On spatial hyperbolicity for Hencky plasticity: the scalar-valued case

Jean-François Babadjian, Gilles A. Francfort

ON SPATIAL HYPERBOLICITY FOR HENCKY PLASTICITY: THE SCALAR-VALUED CASE

JEAN-FRANÇOIS BABADJIAN AND GILLES A. FRANCFORT

Abstract. We investigate a scalar two-dimensional version of the simplest model for plasticity, namely Hencky plasticity. Under the assumption that the stress reaches its threshold on an open set, we analyze the hyperbolicity properties of the equations that govern the plastic strain and give a precise description of the possible geometries of the characteristics, as well as of the possible associated solution fields on that set.

Contents

1. Introduction

1.1. **The context.** When departing from a completely reversible behavior, fluid mechanics essentially follows a unique path, that of viscosity. In its simplest manifestation, Euler equations cede the ground to Navier-Stokes equations which become the template for classical fluid behavior. Even non-Newtonian fluids usually exhibit viscosity, although one that may depend on a variety of kinematic or internal variables. When it comes to solids, while elasticity is the universally adopted reversible behavior, the irreversibility palette is much richer. This is so because solid mechanics encodes geometry and not only flow. As for fluids, viscosity is one expression of dissipation, leading to various kinds of viscoelastic models which, by the way, are mathematically much easier to handle in the case of solids. But many other kinds of dissipative behaviors may occur, together with,

Date: May 19, 2020.

or separate from viscosity. Their essential distinguishing feature is rate-independence: Material response is, up to rescaling, impervious to the loading rate. In that class, the best-established behavior is plasticity, and, within plasticity, Von Mises plasticity. While other rate-independent behaviors are still a modeling challenge, Von Mises plasticity can be thought of as the solid equivalent to Navier-Stokes, that is an admittedly simplistic model that however contains key ingredients for explaining much of the underlying physics at the macroscopic level.

Of course, because of geometry, this assertion should be nuanced: Von Mises plasticity is a perfectly sound model as long as deformations are small, that is as long as the kinematics of the deformation does not result in large changes of shape. Models for large deformations are not completely settled at present, even in the absence of irreversibility. In spite of major advances in the past 40 years spearheaded by the work of J.M. Ball [9], finite elasticity is far from a complete theory, while finite plasticity is a minefield.

Von Mises plasticity, also called Prandtl-Reuss elasto-plasticity with a Von Mises yield criterion, consists in the system of equations below. There, we denote by Ω the three-dimensional domain under consideration and, for simplicity, place ourselves in a quasi-static setting, that is in the absence of inertia; we also assume homogeneity, that is that all material parameters are spatially constant.

• Kinematic decomposition: The displacement field $u(t) : \Omega \to \mathbb{R}^3$ is constrained (for simplicity) by a time-dependent Dirichlet boundary condition $u(t) = w(t)$ on $\partial\Omega_d$, a relatively open subset of $\partial\Omega$, which constitutes the Dirichlet part of the boundary. The associated linearized strain $Eu(t) := \frac{1}{2}(\nabla u(t) + \nabla u(t)^T)$ is additively decomposed into the elastic strain $e(t)$ (a 3×3 symmetric matrix) and the plastic strain $p(t)$ (a trace free 3×3 symmetric matrix), *i.e.*,

$$Eu(t) = e(t) + p(t), \text{ with } \mathrm{tr}(p(t)) = 0.$$

• Constitutive law: The Cauchy stress $\sigma(t)$ is linearly related to $e(t)$ through Hooke's law

$$\sigma(t) = \mathbf{A}e(t),$$

where $\mathbf{A}$ is a fourth order tensor with the usual symmetry and coercivity properties of elasticity, that is, for some α, $\beta > 0$,

$$\mathbf{A}_{ijkl} = \mathbf{A}_{klij} = \mathbf{A}_{jikl} \quad \text{for all } 1 \le i, j, k, l \le 3$$
$$\alpha|\xi|^2 \le \mathbf{A}\xi \cdot \xi \le \beta|\xi|^2 \text{ for all } 3 \times 3 \text{ symmetric matrices } \xi.$$

• Equilibrium: We assume, for simplicity once again, that the only driving mechanism, besides the imposed displacement $w(t)$, is a surface load $g(t)$; there are no body loads. The stress $\sigma(t)$ is in quasi-static equilibrium, *i.e.*,

$$\mathrm{div}\,\sigma(t) = 0 \text{ in } \Omega, \quad \sigma(t)\nu = g(t) \text{ on } \partial\Omega_n := \partial\Omega \setminus \overline{\partial\Omega_d} \quad (\nu \text{ is the outer normal to } \partial\Omega_n),$$

while its deviatoric part $\sigma_D(t) := \sigma(t) - \frac{1}{3}\mathrm{tr}\,(\sigma(t))\mathrm{Id}$ satisfies the Von Mises yield criterion,

$$|\sigma_D(t, x)| \le \sqrt{\frac{2}{3}}\,\sigma_c \text{ at every point } x \in \Omega$$

where the threshhold $\sigma_c > 0$, the yield stress, is a datum.

• Flow rule: The deviatoric stress $\sigma_D(t)$ and the plastic strain rate $\dot{p}(t)$ are related, at every point $x \in \Omega$, through

$$\dot{p}(t, x) = \lambda(t, x)\sigma_D(t, x), \text{ with } \lambda(t, x) \ge 0 \text{ and } \lambda(t, x) = 0 \text{ if } |\sigma_D(t, x)| < \sqrt{\frac{2}{3}}\sigma_c. \tag{1.1}$$

In other words, whenever the (deviatoric part of the) stress reaches the boundary of its admissible set, the plastic strain should flow in the direction normal to that set.

The modern mathematical treatment of Von Mises plasticity finds its roots in the work of P.-M. Suquet [36], later completed by various works (see *e.g.* [37, 30, 4, 5, 6]). That work was revisited some 20 years later by G. Dal Maso, A. De Simone and M. G. Mora [15] within the framework of the variational theory of rate independent evolutions popularized by A. Mielke (see *e.g.* [32]). The basic tenet there is that the evolution can be viewed as a time-parameterized set of minimization problems for the sum of the elastic energy and of the add-dissipation. The minimizers should also be such that an energy conservation statement, amounting to a kind of Clausius-Duhem inequality, is satisfied throughout the evolution.

In any case, for a Lipschitz bounded domain Ω, a prescribed boundary displacement field $w \in AC([0,T];H^{1/2}(\partial\Omega_d;\mathbb{R}^3))$, and a prescribed surface load $g \in AC([0,T];\mathcal{C}^0(\overline{\Omega};\mathbb{R}^3))$ that satisfies what is usually called a safe-load condition, namely there exist $\delta \in (0,1)$ and $\tau \in AC([0,T];\mathcal{C}^0(\overline{\Omega};\mathbb{R}^6))$ with $|\tau| \leq \delta$ and $\operatorname{div}\tau = 0$ in $\Omega\times(0,T)$,

$$g := \tau\cdot\nu \text{ on } \partial\Omega_n\times(0,T)$$

(see e.g. [23, Remark 2.10]), the resulting evolutions $t \mapsto (u(t),e(t),p(t))$ are found to live in $AC([0,T];BD(\Omega)\times L^2(\Omega;\mathbb{R}^6)\times\mathcal{M}(\Omega\cup\partial\Omega_d;\mathbb{R}^6))$. Here, $BD(\Omega)$ stands for the space of functions of bounded deformation, *i.e.*, integrable vector fields $v:\Omega\to\mathbb{R}^n$ whose distributional gradient $Ev = \frac{1}{2}(Dv+Dv^T)$ is a bounded measure in Ω and $\mathcal{M}(\Omega\cup\partial\Omega_d;\mathbb{R}^6)$ stands for the space of bounded Radon measures on $\Omega\cup\partial\Omega_d$ (see [23] under those conditions). Further, uniqueness of $e(t)$, hence of $\sigma(t)$ is guaranteed. Such however is not the case for $p(t)$, hence for $u(t)$. The first example of non-uniqueness was presented in [36, Section 2.1] while [18, Section 10] introduces the first examples of uniqueness. In those references the setting is essentially 1D. To our knowledge, the only examples of determination of uniqueness or non-uniqueness in 3D can be found in [25, 26]. There, the discussion around uniqueness is centered around the equation that the Lagrange multiplier λ in (1.1) must satisfy.

We next formally manipulate the equations at a given fixed time t. Indeed, since $\dot{p} = E\dot{u} - \dot{e}$, or still $\lambda\sigma = E\dot{u} - \mathbf{A}^{-1}\dot{\sigma}$, we can use the compatibility equations for symmetrized gradient, that is

$$\operatorname{curl}\operatorname{curl} E\dot{u} = 0,$$

or, componentwise, for all $1\leq i,j,k,l\leq 3$,

$$\frac{\partial^2(E\dot{u})_{ij}}{\partial x_k\partial x_l} + \frac{\partial^2(E\dot{u})_{kl}}{\partial x_i\partial x_j} - \frac{\partial^2(E\dot{u})_{ik}}{\partial x_j\partial x_l} - \frac{\partial^2(E\dot{u})_{il}}{\partial x_j\partial x_k} = 0.$$

We obtain

$$\operatorname{curl}\operatorname{curl}(\lambda\sigma) = -\operatorname{curl}\operatorname{curl}(\mathbf{A}^{-1}\dot{\sigma}). \tag{1.2}$$

This also reads, for $i,j,k,l\in\{1,2,3\}$, as a system of 6 equations, namely,

$$\sigma_{ij}\frac{\partial^2\lambda}{\partial x_k\partial x_l} + \sigma_{kl}\frac{\partial^2\lambda}{\partial x_i\partial x_j} - \sigma_{ik}\frac{\partial^2\lambda}{\partial x_j\partial x_l} - \sigma_{il}\frac{\partial^2\lambda}{\partial x_j\partial x_k}$$
$$(+\text{ terms of lower order in }\lambda) = -(\operatorname{curl}\operatorname{curl}(\mathbf{A}^{-1}\dot{\sigma}))_{ijkl}.$$

In the example investigated in [25], the stress field σ is spatially homogeneous, so that the lower order terms disappear as well as the right-hand side, and the formal manipulations can be justified. We then have to deal with a *bona fide* system of second order linear partial differential equations for the measure λ of the form

$$D(t)\nabla^2\lambda(t) = 0,$$

with $D(t)$ a spatially constant 6×6 matrix. Whenever the determinant of $D(t)$ is not 0, we obtain that $\nabla^2\lambda(t)\equiv 0$; then, because of the specific setting in that example, $\lambda(t)$ is x-independent, from which it follows that $p(t)$ is an x-independent plastic strain $p(t)$. With that result at hand, $p(t)$ is

easily shown to be unique. (The example in [26], while more intricate, follows a similar path.) Otherwise, the system reduces to a spatial hyperbolic equation for $\lambda(t)$ and then uniqueness depends on whether the associated characteristics coming out of $\partial\Omega_d$ fill the whole domain. Roughly speaking, if they do, then uniqueness is obtained. If they don't, then non-uniqueness can be drastic because plastic strains that are as badly behaved as one desires (for example plastic strains supported on Cantor sets) can appear at any time t in the region not reached by those characteristics. So one could then impose a homogeneous boundary condition and the stress field would consequently be identically null. Yet, non-zero plastic strains could appear at arbitrary times. This is reminiscent of what was recently observed by C. De Lellis and L. Székelyhidi when dealing with non-uniqueness in Euler equations (see [17] and subsequent works), with the important caveat that plastic strains, once turned on, cannot be turned off because of dissipation.

This kind of analysis of uniqueness depends on our ability to deal with a system such as (1.2). In the examples already alluded to, the key observation is the spatial homogeneity of the stress field $\sigma(t)$, a feast that cannot be easily reproduced in a generic problem. Barring this, our toolbox is rather empty. As a matter of fact, it is impossible to even define possible characteristics in a meaningful manner because of the lack of regularity of the stress field $\sigma(t)$. At best, it is a locally H^1 function while $\lambda(t)$ is a measure.

