from using objects such as gradients
 330 or clusterings, which are required in almost all comparable works ((Censi and Murray, 2012; Montone
 331 et al., 2015; Laflaquière, 2017) among others). This knowledge was then used for example to compute
 332 set-theoretic motor kernels (Marcel et al., 2017) which were shown to be a structural invariant of the
 333 sensorimotor interaction (Marcel et al., 2019). By contrast, in this paper slight modifications are applied.
 334 Indeed, from the external point of view we now have \mathcal{B} as a functional analogue to the previous \mathcal{M} ,
 335 that is the set of “parameter” data that entirely determines the state of the interaction between agent
 336 and environment. However, as the definition of \mathcal{B} refers to some explicitly *external* data (i.e. the τ in
 337 $\mathbf{b} = (\mathbf{m}, \tau)$), we cannot assume its knowledge from the point of view of the agent. We could however elect,
 338 on the same basis previous contributions used, to assume internal knowledge of the \mathbf{m} part of $\mathbf{b} = (\mathbf{m}, \tau)$.
 339 Instead, we even assume no direct access to “proprioceptive” data and treat it as unknown to the agent. Our
 340 hypothesis is that the agent should learn to isolate what part of its proprioception lies in its unified sensory
 341 array \mathbf{s} from the statistics of its sensorimotor experience.

342 As for remedies, it is instead where a variational approach, as defined in Section 2.1.1, is preferred: while
 343 configuration data represented by $\mathbf{b} \in \mathcal{B}$ still exists as an *external* object, the agent may only choose a
 344 *motor action* $a \in \mathcal{A}$ which, applied at \mathbf{b} , yields the following configuration $\mathbf{b}' = a\mathbf{b} = a(\mathbf{b})$. The agent is
 345 therefore given the capacity to compare any two elements of \mathcal{A} for equality, so that it may tell whether
 346 at any two steps of its sensorimotor experience it performed two identical or distinct actions. Moreover,
 347 much deeper change in knowledge occurs at the level of sensory readings: in the following we not only
 348 ask that the agent be able to compare its entire sensory output $\mathbf{s} = (s_c)_{c \in \mathcal{C}}$ for equality as a vector, but
 349 that it also can check for equality two values of any given sensel. That is, for every sensel $c \in \mathcal{C}$, for
 350 every values s_c, s'_c this sensel may output, the agent may test whether $s_c == s'_c$. In this contribution, it
 351 will be further assumed that the values output by distinct sensels are themselves *a priori* comparable for
 352 equality. While it is a common property in many classical applications, this limitation has been partially
 353 tackled in (Laflaquière, 2017) via sensory prediction. However, this solution relies on clustering methods

354 implicitly exploiting structure assumptions we do not yet consider available. Therefore, this remains a
 355 current limitation of our approach which will be addressed in a future ongoing work.

356 3.2 Sensorimotor binding: a marketplace for spatial information

357 The formalism introduced in Section 2 makes space appear as a variable in the sensorimotor equations
 358 via the receptive field, which we will use in this section to prove that under some reasonable assumptions
 359 we can talk about the spatial information content of a sensory signal. This in turn is used to form the basis
 360 of a *sensory prediction* the agent can use to try and infer the sensory consequences of its motor actions,
 361 mirroring the psychological construct of forward sensory model which is at the heart of ideomotor theories.
 362 This is the core idea we will further develop in the simulations of Section 4 to see how a naive agent can
 363 derive such a prediction function from its sensorimotor flow.

364 Recall that for any given sensel $c \in \mathcal{C}$ and environment state $\epsilon \in \mathcal{E}$, we introduced F_c the receptive field
 365 of sensel c as the function which given agent configuration $\mathbf{b} \in \mathcal{B}$ yields $\mathcal{X}' = F_c(\mathbf{b}) \subset \mathcal{X}$ the minimal
 366 region of space which entirely determines the output of ψ_c . Therefore we can write

$$\forall \mathbf{b} \in \mathcal{B}, \forall \epsilon \in \mathcal{E}, \psi_c(\mathbf{b}, \epsilon) = f_c(\epsilon|_{F_c(\mathbf{b})}), \quad (6)$$

367 where f_c is a “sensitivity” function (or filter) which converts the physical properties of environment sampled
 368 into a sensory output, both selecting to which property the sensor reacts and how. Equation (6) describes
 369 the sensorimotor dynamics by dissociating the spatial dependency (which is given by F_c) and the sensitivity
 370 one (as seen with f_c), so that the observer can now speak of sensels that *look at the same region of space*.
 371 Let us then consider a particular condition, in which for a given action a , some sensel c_i samples *after* a the
 372 same point some other sensel c_j was sampling before the agent began to move. This is formally described
 373 by the relation

$$\forall \mathbf{b} \in \mathcal{B}, F_{c_i}(a\mathbf{b}) = F_{c_j}(\mathbf{b}). \quad (7)$$

374 This situation where the spatial difference between two sensels can be bridged by a displacement of the
 375 agent over time to make their respective sensory experiences coincide has already proven to yield interesting
 376 structures as in (Montone et al., 2015; Laflaquière, 2017). However, these works largely dealt with the
 377 geometry of sensels and sensors, while we aim to elaborate on how this relates to the structure of actions.
 378 As for us, to have this relation apparent to the agent we also require that the output of these particular
 379 sensels be comparable, as already discussed in §3.1. In the strictest sense, this can be by requiring that their
 380 sensitivity functions f_{c_i} and f_{c_j} are equal. It follows that

$$\begin{aligned} \forall \mathbf{b} \in \mathcal{B}, \quad \forall \epsilon \in \mathcal{E}, \\ \psi_{c_i}(a\mathbf{b}, \epsilon) &= f_{c_i}(\epsilon|_{F_{c_i}(a\mathbf{b})}) \\ &= f_{c_j}(\epsilon|_{F_{c_j}(\mathbf{b})}) = \psi_{c_j}(\mathbf{b}, \epsilon). \end{aligned} \quad (8)$$

381 While there are reasons to hope that a working relation could be found even for dissimilar f_{c_i} and f_{c_j} , one
 382 should remember that at the moment the outputs of sensels c_i and c_j lie in some sets totally devoid of
 383 structure. Therefore, even though a conversion function $C_{i,j}$ such that $\psi_{c_i}(a\mathbf{b}, \epsilon) = C_{i,j}(\psi_{c_j}(\mathbf{b}, \epsilon))$ might
 384 exist, we would lack the means to represent it in any way but the collection of the related sensory outputs,
 385 e.g. as opposed to the already resource heavy clustering done in (Laflaquière, 2017).

386 Equations (7) and (8) are both illustrated in Figure 3, where a 1D (infinite) pixel array is placed in front
 387 of a 1D colored line along which the sensor can translate itself thanks to actions a . Equation (7) is captured

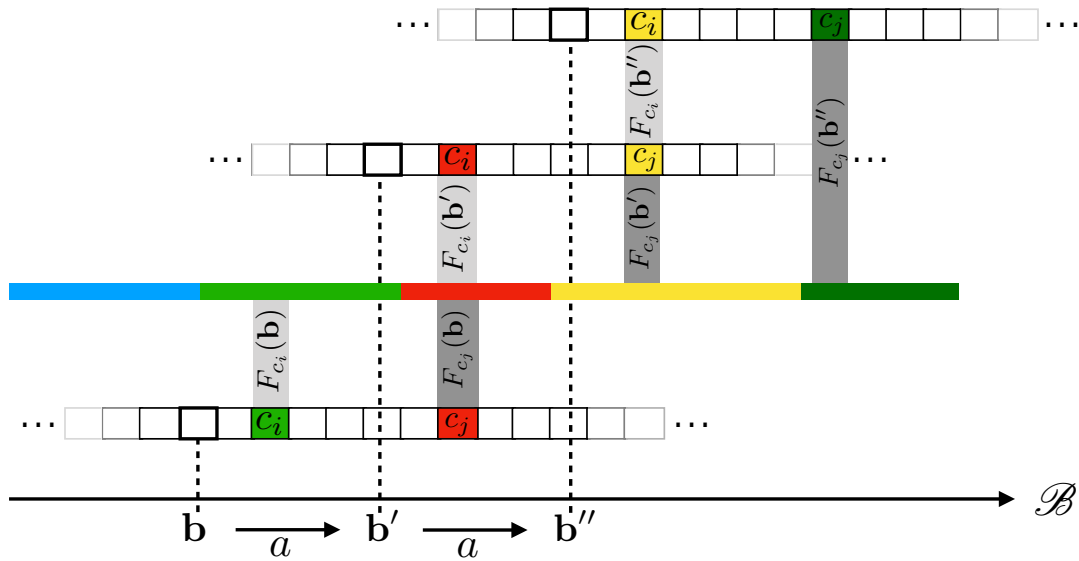


Figure 3. Illustration of how the underlying 1D space induces transitions between cross-sensory outputs. In this case, under action a , sensel c_i takes the place of sensel c_j : the output of c_i after a (red) is the same as the output of c_j before a (red). The same applies when performing a a second time: the yellow color is transferred from c_j to c_i .

388 by the fact that both receptive fields $F_{c_j}(\mathbf{b})$ and $F_{c_i}(\mathbf{b}')$, drawn as two rectangular shapes, project on the
 389 same area on the environment. Then, Equation (8) explains how, provided the environment state ϵ at these
 390 locations stays constant through any one execution of a , it causes both sensels to actually generate the same
 391 sensory (red) output. It is clear that the spatial relation being forwarded to sensory transitions depends on
 392 the sensels actually outputting the same (red) value. This might be argued to be a restrictive assumption.
 393 Nevertheless, being able to deal with different sensitivity functions is a sizable development to which an
 394 ongoing contribution shall be devoted. To conclude, a key point here is that a property entirely defined from
 395 the external point of view through receptive fields is accessible from the internal one by the constraints it
 396 imposes on the sensels outputs values during exploration. Equation (8) therefore shows how space, insofar
 397 as it is common to all sensels and actions, makes this phenomenon of shifts of receptive fields into an
 398 observable contingency of the agent’s sensorimotor experience.