This is why, in an attempt to simplify the problem, we address in the present paper a scalar-valued version of Von Mises plasticity in the simplest setting where geometry will play its part, that is in 2D (see [7, Section 3.1] for a formal derivation). Furthermore, since time evolution seems to be a complicating feature but not one that uniqueness hinges on, we propose to investigate a time independent (static) version of Von Mises plasticity, that of Hencky plasticity which actually predates evolutionary plasticity à la Von Mises. Assuming for simplicity that all relevant material constants are set to 1, the system of time-independent equations becomes, in its formal version (see (3.2) for a more precise formulation)

$$\begin{cases} \mathrm{div}\sigma = 0 \quad \text{in } \Omega, \\ |\sigma| \le 1 \quad \text{in } \Omega, \\ Du = \sigma + p \text{ in } \Omega, \\ u = w \text{ on } \partial\Omega_d, \quad \sigma\cdot\nu = g \text{ on } \partial\Omega_n, \\ p = \lambda\sigma \text{ in } \Omega \text{ with } \lambda \ge 0 \text{ and } \lambda(x) \equiv 0 \text{ if } |\sigma(x)| < 1. \end{cases}$$

Once again existence of (u,σ,p) (for a slightly relaxed problem) is guaranteed, this time through a straightforward minimization process (see Section 3). The triplet (u,σ,p) belongs to $BV(\Omega)\times L^2(\Omega;\mathbb{R}^2)\times\mathcal{M}(\Omega\cup\partial\Omega_d;\mathbb{R}^2)$ and σ is unique and actually belongs to $H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ (see below).

In that setting as well, uniqueness issues are intimately tied to the solving of a first order hyperbolic equation. Let us illustrate this with a very simple example. For $0<d<\ell$, take $\Omega := \{(x,y)\in\mathbb{R}^2:\ 0<x<d,\ 0<y<\ell-x\}$, $\partial\Omega_d = (0,d)\times\{0\}\cup\{(x,y)\in\mathbb{R}^2:\ 0<x<d,\ x+y=\ell\}$ and $\partial\Omega_n = (\{0\}\times(0,\ell))\cup(\{d\}\times(0,\ell-d))$. We set

$$\begin{cases} w(x,0) = \dfrac{x}{\sqrt{2}}, \quad w(x,\ell-x) = \dfrac{a\ell}{\sqrt{2}} \quad \text{for all } x\in(0,d) \\ g(0,y) = -\dfrac{1}{\sqrt{2}} \quad \text{for all } y\in(0,\ell), \text{ and } g(d,y) = \dfrac{1}{\sqrt{2}} \quad \text{for all } y\in(0,\ell-d) \end{cases}$$

with $a>1$. It is then easily seen that the unique stress field is given by $\sigma(x,y) = \frac{1}{\sqrt{2}}(1,1)$, and that λ satisfies $\frac{\partial\lambda}{\partial x} - \frac{\partial\lambda}{\partial y} = 0$ from which we conclude that λ reads as $\zeta(x+y)$, the push forward of

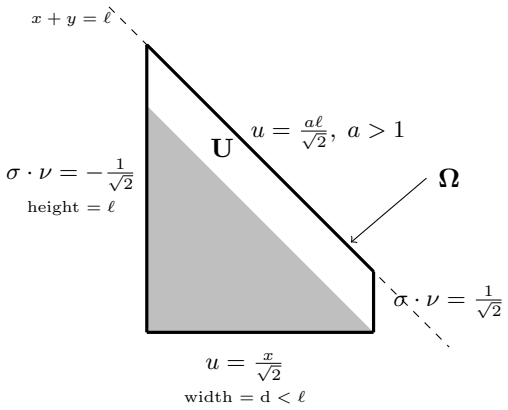

FIGURE 1. Example of non-uniqueness.

a nonnegative bounded Radon measure on $\zeta \in \mathcal{M}(\mathbb{R})$ by the map $(x,y) \mapsto x+y$. Consequently,

$$u(x,y) = \frac{x+y+Z(x+y)}{\sqrt{2}} \quad \text{for all } (x,y) \in \Omega \tag{1.3}$$

for some $Z \in BV(\mathbb{R})$ with $DZ = \zeta$.

In view of Remark 3.1 below, there can be no jumps of u on $(0,d)\times\{0\}$ since $\sigma\cdot\nu \neq \pm 1$ on that set. Thus $\lambda = 0$ on $(0,d)\times\{0\}$, which means that $\zeta = 0$ in $(0,d)$. Using again the boundary condition and the fact that u does not jump on $(0,d)\times\{0\}$, we obtain

$$Z(t) = 0 \quad \text{for all } t \in (0,d).$$

However, there can be a jump on $\{(x,y)\in\mathbb{R}^2 :\ 0<x<d,\ x+y=\ell\}$ since, in that case, $\sigma\cdot\nu = 1$. Since $d<\ell$, the part $U := \{(x,y)\in\Omega :\ d-x<y<\ell-x\}$ is not traversed by characteristic lines intersecting $(0,d)\times\{0\}$ (see Figure 1). So, $\lambda(x,y) = \zeta(x+y)$, where ζ is any nonnegative Radon measure. Because $w = \frac{a\ell}{\sqrt{2}}$ on $\{(x,y)\in\mathbb{R}^2 :\ 0<x<d,\ x+y=\ell\}$, we must have $Z^+(\ell) = (a-1)\ell$. In conclusion, Z can be any monotonically increasing function such that $Z\equiv 0$ on $(0,d)$ and $Z^+(\ell) = (a-1)\ell$. This in turn will give rise to many possible u's in U through (1.3).

Note that, for $d=\ell$ the domain degenerates to a triangle, the Neumann boundary condition is only on $\{0\}\times(0,\ell)$ and the solution is unique. It corresponds to $Z\equiv 0$ on $(0,\ell)$ and $Z^+(\ell) = (a-1)\ell$.

In the spirit our prior discussion, one should investigate the compatibility equation curl $Du = 0$. This yields

$$\text{curl}\,(\lambda\sigma) + \text{curl}\,\sigma = 0.$$

Formally once more, this reads as

$$-\sigma^\perp\cdot D\mu + \mu\,\text{curl}\,\sigma = 0 \tag{1.4}$$

with $\mu := \mathcal{L}^2 + \lambda$.

Any attempt to rigorously handle an equation such as (1.4) must start with a good definition of the characteristics, that is of the solutions to the ordinary differential equation

$$\begin{cases} \dfrac{dX}{ds} = \sigma^\perp(X(s)) \\ X(0) = X_0 \end{cases} \tag{1.5}$$

and it is tempting to resort to the theory of Lagrangian flows initiated by R. DiPerna and P.L. Lions [20] and pursued in *e.g.* [1, 14]. Unfortunately, the method falters, essentially because the

lack of integrability of $\operatorname{div}\sigma^\perp = -\operatorname{curl}\sigma$, which is *a priori* no better than a $L^2_{\rm loc}(\Omega)$-function, prevents any control of the so called compressibility constant.

We thus have to further specialize the setting by assuming that there is an open set Ω_p on which $|\sigma| \equiv 1$. This is the case in the examples that were investigated in [25, 26] in a vectorial setting. This assumption is also related to the so-called slip line theory, widely used in mechanics to get exact solutions to plane-stress rigid-plasticity problems. There, the spatial hyperbolic structure of the stress equations is used in combination with the method of characteristics to construct non trivial solutions to the stress problem and to determine the associated velocities [31, Section 5.1]. Doing so allows us to appeal to various works devoted to divergence free unit vector fields, and in particular to [29].

Our results are briefly described in the next subsection.

1.2. **The results.** Section 4 focuses mainly on the $H^1_{\rm loc}(\Omega;\mathbb{R}^2)$-regularity of the stress field σ. That regularity was first demonstrated in [35], then in [10], in both instances in the vectorial setting. This is the object of Theorem 4.1 which offers a short derivation in the current scalar setting using a Perzyna type approximation of the problem as in [8] (see (4.1)), in lieu of the so-called Norton-Hoff approximation used in [10], or of the Kelvin-Voigt visco-elastic approximation used in [35]. It is our belief that the proof of that result is rather convoluted in those prior works; we strive to give a hopefully more transparent and self-contained proof.

In Subsection 5.1, we quickly revisit the main result in [29] adapted to our context, that is the locally Lipschitz regularity of the stress in Ω_p, provided that this set is convex (see Theorem 5.5). In particular, those results imply that σ remains constant along straight lines in Ω_p with direction $\sigma^\perp$ which are precisely the characteristics inside Ω_p (see Proposition 5.3). This is done by adapting the results of [29] (see also [11, 28]) which use the notion of entropies introduced in [19]. Again, the results of this Subsection are not new. However, the proofs in prior works do not take into account the specificities of the case at hand, that is the *a priori* knowledge of the Sobolev regularity of the stress. In Subsection 5.2, we prove that any displacement field u must remain constant along the characteristic lines in Ω_p (see Theorem 5.7).

In Section 6 we first consider boundary fans that are solutions that correspond to vortices for σ (see Subsection 6.1). This happens when two distinct characteristic lines intersect on the boundary. Note that such an intersection is impossible inside Ω_p because it would contradict the continuity of σ at that point. Those can be anywhere in Ω_p. We take their union $\mathscr{F}$ and consider the complementary set $\mathscr{C}$ within Ω_p. We show that each connected component of $\mathscr{C}$ intersects the boundary $\partial\Omega_p$. If its interior is empty, it is a characteristic line. Otherwise, its intersection with $\partial\Omega_p$ has one or two connected components. Furthermore, if it has two connected components, then all points in those are traversed by a characteristic line, whereas if it has only one component, then it might be so that a single line segment within that component is not traversed by any characteristic. This is the object of Theorem 6.12 which is our main rigidity result. As far as the stress is concerned, we show continuity of the stress at all points of the boundary that are traversed by a characteristic (see Theorem 6.23). Finally, we demonstrate that, besides boundary fans, exterior fans (that are fans with an apex outside $\overline{\Omega}_p$), and areas of constant σ (which correspond to parallel characteristic lines), one can also have areas where the characteristic lines look like a "continuous" one parameter family of lines, *e.g.* of the form $y = x/t - t$, for $t > 0$ (see Paragraph 6.3.3). We conjecture that those four situations are the only possible ones for σ in the region on which $|\sigma| = 1$.

The behavior of any solution field (σ, u) along the characteristic lines in Ω_p (see Proposition 5.3 and Theorem 5.7) seems beyond reach for now, even in the scalar-valued setting, absent an additional assumption like the existence of a set with non empty interior where $|\sigma| = 1$. But even in our restrictive setting this result falls short of adjudicating uniqueness of the plastic strain p.

This is so because the set $P := \{x \in \Omega : \ |\sigma| = 1\}$ is a closed set in Ω while we have to assume that Ω_p is a convex open set in the interior of P. In particular, we have no systematic way of relating the values of u on the boundary of $\partial\Omega$ to those on $\partial\Omega_p$, except for very particular settings (see Propositions 6.3 and 6.24). If Ω was a convex domain and $P = \Omega$, then uniqueness could be obtained, at least in the case of Dirichlet boundary conditions throughout $\partial\Omega$. For more details see Remark 6.25.

For the reader's convenience, we concatenate the main results in a unique Theorem which, in its concision, somewhat hides the hyperbolic nature of the questions that are central to this paper (see Figure 2 for an illustration of the geometric structure of the solutions).

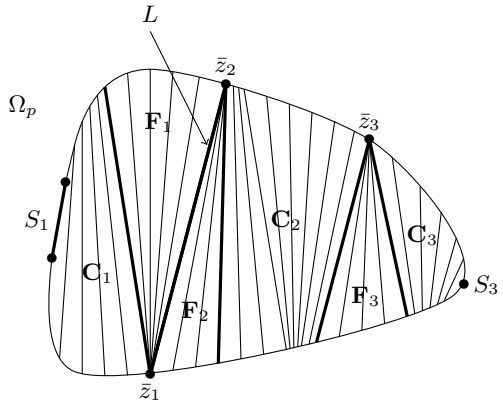

FIGURE 2. An example of geometry with three boundary fans $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ with apexes, respectively, $\bar{z}_1$, $\bar{z}_2$, $\bar{z}_3$, three connected components $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_3$, and a characteristic line L which is the intersecting characteristic line segment between the (open) fans $\mathbf{F}_1$ and $\mathbf{F}_2$. The connected components $\mathbf{C}_1$ and $\mathbf{C}_3$ have one characteristic line segment on their boundaries, the characteristic boundary set S_1 is a closed line segment, while S_3 is a single point. The connected component $\mathbf{C}_2$ has two characteristic line segments on its boundary.

Theorem 1.1 (Main results)**.** *Assume that Ω is a Lipschitz bounded domain of $\mathbb{R}^2$ and that $w \in H^1(\Omega)$. The minimization problem*

$$\inf\Big\{\frac{1}{2}\int_\Omega |\sigma|^2\,dx + |p|(\overline{\Omega}) : \ (u,\sigma,p) \in \mathcal{A}_w\Big\}$$

with

$$\mathcal{A}_w := \{(u,\sigma,p) \in BV(\Omega)\times L^2(\Omega;\mathbb{R}^2)\times \mathcal{M}(\overline{\Omega};\mathbb{R}^2) : Du = \sigma + p \text{ in } \Omega,\ p = (w-u)\nu\mathcal{H}^1 \text{ on } \partial\Omega\}$$

has at least one minimizer (u,σ,p). Furthermore, σ is unique and belongs to $H^1_{\rm loc}(\Omega;\mathbb{R}^2)$.