399 3.3 A motor and sensory account of spatial conservation

400 3.3.1 Conservation through permutation: conservative actions

401 The result obtained in the previous subsection exhibits an important property making internally available
 402 spatial matching between receptive fields at different timesteps of motor exploration. But given that the
 403 actual motor exploration follows the algebraic structure of actions \mathcal{A} , it still remains to be shown how these
 404 two structures are consistent. This can be made apparent by introducing *conservative actions* as those a of
 405 \mathcal{A} for which *all* sensels of the agent exchange the places they sample: there is conservation of the (spatial)
 406 information available. In terms of the formalism, $a \in \mathcal{A}$ is conservative if it verifies

$$\forall c \in \mathcal{C}, \exists c' \in \mathcal{C} \text{ such that } \forall \mathbf{b} \in \mathcal{B}, F_c(a\mathbf{b}) = F_{c'}(\mathbf{b}), \quad (9)$$

407 generalizing somewhat Equation (7). This characterization makes apparent that many actions can’t be
 408 conservative: for example, “turning back” may only be conservative for the rare agent that “sees” precisely
 409 as much forwards as it does backwards. In fact, the spatiality of the condition on receptive fields makes it

410 so that all readily found conservative actions correspond to displacements of the body of the agent. In the
 411 following, “ $\forall \mathbf{b} \in \mathcal{B}, F_c(a\mathbf{b}) = F_{c'}(\mathbf{b})$ ” will be shortened to the more legible $c \xrightarrow{a} c'$, and c (resp. c') is said
 412 to be the *predecessor* (resp. *successor*) of c' (resp. c) by a . It is proven in Appendix 1 that for conservative
 413 actions a , the relation \xrightarrow{a} can be made into a successor function

$$\begin{aligned} \sigma_a: \mathcal{C} &\rightarrow \mathcal{C} \\ c &\mapsto c' \end{aligned} \quad (10)$$

414 where $c' = \sigma_a(c)$ is a sensel verifying $c \xrightarrow{a} c'$. Therefore, conservative actions can equivalently be thought
 415 of as *permutation of sensels*. Importantly, conservative actions provide a natural framework for exploiting
 416 Equations (7) and (8) during motor exploration. Indeed, it is proven in Appendix 2 that conservative actions
 417 form a *subgroup* $\mathcal{A}_{\mathcal{C}} \subset \mathcal{A}$ for its composition operation. That is to say, chaining conservative actions yield
 418 other actions which are necessarily conservative, and the inverses of conservative actions are themselves
 419 conservative.

420 At this stage, it has been shown how the spatial property of permutation of the receptive fields relates to
 421 the intrinsic motor structure of the agent. However, this does not suffice to make this group structure of
 422 conservative actions accessible to the agent given the *a priori* knowledge we discussed in Subsection 3.1,
 423 since the dependency of the sensorimotor process on the spatial variable is implicit. We must therefore
 424 go through one final step to relate the available informational content (i.e. sensory reading) to the motor
 425 structure.

426 3.3.2 From permutation to prediction: making it into sensory territory!

427 Let us consider the agent at any point (\mathbf{b}, \mathbf{s}) of its sensorimotor experience. Its sensory output is
 428 $\mathbf{s} = \psi_{\mathcal{C}}(\mathbf{b}, \epsilon) = (s_c)_{c \in \mathcal{C}}$, and for any action a it may perform this sensory output should shift to
 429 $\mathbf{s}' = \psi(\mathbf{b}' = a\mathbf{b}, \epsilon)$ provided the environment state stays constant throughout the action. If we now restrict
 430 ourselves to the case of conservative actions, we get

$$\begin{aligned} \mathbf{s}' &= (s'_c)_{c \in \mathcal{C}} \\ &= (\psi_c(a\mathbf{b}, \epsilon))_{c \in \mathcal{C}} = (f_c(\epsilon|_{F_c(a\mathbf{b})}))_{c \in \mathcal{C}} \\ &= (f_c(\epsilon|_{F_{\sigma_a(c)}(\mathbf{b})}))_{c \in \mathcal{C}} = (\psi_{\sigma_a(c)}(\mathbf{b}, \epsilon))_{c \in \mathcal{C}} \\ &= (s_{\sigma_a(c)})_{c \in \mathcal{C}} \end{aligned} \quad (11)$$

431 so that performing motor action a only results in a permutation of the components of the sensory output.
 432 This permutation is exactly σ_a , and therefore is a constant of the agent which does not depend on the actual
 433 current configuration (\mathbf{b}, ϵ) . Equation (11) shows that any *conservative* action $a \in \mathcal{A}_{\mathcal{C}}$ corresponds to a
 434 *sensory* function

$$\begin{aligned} \Pi_a: \mathcal{S} &\rightarrow \mathcal{S} \\ (s_c)_{c \in \mathcal{C}} &\mapsto (s_{\sigma_a(c)})_{c \in \mathcal{C}} \end{aligned} \quad (12)$$

435 which verifies the property

$$\forall \mathbf{b} \in \mathcal{B}, \forall \epsilon \in \mathcal{E}, \psi_{\mathcal{C}}(a\mathbf{b}, \epsilon) = \Pi_a(\psi_{\mathcal{C}}(\mathbf{b}, \epsilon)). \quad (13)$$

436 Per this property, Π_a is a function which given *any* starting sensory reading of the agent can determine
 437 the sensation it would experience after performing action a (provided the environment state stays constant

438 during a). It must be reiterated that a crucial part is that this function operates on *sensory* data, which is
 439 precisely the only data available to the agent.

440 **3.4 Prediction as an internal proxy of the action group**

441 From there, let us now consider

$$\begin{aligned} \Pi: \mathcal{A}_{\mathcal{E}} &\rightarrow \text{Bij}(\mathcal{S}) \\ a &\mapsto \Pi_a \end{aligned} \tag{14}$$

442 with $\text{Bij}(\mathcal{S})$ the set of all bijections from \mathcal{S} onto itself, i.e. Π maps abstract motor actions to their sensory
 443 prediction functions. As proven in Appendix 3, it establishes a group isomorphism between conservative
 444 actions $a \in \mathcal{A}_{\mathcal{E}}$ and their associated sensory prediction maps $\Pi_a \in \Pi(\mathcal{A}_{\mathcal{E}})$, so that

$$\mathcal{A}_{\mathcal{E}} \cong \Pi(\mathcal{A}_{\mathcal{E}}). \tag{15}$$

445 While Equation (15) written as is might easily pass as benign, it is actually a very powerful result and the
 446 centerpiece of our argument. In a similar fashion to Equation (8) before it, this specifies how the algebraic
 447 structure of (conservative) actions—which largely governs the sensorimotor experience—appears as
 448 a contingency of the sensorimotor flow which can be picked up on by an agent as naive as outlined in
 449 Section 3.1. Using the terminology introduced there, it shows how some *external* structures describing the
 450 interaction between agent and environment can be captured from the *internal* point of view. In turn, it is the
 451 enunciation—and the proof—of this result that motivate developing the formalism as in Section 2, going
 452 as far back as absolute configurations $\mathbf{b} \in \mathcal{B}$ and ambient space \mathcal{X} . Equation (15) will be leveraged as part
 453 of the simulations in the following.

4 SIMULATING A 2D VERSION OF OUR TOY MODEL

454 Up until this point, the discussion has been kept to a purely theoretical level. The following section is now
 455 devoted to a simulated experiment illustrating the new proposed formalism. To this end it starts with a
 456 description of the experimental setup, highlighting how it manifests in the proposed formalism of Section
 457 2. Then, we describe what steps the agent goes through and how they relate to the theoretical results we put
 458 forth in the previous section. Finally, we review the observable results of these experiments to inspect how
 459 our earlier theoretical claims appear in practical cases.

460 **4.1 Description of the experimental setup**

461 In the following, we will consider the 2D generalization of the illustrating case used in the previous sec-
 462 tions. That is, the studied agent body is now made of a planar, rectangular camera sat atop omnidirectional
 463 wheels, see Figure 4. These allow for translations along both x and y coordinates, as well as rotations in
 464 the plane. The pixels of the camera are sensitive to the luminance of the ambient stimulus, which for our
 465 experimental purposes is a fixed grayscale image placed above the moving camera. Describing the problem
 466 in the terms of the developed formalism gives:

- 467 • the ambient space \mathcal{X} is the plane \mathbb{R}^2 ;
- 468 • the set of physical properties of space \mathcal{P} is $[0; 255]$ the set of luminance values. Therefore, a state of
 469 the environment $\epsilon \in \mathcal{E}$ is a function which takes points (x, y) of the ambient plane and map them to
 470 luminances as given by the data of the acquired image;

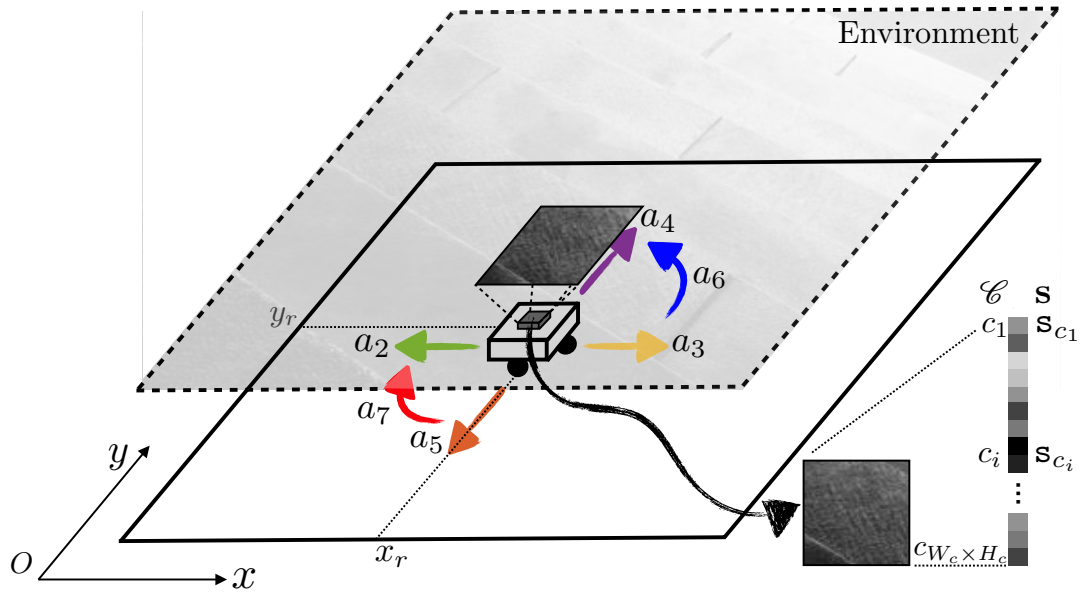


Figure 4. Experimental setup used in simulation to assess the proposed formalism. A holonomic agent is placed in a 2D environment which ceiling is made of a fixed grayscale image. The agent can move in this environment by applying seven different actions a_k . A 10×10 camera pointed towards the ceiling is placed on the top of the agent and generates a sensory array $\mathbf{s} = (s_{c_i})_i$.

- 471 • the configuration space \mathcal{B} is $\mathbb{R}^2 \times S_1 \cong \mathbb{R}^2 \times]-\pi; \pi]$ to account for both position (x, y) and orientation
 472 θ of the robot on the plane;
- 473 • the sensory output of the agent is an array $\mathbf{s} \in [0; 255]^{W_c \times H_c}$, with W_c (resp. H_c) the number of
 474 sensels/pixels in one row (resp. one column) of the camera. In the simulation, the image dimension is
 475 set to $W_c = H_c = 10$. Each of the components s_{c_i} of \mathbf{s} are the sensory output of pixel c_i , given by the
 476 luminance of the spatial location in the environment it is currently looking at. Importantly, the order of
 477 each pixel in \mathbf{s} is chosen arbitrarily.

478 Let us define a set A of *seven* basic actions $a_k, k = 1, \dots, 7$:

- 479 1. one identity action a_1 , mapping any current absolute configuration to itself;
- 480 2. four translations a_2, a_3, a_4, a_5 , one for each direction of the basis axes on the plane, all of amplitude
 481 the size of 1 pixel. These are defined relative to the *current orientation of the agent*, which can end up
 482 distinct from external systems of axes when the agent rotates;
- 483 3. two 90° rotations a_6, a_7 , to account for both clockwise and counter-clockwise turns.

484 These actions are depicted in Figure 4 with colored arrows. Note that the color convention used in this
 485 figure is the same used in the forthcoming figures for coherence.

486 Relative to the prior discussion about properties of motor actions, these are not strictly *conservative* as
 487 per the definition (9): indeed, consider a_5 the elementary “forward” translation. While inner pixels of the
 488 camera will certainly exchange receptive fields, those in the front row will necessarily observe new areas
 489 of space after the agent has moved forward. Therefore none of these front row pixels has any successor for
 490 a_5 , which precludes it from being strictly conservative. The same phenomenon of border unpredictability
 491 occurs for all translations, each with their respective side failing to verify the conservation property. We
 492 nevertheless proceed with the formalism on the basis that actions are at worst, informally speaking, “quasi”

493 conservative. This is based on the quick analysis that, for a N -by- N square camera, this defect only occurs
 494 in N pixels which remains an order of magnitude fewer than the N^2 total.

495 Representing the sensory configuration as numerical arrays makes the permutation of sensels into N_c -by-
 496 N_c sparse matrices, where $N_c = W_c \times H_c$ is the number of sensels. Indeed, starting with any permutation
 497 $\phi : \llbracket 1, N_c \rrbracket \rightarrow \llbracket 1, N_c \rrbracket$ we can define a matrix $M_\phi \in M_{N_c, N_c}(\mathbb{R})$ by

$$M_{\phi,i,j} = \begin{cases} 1 & \text{iff } j = \phi(i), \\ 0 & \text{else.} \end{cases} \quad (16)$$

498 It can then be checked that for any array $\mathbf{s} = (s_i)_{i \in \llbracket 1, N_c \rrbracket}$, the array $\mathbf{s}_\phi = (s_{\phi(i)})_{i \in \llbracket 1, N_c \rrbracket}$ obtained by
 499 permutating the components of \mathbf{s} by ϕ verifies $\mathbf{s}_\phi = M_\phi \mathbf{s}$. It is clear that working with such a representation
 500 incurs a large memory overhead (with only N_c of all N_c^2 coefficients being non null). Furthermore, finding
 501 a permutation is known to be a problem of exponential complexity. However we do not aim to propose a
 502 scalable implementation in the following, but rather to illustrate as a proof of concepts the developments in
 503 Section 3.

504 4.2 Description of the experiments

505 The proposed simulation can be decomposed as a sequence of 2 related, successive, experiments. First,
 506 these are briefly described in a global manner so as to go through the flow of the experiment. Then, each
 507 experiment is described in greater detail with respect to its implementation. It is in this second part that
 508 relevant proofs ensuring both completion and correctness of the endeavor are provided. In this setup, the
 509 robot is given a set $\mathcal{A}_{\text{init}}$ of $n_{\mathcal{A}}$ unknown actions drawn in the set of *combinations* of actions of A . Although
 510 A was designed for convenience from an external point of view, $\mathcal{A}_{\text{init}}$ may not accurately reflect it. Indeed,
 511 for random draws there is a high likelihood of missing actions when $n_{\mathcal{A}}$ is small, of duplicate actions
 512 when it is large. However, as discussed previously these notions do not yet make sense to the agent, which
 513 can only “run” actions drawn. Importantly, at first the considerations will be restricted to the case where
 514 $\mathcal{A}_{\text{init}} = A$. This is a possibly strong assumption about the initial fitness of readily available commands
 515 to the “objective” capabilities of the agent. The influence of this choice and the effect of less optimally
 516 designed starting command shall be discussed in the final part of this section.

517 The first part of the experiment is one of *motor babbling*. During it, the agent effectively runs its available
 518 actions $a_k \in \mathcal{A}_{\text{init}}$ multiple times and tries to figure out whether they are *conservative* by computing
 519 their associated sensel permutation map. This is realized as a sequential process: at timestep t_n , the agent
 520 chooses and runs an action $a_k = a[t_n] \in \mathcal{A}_{\text{init}}$, and the absolute configuration $\mathbf{b}[t_n] = (x[t_n], y[t_n], \theta[t_n])$
 521 is accordingly changed to $\mathbf{b}[t_{n+1}] = a_k \mathbf{b}[t_n]$. Corresponding sensory array $\mathbf{s}[t_{n+1}] = (s_i[t_{n+1}])_i$ is then
 522 used to proceed in the computation of the (candidate) permutation matrix M_{a_k} of a_k , with the details of
 523 the update rule discussed in the following subsection. It must again be stressed that we do not consider
 524 the actual time $t_{n+1} - t_n$ required to perform the action a as a relevant information of the proposed
 525 sensorimotor framework. It may vary for distinct actions without it affecting whatsoever the sequence
 526 of experienced absolute configurations \mathbf{b} . During this exploration, the state of the environment ϵ is also
 527 allowed to vary with time so long as it is not updated during the generation of a_k , i.e. ϵ can change *between*
 528 actions. This is achieved during the simulation by entirely changing the grayscale image presented to the
 529 agent between each action. In the following, the choice of action is randomly made at each timestep. This
 530 may at times slow the learning of the permutation matrices and could certainly be improved, for instance

531 by introducing a necessarily intrinsic criterion like curiosity as in (Oudeyer et al., 2005). However, the
532 specific case studied here is simple enough that a random strategy suffices to obtain good results.

533 Once this first step is complete, the agent computes all products of (quasi) permutation matrices to make
534 the resulting set of matrices. As per Equation (4), this set is precisely the one of all matrices that decompose
535 over the M_{a_k} , $a_k \in \mathcal{A}_{\text{init}}$. Following our argument about the groups of prediction functions and motor
536 actions being isomorphic, this set can be taken as the *global* understanding of its motor capabilities the
537 agent has acquired. Here “global” denotes that new structure, absent from the first empirical phase which
538 was limited to $\mathcal{A}_{\text{init}}$, emerged from the computation of products. Finally, the effect of changing the set of
539 actions available at start on the structure graph discovered in the second experiment is studied in a third
540 part, see 4.3.3.

541 4.2.1 Learning the prediction through sensorimotor interaction

542 The first experiment performed by the agent is computing, where possible, the permutation matrix
543 associated to each of its available motor actions. This is done according to the following procedure: at the
544 beginning of the sensorimotor experience, to each starting action $a_k \in \mathcal{A}_{\text{init}}$ associate a $N_c \times N_c$ matrix
545 M_{a_k} where N_c is the number of sensels. This matrix is initialized so that all of its coefficients are 1. Then,
546 at the end of timestep t_n where it performed action a_k (that is $a[t_n] = a_k$), the agent uses its sensory output
547 arrays both previous ($\mathbf{s}[t_n]$) and current ($\mathbf{s}[t_{n+1}]$) as per the update rule:

$$(M_{a_k}[t_{n+1}])_{i,j} = \begin{cases} 1 & \text{iff } s_j[t_{n+1}] = s_i[t_n] \text{ and } (M_{a_k}[t_n])_{i,j} = 1 \\ 0 & \text{else.} \end{cases} \quad (17)$$

548 Let us first observe that in this rule the only possible change in coefficients is going from 1 to 0: whenever
549 a coefficient $(M_{a_k}[t_n])_{i,j}$ is already 0, the condition of the first case automatically fails so that its value
550 stays at 0. Therefore, the rough dynamics of the update is that while all coefficients start at 1, some are
551 eventually switched to 0 upon exploration until matrices converge to a final (possibly null) form.

552 One can note that this is a very drastic choice compared to the more usual soft incremental rules. This
553 offers increased simplicity such as in Appendix 4 where an argument is provided that for any conservative
554 motor action this algorithm makes the empirical matrix M_{a_k} converge to the associated permutation matrix
555 $M_{\sigma_{a_k}}$. Moreover, we argue that obtaining said convergence with such an unforgiving rule is strong evidence
556 towards the systematic, rather than statistical, nature of the supporting mechanism. The argument also
557 proves that for non conservative actions, under the same richness hypothesis the associated empirical matrix
558 will converge to the null matrix. This fact allows the robot to naively distinguish between conservative and
559 non conservative actions, should he be given the capability to perform both on startup.

560 4.2.2 Inferring motor structure from learned interaction

561 In the second phase of the experiment, the agent uses the prediction functions it discovered for elementary
562 conservative moves to infer how combinations of these moves relate to each other. Indeed, it was proved in
563 the previous part that for any conservative actions a and a' with associated permutation matrices M_{σ_a} and
564 $M_{\sigma_{a'}}$, it is true that

$$M_{\sigma_{a'}} M_{\sigma_a} = M_{\sigma_{a'a}}. \quad (18)$$

565 In the case of actions which are not strictly conservative such as those in the simulation, equality in the
566 previous equation is not guaranteed. This happens because in the $M_{\sigma_{a'}} M_{\sigma_a}$ expression, all the loss of
567 information of a and a' on their respective boundaries is accumulated, whereas $a'a$ might recoup some of

568 it e.g. when $a' = a^{-1}$. However, multiple expressions should at least yield non contradictory prediction,
 569 that is whenever one specifies a pair $c_i \rightarrow c_j$ another other cannot assert $c_i \rightarrow c_k$ with $j \neq k$. As long
 570 as these combinations are kept short enough to limit the accumulation, this non contradiction criterion
 571 can be used by the agent to internally infer the sensory prediction of *any* combination of the moves it
 empirically learned. This is used in a Dijkstra-like process to build a graph of prediction matrices, which

Algorithm 1 Dijkstra-like algorithm for live construction of action group graph.

Input

- A The set of all matrices learned in exp.1
- D A bound on length of matrix combinations used
- O A reference matrix around which to explore

Output

- G A local view of the combinatorial graph of matrix products around O, using edges in A

Add O to collection U

O.depth \leftarrow 0

Add node O to G

while U is not empty **do** ▷ True iff the neighborhood of some node K is still Unexplored

 K \leftarrow node in U

for all M_a in A **do** ▷ Test all learned predictions starting from node K

 P $\leftarrow M_a K$

 P.depth \leftarrow K.depth + 1

if P.depth \leq D **then**

 B \leftarrow False

for all node C in G **do** ▷ Test previously discovered nodes for equality

if predictions for P and C match **then**

 B \leftarrow True

 Set edge $M_a: K \rightarrow P$ in G

end if

end for ▷ END for all node C in G

if B is False **then** ▷ Branch taken iff P := $M_a K$ was not previously discovered

 Add P to U

 Add node P to G

 Set edge $M_a: K \rightarrow P$ in G

end if

end if ▷ END if P.depth \leq D

end for ▷ END for all a in A

 Remove K from U

end while ▷ END while U is not empty

572 runs as follows, see Algorithm 1: starting from a prediction matrix M_0 corresponding to any origin action
 573 a_0 , each of the known matrices M_{a_k} , $a_k \in \mathcal{A}_{\text{init}}$ are applied to yield both a set of new neighboring “end
 574 points” $N_{M_0} := \{M_{a_k} M_0, a_k \in \mathcal{A}_{\text{init}}\}$ and for each pair $(M_0, M_{a_k} M_0)$ a directed edge M_{a_k} . This is then
 575 recursively applied to all newly discovered end points, while those that were previously visited (as the
 576 prediction matrices *can* be compared for equality) are discarded. However, the resulting graph would in
 577 most cases be infinite, therefore a stopping rule must be chosen. In our case, we chose to explore up to a
 578 given depth parameter in graph edge distance.