Assume moreover that there exists a non empty convex open set $\Omega_p \subset \Omega$ such that $|\sigma| = 1$ a.e. on Ω_p. Then σ is locally Lipschitz on Ω_p and remains constant along each open line segment

$$L_x := (x + \mathbb{R}\sigma^\perp(x)) \cap \Omega_p, \qquad \text{with } x \in \Omega_p,$$

called a characteristic *line segment. Moreover, u remains constant along $L_x \cap \Omega_p$ for all $x \in \Omega_p \setminus \big(\bigcup_{z\in Z} L_z\big)$ for some $\mathcal{H}^1$-negligible set $Z \subset \Omega_p$ with $\mathcal{L}^2\big(\bigcup_{z\in Z} L_z \cap \Omega_p\big) = 0$.*

Geometrically, Ω_p can be decomposed as the following disjoint union

$$\Omega_p = \bigcup_{i\in I} \mathbf{F}_i \cup \bigcup_{\lambda\in\Lambda} (L_{x_\lambda} \cap \Omega_p) \cup \bigcup_{j\in J} \mathbf{C}_j,$$

for some countable sets I and J, and some (possibly) uncountable set Λ. For all $i \in I$, $\mathbf{F}_i$ is a boundary fan, i.e., the intersection of Ω_p with an open cone with apex $\bar{z}_i \in \partial\Omega_p$ and two characteristic line segments as generatrices; for all $\lambda \in \Lambda$, L_{x_λ} is a characteristic line segment passing through $x_\lambda \in \Omega_p$ and we set $P_\lambda := L_{x_\lambda} \cap \partial\Omega_p$ (a set made of two points); and for all $j \in J$, $\mathbf{C}_j$ is a convex set, closed in the relative topology of Ω_p and with non empty interior, endowed with one of the following two properties:

- *Either $\partial\mathbf{C}_j = L_j \cup \Gamma_j$ with $L_j \subset \Omega_p$ an open characteristic line segment and Γ_j a connected closed set in $\partial\Omega_p$. In that case, $\Gamma_j = \Gamma_j^1 \cup \Gamma_j^2 \cup S_j$ where Γ_j^1 and Γ_j^2 are connected and S_j is a closed line segment (possibly reduced to single point) that separates Γ_j^1 and Γ_j^2. Further each point of Γ_j^1 (resp. Γ_j^2) is traversed by a characteristic line segment which will re-intersect $\partial\Omega_p$ on Γ_j^2 (resp. Γ_j^1);*
- *Or $\partial\mathbf{C}_j = L_j \cup L_j' \cup \Gamma_j \cup \Gamma_j'$ where L_j and $L_j' \subset \Omega_p$ are open characteristic line segments, while Γ_j and Γ_j' are two disjoint connected closed sets in $\partial\Omega_p$. Further each point of Γ_j (resp. Γ_j') is traversed by a characteristic line segment which will re-intersect $\partial\Omega_p$ on Γ_j' (resp. Γ_j). In that case we set $S_j = \emptyset$.*

Finally, σ is continuous on $\overline{\Omega}_p \setminus \big(\bigcup_{\lambda\in\Lambda} P_\lambda \cup \bigcup_{j\in J} S_j \cup \bigcup_{i\in I}\{\bar{z}_i\}\big)$.

Remark 1.2. The reader will not have failed to note that our results only pertain to the case $\partial\Omega_d = \partial\Omega$ and $\partial\Omega_n = \emptyset$. There are no obstacles in treating the more general case of a surface load g on a Neumann part of the boundary $\partial\Omega_n$. This would simply add the term $-\int_{\partial\Omega_n} gu\, d\mathcal{H}^1$ in the minimization problem (3.1) and the term $\int_{\partial\Omega_n} \varphi g(u-w)\, d\mathcal{H}^1$ in the definition (2.1) of the duality. However, one would have to spell out the safe-load conditions on g evoked earlier, as well as add technical conditions on $\partial_{|\partial\Omega}\partial\Omega_d$ (the boundary of the Dirichlet part $\partial\Omega_d$ in $\partial\Omega$) (see [23, Section 6]). Barring this, all results are local in nature and would not be affected. With that caveat in mind, all our results equally hold in the enlarged setting of a Neumann condition on part of the boundary of the domain. ¶

2. Notation and preliminaries

The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$ and the s-dimensional Hausdorff measure by $\mathcal{H}^s$.

From here onward the space dimension is set to 2. If a and $b \in \mathbb{R}^2$, we write $a \cdot b$ for the Euclidean scalar product, and we denote the norm by $|a| = \sqrt{a \cdot a}$. The open (resp. closed) ball of center x and radius ρ is denoted by $B_\rho(x)$ (resp. $\overline{B}_\rho(x)$). If $K \subset \mathbb{R}^2$ is a closed and convex set, we denote by $\mathscr{N}_K(x) = \{\xi \in \mathbb{R}^2 :\ \xi \cdot (y - x) \le 0 \text{ for all } y \in K\}$ the normal cone to K at $x \in \partial K$, and by $\mathscr{T}_K(x) = \{\zeta \in \mathbb{R}^2 :\ \xi \cdot \zeta \le 0 \text{ for all } \xi \in \mathscr{N}_K(x)\}$ the tangential cone to K at $x \in \partial K$. If $z = (z_1, z_2) \in \mathbb{R}^2$, we denote by $z^\perp = (-z_2, z_1)$ the rotation of z of an angle $\pi/2$. Given two vectors u and $v \in \mathbb{R}^2$, we denote by $C(u, v) := \{\alpha u + \beta v :\ \alpha > 0,\, \beta > 0\}$ the open cone generated by u and v with apex at the origin. Also, if $f : \mathbb{R} \to \overline{\mathbb{R}}$ is a proper, convex function, we denote by $\boldsymbol{\partial} f(x)$ (or $\boldsymbol{\partial} f(\cdot)(x)$) the subdifferential of f at the point $x \in \mathbb{R}^2$ which is not empty at all points in the interior of its domain.

In all that follows, $\Omega \subset \mathbb{R}^2$ is a bounded and Lipschitz open set. We use standard notation for Lebesgue and Sobolev spaces. We write $\mathcal{M}(\Omega; \mathbb{R}^2)$ (resp. $\mathcal{M}(\Omega)$) for the space bounded Radon measures in Ω with values in $\mathbb{R}^2$ (resp. $\mathbb{R}$), endowed with the norm $|\mu|(\Omega)$, where $|\mu| \in \mathcal{M}(\Omega)$ is the total variation of the measure μ. The space $BV(\Omega)$ of functions of bounded variation in Ω is made of all functions $u \in L^1(\Omega)$ such that its distributional gradient $Du \in \mathcal{M}(\Omega; \mathbb{R}^2)$. Then $BV(\Omega) \subset L^2(\Omega)$.

Given a map $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$, we set $\operatorname{div}\sigma := \frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2}$ and denote by $\operatorname{curl}\sigma$ the scalar quantity $\frac{\partial \sigma_2}{\partial x_1} - \frac{\partial \sigma_1}{\partial x_2}$. We denote by $H(\operatorname{div},\Omega)$ the Hilbert space of all $\sigma \in L^2(\Omega;\mathbb{R}^2)$ such that $\operatorname{div}\sigma \in L^2(\Omega)$. We recall that if Ω is bounded with Lipschitz boundary and $\sigma \in H(\operatorname{div},\Omega)$, its normal trace, denoted by $\sigma\cdot\nu$, is well defined as an element of $H^{-1/2}(\partial\Omega)$. If further $\sigma \in H(\operatorname{div},\Omega)\cap L^\infty(\Omega;\mathbb{R}^2)$, then $\sigma\cdot\nu \in L^\infty(\partial\Omega)$ with $\|\sigma\cdot\nu\|_{L^\infty(\partial\Omega)} \le \|\sigma\|_\infty$ (see *e.g.* [3, Theorem 1.2]). Moreover, according to [13, Theorem 2.2 (iii)], if Ω is of class $\mathcal{C}^2$, then for all $\varphi \in L^1(\partial\Omega)$,

$$\lim_{\varepsilon\to 0}\int_0^1\int_{\partial\Omega}\big(\sigma(y-\varepsilon s\nu(y))\cdot\nu(y) - (\sigma\cdot\nu)(y)\big)\varphi(y)\,d\mathcal{H}^1(y)\,ds = 0, \tag{2.1}$$

where ν denotes the outer unit normal to $\partial\Omega$.

According to [3, Definition 1.4], we define a generalized notion of stress/strain duality pairing between stresses and plastic strains as follows (see also [23, Section 6]).

Definition 2.1. *Let $\sigma \in H(\operatorname{div},\Omega)\cap L^\infty(\Omega;\mathbb{R}^2)$, $(u,e,p) \in BV(\Omega)\times L^2(\Omega;\mathbb{R}^2)\times\mathcal{M}(\overline{\Omega};\mathbb{R}^2)$ and $w \in H^1(\Omega)$ be such that $Du = e+p$ in Ω and $p = (w-u)\nu\mathcal{H}^1$ on $\partial\Omega$. We define the distribution $[\sigma\cdot p] \in \mathcal{D}'(\mathbb{R}^2)$ by*

$$\langle[\sigma\cdot p],\varphi\rangle = \int_\Omega \varphi\sigma\cdot(\nabla w - e)\,dx + \int_\Omega (w-u)\sigma\cdot\nabla\varphi\,dx + \int_\Omega (w-u)(\operatorname{div}\sigma)\varphi\,dx \quad \text{for all } \varphi\in\mathcal{C}_c^\infty(\mathbb{R}^2). \tag{2.2}$$

If Ω has Lipschitz boundary, approximating σ by smooth functions, and using the integration by parts formula in BV, one can show that $[\sigma\cdot p]$ is actually a bounded Radon measure supported in $\overline{\Omega}$ satisfying

$$|[\sigma\cdot p]| \le \|\sigma\|_\infty |p| \quad \text{in } \mathcal{M}(\overline{\Omega}) \tag{2.3}$$

and with total mass that obtained by taking $\varphi\equiv 1$ in (2.2). Moreover, if $\sigma \in \mathcal{C}(\Omega;\mathbb{R}^2)$ we can show that

$$\langle[\sigma\cdot p],\varphi\rangle = \int_\Omega \varphi\sigma\cdot\,dp = \int_\Omega \varphi\sigma\cdot\frac{dp}{d|p|}\,d|p| \quad \text{for all } \varphi\in\mathcal{C}_c(\Omega),$$

where $\frac{dp}{d|p|}$ stands for the Radon-Nikodým derivative of p with respect to its total variation $|p|$. For all of the above, see [23, Section 6] in the vectorial case.

3. Hencky plasticity

We recall in a few lines the basic tenet of Hencky plasticity, which is the ancestor of modern (small strain) plasticity. In Hencky plasticity (see *e.g.* [37]), time is absent, so that the plasticity problem reduces to a problen of statics which can be tackled as a minimization problem. In this paper, we even go further and only address what is usually called the anti-plane shear strain case for which the displacement field (generally a vector-valued field in 3d) is along the x_3-direction and only depends on the planar variables (x_1,x_2).