580 **4.3 Results**

581 This subsection is devoted to the evaluation in simulation of the previous points, divided in three successive
 582 experiments. The first one illustrates how the agent can build the permutation matrices associated to each

583 of its conservative actions; a discussion about the convergence and the statistics of this experiment is then
 584 proposed. The second one exploits the permutation matrices just obtained to structure its own actions
 585 through a graph of their combinations; a discussion about its fidelity as a representation of the action
 586 group \mathcal{A} is proposed. In the third and final one, the effect of varying starting action sets on the structure
 587 discovered is studied, concluding the subsection.

588 4.3.1 Experiment 1: discovering the permutations

589 4.3.1.1 Building the permutation matrices

590 To begin with, the simulated robot in Figure 4 is placed at a random 2D position inside the image to be
 591 explored. The available action set is defined as $\mathcal{A}_{\text{init}} = A$ so that $n_{\mathcal{A}} = 7$. Then, at each time step t_n , a
 592 random action $a_k = a[t_n] \in \mathcal{A}_{\text{init}}$ is run, and the associated permutation matrix M_{a_k} is updated according
 593 to (17). After this update, the agent is able to evaluate if these matrices have finished converging and
 594 therefore can decide when to stop the exploration. An entropy-like internal criterion is proposed to quantify
 595 this convergence, along

$$C(M) = 1 - \frac{1}{N_c \log_2(N_c)} \sum_{i=1}^{N_c} H_i,$$

$$\text{where } H_i = - \sum_{j=1}^{N_c} \frac{M_{i,j}}{\mu_i} \log_2\left(\frac{M_{i,j}}{\mu_i}\right), \quad (19)$$

$$\text{and } \mu_i = \frac{1}{\max(1, \sum_{j=1}^{N_c} M_{i,j})}.$$

596 In this criterion, H_i is the entropy of the post-action output of sensel c_i as a random variable of the pre-
 597 action outputs of all sensels c_j . Therefore it measures which degree of surprise remains in the determination
 598 of which (if any) sensel is successor to c_i . Finally, this makes C into an average measure of certainty in
 599 the discovery of successor sensel pairs, going in nondecreasing trajectories from 0 at initialization to 1 at
 600 permutation matrices. Consequently when it obtains the updated matrices $M_{a_k}[t_{n+1}]$, $a_k \in \mathcal{A}_{\text{init}}$ the agent
 601 computes all $C_k[t_{n+1}] = C(M_{a_k}[t_{n+1}])$ to assess the state of its discovery, stopping its exploration when
 602 all the C_k have reached 1.

603 After convergence, the resulting matrices for all seven actions shown in Figure 4 are depicted in Figure 5.
 604 In this figure, a 0 (resp. 1) is represented in black (resp. white). Since the agent has no knowledge of
 605 its sensor geometry, the position of its sensels (i.e. pixels) inside the sensory array s (i.e. the flattened
 606 image) is randomly chosen. In this case, the resulting permutation matrices for each action is depicted in
 607 Figure 5 (top), demonstrating the fact that those matrices are not easy to understand from an external point
 608 of view. If one now selects a more natural ordering of the pixels inside s , like a line by line arrangement, one
 609 then gets the permutation matrices in Figure 5 (middle). With such an arrangement, an external observer is
 610 now able to get a clearer intuition about the effects of each action on the pixels permutations. Nevertheless,
 611 these two different sets of matrices, as two contingent images of the same underlying structure, are purely
 612 equivalent from an internal point of view. This can be illustrated by mapping the permutation on the
 613 overall sensor to better catch how the agent has been able to discover the underlying spatial transfer
 614 between sensels. This is done by plotting the sensel pairs along which values are transferred as proposed in
 615 Figure 5 (bottom). In this figure, the 10×10 pixel grid of the simulated camera is represented together with
 616 arrows connecting each sensel to its successor. While such a representation requires external knowledge in
 617 the sensor geometry, the arrows are entirely determined by the internal permutation matrices from either of

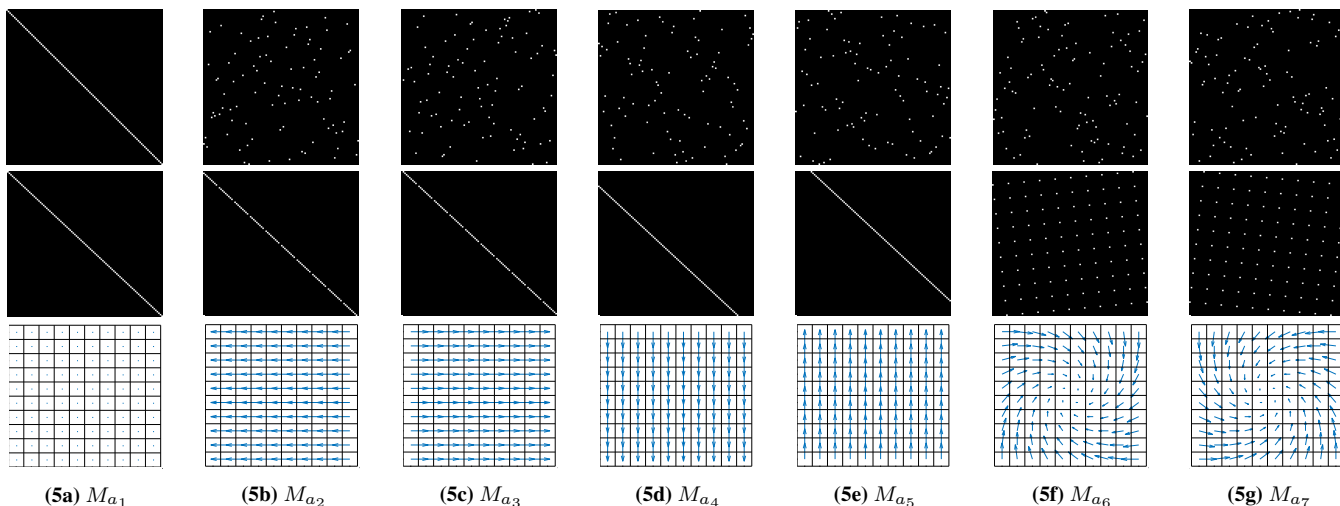


Figure 5. Representation of the seven binary $10^2 \times 10^2$ permutation matrices M_{a_k} corresponding to the actions a_k possibly generated by the agent, where a 0 (resp. a 1) is represented as black (resp. white). **(first row)** Matrices obtained for a random organization of the sensels outputs s_i inside the sensory array \mathbf{s} . **(second row)** Matrices obtained for a well chosen sensels arrangement, where each pixel values are stored line by line in \mathbf{s} . **(third row)** Interpretation of the permutation matrices (either from the first or second row) directly on the physical 10×10 pixel array: if a 1 is present at line i and column j of a matrix M_{a_k} , then an arrow joining pixel i to j is plotted. Note that the arrow length has been resized for actions 6 and 7 (i.e. rotations) to enhance readability.

618 the two sets presented. It is thus a convenient external way to display that each matrix has actually captured
 619 the pixel shift induced by each action. For instance, with such a visualization, it is now very clear that
 620 (a_2, a_3) , (a_4, a_5) or even (a_6, a_7) are all found to be pairs of inverse actions; this specific capability will
 621 actually be exploited in §4.3.2 to structure the agent set of actions.

622 4.3.1.2 A discussion about the dynamics of convergence

623 It is clear from Figure 5 that at some point the agent captured the permutation to the best of its capabilities.
 624 One therefore proposes to study the dynamics of the convergence of the approach w.r.t. the experimental
 625 time step t_n . For the remainder of experiment 1, we now keep the image constant during all the simulation,
 626 so as to better assess the influence of the experienced environment on the results. First, the internal criterion
 627 $C_k = (M_{a_k})$ defined in Equation (19) is evaluated at each t_n and each a_k , resulting in the plot in Figure 6.
 628 One can then confirm that the C_k increase from 0 (all elements in the matrices are initialized at 1) to 1 (all
 629 successor pairs have been discovered). It also appears that for each particular action a_k , the associated
 630 criterion increases in sparse jumps because its matrix M_{a_k} is only actually updated at the random time steps
 631 when a_k is drawn. Figure 6 also illustrates the fact that the amplitude of these jumps decreases over the
 632 experiment. For the starting conditions of this experiment, a detailed analysis shows that about 7 realizations
 633 of each action are necessary to fully discover the target permutation matrices. But it also appears that most
 634 of the initial 1s in the matrices are wiped out very early, with a criterion value $C_k[t_n] \approx 0.7$ after only
 635 one execution of the corresponding action a_k . However, one still questions whether the differences in the
 636 dynamic of all actions is a random occurrence of this particular exploration, or there is an intrinsic variance
 637 in difficulty in learning between actions.

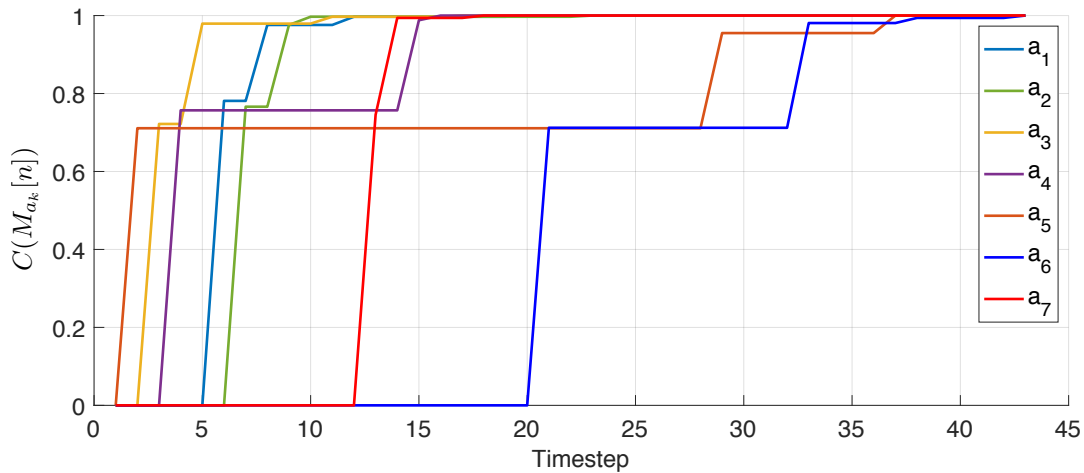


Figure 6. Representation of the criterion $C_k = C(M_{a_k})$ for the seven actions a_k . Each jump in this figure corresponds to a reevaluation of the criterion happening at a timestep when the corresponding action has been drawn in the set in $\mathcal{A}_{\text{init}}$. As expected, the criterion starts from 0 to reach 1, indicating that all possible permutations have been found.