Let Ω be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary and $w \in H^1(\Omega)$ be a boundary data. We consider the following minimization problem

$$\inf\Big\{\frac{1}{2}\int_\Omega |\sigma|^2\,dx + |p|(\overline{\Omega}) :\ (u,\sigma,p)\in\mathcal{A}_w\Big\} \tag{3.1}$$

with

$$\mathcal{A}_w := \{(u,\sigma,p)\in BV(\Omega)\times L^2(\Omega;\mathbb{R}^2)\times\mathcal{M}(\overline{\Omega};\mathbb{R}^2) : Du = \sigma + p \text{ in } \Omega,\ p = (w-u)\nu\mathcal{H}^1 \text{ on } \partial\Omega\}.$$

The direct method in the calculus of variations ensures the existence of minimizers $(u,\sigma,p)\in BV(\Omega)\times L^2(\Omega;\mathbb{R}^2)\times\mathcal{M}(\overline{\Omega};\mathbb{R}^2)$. In addition, the stress σ is unique and all minimizers (u,σ,p) satisfy the following first order conditions

$$\begin{cases} \mathrm{div}\sigma = 0 \quad \text{in } H^{-1}(\Omega),\\ |\sigma|\le 1 \quad \text{a.e. in } \Omega,\\ Du = \sigma + p \quad \text{in } \mathcal{M}(\Omega;\mathbb{R}^2),\\ p = (w-u)\nu\mathcal{H}^1 \text{ in } \mathcal{M}(\partial\Omega;\mathbb{R}^2),\\ |p| = [\sigma\cdot p] \quad \text{in } \mathcal{M}(\overline{\Omega}). \end{cases} \tag{3.2}$$

The uniqueness of σ is a immediate consequence of the strict convexity of $\sigma\mapsto\int_\Omega|\sigma|^2\,dx$. We quickly check (3.2). Performing variations of the form $(u,\sigma,p)+t(v,\tau,q)$ where $t\in(0,1)$ and $(v,\tau,q)\in\mathcal{A}_0$ and letting $t\to 0^+$ leads to

$$|p|(\overline{\Omega})\le\int_\Omega\sigma\cdot\tau\,dx+|p+q|(\overline{\Omega}). \tag{3.3}$$

Choosing first $(v,\tau,q)=\pm(\varphi,\nabla\varphi,0)$, with $\varphi\in\mathcal{C}_c^\infty(\Omega)$ arbitrary, as test function in (3.3) gives

$$\int_\Omega\sigma\cdot\nabla\varphi\,dx=0$$

hence $\mathrm{div}\sigma=0$ in $H^{-1}(\Omega)$. Next, denoting by $p^a=\frac{dp}{d\mathcal{L}^2}$ the Radon-Nikodým derivative of p with respect to $\mathcal{L}^2$ and considering $(v,\tau,q)=(0,p^a-\eta,\eta-p^a)$ as test function in (3.3), where $\eta\in L^1(\Omega;\mathbb{R}^2)$ is arbitrary, leads to

$$\int_\Omega|\eta|\,dx\ge\int_\Omega|p^a|\,dx+\int_\Omega\sigma\cdot(\eta-p^a)\,dx.$$

Localizing this inequality yields $\sigma(x)\in\boldsymbol{\partial}|\cdot|(p^a(x))\subset\boldsymbol{\partial}|\cdot|(0)$ for a.e. in $x\in\Omega$, hence

$$|\sigma|\le 1 \text{ a.e. in } \Omega. \tag{3.4}$$

It remains to prove the "flow rule". From (2.3) and (3.4) the first inequality $|p|\ge[\sigma\cdot p]$ in $\mathcal{M}(\overline{\Omega})$ holds. To prove the reverse inequality take $(v,\tau,q)=(w-u,\nabla w-\sigma,-p)$ as test function in (3.3), and use the definition (2.2) of duality. This gives

$$|p|(\overline{\Omega})\le\int_\Omega\sigma\cdot(\nabla w-\sigma)\,dx=\langle[\sigma\cdot p],1\rangle.$$

Hence, the nonnegative measure $|p|-[\sigma\cdot p]$ has zero total mass which leads to the flow rule $|p|=[\sigma\cdot p]$ in $\mathcal{M}(\overline{\Omega})$.

Remark 3.1. Exactly as in [23, Lemma 3.8], if ω is an open subset of Ω with Lipschitz boundary $\Gamma=\partial\omega$ and such that $\overline{\omega}\subset\Omega$, then $\sigma\cdot\nu\in L^\infty(\Gamma)$ and

$$[\sigma\cdot p]\llcorner\Gamma=(\sigma\cdot\nu)(u^+-u^-)\mathcal{H}^1\llcorner\Gamma,$$

where u^+ and u^- are the outer and inner traces of u on Γ and $\sigma\cdot\nu$ is the normal trace of σ on Γ. Thus, the flow rule localized on Γ reads

$$(\sigma\cdot\nu)(u^+-u^-)=|u^+-u^-| \quad \mathcal{H}^1\text{-a.e. on } \Gamma.$$

Since by definition $u^+\neq u^-$ on J_u, we infer that $\sigma\cdot\nu=\pm1$ $\mathcal{H}^1$-a.e. on $\Gamma\cap J_u$. This applies also if $\mathcal{H}^1(\partial\omega\cap\partial\Omega)>0$, replacing u^+ by w on that part of $\partial\omega$. ¶

4. Sobolev regularity of the stress

This section revolves around the Seregin/Bensoussan-Frehse's Sobolev regularity property of the (unique) stress field σ in the minimization problem (3.1) (or equivalently in the system (3.2)).

Theorem 4.1 (Seregin/Bensoussan-Frehse regularity)**.** *The unique stress σ such that the triplet $(u,\sigma,p) \in BV(\Omega)\times L^2(\Omega;\mathbb{R}^2)\times \mathcal{M}(\overline{\Omega};\mathbb{R}^2)$ is a minimizer of* (3.1) *belongs to $H^1_{\rm loc}(\Omega;\mathbb{R}^2)$.*

Proof. Step 1. In a first step we perform a so-called Perzyna approximation of the minimization problem (3.1). We thus consider, for $\varepsilon>0$,

$$\inf\Big\{\int_\Omega \Big(\frac12|\sigma|^2+|p|+\frac{\varepsilon}{2}|p|^2\Big)\,dx:\ (u,\sigma,p)\in H^1(\Omega)\times L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2) \text{ such that } \nabla u=\sigma+p \text{ a.e. in } \Omega \text{ and } u=w\ \mathcal{H}^1\text{-a.e. on } \partial\Omega\Big\}. \quad (4.1)$$

By strict convexity, there exists a unique minimizing triplet $(u_\varepsilon,\sigma_\varepsilon,p_\varepsilon)$.

Further, for a subsequence (still labeled by ε), it is straightforward to show that

$$\begin{cases} u_\varepsilon \rightharpoonup u \text{ weak-* in } BV(\Omega),\\ \sigma_\varepsilon \rightharpoonup \sigma \text{ weak in } L^2(\Omega;\mathbb{R}^2),\\ (\sqrt{\varepsilon}p_\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^2(\Omega;\mathbb{R}^2),\\ p_\varepsilon \rightharpoonup p \text{ weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{R}^2), \end{cases} \quad (4.2)$$

where $(u,\sigma,p)\in BV(\Omega)\times L^2(\Omega;\mathbb{R}^2)\times\mathcal{M}(\overline{\Omega};\mathbb{R}^2)$ is a minimizer of (3.1) as can be seen through direct application of an approximation result found in [33, Theorem 3.5]; see also [8, Proposition 3.3].

Now, testing minimality with triplets $(u_\varepsilon+tv,\sigma_\varepsilon+t\tau,p_\varepsilon+tq)$ with $t\in(0,1)$ and $(v,\tau,q)\in H^1_0(\Omega)\times L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2)$ where $\nabla v=\tau+q$, it is easily seen – choosing either $q\equiv 0$ and $\tau=\nabla v$ with $v\in H^1_0(\Omega)$ arbitrary, or $v\equiv 0$ and $\tau=-q\mathbf{1}_E$ where $E\subset\Omega$ is measurable and $q\in\mathbb{R}^2$ arbitrary – that the minimizing triplet $(u_\varepsilon,\sigma_\varepsilon,p_\varepsilon)$ satisfies the following Euler-Lagrange equations:

$$\begin{cases} \operatorname{div}\sigma_\varepsilon=0 \quad \text{in } H^{-1}(\Omega),\\ \nabla u_\varepsilon=\sigma_\varepsilon+p_\varepsilon \quad \text{a.e. in } \Omega,\\ u_\varepsilon=w \quad \mathcal{H}^1\text{-a.e. on } \partial\Omega,\\ \sigma_\varepsilon-\varepsilon p_\varepsilon\in\partial H(p_\varepsilon) \quad \text{a.e. in } \Omega, \end{cases} \quad (4.3)$$

where $H(q):=|q|$ for all $q\in\mathbb{R}^2$.

Remark 4.2. Observe that the fourth relation in (4.3) reads, by convex duality, as

$$p_\varepsilon\in\boldsymbol{\partial} I(\sigma_\varepsilon-\varepsilon p_\varepsilon) \quad \text{a.e. in } \Omega,$$

where I is the indicator function of the closed unit ball, *i.e.*,

$$I(q):=\begin{cases}0 & \text{if } |q|\le 1,\\ +\infty & \text{if } |q|>1.\end{cases}$$

Thus $\nabla u_\varepsilon-\varepsilon p_\varepsilon=p_\varepsilon+(\sigma_\varepsilon-\varepsilon p_\varepsilon)\in\boldsymbol{\partial}\Psi(\sigma_\varepsilon-\varepsilon p_\varepsilon)$ where $\Psi:\mathbb{R}^2\to\mathbb{R}$ is defined by

$$\Psi(q):=\frac12|q|^2+I(q) \quad \text{for all } q\in\mathbb{R}^2,$$

or still

$$\sigma_\varepsilon-\varepsilon p_\varepsilon=D\Psi^*(\nabla u_\varepsilon-\varepsilon p_\varepsilon) \quad (4.4)$$

where $\Psi^* : \mathbb{R}^2 \to \mathbb{R}$, the convex conjugate of Ψ, is given by

$$\Psi^*(q) = \begin{cases} \frac{1}{2}|q|^2 & \text{if } |q| \leq 1, \\ |q| - \frac{1}{2} & \text{if } q| > 1. \end{cases}$$

Remark that $\Psi^* \in \mathcal{C}^1(\mathbb{R}^2)$ with

$$D\Psi^*(q) = \begin{cases} q & \text{if } |q| \leq 1, \\ q/|q| & \text{if } |q| > 1, \end{cases}$$

and that $D\Psi^* \in \mathrm{Lip}(\mathbb{R}^2) \cap \mathcal{C}^1(\mathbb{R}^2 \setminus \mathbb{S}^1)$ with

$$D^2\Psi^*(q) = \begin{cases} \mathrm{Id} & \text{if } |q| < 1, \\ \dfrac{1}{|q|}\mathrm{Id} - \dfrac{1}{|q|^3} q \otimes q & \text{if } |q| > 1. \end{cases}$$

Finally, the expression for $D^2\Psi^*$ implies that, for $|q| > 1$ and for all $r \in \mathbb{R}^2$,

$$D^2\Psi^*(q)[r] = \frac{1}{|q|} P_{q^\perp}(r)$$

where $P_{q^\perp}$ is the orthogonal projection in the direction $q^\perp$.

Those projection properties, which are specific to the Von-Mises criterion, will be instrumental in the proof of the Sobolev regularity of the stress. ¶

For ε fixed, a usual translation argument yields the classical local elliptic regularity of the fields (see *e.g.* [8, Proposition 3.4] in the vectorial evolution case), that is

$$u_\varepsilon \in H^2_{\mathrm{loc}}(\Omega), \quad \sigma_\varepsilon, p_\varepsilon \in H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^2) \tag{4.5}$$

with corresponding ε-dependent bounds.

Step 2. Let $k \in \{1, 2\}$ and $\varphi \in \mathcal{C}^\infty_c(\Omega)$. Taking $\varphi^2 \partial_k u_\varepsilon$ (which belongs to $H^1_0(\Omega)$ thanks to (4.5)) as test function for the equation

$$\mathrm{div}(\partial_k \sigma_\varepsilon) = 0 \quad \text{ in } H^{-1}(\Omega),$$

we obtain

$$0 = \int_\Omega \varphi^2 \partial_k \sigma_\varepsilon \cdot \partial_k \nabla u_\varepsilon \, dx + \int_\Omega \partial_k \sigma_\varepsilon \cdot \nabla \varphi^2 \partial_k u_\varepsilon \, dx =: I_1 + I_2.$$

In the sequel, C_φ will stand for a positive constant which may vary from line to line; it may depend on φ and on the bounds coming from (4.2), but it is independent of ε.