638 4.3.1.3 A statistical analysis about richness of the environment

639 The answer to the previous question can be obtained by performing an empirical survey (i) by averaging
 640 over random explorations for given starting conditions, and (ii) by varying these starting conditions and
 641 comparing the resulting performances. With such a study, (i) will allow to quantify the influence of the
 642 randomness in exploration, while (ii) lets us assess how the properties of the environment influence the
 643 discovery of permutations. For this experiment, the environment is made of the image shown in Figure 7a,
 644 where the starting points of each exploration is depicted as a grid of points on it. At each of these points,
 645 1000 random explorations are conducted, each of them consisting in a random run of actions a_k as in
 646 §4.3.1.1, resulting in 1000 sets of seven C_k curves as in Figure 6. For each random exploration l and each
 647 action a_k , the number of jumps J_{l,a_k} in the C_k curve obtained is taken as a measure of difficulty in learning
 648 the permutation. The average $J = \frac{1}{L} \sum_l \sum_k J_{l,a_k}$ of J_{l,a_k} over all actions a_k and explorations l at a given
 649 starting position is depicted as the color of the grid in Figure 7a, with $L = 1000$ (runs) \times 7 (actions). Green
 650 points correspond to a low number of jumps J , while red ones are representing higher values. One can
 651 observe that the points are overwhelmingly green, and that the red ones are restricted to precise areas in the
 652 picture. These correspond to areas with locally low contrast, such as the sky (in the top left corner) or its
 653 reflection (in the bottom). The extremal conditions corresponding to the two green and red highlighted
 654 points are further compared. For each of them, the distribution of the J_{l,a_k} is plotted as an histogram in
 655 Figure 7b. Clearly, green points correspond to areas in the environment where the permutation matrices
 656 can be discovered in at most 5 executions of actions. On the contrary, at red points the agent must wait
 657 for about 17 on average, and up to 35, executions before it has obtained the same results. This illustrates
 658 how the richness of the environment might influence the agent ability to capture the structure of its sensory
 659 prediction. On a more global scale, Figure 7c shows the distribution of the J_{l,a_k} for all random explorations,
 660 indiscriminately of the starting position. This corroborates the observation that most positions in the image
 661 are green, i.e. lead to easy convergence. It appears that for a randomly selected starting position, there
 662 is more than 66% of chance of permutation matrices being discovered in less than 4 executions of their
 663 corresponding actions.

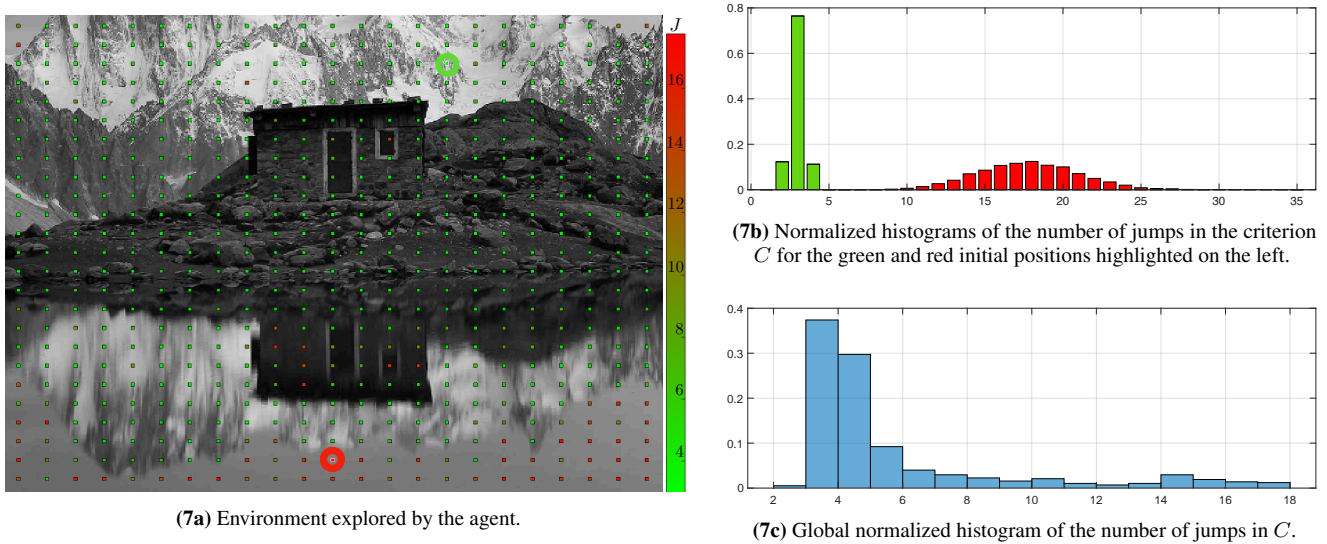


Figure 7. Statistical analysis of the permutation matrices building process. **(left)** Environment explored by the agent. Each point in this environment corresponds to a starting position around which the agent draws actions to build the permutation matrices M_{a_k} . Counting the mean number of jumps in the criterion curves C_k for each realization of the exploration around a given starting point and each action leads to the value J representing the difficulty to build the corresponding matrices. A high (resp. low) J value in red (resp. in green) corresponds to areas in the environment harder (resp. easier) to exploit for sensory prediction. **(right)** Normalized histograms of the number of jumps in the criterion curves C_k averaged across actions. **(top right)** Focus on the histograms obtained around two different starting conditions corresponding to low (resp. high) J value highlighted by a green (resp. red) circle in the environment. **(bottom right)** Overall normalized histogram for all actions and all starting positions in the environment, showing that most of the permutation matrices are correctly obtained after a low number of action run.

664 4.3.2 Experiment 2: structuring actions by combination

665 From the previous experiment, we now have as many permutation matrices M_{a_k} as we have actions in
 666 $\mathcal{A}_{\text{init}}$. As outlined in §4.2.2, one can then use them to build a *graph of prediction matrices* by following
 667 Algorithm 1. Recall that in this graph, a node is a permutation matrix obtained as a combination of the
 668 M_{a_k} matrices, while there is a M_{a_k} edge from matrix M to M' iff $M' = M_{a_k}M$. Therefore, all edges in
 669 the graph correspond to the permutation matrices built during experiment 1. According to Equation (15),
 670 this graph is isomorphic to the graph of corresponding actions, meaning all properties discovered of any
 671 combination of matrices holds true for the corresponding combination of actions. As an example, if one
 672 discovers that $M_{a_1} = M_{a_2}M_{a_3}$, then one also has $a_1 = a_2a_3$.

673 As a first step, let us consider only the actions corresponding to translations in the environment, i.e. a_6
 674 and a_7 are discarded from $\mathcal{A}_{\text{init}}$. This *a priori* selection is only made to simplify the visualization of the
 675 graph at first. After applying Algorithm 1 to the matrices shown in Figure 5, one gets the directed graph in
 676 Figure 8a, where all the color conventions are consistent with experiment 1. This particular graph has been
 677 built for a maximum depth set to 3 and with M_{a_1} taken as the origin of the graph. Note that the depth of
 678 this graph has been maintained voluntarily low so as to help in the reading of the graph. Note also that the
 679 arbitrary choice of origin makes all of its neighbors themselves correspond to one of the M_{a_k} discovered in
 680 experiment 1 since they all occurred as $M_{a_k}M_{a_1} = M_{a_k}$ products, whereas all other nodes are indeed new
 681 matrices.

682 This graph mirrors many algebraic properties of the M_{a_k} as captured by the internal experience. Indeed
 683 one can first observe that the light blue arrow leads from any given node M to itself, which corresponds to

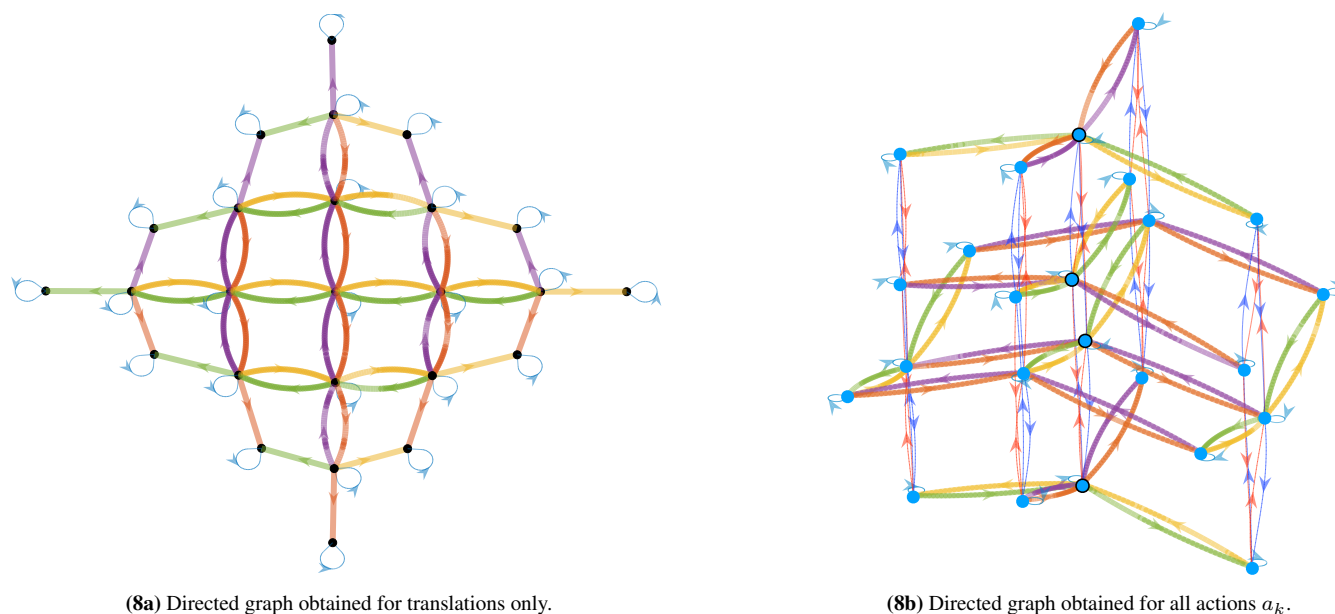


Figure 8. Directed graph of permutation matrices M_{a_k} —and thus also of corresponding actions a_k as per Equation (15)—obtained by combination of these matrices. Color conventions for edges match the color of each action in Figure 4. The depth up to which new nodes are explored has been limited for visualization purposes by setting low depth parameters in Algorithm 1

684 M_{a_1} being the identity matrix I_{N_c} . Furthermore one can note that the graph obtained is, up to its borders,
 685 completely homogeneous; that is the neighborhoods of each interior nodes share the same geometry. This
 686 even extends to the color of edges matching, so that some of them form pairs. One can for example verify
 687 that whenever a yellow edge goes from node M to M' , there is a green edge from M' to M and no other
 688 one. This identifies the corresponding actions to be inverses w.r.t. successive execution since from any
 689 starting node, taking first the green (resp. yellow) edge then the yellow (resp. green) one forms a loop. The
 690 same can be said of the red and purple colors, which are found to correspond to another pair of inverse
 691 actions. At last, the four central squares correspond to the commutativity of the a_k used: indeed one can
 692 see on the graph that taking the red edge first, then the green one always leads to the same node as green
 693 first, red second.

694 While those observations were discussed as properties of the permutation matrices, the actual result is
 695 their representing properties of the abstract motor actions a_k . And indeed one can check that the blue arrow
 696 corresponds to the identity action a_1 , that the inverse pairs (yellow, green) and (orange, purple) respectively
 697 correspond to (rightward, leftward) and (forward, backward) translations, and that the commutativity
 698 discussed is that of “forward then left” being the same as “left then forward”. While these facts seem
 699 obvious from an external point of view, they were not part of the initial knowledge of the agent discussed
 700 in §3.1. This only appears as a consequence of the agent capability to predict the sensory consequences of
 701 its own actions built during experiment 1. On a functional level, this is very similar to the property of motor
 702 sequence compression exhibited by RNNs performing sensorimotor prediction in (Ortiz and Laflaquière,
 703 2018); in fact we argue that it is the same phenomenon that is picked up on by the neural networks and that
 704 it is intrinsically related to sensorimotor prediction as developed in Section 3.

705 This also applies to the graph shown in Figure 8b obtained when considering all seven actions, i.e. the
 706 two rotations corresponding to actions a_6 and a_7 are now included in the analysis. This plot, obtained
 707 through a classical force-directed algorithm, shows the same 2D graph of translations obtained before, but

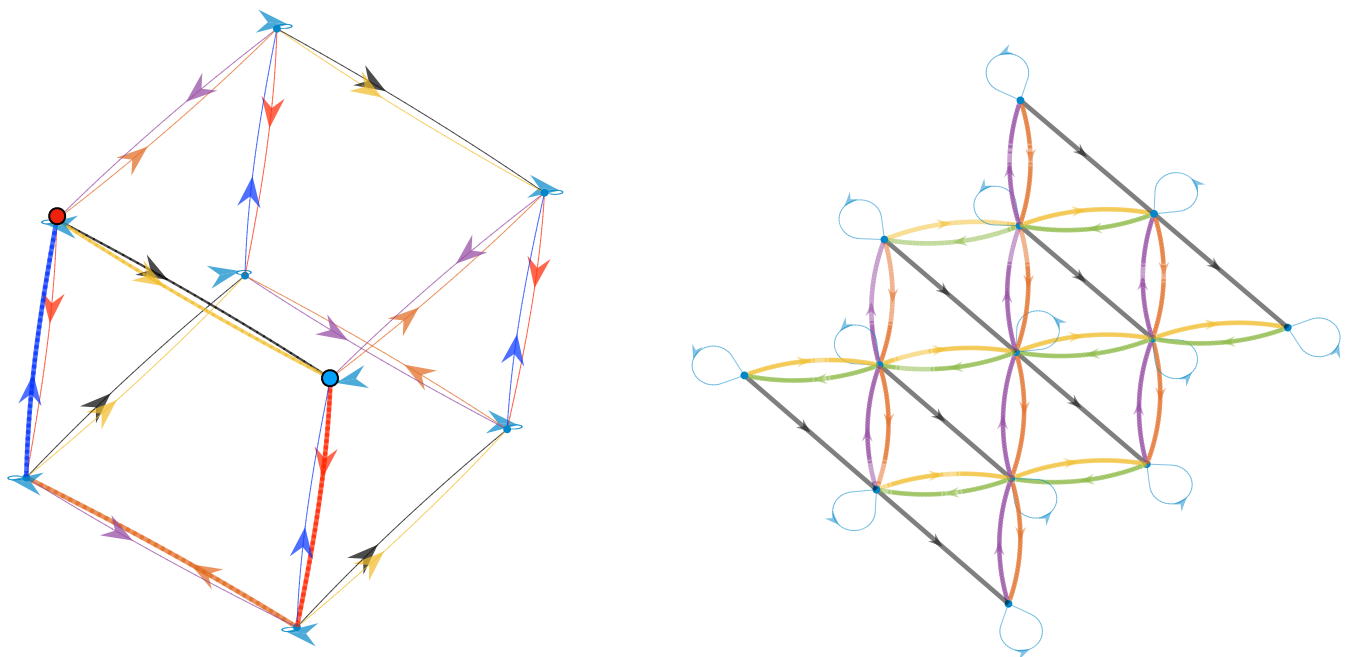
708 enriched with a third dimension supporting the change of orientation induced by rotations. Again, the depth
 709 of the graph is maintained low to keep things legible. The global structure of the graph can be described as
 710 a disjunction of 2D subgraphs corresponding to translations at fixed orientation. Each subgraph is therefore
 711 equivalent to each other up to a rotation as can be seen by the edges colors shifting between the planes.
 712 As an example, one can see that the same node in the graph can be reached by following either (green,
 713 dark blue) and (dark blue, purple), or (left, turn left) and (turn left, forward) in terms of actions seen from
 714 an external point of view. Figure 8b also shows that rotations are limited to the third vertical dimension
 715 in which they form cycles at constant position in the planar subgraph. This property is highlighted in the
 716 graph by the 4 circled nodes which figure the same agent position for the four possible orientations. The
 717 cycle simply mirrors the external observation that taking four $\pi/2$ rotations successively takes one back to
 718 the initial orientation. Importantly, this could constitute an internal signature of rotations as opposed to
 719 translations.

720 In the end of this second experiment, the agent has thus been able to discover a structure of its actual group
 721 of motor actions. The agent now has access to algebraic relations between its own actions which relate to
 722 its motor capabilities. This knowledge also allows it to generalize the sensory prediction it discovered in
 723 experiment 1 to all the combinations considered in the graph. Nevertheless, one has to keep in mind that all
 724 the actions considered in these experiments are not exactly conservative in the sense of Equation (9). Indeed,
 725 they fail at conserving spatial information on the border of the simulated camera. However, results show
 726 that conservativity holding true for all internal pixels allows for discovering the aforementioned properties.
 727 Because all calculations are made on actual current outputs of the pixels, one obvious consequence is that
 728 the agent has no way to predict what happens outside of its field of view, and so far it keeps no memory of
 729 it. Therefore, the results only hold for a very local movement w.r.t. the dimension of the agent. However,
 730 the discovered structure stays true whatever the initial position of the exploration.

731 4.3.3 Experiment 3: exploiting the graph to improve the representation of the action set

732 In the previous results, we have run the experiments with an experimental starting action set $\mathcal{A}_{\text{init}}$
 733 conveniently set to A , the set of externally defined movements. While this has been useful at first to yield
 734 easily recognized structure in Experiment 2, it is a crucial point that the results do not depend on this strong
 735 assumption. Therefore the same two part experiment is conducted with the difference that the starting
 736 action set $\mathcal{A}_{\text{init}}$ the agent can run is not arbitrarily set to A anymore. Instead, it is now drawn in the set
 737 of *combinations of* actions $a_k \in A$. Three important cases are now possible : first, it may be that some
 738 of the a_k are “missing” in $\mathcal{A}_{\text{init}}$; on the contrary, duplicates may have been drawn so that the agent can
 739 run $a, a' \in \mathcal{A}_{\text{init}}$ which are effectively the same action (i.e. $\forall \mathbf{b} \in \mathcal{B}, a\mathbf{b} = a'\mathbf{b}$). Finally, it may even have
 740 drawn “complex” actions $a \notin A$, that is actions that can only be obtained by combining some of the a_k .

741 The three situations are illustrated in Figure 9, where the graphs obtained at the end of Experiment 2
 742 are drawn for various starting action sets. In both cases, both the complexity of the starting action set and
 743 the depth of the depicted graph have been limited to keep the discussed structure as readable as possible.
 744 Figure 9a depicts the first two cases: the agent was given a duplicate action from A as well as missing one.
 745 This can be assessed in the resulting prediction graph by the yellow and black arrows which relate the
 746 same ordered pairs of nodes, e.g. from the highlighted red to blue nodes, and the lack of an inverse arrow
 747 that would match them. Note that while the absence of this inverse (green) arrow represents the lack of a
 748 “direct” inverse action *in* $\mathcal{A}_{\text{init}}$, the emerging structure from the graph allows for the determination of an
 749 inverse *path* as highlighted by the bold (red, orange, blue) arrows. From an external point of view, this
 750 basically means that if the agent has no action to translate itself to its left, it can instead rotate clockwise,
 751 then move backward, and finally rotate counter-clockwise to reach the correct orientation. Interestingly, this



(9a) Graph obtained with a duplicate starting action (depicted in black) and a missing action (a_2 , green in other figures). A path made of 3 edges (red, orange, blue) equivalent to the missing starting action is highlighted, providing an inverse to the (also highlighted) yellow edge.

(9b) Graph obtained by adding a combination (depicted in black) to the set of starting actions. Here, the action set was limited to translations to keep a clear visual.

Figure 9. Illustration of the effect of drawing starting actions at random on the discovered structure.

752 phenomenon where a missing inverse can be otherwise obtained by combination of other actions can *only*
 753 occur when the agent is able to rotate. The third situation is depicted in Figure 9b, where the experiment
 754 was conducted with the robot given the additional action a_8 “forward then rightward” along $a_8 = a_5a_3$ in
 755 addition to the translations of A . The choice not to give the agent its defined rotations a_6 and a_7 serves
 756 only to get an easily legible picture of the resulting graph, much like in Figure 8a, and does not impact the
 757 following. This additional move a_8 , which we as an external observer know to be a combination, is studied
 758 like all other basic actions by the agent during the motor babbling phase. It means that the agent has no cue
 759 about a_8 being an actual combination of two other actions. In the end of the experiment, the obtained graph
 760 of action exhibits this additional action a_8 as black edges, as shown in Figure 9b. From this graph, one can
 761 easily see that, from any point, it is indeed equivalent to follow either the black arrow or first the orange
 762 and then the yellow ones. The agent has thus been able to discover the action combination property.