We now rewrite I_1 and I_2 as follows.

$$I_1 = \int_\Omega \varphi^2 |\partial_k \sigma_\varepsilon|^2 \, dx + \int_\Omega \varphi^2 \left(\partial_k \sigma_\varepsilon - \varepsilon \partial_k p_\varepsilon\right) \cdot \partial_k p_\varepsilon \, dx + \varepsilon \int_\Omega \varphi^2 |\partial_k p_\varepsilon|^2 \, dx, \tag{4.6}$$

while

$$\begin{aligned} |I_2| &\leq 2 \int_\Omega |\partial_k \sigma_\varepsilon \cdot \nabla \varphi| \, |(\sigma_\varepsilon)_k \varphi| \, dx + \int_\Omega |\partial_k \sigma_\varepsilon| |\nabla \varphi^2| |(p_\varepsilon)_k| \, dx \\ &\leq C_\varphi \|\sigma_\varepsilon\|_{L^2(\Omega)} \|\varphi \partial_k \sigma_\varepsilon\|_{L^2(\Omega)} + \int_\Omega |\partial_k \sigma_\varepsilon| |\nabla \varphi^2| |p_\varepsilon| \, dx. \end{aligned} \tag{4.7}$$

Using Young's inequality and reassembling (4.6), (4.7) we thus get, in view of the bound coming from the second convergence in (4.2),

$$\frac{1}{2}\int_\Omega \varphi^2|\partial_k\sigma_\varepsilon|^2\,dx+\varepsilon\int_\Omega \varphi^2|\partial_k p_\varepsilon|^2\,dx+\int_\Omega \varphi^2\left(\partial_k\sigma_\varepsilon-\varepsilon\partial_k p_\varepsilon\right)\cdot\partial_k p_\varepsilon\,dx \\ \leq C_\varphi+\int_\Omega |\partial_k\sigma_\varepsilon||\nabla\varphi^2||p_\varepsilon|\,dx, \quad (4.8)$$

or still, adding and subtracting $\partial_k\sigma_\varepsilon-\varepsilon\partial_k p_\varepsilon$ to $\partial_k p_\varepsilon$ in the third integral on the left hand-side of (4.8) above and recalling that $\nabla u_\varepsilon=\sigma_\varepsilon+p_\varepsilon$,

$$\frac{1}{2}\int_\Omega \varphi^2|\partial_k\sigma_\varepsilon|^2\,dx+\varepsilon\int_\Omega \varphi^2|\partial_k p_\varepsilon|^2\,dx+\int_\Omega \varphi^2(\partial_k\sigma_\varepsilon-\varepsilon\partial_k p_\varepsilon)\cdot(\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon)\,dx \\ \leq C_\varphi+\int_\Omega \varphi^2|\partial_k\sigma_\varepsilon-\varepsilon\partial_k p_\varepsilon|^2\,dx+\int_\Omega |\partial_k\sigma_\varepsilon||\nabla\varphi^2||p_\varepsilon|\,dx. \quad (4.9)$$

We now rewrite (4.9) using Remark 4.2. Since, in view of (4.5), $\nabla u_\varepsilon-\varepsilon p_\varepsilon\in H^1_{\rm loc}(\Omega;\mathbb{R}^2)$, we can apply the generalized chain rule formula from [34, Theorem 2.1] to (4.4). We obtain

$$\partial_k\sigma_\varepsilon-\varepsilon\partial_k p_\varepsilon=D^2\Psi^*(\nabla u_\varepsilon-\varepsilon p_\varepsilon)\left[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon\right]. \quad (4.10)$$

Remark 4.3. Relation (4.10) has to be understood as follows (see [34]):

$$\partial_k\sigma_\varepsilon-\varepsilon\partial_k p_\varepsilon=\begin{cases}\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon & \text{if } |\nabla u_\varepsilon-\varepsilon p_\varepsilon|\leq 1,\\ \dfrac{1}{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|}P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}(\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon) & \text{if } |\nabla u_\varepsilon-\varepsilon p_\varepsilon|>1.\end{cases} \quad (4.11)$$

Note that $(\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon)\cdot(\nabla u_\varepsilon-\varepsilon p_\varepsilon)=0$ a.e. on $\{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|=1\}$; see [34, Proposition 2.2] applied to the $\mathcal{C}^1$-Lipschitz functions

$$q\mapsto\begin{cases}|q|^2 & \text{if } |q|\leq 1\\ 2|q|-1 & \text{if } |q|>1.\end{cases}$$

Consequently,

$$\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon=\frac{1}{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|}P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}(\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon) \quad \text{a.e. on } \{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|=1\},$$

which implies that (4.11) can be changed into

$$\partial_k\sigma_\varepsilon-\varepsilon\partial_k p_\varepsilon=\begin{cases}\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon & \text{if } |\nabla u_\varepsilon-\varepsilon p_\varepsilon|<1,\\ \dfrac{1}{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|}P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}(\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon) & \text{if } |\nabla u_\varepsilon-\varepsilon p_\varepsilon|\geq 1.\end{cases}$$

¶

In view of (4.10), (4.9) reads as

$$\frac{1}{2}\int_\Omega \varphi^2|\partial_k\sigma_\varepsilon|^2\,dx+\varepsilon\int_\Omega \varphi^2|\partial_k p_\varepsilon|^2\,dx+\int_\Omega \varphi^2 D^2\Psi^*(\nabla u_\varepsilon-\varepsilon p_\varepsilon)[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]\cdot(\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon)\,dx \\ \leq C_\varphi+\int_\Omega \varphi^2|D^2\Psi^*(\nabla u_\varepsilon-\varepsilon p_\varepsilon)[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]|^2\,dx+\int_\Omega |\partial_k\sigma_\varepsilon||\nabla\varphi^2||p_\varepsilon|\,dx. \quad (4.12)$$

Step 3. We next exploit inequality (4.12) obtained at the end of Step 2 by splitting Ω into three ε-dependent subsets as follows,

$$\begin{aligned}
\Omega_\varepsilon^- &:= \{x \in \Omega : \ |\nabla u_\varepsilon(x) - \varepsilon p_\varepsilon(x)| \leq 1\},\\
\Omega_{\varepsilon>}^+ &:= \{x \in \Omega : \ |\nabla u_\varepsilon(x) - \varepsilon p_\varepsilon(x)| > 2\},\\
\Omega_{\varepsilon<}^+ &:= \{x \in \Omega : \ 1 < |\nabla u_\varepsilon(x) - \varepsilon p_\varepsilon(x)| \leq 2\},\\
\Omega_\varepsilon^+ &:= \Omega_{\varepsilon<}^+ \cup \Omega_{\varepsilon>}^+.
\end{aligned}$$

First note that, on Ω_ε^-, (4.11) in Remark 4.3 implies that

$$\partial_k \sigma_\varepsilon - \varepsilon \partial_k p_\varepsilon = D^2\Psi^*(\nabla u_\varepsilon - \varepsilon p_\varepsilon)\left[\partial_k \nabla u_\varepsilon - \varepsilon \partial_k p_\varepsilon\right] = \partial_k \nabla u_\varepsilon - \varepsilon \partial_k p_\varepsilon.$$

Consequently, the contributions of the integrals involving the term $D^2\Psi^*(\nabla u_\varepsilon - \varepsilon p_\varepsilon)$ cancel out on that set in (4.12). Further, writing $p_\varepsilon = (\nabla u_\varepsilon - \varepsilon p_\varepsilon) - \sigma_\varepsilon + \varepsilon p_\varepsilon$ and by virtue of the bounds coming from the second and third convergences in (4.2), we have that,

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon^- \cup \Omega_{\varepsilon<}^+)} \leq C$$

with $C > 0$ a constant independent of ε (and actually also independent of φ). Using once again Young's inequality we can thus absorb the contribution of the term $\int_{\Omega_\varepsilon^- \cup \Omega_{\varepsilon<}^+} |\partial_k \sigma_\varepsilon|\,|\nabla \varphi^2||p_\varepsilon|\,dx$ in the term $\frac{1}{2}\int_\Omega \varphi^2 |\partial_k \sigma_\varepsilon|^2\,dx$ in (4.12) at the possible expense of changing C_φ. In lieu of (4.12), we are thus left with

$$\begin{aligned}
\frac{1}{4}\int_\Omega \varphi^2 |\partial_k \sigma_\varepsilon|^2\,dx + \varepsilon \int_\Omega \varphi^2 |\partial_k p_\varepsilon|^2\,dx&\\
+ \int_{\Omega_\varepsilon^+} \varphi^2 D^2\Psi^*(\nabla u_\varepsilon - \varepsilon p_\varepsilon)[\partial_k \nabla u_\varepsilon - \varepsilon \partial_k p_\varepsilon] \cdot (\partial_k \nabla u_\varepsilon - \varepsilon \partial_k p_\varepsilon)\,dx&\\
\leq C_\varphi + \int_{\Omega_\varepsilon^+} \varphi^2 |D^2\Psi^*(\nabla u_\varepsilon - \varepsilon p_\varepsilon)[\partial_k \nabla u_\varepsilon - \varepsilon \partial_k p_\varepsilon]|^2\,dx + \int_{\Omega_{\varepsilon>}^+} |\partial_k \sigma_\varepsilon||\nabla \varphi^2||p_\varepsilon|\,dx,&
\end{aligned}$$

or still, in view of (4.10), with

$$\begin{aligned}
\frac{1}{4}\int_\Omega \varphi^2 |\partial_k \sigma_\varepsilon|^2\,dx + \varepsilon \int_\Omega \varphi^2 |\partial_k p_\varepsilon|^2\,dx&\\
+ \int_{\Omega_\varepsilon^+} \varphi^2 D^2\Psi^*(\nabla u_\varepsilon - \varepsilon p_\varepsilon)[\partial_k \nabla u_\varepsilon - \varepsilon \partial_k p_\varepsilon] \cdot (\partial_k \nabla u_\varepsilon - \varepsilon \partial_k p_\varepsilon)\,dx&\\
\leq C_\varphi + \int_{\Omega_\varepsilon^+} \varphi^2 |D^2\Psi^*(\nabla u_\varepsilon - \varepsilon p_\varepsilon)[\partial_k \nabla u_\varepsilon - \varepsilon \partial_k p_\varepsilon]|^2\,dx&\\
+ \int_{\Omega_{\varepsilon>}^+} |D^2\Psi^*(\nabla u_\varepsilon - \varepsilon p_\varepsilon)[\partial_k \nabla u_\varepsilon - \varepsilon \partial_k p_\varepsilon]||\nabla \varphi^2||p_\varepsilon|\,dx + \varepsilon \int_{\Omega_{\varepsilon>}^+} |\partial_k p_\varepsilon||\nabla \varphi^2||p_\varepsilon|\,dx.&
\end{aligned}$$

Because of the third relation in (4.2) and of Young's inequality, the third integral in the right hand-side of the last inequality above can be controled by the term $\varepsilon \int_\Omega \varphi^2 |\partial_k p_\varepsilon|^2\,dx$ in the left

hand-side at the expense of changing C_φ, *i.e.* ,

$$\begin{aligned}\frac{1}{4}\int_\Omega \varphi^2|\partial_k\sigma_\varepsilon|^2\,dx+\frac{\varepsilon}{2}\int_\Omega \varphi^2|\partial_k p_\varepsilon|^2\,dx\\ +\int_{\Omega_\varepsilon^+}\varphi^2 D^2\Psi^*(\nabla u_\varepsilon-\varepsilon p_\varepsilon)[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]\cdot(\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon)\,dx\\ \le C_\varphi+\int_{\Omega_\varepsilon^+}\varphi^2|D^2\Psi^*(\nabla u_\varepsilon-\varepsilon p_\varepsilon)[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]|^2\,dx\\ +\int_{\Omega_{\varepsilon>}^+}|D^2\Psi^*(\nabla u_\varepsilon-\varepsilon p_\varepsilon)[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]||\nabla\varphi^2|\big\{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|+|\sigma_\varepsilon+\varepsilon p_\varepsilon|\big\}\,dx,\end{aligned}\tag{4.13}$$

where we used $|p_\varepsilon|\le|\nabla u_\varepsilon-\varepsilon p_\varepsilon|+|\sigma_\varepsilon+\varepsilon p_\varepsilon|$ in the last term of the right hand side.

Recalling Remark 4.2 and noting that $P_{q^\perp}(\xi)\cdot\xi=|P_{p^\perp}(\xi)|^2$, (4.13) now reads as

$$\begin{aligned}\frac{1}{4}\int_\Omega \varphi^2|\partial_k\sigma_\varepsilon|^2\,dx+\frac{\varepsilon}{2}\int_\Omega \varphi^2|\partial_k p_\varepsilon|^2\,dx+\int_{\Omega_\varepsilon^+}\frac{\varphi^2}{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]|^2\,dx\\ \le C_\varphi+\int_{\Omega_\varepsilon^+}\frac{\varphi^2}{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|^2}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]|^2\,dx\\ +\int_{\Omega_{\varepsilon>}^+}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]||\nabla\varphi^2|\,dx\\ +\int_{\Omega_{\varepsilon>}^+}\frac{|\nabla\varphi^2|}{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]||\sigma_\varepsilon+\varepsilon p_\varepsilon|\,dx.\end{aligned}$$

In view of the bounds coming from the second and third relations in (4.2), application of the Cauchy-Schwarz and Young inequalities to the last term in the inequality above yield, at the possible expense of changing C_φ,

$$\begin{aligned}\frac{1}{4}\int_\Omega \varphi^2|\partial_k\sigma_\varepsilon|^2\,dx+\frac{\varepsilon}{2}\int_\Omega \varphi^2|\partial_k p_\varepsilon|^2\,dx+\int_{\Omega_\varepsilon^+}\frac{\varphi^2}{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]|^2\,dx\\ \le C_\varphi+\int_{\Omega_{\varepsilon<}^+}\frac{\varphi^2}{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|^2}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]|^2\,dx\\ +\int_{\Omega_{\varepsilon>}^+}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]||\nabla\varphi^2|\,dx\\ +\frac{3}{2}\int_{\Omega_{\varepsilon>}^+}\frac{\varphi^2}{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|^2}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]|^2\,dx.\end{aligned}\tag{4.14}$$

The second integral in the right hand-side of inequality (4.14) can in turn be estimated as follows with the help, once more, of the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{\Omega_{\varepsilon>}^+}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]||\nabla\varphi^2|\,dx\\ \le\|\nabla u_\varepsilon-\varepsilon p_\varepsilon\|_{L^1(\Omega)}\left\{\int_{\Omega_{\varepsilon>}^+}\frac{|\nabla\varphi^2|^2}{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]|^2\,dx\right\}^{1/2}\\ \le C'_\varphi\left\{\int_{\Omega_{\varepsilon>}^+}\frac{\varphi^2}{|\nabla u_\varepsilon-\varepsilon p_\varepsilon|}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon-\varepsilon\partial_k p_\varepsilon]|^2\,dx\right\}^{1/2},\end{aligned}$$

where the last inequality holds because $\|\nabla u_\varepsilon - \varepsilon p_\varepsilon\|_{L^1(\Omega)} \le C$ in view of the bounds coming from (4.2), and $C'_\varphi > 0$ is another φ-dependent constant.

Thus inequality (4.14) becomes

$$\begin{aligned}\frac{1}{4}\int_\Omega \varphi^2|\partial_k\sigma_\varepsilon|^2\,dx + \frac{\varepsilon}{2}\int_\Omega \varphi^2|\partial_k p_\varepsilon|^2\,dx + \int_{\Omega^+_{\varepsilon>}} \frac{\varphi^2}{|\nabla u_\varepsilon - \varepsilon p_\varepsilon|}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon - \varepsilon\partial_k p_\varepsilon]|^2\,dx \\ + \int_{\Omega^+_{\varepsilon<}} \frac{\varphi^2}{|\nabla u_\varepsilon - \varepsilon p_\varepsilon|}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon - \varepsilon\partial_k p_\varepsilon]|^2\,dx \\ \le C_\varphi + \frac{3}{2}\int_{\Omega^+_{\varepsilon>}} \frac{\varphi^2}{|\nabla u_\varepsilon - \varepsilon p_\varepsilon|^2}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon - \varepsilon\partial_k p_\varepsilon]|^2\,dx \\ + \int_{\Omega^+_{\varepsilon<}} \frac{\varphi^2}{|\nabla u_\varepsilon - \varepsilon p_\varepsilon|^2}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon - \varepsilon\partial_k p_\varepsilon]|^2\,dx \\ + C'_\varphi\left\{\int_{\Omega^+_{\varepsilon>}} \frac{\varphi^2}{|\nabla u_\varepsilon - \varepsilon p_\varepsilon|}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon - \varepsilon\partial_k p_\varepsilon]|^2\,dx\right\}^{1/2}. \quad (4.15)\end{aligned}$$

But $|\nabla u_\varepsilon - \varepsilon p_\varepsilon| > 2$ on $\Omega^+_{\varepsilon>}$, while $|\nabla u_\varepsilon - \varepsilon p_\varepsilon|^2 \ge |\nabla u_\varepsilon - \varepsilon p_\varepsilon|$ on $\Omega^+_{\varepsilon<}$, so that, finally, (4.15) implies that

$$\begin{aligned}\frac{1}{4}\int_\Omega \varphi^2|\partial_k\sigma_\varepsilon|^2\,dx + \frac{\varepsilon}{2}\int_\Omega \varphi^2|\partial_k p_\varepsilon|^2\,dx + \frac{1}{4}\int_{\Omega^+_{\varepsilon>}} \frac{\varphi^2}{|\nabla u_\varepsilon - \varepsilon p_\varepsilon|}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon - \varepsilon\partial_k p_\varepsilon]|^2\,dx \\ \le C_\varphi + C'_\varphi\left\{\int_{\Omega^+_{\varepsilon>}} \frac{\varphi^2}{|\nabla u_\varepsilon - \varepsilon p_\varepsilon|}|P_{(\nabla u_\varepsilon-\varepsilon p_\varepsilon)^\perp}[\partial_k\nabla u_\varepsilon - \varepsilon\partial_k p_\varepsilon]|^2\,dx\right\}^{\frac{1}{2}}. \quad (4.16)\end{aligned}$$

From a last application of Young's inequality, it is immediately deduced from (4.16) that the sequence $(\partial_k\sigma_\varepsilon)_{\varepsilon>0}$ is bounded in $H^1_{\rm loc}(\Omega;\mathbb{R}^2)$ independently of ε, which implies the desired regularity for $\partial_k\sigma$. The proof of Theorem 4.1 is complete. □

Remark 4.4. Since $\sigma \in H^1_{\rm loc}(\Omega;\mathbb{R}^2)$, then it admits a precise representative defined $\mathrm{Cap}_{\rm p}$-quasi everywhere for any $p < 2$ hence $\mathcal{H}^s$-almost everywhere in Ω for any $s > 0$ (see *e.g.* [21, Sections 4.7, 4.8]). In particular, $\mathcal{H}^1$-almost every point in Ω is a Lebesgue point for σ, *i.e.*,

$$\lim_{\rho\to 0}\frac{1}{\rho^2}\int_{B_\rho(x_0)}|\sigma(y)-\sigma(x)|^2\,dy = 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } x\in\Omega.$$

In the sequel we will identify σ to be its precise representative which is thus defined $\mathcal{H}^1$-almost everywhere in Ω (it is actually defined outside a set of zero Hausdorff dimension). ¶

Remark 4.5. Arguing as in [4, 15, 24, 8], it is possible to express the flow rule by means of the quasi-continuous representative of the stress, still denoted by σ, which is $|p|$-measurable, in a pointwise sense:

$$\sigma(x)\cdot\frac{dp}{d|p|}(x) = 1 \quad \text{for } |p|\text{-a.e. } x\in\Omega$$

or still

$$p = \sigma|p| \quad \text{in } \mathcal{M}(\Omega_p).$$

In particular, the measure $|p|$ is concentrated in the plastic part of the domain, *i.e.*, $|p|(\{x\in\Omega : |\sigma(x)|<1\}) = 0$. ¶

Remark 4.6. Let S be a segment such that $\overline{S} \subset \Omega$, then $\sigma \in H^{1/2}(S;\mathbb{R}^2)$. In addition, if $x_0 \in S$ is a Lebesgue point of σ, then

$$\lim_{\rho\to 0} \frac{1}{\rho} \int_{B_\rho(x)\cap S} |\sigma(y) - \sigma(x_0)|^2 \, d\mathcal{H}^1(y) = 0.$$

In other words, x_0 is also a Lebesgue point for the trace of σ on S.

Indeed, the trace theorem in Sobolev spaces states that σ has a trace on S, denoted by $\sigma_{|S}$, which belongs to $H^{1/2}(S;\mathbb{R}^2)$. For simplicity, we will assume that $S = (-1,1)\times\{0\}$. Arguing by approximation, we first observe that, for all $x \in S$,

$$\frac{1}{4\rho^2} \int_{x+(-\rho,\rho)^2} |\sigma(y) - \sigma_{|S}(y_1,0)|^2 \, dy \le C \int_{x+(-\rho,\rho)^2} |\nabla\sigma(y)|^2 \, dy \to 0 \tag{4.17}$$

as $\rho \to 0$, since $\nabla\sigma \in L^2_{\rm loc}(\Omega;\mathbb{M}^{2\times 2})$.

Let us consider $x \in S$ which is a Lebesgue point of σ (as an element of $L^2_{\mathcal{L}^2}(\Omega;\mathbb{R}^2)$), *i.e.*,

$$\lim_{\rho\to 0} \frac{1}{\rho^2} \int_{x+(-\rho,\rho)^2} |\sigma(y) - \sigma(x)|^2 \, dy = 0 \tag{4.18}$$

as well as a Lebesgue point of $\sigma_{|S}$ (as an element of $L^2_{\mathcal{H}^1}(S;\mathbb{R}^2)$), *i.e.*,

$$\lim_{\rho\to 0} \frac{1}{\rho} \int_{B_\rho(x)\cap S} |\sigma_{|S}(y) - \sigma_{|S}(x)|^2 \, d\mathcal{H}^1(y) = 0. \tag{4.19}$$

Observe that $\mathcal{H}^1$-almost every point x in S satisfy these properties. As a consequence of (4.17) and (4.18), we have

$$\begin{aligned}\frac{1}{\rho} \int_{B_\rho(x)\cap S} |\sigma_{|S}(y) - \sigma(x)|^2 \, d\mathcal{H}^1(y) &= \frac{1}{\rho} \int_{-\rho}^{\rho} |\sigma_{|S}(y_1,0) - \sigma(x)|^2 \, dy_1 \\ &\le \frac{1}{\rho^2} \int_{x+(-\rho,\rho)^2} |\sigma(y) - \sigma_{|S}(y_1,0)|^2 \, dy + \frac{1}{\rho^2} \int_{x+(-\rho,\rho)^2} |\sigma(y) - \sigma(x)|^2 \, dy \to 0.\end{aligned}$$

Using next (4.19), we infer that $\sigma(x) = \sigma_{|S}(x)$ which shows that $\sigma = \sigma_{|S}$ $\mathcal{H}^1$-a.e. on S, and that $\sigma \in H^{1/2}(S;\mathbb{R}^2)$.

If now $x_0 \in S$ is only a Lebesgue of σ, then by (4.17) (which holds for all $x_0 \in S$) and (4.18), we have similarly

$$\begin{aligned}\frac{1}{\rho} \int_{B_\rho(x_0)\cap S} |\sigma(y) - \sigma(x_0)|^2 \, d\mathcal{H}^1(y) &= \frac{1}{\rho} \int_{-\rho}^{\rho} |\sigma(y_1,0) - \sigma(x_0)|^2 \, dy_1 \\ &\le \frac{1}{\rho^2} \int_{x_0+(-\rho,\rho)^2} |\sigma(y) - \sigma(y_1,0)|^2 \, dy + \frac{1}{\rho^2} \int_{x_0+(-\rho,\rho)^2} |\sigma(y) - \sigma(x_0)|^2 \, dy \to 0,\end{aligned}$$

which completes the proof of the result. ¶

5. Rigidity properties of the solutions

The goal of this section is to take advantage of the hyperbolic equations satisfied by σ and $|p|$ in the plastic zone $\{x \in \Omega : \ |\sigma(x)| = 1\}$ in order to derive rigidity properties of the solutions σ and u in that zone. The equations are

$$\mathrm{div}\sigma = 0, \quad |\sigma| = 1, \tag{5.1}$$

and

$$\mathrm{curl}\,(\sigma(1+|p|)) = \mathrm{curl}\, Du = 0. \tag{5.2}$$

We will need the the following

Hypothesis (H). *The set* $\{x \in \Omega : \ |\sigma(x)| = 1\}$ *has a nonempty interior.*

As already discussed in the Introduction, hypothesis (H) is fulfilled in several vectorial examples. It could also be the case in particular in simple scalar settings, for example when Ω itself is a (countable union of) boundary fans (see Subsection 6.1).

In the sequel, we consider a convex open subset Ω_p of the interior of $\{x \in \Omega : \ |\sigma(x)| = 1\}$. Note, for future use, that, in such a setting $\partial\Omega_p$ has Lipschitz boundary (see *e.g.* Propositions 2.4.4 and Proposition 2.4.7 in [27]).

5.1. **Lipschitz regularity and rigidity of the stress.** In this subsection we improve the Sobolev regularity of the stress field in the plastic region. We show that under assumption (H), the stress is actually locally Lipschitz continuous in Ω_p, and that it is constant along all the characteristic lines $L_x = x + \mathbb{R}\sigma^\perp(x)$ associated with the hyperbolic conservation law (5.1) solved by σ in Ω_p.