763 These graphs therefore show two important results. The first conceptual comment is that the validity of
 764 the proposed experiment is not conditional to a perfect match between ideal, “objective” moves of the
 765 agent and actions it is effectively able to perform at start. This is a desirable property for genericity and our
 766 goal of bootstrapping, for it allows to avoid justifying said match. The second, more practical, comment is
 767 that the graph resulting from the experiment can be used by the agent to select an action set “better” than
 768 \mathcal{A}_{init} . Indeed, the redundancy between edges (or paths of edges) in Figures 9a and 9b represent the agent
 769 discovering it can discard the actions corresponding to black edges *without losing capabilities*, i.e. while
 770 keeping all nodes reachable. It can then be used to prune the available action set to a *minimal* set generating
 771 the same group, in the sense of Equation (4). Determining a criterion for selecting which actions are kept
 772 and which are discarded could functionally correspond to a basis for invariant principles in motor actions
 773 of the agent (Flash and Hogan, 1998). The agent may also *expand* its action set with new actions that verify

774 useful properties: for example, in the case depicted in Figure 9a it can package the bolded path of edges
775 into a single action so that it gets a missing inverse.

5 CONCLUSION

776 This paper was devoted to the introduction of a variational extension of a previous framework into the
777 sensorimotor framework, extending the scope of such approaches to naive agents able to move freely in
778 their environment. We demonstrated how, despite their extremely limited starting capabilities, these agents
779 could exploit said framework not only to perform sensory prediction, but also to structure their own actions.
780 The proposed formalism has been assessed in simulation as a proof of concept, with a naive agent able to (i)
781 build for each of its action some permutation matrices associated to its own sensory array, and (ii) exploit
782 them to structure its own set of actions. These experiments were conducted here in a somewhat simplistic
783 experimental setting to keep the simulated situation as close as possible to the theoretical exposition.
784 However, their transparent exploitation of the formal mechanisms we explicitly isolated yields valuable
785 insight as to similar results, which related works otherwise achieved in more realistic conditions.

786 Implementing a formal version of sensory prediction comes with many interesting perspectives, as it was
787 shown to yield crucial properties both in the original cognitive psychology literature and in the previous
788 robotic contexts. We hypothesize that it can be used to better understand the emergence and properties
789 of capabilities often related to that of sensory prediction, both from robotics and from cognitive sciences,
790 such as those mentioned in the Introduction. These include e.g. motor control, motor planning, isolating
791 proprioception, suppression of self-induced changes or object perception. These capabilities therefore
792 constitute potential applications to which further study could be devoted from there.

793 Nevertheless, the applicability of the proposed paradigm to real agents or robots is still an opened
794 question. First, it is clear that most of the actions an agent will be dealing with are not strictly conservative,
795 but rather *quasi-conservative* like in the simulations conducted in this paper. While not extensively studied
796 in this paper, some ongoing mathematical developments show that their properties still allow to reach the
797 same concepts of sensory prediction and action structuration. Then, the fact that the sensory prediction
798 relies on exact sensory values shifts inside the un-noisy sensory array is not very realistic. Introducing
799 stochastic matrices instead of permutation ones constitutes a promising way to deal with such an issue,
800 also pulling all these developments inside a probability territory (Rao and Ballard, 2005; Seth, 2014) in
801 which a lot of development still needs to be done. Moreover, the way this framework can be extended
802 to agents exhibiting dynamical effects, e.g. when performing kinematic or dynamical control, must still
803 be investigated. This requires some clarification about the structure and role of time in the sensorimotor
804 experience, a point which is still largely eluded in the SMCT context. Finally, actions can also be noisy, and
805 the question of their repeatability over time needs to be addressed so as to face realistic conditions. This
806 poses significant challenges in the SMCT context of minimal *a priori* knowledge outlined in the present
807 contribution. However, ongoing exploratory work tends to show that topological structure grounding
808 some continuity of the sensorimotor experience can be found as a contingency in said naive context. All
809 these paths constitute future promising works in the field and will undoubtedly extend the scope of these
810 approaches to naive adaptive and robust agents able to build by themselves their own understanding of
811 their interaction with their environment.

APPENDIX

812 In the following proofs, we will assume

$$(\forall \mathbf{b} \in \mathcal{B}, \forall \epsilon \in \mathcal{E}, \psi_{\mathcal{E}}(a'\mathbf{b}, \epsilon) = \psi_{\mathcal{E}}(a\mathbf{b}, \epsilon)) \Rightarrow a = a' \quad (20)$$

813 and

$$\forall c, c' \in \mathcal{C}, F_c = F_{c'} \Rightarrow c = c'. \quad (21)$$

814 These assumptions are of minimal importance for two reasons

- 815 • Eq. (21) only mandates that any two distinct sensels have different receptive fields *if only one time*,
816 while Eq. (20) asks for actions to have no difference (as denoted by =) except that which can be
817 assessed by the sensory capabilities $\psi_{\mathcal{E}}$. These conditions only fail to hold in very particular cases and
818 can be found to be true in the presented examples.
- 819 • In any case where they indeed *fail* to hold, the exact same results can be found with suitable equivalence
820 relations for actions (for Eq. (20)) and sensels (for Eq. (21)) at the cost of more loaded notations.

821 Therefore these conditions only serve as a way to streamline the presentation of the results with an at most
822 negligible impact on generality.

823 1 Equivalency between conservative and permutative

824 Here is provided a proof that conservative actions can be described as permutations of sensels, as
825 discussed in Section 3.3.1.

826 PROPOSITION 1. *Let a be a conservative action $\in \mathcal{A}$, there exists a unique map*

$$\begin{aligned} \sigma_a: \mathcal{C} &\rightarrow \mathcal{C} \\ c &\mapsto c' \end{aligned} \quad (22)$$

827 such that

$$\sigma_a(c) = c' \Leftrightarrow c \xrightarrow{a} c' \quad (23)$$

PROOF. Let $a \in \mathcal{A}$ conservative and $c \in \mathcal{C}$. By conservativity $\exists c' \in \mathcal{C}$ such that $c \xrightarrow{a} c'$. Let $c'' \in \mathcal{C}$ such that $c \xrightarrow{a} c''$, then

$$\forall \mathbf{b} \in \mathcal{B}, F_c(\mathbf{b}) = F_{c'}(a\mathbf{b}) \text{ and } F_c(\mathbf{b}) = F_{c''}(a\mathbf{b})$$

so that

$$\forall \mathbf{b} \in \mathcal{B}, F_{c'}(a\mathbf{b}) = F_{c''}(a\mathbf{b}).$$

828 But $a: \mathcal{B} \rightarrow \mathcal{B}$ must be surjective because it is bijective, so that *all* $\mathbf{b} \in \mathcal{B}$ can be written $a\mathbf{b}'$ for some
829 $\mathbf{b}' \in \mathcal{B}$. Therefore $F_{c'} = F_{c''}$, from which $c' = c''$: successor sensels are necessarily unique. We therefore
830 declare σ_a to be the map that takes each sensel $c \in \mathcal{C}$ to its *unique* successor sensel.

831 PROPOSITION 2. *For any conservative action $a \in \mathcal{A}$, its successor map σ_a is bijective.*

PROOF. Let a be a conservative action, and let $c, c' \in \mathcal{C}$ be sensels such that $\sigma_a(c) = \sigma_a(c')$. From this it follows that

$$\forall \mathbf{b} \in \mathcal{B}, F_c(\mathbf{b}) = F_{c''}(a\mathbf{b}) \text{ and } F_{c'}(\mathbf{b}) = F_{c''}(a\mathbf{b})$$

for some common successor $c'' \in \mathcal{C}$. But it entails in particular

$$\forall \mathbf{b} \in \mathcal{B}, F_c(\mathbf{b}) = F_{c'}(\mathbf{b})$$

832 that is $F_c = F_{c'}$, which further yields $c = c'$: σ_a is injective.

833 From injectivity of σ_a , it follows that $|\sigma_a(\mathcal{C})| = |\mathcal{C}|$. But because \mathcal{C} is finite it in turns follows from this
834 equality that $\sigma_a(\mathcal{C}) = \mathcal{C}$, i.e. σ_a is also surjective.

835 2 Conversing of conserving

836 We provide here the proof, as used in Section 3.3.1 and following, that conservative actions are themselves
837 a subgroup of \mathcal{A} for its succession operation.

838 PROPOSITION 3. Let $\mathcal{A}_{\mathcal{C}} \subset \mathcal{A}$ be the subset of all conservative actions. Then $\mathcal{A}_{\mathcal{C}}$ is in fact a subgroup
839 of \mathcal{A} .

840 PROOF. $\mathcal{A}_{\mathcal{C}} \subset \mathcal{A}$ by its very definition, therefore we only need prove it is actually a group.

- 841 • $\forall c \in \mathcal{C}, c \xrightarrow{e} c$ with e the identity action: e is conservative.
 • Let a and a' be conservative actions, and $c \in \mathcal{C}$: since $a \in \mathcal{A}_{\mathcal{C}}, \exists c' \in \mathcal{C}$ such that $c \xrightarrow{a} c'$. But since
 842 $a' \in \mathcal{A}_{\mathcal{C}}$ too, there also exists $c'' \in \mathcal{C}$ verifying $c' \xrightarrow{a'} c''$, so that finally

$$\forall c \in \mathcal{C}, \forall a, a' \in \mathcal{A}_{\mathcal{C}}, \exists c'' \text{ such that } c \xrightarrow{a'a} c''$$

842 that is $a'a$ is conservative itself.

- Let $a \in \mathcal{A}_{\mathcal{C}}$ and let σ_a be its successor map $\mathcal{C} \rightarrow \mathcal{C}$. $\forall c \in \mathcal{C}$ since σ_a is surjective (see proof in 1) we
 843 have $c = \sigma_a(c')$ for some $c' \in \mathcal{C}$, or equivalently

$$\forall c \in \mathcal{C}, \exists c' \text{ such that } c' \xrightarrow{a} c.$$

843 Finally, since $c' \xrightarrow{a} c \Leftrightarrow c \xrightarrow{a^{-1}} c'$ it follows that a^{-1} is conservative too.

844 3 Conservation or prediction, it is all the same

845 Here is provided a proof that mapping conservative actions a to their respective *sensory prediction*
 846 *functions* Π_a provides a group isomorphism, as per Equation (15). To this end, let us recall the essential
 847 property of these functions:

$$\forall \mathbf{b} \in \mathcal{B}, \forall \epsilon \in \mathcal{E}, \psi_{\mathcal{C}}(a\mathbf{b}, \epsilon) = \Pi_a(\psi_{\mathcal{C}}(\mathbf{b}, \epsilon)). \quad (24)$$

848 From this we get:

849 PROPOSITION 4. The map

$$\begin{aligned} \Pi: \mathcal{A}_{\mathcal{C}} &\rightarrow \text{Bij}(\mathcal{S}) \\ a &\mapsto \Pi_a \end{aligned} \quad (25)$$

850 is a group morphism. Moreover it is injective, so that it induces a group isomorphism $\mathcal{A}_{\mathcal{C}} \cong \Pi(\mathcal{A}_{\mathcal{C}})$.