Per hypothesis (H), the system

$$\begin{cases} \sigma \in H^1_{\rm loc}(\Omega;\mathbb{R}^2), \\ {\rm div}\sigma = 0, \quad |\sigma| = 1 \quad \text{a.e. in } \Omega_p \end{cases}$$

possesses a solution.

Definition 5.1. *A function* $\Phi \in \mathcal{C}^\infty_c(\mathbb{R}^2;\mathbb{R}^2)$ *is called an entropy if, for all* $z \in \mathbb{R}^2$,

$$z \cdot [D\Phi(z)z^\perp] = 0, \quad \Phi(0) = 0, \quad D\Phi(0) = 0.$$

According to [19, Lemma 2.3], for all entropies Φ, we have ${\rm div}[\Phi(\sigma)] = 0$ a.e. in Ω_p, or still,

$$\int_{\Omega_p} \Phi(\sigma) \cdot \nabla\varphi \, dx = 0 \tag{5.3}$$

for all test functions $\varphi \in \mathcal{C}^\infty_c(\Omega_p)$. Following [19], we introduce the following family of parameterized generalized entropies:

$$\Phi^{(\xi)}(z) := \begin{cases} |z|^2\xi & \text{if } z \cdot \xi > 0, \\ 0 & \text{if } z \cdot \xi \le 0, \end{cases}$$

where $\xi \in \mathbb{S}^1$. According to [19, Lemma 2.5], there exists a sequence $(\Phi_n)_{n\in\mathbb{N}}$ of entropies in $\mathcal{C}^\infty_c(\mathbb{R}^2;\mathbb{R}^2)$ which is locally uniformly bounded, and such that $\Phi_n \to \Phi^{(\xi)}$ pointwise in $\mathbb{R}^2$. Specializing (5.3) to Φ_n and passing to the limit as $n \to +\infty$ with the help of Lebesgue's dominated convergence theorem,

$$\int_{\Omega_p} \Phi^{(\xi)}(\sigma) \cdot \nabla\varphi \, dx = 0$$

for all $\xi \in \mathbb{S}^1$ and all $\varphi \in \mathcal{C}^\infty_c(\Omega_p)$. Now following [29], we introduce, for a.e. $x \in \Omega_p$ and for any $\xi \in \mathbb{S}^1$,

$$\chi(x,\xi) = \begin{cases} 1 & \text{if } \sigma(x) \cdot \xi > 0, \\ 0 & \text{if } \sigma(x) \cdot \xi \le 0. \end{cases}$$

The considerations above establish that

$${\rm div}[\xi\chi(\cdot,\xi)] = 0 \quad \text{ in } \mathcal{D}'(\Omega_p)$$

for all $\xi \in \mathbb{S}^1$. This can be rewritten as a so-called *kinetic formulation* as follows:

$$D_\xi\chi(\cdot,\xi) = 0 \quad \text{ in } \mathcal{D}'(\Omega_p).$$

The previous kinetic formulation entails the ordering property below whose proof is a direct adaptation of [29, Proposition 3.1] (see also [11, Corollary 4.7]).

Proposition 5.2. *Assume that hypothesis (H) holds. Let x_0 and $y_0 \in \Omega_p$ be two Lebesgue points of σ. Then*

$$\sigma(x_0)\cdot(y_0-x_0)>0 \quad \Longrightarrow \quad \sigma(y_0)\cdot(y_0-x_0)\geq 0,$$

and

$$\sigma(x_0)\cdot(y_0-x_0)<0 \quad \Longrightarrow \quad \sigma(y_0)\cdot(y_0-x_0)\leq 0.$$

Proof. We only prove the first implication. Let us set $\xi=\frac{y_0-x_0}{|y_0-x_0|}$. Then

$$\begin{aligned}\limsup_{\rho\to 0}\frac{\mathcal{L}^2(\{\sigma\cdot\xi\leq 0\}\cap B_\rho(x_0))}{\pi\rho^2}\sigma(x_0)\cdot\xi &\leq \limsup_{\rho\to 0}\frac{1}{\pi\rho^2}\int_{\{\sigma\cdot\xi\leq 0\}\cap B_\rho(x_0)}[\sigma(x_0)-\sigma(z)]\cdot\xi\,dz\\ &\leq \limsup_{\rho\to 0}\frac{1}{\pi\rho^2}\int_{B_\rho(x_0)}|\sigma(x_0)-\sigma(z)|\,dz=0.\end{aligned}$$

Thus, since $\sigma(x_0)\cdot\xi>0$,

$$\lim_{\rho\to 0}\frac{\mathcal{L}^2(\{\sigma\cdot\xi\leq 0\}\cap B_\rho(x_0))}{\pi\rho^2}=0,$$

hence

$$\lim_{\rho\to 0}\frac{\mathcal{L}^2(\{\sigma\cdot\xi> 0\}\cap B_\rho(x_0))}{\pi\rho^2}=1.$$

It shows that x_0 is a Lebesgue point of $\chi(\cdot,\xi)$ with $\chi(x_0,\xi)=1$. Note that the same argument shows that if $\sigma(x_0)\cdot\xi<0$ then x_0 is also a Lebesgue point of $\chi(\cdot,\xi)$ with $\chi(x_0,\xi)=0$.

Consider the segment $S=[x_0,y_0]$ and $U=\{z\in\Omega_p:\ \mathrm{dist}(z,S)<\varepsilon\}$ be a (connected) ε-neighborhood of S, where $\varepsilon>0$ is small enough so that $S\subset U\subset\subset\Omega_p$ (which is always possible thanks to (H)). Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be a standard family of mollifiers and set $\chi_\varepsilon:=\eta_\varepsilon*\chi(\cdot,\xi)$. Because of the kinetic formulation $D_\xi\chi(\cdot,\xi)=0$ in $\mathcal{D}'(\Omega_p)$, then $D_\xi\chi_\varepsilon=0$ in U. Thus $\chi_\varepsilon(z)=\chi_\varepsilon(z+\xi)$ for all $z\in U$ with $z+\xi\in U$. Now, x_0 is a Lebesgue point of $\chi(\cdot,\xi)$, so that $\chi_\varepsilon(x_0)\to\chi(x_0,\xi)$. But $y_0=x_0+\xi$, so $\chi_\varepsilon(y_0)=\chi_\varepsilon(x_0+\xi)=\chi_\varepsilon(x_0)\to\chi(x_0,\xi)=1$ which implies that $\sigma(y_0)\cdot\xi\geq 0$. Indeed, if we had $\sigma(y_0)\cdot\xi<0$, then the above argument would show that y_0 is a Lebesgue point of $\chi(\cdot,\xi)$ with $\chi(y_0,\xi)=0$, and thus $\chi_\varepsilon(y_0)\to\chi(y_0,\xi)=0$ which is a contradiction. □

Thanks to the previous ordering property, we will show that σ is constant on every line

$$L_x:=x+\mathbb{R}\sigma^\perp(x) \tag{5.4}$$

for all Lebesgue points x of σ. Those will be called the *characteristic lines*. We adopt henceforth the following

Notation. *L_x is the characteristic line that passes through x defined as* (5.4)*, this for all $x\in\Omega_p$.*

The following result is an adaptation and an improvement of [29, Proposition 3.2] (see also [11, Proposition 4.8]).

Proposition 5.3. *Assume that hypothesis (H) holds. Let $x_0\in\Omega_p$ be a Lebesgue point of σ and $L_{x_0}=x_0+\mathbb{R}\sigma^\perp(x_0)$, then $\sigma(x)=\sigma(x_0)$ for $\mathcal{H}^1$-a.e. $x\in L_{x_0}\cap\Omega_p$.*

Proof. Up to a change of coordinate system and to a scaling argument, we can assume without loss of generality that $x_0=(0,0)$, $L_{x_0}=\mathbb{R}e_2$, $L_{x_0}\cap\overline{\Omega}_p\supset\{0\}\times[-1,1]:=L$, that $\sigma(x_0)=e_1:=(1,0)$. Let us consider the triangle

$$T_\varepsilon=\left\{y=(y_1,y_2)\in\mathbb{R}^2:\ 0<y_2<1 \text{ and } 0<y_1<\frac{\varepsilon}{1-\varepsilon}y_2\right\}.$$

For all Lebesgue points $y\in T_\varepsilon$ of σ, we have

$$\sigma(0)\cdot y=y_1>0$$

which implies, according to Proposition 5.2, that $\sigma(y)\cdot y\geq 0$. Thus

$$\sigma_2(y)\geq -\sigma_1(y)\frac{y_1}{y_2}\geq -\frac{\varepsilon}{1-\varepsilon} \tag{5.5}$$

since $|\sigma_1(y)|\leq 1$ and $0<y_1/y_2\leq \varepsilon/(1-\varepsilon)$.

Fix $\eta\in(0,1/4)$ and define $S_\eta:=\{0\}\times[\eta,1-\eta]$. Let $(0,x_2)\in S_\eta$ be a Lebesgue point of σ, and consider the half-ball centered at $(0,x_2)$ and radius εx_2 with $\varepsilon\in(0,\eta)$, that is

$$B_\varepsilon^+(x_2)=\{y=(y_1,y_2)\in\mathbb{R}^2:\ y_1>0 \text{ and } |y-(0,x_2)|<\varepsilon x_2\}.$$

It is immediately checked that $B_\varepsilon^+(x_2)\subset T_\varepsilon$ so that, in view of (5.5),

$$\sigma_2(y)\geq -\frac{\varepsilon}{1-\varepsilon}\ \text{ for a.e. } y\in B_\varepsilon^+(x_2).$$

Then,

$$-\frac{\varepsilon}{1-\varepsilon}-\sigma_2(0,x_2)\leq \frac{2}{\pi\varepsilon^2x_2^2}\int_{B_\varepsilon^+(x_2)}|\sigma_2(y)-\sigma_2(0,x_2)|\,dy$$

so that, because $(0,x_2)$ is a Lebesgue point of σ we can pass to the $\varepsilon\searrow 0$ limit and we conclude that $\sigma_2(0,x_2)\geq 0$. A similar argument would show that $\sigma_2(0,x_2)\leq 0$, hence that $\sigma_2(0,x_2)=0$. Recalling Remark 4.4, we have established that $\sigma_2(0,x_2)=0$ for $\mathcal{H}^1$-a.e. $(0,x_2)\in S_\eta$. Since $\eta>0$ is arbitrary, we infer that

$$\sigma_2=0 \text{ for } \mathcal{H}^1\text{-a.e. in } \{0\}\times(0,1).$$

The same type of argument would show that

$$\sigma_2=0 \text{ for } \mathcal{H}^1\text{-a.e. in } \{0\}\times(-1,0).$$

As a consequence, since $|\sigma|=1$ $\mathcal{H}^1$-a.e. in L, we deduce that

$$\sigma=e_1\mathbf{1}_A-e_1\mathbf{1}_{L\setminus A}$$

for some $\mathcal{H}^1$-measurable set $A\subset L$. According to Lemma A.1, the Sobolev space $H^{1/2}(L;\mathbb{R}^2)$ is continuously embedded into $\mathrm{VMO}(L;\mathbb{R}^2)$. Then, we deduce from Lemma A.2 that either $\mathcal{H}^1(A)=0$ or $\mathcal{H}^1(L\setminus A)=0$. If $\mathcal{H}^1(A)=0$, then $\sigma=-e_1$ $\mathcal{H}^1$-a.e. in L so that

$$-e_1=\frac{1}{2\rho}\int_{L\cap B_\rho(x_0)}\sigma(y)\,d\mathcal{H}^1(y).$$

But according to Remark 4.6,

$$\frac{1}{2\rho}\int_{L\cap B_\rho(x_0)}\sigma(y)\,d\mathcal{H}^1(y)\to\sigma(x_0)=e_1$$

which is impossible. As a consequence $\mathcal{H}^1(L\setminus A)=0$ which implies that $\sigma=e_1$ $\mathcal{H}^1$-a.e. in L as required. □

We next show that two distinct characteristic lines cannot intersect inside Ω_p.