PROOF. Let a, a' be two conservative actions, we have

$$\begin{aligned}\forall \mathbf{b} \in \mathcal{B}, \forall \epsilon \in \mathcal{E}, \\ \Pi_{a'^{-1}a}(\psi_{\mathcal{E}}(\mathbf{b}, \epsilon)) &= \psi_{\mathcal{E}}(a'^{-1}a\mathbf{b}, \epsilon) \\ &= \Pi_{a'^{-1}}(\psi(a\mathbf{b}, \epsilon)) \\ &= \Pi_{a'^{-1}}(\Pi_a(\psi(\mathbf{b}, \epsilon))) \\ &= (\Pi_{a'^{-1}} \circ \Pi_a)(\psi(\mathbf{b}, \epsilon))\end{aligned}$$

851 so that $\Pi(a'^{-1}a) = \Pi(a'^{-1}) \circ \Pi(a)$: Π is a group morphism.

Now let $a, a' \in \mathcal{A}_{\mathcal{E}}$ such that $\Pi_a = \Pi_{a'}$. It follows from Equation (24) that

$$\forall \mathbf{b} \in \mathcal{B}, \forall \epsilon \in \mathcal{E}, \psi_{\mathcal{E}}(a'\mathbf{b}, \epsilon) = \psi_{\mathcal{E}}(a\mathbf{b}, \epsilon)$$

852 so that under hypothesis (20) Π is indeed injective.

853 4 Convergence of experiment 1

854 This part is devoted to the proof of the relevance of Equation (17) in §4.2.1, that is the convergence of
855 matrices M_a towards the associated permutation matrices M_{σ_a} for all conservative actions a .

856 LEMMA 1. For any coefficient $m_{a_{i,j}}$ of M_{σ_a} , the associated sequence $(m_{a_{i,j}}[t_n])_n$ of values taken in
857 $(M_a[t_n])_n$ during exploration is nonincreasing with values in $\{0, 1\}$.

Proof. Let us consider an arbitrary timestep t_n , $n \in \mathbb{N}$ in the exploration. If a is not drawn at this timestep, then

$$m_{a_{i,j}}[t_{n+1}] = m_{a_{i,j}}[t_n] \leq m_{a_{i,j}}[t_n].$$

858 If it is instead chosen, assuming $m_{a_{i,j}}[t_n] \in \{0, 1\}$ then as per the update rule of M_a , either

- 859 • $m_{a_{i,j}}[t_n] = 0$ and then $m_{a_{i,j}}[t_{n+1}] = 0$ too,
- 860 • or $m_{a_{i,j}}[t_n] = 1$ and $m_{a_{i,j}}[t_{n+1}] \in \{0, 1\}$

861 so that the lemma follows by induction on n .

862 LEMMA 2. For any coefficient $m_{a_{i,j}} = 1$ in M_{σ_a} , the associated sequence $(m_{a_{i,j}}[t_n])_n$ is constant with
863 value $m_{a_{i,j}}[t_n] = 1$.

Proof. At any timestep t_n of the exploration, if a is not chosen then $m_{a_{i,j}}[t_{n+1}] = m_{a_{i,j}}[t_n]$.
If it is instead drawn, then by Equation (11) we know that

$$s_j[t_{n+1}] = s_i[t_n]$$

864 because $m_{a_{i,j}} = 1$ implies that $j = \sigma_a(i)$ as per the definition of M_{σ_a} . Then by the update rule of $M_a[t_n]$,
865 $m_{a_{i,j}}[t_{n+1}] = 1$.

866 The lemma then follows by induction on n .

867 We now proceed with the last part of our argument, that is showing that coefficients of the empirical
868 matrices M_{a_k} which do not correspond to successor sensel pairs will actually be nulled during exploration.
869 This specific part is provided in the specific case of the simulated experiment presented, allowing us to

870 formulate the relevant equations in the vector geometry of $\mathcal{X} = \mathbb{R}^2$. The same idea could also be adapted
 871 for generalized spaces \mathcal{X} , \mathcal{B} and actions $\mathcal{A}_{\text{init}}$, more in line with the previous theoretical descriptions.
 872 However, such a development is out of the scope of this contribution.

Let us define

$$\forall \theta \in \mathbb{R}, R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

the matrix corresponding to the rotation in \mathbb{R}^2 of angle θ . As per the definitions provided for our particular example, we may assume the properties:

$$\forall c \in \mathcal{C}, \forall (x, y, \vec{\theta}) \in \mathcal{B}, F_c(x, y, \vec{\theta}) = \begin{pmatrix} x \\ y \end{pmatrix} + R_\theta F_c(0, 0, \vec{0})$$

and

$$\forall a \in \mathcal{A}_{\text{init}}, \exists \vec{u}_a = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \in \mathbb{R}^2, \exists \theta_a \in \mathbb{R}$$

such that $\forall \mathbf{b} = (x, y, \vec{\theta}) \in \mathcal{B}$, $\mathbf{ab} = (x', y', \vec{\theta}')$

where $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + R_\theta \begin{pmatrix} x_a \\ y_a \end{pmatrix}$ and $\theta' = \theta + \theta_a$.

873 Therefore we have

LEMMA 3. Let $a \in \mathcal{A}_{\text{init}}$, $c, c' \in \mathcal{C}$. There exists a unique vector $\vec{d}_{a,c,c'} \in \mathbb{R}^2$ such that

$$\forall \mathbf{b} = (x, y, \vec{\theta}) \in \mathcal{B}, \overrightarrow{F_c(\mathbf{ab})F_{c'}(\mathbf{b})} = R_\theta \vec{d}_{a,c,c'}.$$

874 PROOF. Let $\mathbf{b} = (x, y, \vec{\theta}) \in \mathcal{B}$. We therefore have

$$875 \quad 1. F_{c'}(\mathbf{b}) = \begin{pmatrix} x \\ y \end{pmatrix} + R_\theta F_{c'}(0, 0, \vec{0}),$$

$$876 \quad 2. F_c(\mathbf{ab}) = F_c\left(\begin{pmatrix} x \\ y \end{pmatrix} + R_\theta \begin{pmatrix} x_a \\ y_a \end{pmatrix}, \vec{\theta} + \vec{\theta}_a\right) \\ = \begin{pmatrix} x \\ y \end{pmatrix} + R_\theta \begin{pmatrix} x_a \\ y_a \end{pmatrix} + R_{\theta+\theta_a} F_c(0, 0, \vec{0})$$

so that

$$\overrightarrow{F_c(\mathbf{ab})F_{c'}(\mathbf{b})} = R_\theta \left(F_{c'}(0) - \begin{pmatrix} x_a \\ y_a \end{pmatrix} - R_{\theta_a} F_c(0) \right).$$

877 which proves taking $\vec{d}_{a,c,c'} = \left(F_{c'}(0) - \begin{pmatrix} x_a \\ y_a \end{pmatrix} - R_{\theta_a} F_c(0) \right)$ satisfies the property.

878 It should be noted that $\vec{d}_{a,c,c'}$ captures some geometry of conservation: indeed, from the definition of \xrightarrow{a}
 879 it can easily be shown that

$$\forall a \in \mathcal{A}_{\text{init}}, \forall c, c' \in \mathcal{C}, \left(c \xrightarrow{a} c' \Leftrightarrow \vec{d}_{a,c,c'} = 0 \right). \quad (26)$$

880 This greatly serves the conclusion of our argument with

881 PROPOSITION 5. Let $a \in \mathcal{A}_{\text{init}}$, $c_i, c_j \in \mathcal{C}$. Then for given environment configuration $\epsilon \in \mathcal{E}$ the two
882 statements

883 1. There exists an absolute configuration $\mathbf{b} \in \mathcal{B}$ such that if $\mathbf{b}[t_n] = \mathbf{b}$ and $a[t_n] = a$ for some $n \in \mathbb{N}$,
884 then $m_{a_i,j}[t_k] = 0 \forall k \geq n + 1$

885 2. ϵ is not doubly periodic with periods \vec{d}_{a,c_i,c_j} and $R_{\frac{\pi}{2}}\vec{d}_{a,c_i,c_j}$

886 are equivalent.

887 PROOF.

888 • Assume that ϵ is both \vec{d}_{a,c_i,c_j} - and $R_{\frac{\pi}{2}}\vec{d}_{a,c_i,c_j}$ -periodic. Let $n \in \mathbb{N}$ such that $a[t_n] = a$, let $\mathbf{b} = \mathbf{b}[t_n]$.

889 We have $s_i[t_n] = \epsilon(F_c(\mathbf{b}[t_n]))$ and $s_j[t_{n+1}] = \epsilon(F_{c'}(\mathbf{b}[t_{n+1}])) = \epsilon(F_c(\mathbf{b}[t_n]) + R_{\theta}\vec{d}_{a,c_i,c_j})$. But since
890 $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, $R_{\theta} = \pm I_2$ or $R_{\theta} = \pm R_{\frac{\pi}{2}}$. Therefore by periodicity of ϵ we have $s_j[t_{n+1}] = s_i[t_n]$,
891 from which by induction on n we get $\forall n \in \mathbb{N}$, $m_{a_i,j}[t_n] = 1: 1) \Rightarrow 2)$.

892 • Without loss of generality, let us assume that ϵ is not \vec{d}_{a,c_i,c_j} -periodic (if it is instead only not $R_{\frac{\pi}{2}}\vec{d}_{a,c_i,c_j}$ -
893 periodic, the same argument follows up to a rotation).

894 Let $X_0 \in \mathbb{R}^2$ such that $\epsilon(X_0) \neq \epsilon(X_0 + \vec{d}_{a,c_i,c_j})$, $\mathbf{b}_0 = (x, y, \vec{0}) \in \mathcal{B}$ such that $F_c(\mathbf{b}_0) = X_0$. By
895 definition $F_{c'}(a\mathbf{b}_0) = X_0 + \vec{d}_{a,c_i,c_j}$ so that if $\mathbf{b}[t_n] = \mathbf{b}_0$ for some $t_n \in \mathbb{N}$, $s_j[t_{n+1}] \neq s_i[t_n]$. From the
896 update rule of M_a we then get $m_{a_i,j}[t_{n+1}] = 0$, which by Lemma 1 concludes the proof.

897 Finally, simultaneously applying this proof to *all* actions and pair of sensels of the agent has us deduce:

COROLLARY. If $\epsilon : \mathbb{R}^2 \rightarrow \mathcal{P}$ is aperiodic, then there exists a sequence of drawings of actions $(a[t_n])_{n \in \mathbb{N}}$
such that

$$\forall a_k \in \mathcal{A}_{\text{init}}, \lim_n M_{a_k}[t_n] = M_{\sigma_{a_k}}.$$

898 While the converse strictly speaking is not true, we can see from the preliminary lemma that problems in
899 the algorithm arise from very particular periodicity properties which relate to the geometry of (receptive
900 fields of) sensels. It therefore should be noted already how most experiments in live specimens made
901 use of specifically engineered symmetric and *periodic* environments to try and impair the development
902 of perception (Held and Hein, 1963). Future works could expand on the effects of such “pathological”
903 environment configurations on the proposed algorithm.

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