Proposition 5.4. *Assume that hypothesis (H) holds. Let x_0 and $y_0\in\Omega_p$ be two Lebesgue points of σ. Then either L_{x_0} and L_{y_0} are colinear, or $L_{x_0}\cap L_{y_0}=\{z_0\}$ where $z_0\notin\Omega_p$.*

Proof. If L_{x_0} and L_{y_0} are not colinear, we have in particular that $\sigma(x_0)\neq\sigma(y_0)$, $\sigma(x_0)\neq-\sigma(y_0)$ and there exists a unique $z_0\in L_{x_0}\cap L_{y_0}$. Assume that $z_0\in\Omega_p$, and let $z_1\in L_{x_0}\cap\Omega_p$ and $z_2\in L_{y_0}\cap\Omega_p$ be such that the triangle T with vertices z_0, z_1 and z_2 satisfies $\overline{T}\subset\Omega_p$. Since $\sigma\in H^1(T;\mathbb{R}^2)$, its trace $\sigma_{|\partial T}\in H^{1/2}(\partial T;\mathbb{R}^2)$. Let us denote by $S_{x_0}:=\partial T\cap L_{x_0}$, $S_{y_0}:=\partial T\cap L_{y_0}$ and $\Gamma=S_{x_0}\cup S_{y_0}$ so that

$$\sigma_{|\partial T}\in H^{1/2}(\Gamma;\mathbb{R}^2).$$

On the other hand, Proposition 5.3 implies that $\sigma_{|\partial T} = \sigma_{|S_{x_0}} = \sigma(x_0)$ $\mathcal{H}^1$-a.e. on S_{x_0} and $\sigma_{|\partial T} = \sigma_{|S_{y_0}} = \sigma(y_0)$ $\mathcal{H}^1$-a.e. on S_{y_0}, which is impossible in view of Lemma A.3. We thus deduce that $z_0 \notin \Omega_p$. □

A concatenation of the previous results implies the announced local Lipschitz regularity for σ (see [11, Theorem 1.5]).

Theorem 5.5 (Jabin-Otto-Perthame regularity)**.** *Assume that hypothesis (H) holds. Let ω be a bounded and convex open set such that $\overline{\omega} \subset \Omega_p$ and let $d := \mathrm{dist}(\omega, \partial\Omega_p) > 0$. Then, for every Lebesgue points x_0 and $y_0 \in \omega$ of σ,*

$$|\sigma(x_0) - \sigma(y_0)| \leq \frac{1}{d}|x_0 - y_0|.$$

In particular, σ admits a representative which is locally Lipschitz in Ω_p.

Proof. Case I: If $\sigma(x_0) = \sigma(y_0)$ then the result follows.

Case II: We now prove that the case $\sigma(x_0) = -\sigma(y_0)$ cannot happen. If $\sigma(x_0) \cdot (y_0 - x_0) > 0$, according to Proposition 5.2, we have $\sigma(y_0) \cdot (y_0 - x_0) \geq 0$ hence $\sigma(x_0) \cdot (y_0 - x_0) \leq 0$ which is impossible. A similar argument shows that $\sigma(x_0) \cdot (y_0 - x_0) < 0$ is impossible. It remains to consider the case where $\sigma(x_0) \cdot (y_0 - x_0) = 0$ which means that $y_0 - x_0 \in L_{x_0} =: x_0 + \mathbb{R}\sigma^\perp(x_0)$ or still that $L_{y_0} = L_{x_0}$. Since x_0 and y_0 are Lebesgue points of σ, we obtain a contradiction with the result of Proposition 5.3.

Case III: Assume now that $\sigma(x_0)$ and $\sigma(y_0)$ are not colinear. Then both lines $L_{x_0} = x_0 + \mathbb{R}\sigma^\perp(x_0)$ and $L_{y_0} = y_0 + \mathbb{R}\sigma^\perp(y_0)$ intersect at a single point $z_0 \notin \Omega_p$ by Proposition 5.4. Note that L_{x_0} (resp. L_{y_0}) is colinear with $x_0 - z_0$ (resp. $y_0 - z_0$) so that there is no loss of generality in assuming that *e.g.*

$$\sigma^\perp(x_0) = \frac{x_0 - z_0}{|x_0 - z_0|}, \quad \sigma^\perp(y_0) = \pm\frac{y_0 - z_0}{|y_0 - z_0|}.$$

We claim that actually

$$\sigma^\perp(y_0) = \frac{y_0 - z_0}{|y_0 - z_0|}. \tag{5.6}$$

Since $y_0 \notin L_{x_0}$,

$$\sigma(x_0) \cdot \frac{y_0 - x_0}{|y_0 - x_0|} \neq 0$$

so that Proposition 5.2 ensures that

$$\mathrm{sign}\left(\sigma(x_0) \cdot \frac{y_0 - x_0}{|y_0 - x_0|}\right) = \mathrm{sign}\left(\sigma(y_0) \cdot \frac{y_0 - x_0}{|y_0 - x_0|}\right). \tag{5.7}$$

Since $\frac{y_0 - z_0}{|y_0 - z_0|}$ belongs to the convex cone $C(\frac{x_0 - z_0}{|x_0 - z_0|}, \frac{y_0 - x_0}{|y_0 - x_0|})$, there exist $\alpha > 0$ and $\beta > 0$ such that

$$\frac{y_0 - z_0}{|y_0 - z_0|} = \alpha\frac{x_0 - z_0}{|x_0 - z_0|} + \beta\frac{y_0 - x_0}{|y_0 - x_0|},$$

hence

$$\frac{y_0 - z_0}{|y_0 - z_0|} \cdot \frac{(y_0 - x_0)^\perp}{|y_0 - x_0|} = \alpha\frac{x_0 - z_0}{|x_0 - z_0|} \cdot \frac{(y_0 - x_0)^\perp}{|y_0 - x_0|}.$$

As a consequence

$$\pm\sigma^\perp(y_0) \cdot \frac{(y_0 - x_0)^\perp}{|y_0 - x_0|} = \alpha\sigma^\perp(x_0) \cdot \frac{(y_0 - x_0)^\perp}{|y_0 - x_0|}$$

or still

$$\pm\sigma(y_0) \cdot \frac{y_0 - x_0}{|y_0 - x_0|} = \alpha\sigma(x_0) \cdot \frac{y_0 - x_0}{|y_0 - x_0|} \tag{5.8}$$

Gathering (5.7) and (5.8) yields (5.6).

Since $\mathrm{dist}(z_0,\omega) \geq d$, then it follows that $|z_0 - x_0| \geq d$ and $|z_0 - y_0| \geq d$. Therefore the projections of $x_0 - z_0$ and $y_0 - z_0$ onto the closed ball $\overline{B}_d(z_0)$ are given, respectively, by $d(x_0 - z_0)/|x_0 - z_0|$ and $d(y_0 - z_0)/|y_0 - z_0|$. Since the projection is 1-Lipschitz, we deduce that

$$|\sigma^\perp(x_0) - \sigma^\perp(y_0)| = \left| \frac{x_0 - z_0}{|x_0 - z_0|} - \frac{y_0 - z_0}{|y_0 - z_0|} \right| \leq \frac{1}{d}|x_0 - y_0|,$$

and the conclusion follows. □

Remark 5.6. In the sequel, we will identify σ with its locally Lipschitz representative. In particular, the conclusion of Proposition 5.2 now holds *for all* x_0 and $y_0 \in \Omega_p$, while that of Proposition 5.3 states that *for all* $x_0 \in \Omega_p$, then $\sigma(x) = \sigma(x_0)$ *for all* $x \in L_{x_0} \cap \Omega_p$. ¶

5.2. **Rigidity of the displacement.** We now demonstrate that the displacement field(s), like the stress field, is (are) severely constrained by assumption (H) and conform(s) to what formal manipulations of the hyperbolic equation (5.2) would entail, that is that the displacement must remain constant on (almost) every characteristic line in Ω_p. Formally, the argument goes as follows. Compute the derivative of u along the characteristics. The chain rule gives

$$\frac{d}{ds} u(x + s\sigma^\perp(x)) = Du(x + s\sigma^\perp(x)) \cdot \sigma(x).$$

Using that σ is constant along the characteristics, we get

$$\frac{d}{ds} u(x + s\sigma^\perp(x)) = Du(x + s\sigma^\perp(x)) \cdot \sigma(x + s\sigma^\perp(x)) = 0,$$

since, thanks to the flow rule, $Du = \sigma(1 + |p|)$ is colinear with σ. Unfortunately, this argument cannot be made rigorous for want of a general chain rule formula for the composition of a BV function with a (locally) Lipschitz function.

We will first show that such a property actually holds locally, *i.e.*, for small values of s. Using a well-suited covering we then establish the global result in Ω_p, resulting in the

Theorem 5.7. *Assume that hypothesis (H) holds. There exists an* $\mathcal{H}^1$*-negligible set* $Z \subset \Omega_p$ *with* $\mathcal{L}^2\left(\left(\bigcup_{z\in Z} L_z\right) \cap \Omega_p\right) = 0$ *such that* u *is constant on* $L_x \cap \Omega_p$ *for all* $x \in \Omega_p \setminus \left(\bigcup_{z\in Z} L_z\right)$.

The rest of the section is devoted to the proof of Theorem 5.7. Let ω be a convex open set such that $\overline{\omega} \subset \Omega_p$. Let $x_0 \in \overline{\omega}$ and $R > 0$ small enough so that $\overline{B_{2R}(x_0)} \subset \Omega_p$. Let us define the mapping $\Phi_{x_0} : [-R, R] \times \mathbb{R} \to \mathbb{R}^2$ by

$$\Phi_{x_0}(t, s) := x_0 + t\sigma(x_0) + s\sigma^\perp(x_0 + t\sigma(x_0)) \quad \text{for all } (t, s) \in [-R, R] \times \mathbb{R}.$$

Clearly, Φ_{x_0} is locally Lipschitz in $[-R, R] \times \mathbb{R}$ as composition of locally Lipschitz mappings.

Proposition 5.8. *There exists* $0 < r < R$ *and an open neighborhood* U_{x_0} *of* x_0 *such that the function* Φ_{x_0} *is bi-Lipschitz from* $Q_r := (-r, r)^2$ *onto* U_{x_0}.

Proof. Since σ is locally Lipschitz in Ω_p and $\overline{B_{2R}(x_0)} \subset \Omega_p$, it follows that $t \in [-R, R] \mapsto \sigma^\perp(x_0 + t\sigma(x_0))$ is Lipschitz. As a consequence of Rademacher's Theorem, it is differentiable almost everywhere and there exists $M > 0$ such that

$$\left| \frac{d}{dt} \sigma^\perp(x_0 + t\sigma(x_0)) \right| \leq M \quad \text{for a.e. } t \in [-R, R]. \tag{5.9}$$

Hence, Φ_{x_0} is Lipschitz in $\overline{Q}_R$ as composition of Lipschitz functions.

Further, Φ_{x_0} is injective on $\overline{Q_R}$. Indeed, if $(t,s) \neq (t',s') \in \overline{Q_R}$ are such that $\Phi_{x_0}(t,s) = \Phi_{x_0}(t',s')$, then

$$z := x_0 + t\sigma(x_0) + s\sigma^\perp(x_0 + t\sigma(x_0)) = x_0 + t'\sigma(x_0) + s'\sigma^\perp(x_0 + t'\sigma(x_0)).$$

Clearly $t \neq t'$ and thus $x_0 + t\sigma(x_0) \neq x_0 + t'\sigma(x_0)$. The point z would then belong to both $L_{x_0+t\sigma(x_0)}$ and $L_{x_0+t'\sigma(x_0)}$ so that, by Proposition 5.4, we would have $L_{x_0+t\sigma(x_0)} = L_{x_0+t'\sigma(x_0)} = x_0 + \mathbb{R}\sigma(x_0)$, hence, by Proposition 5.3, $\sigma(x_0 + t\sigma(x_0)) = \sigma(x_0 + t'\sigma(x_0))$. But then $\sigma(x_0)$ and $\sigma^\perp(x_0 + t\sigma(x_0))$ are colinear, and, because they are both unit vectors,

$$\sigma^\perp(x_0 + t\sigma(x_0)) = \pm\sigma(x_0). \tag{5.10}$$

Consequently, we would have $x_0 = x_0 + t\sigma(x_0) \mp t\sigma^\perp(x_0 + t\sigma(x_0))$, and the characteristic line $L_{x_0+t\sigma(x_0)}$ would intersect L_{x_0} at the point x_0 which is impossible, owing again to Proposition 5.4, unless $\sigma(x_0) = \pm\sigma(x_0 + t\sigma(x_0))$. But this last relation contradicts (5.10). The injectivity of Φ_{x_0} in $\overline{Q_R}$ is established, and, as a consequence of Brouwer's Invariance Domain Theorem (see [22, Theorem 3.30]), Φ_{x_0} is a homeomorphism from the open square Q_R onto its range which is open.

We now compute the Jacobian determinant of Φ_{x_0}. For a.e. $(t,s) \in Q_R$, we have

$$\begin{aligned} \det\nabla\Phi_{x_0}(t,s) &= \sigma(x_0)\cdot\sigma(x_0 + t\sigma(x_0)) - s\sigma^\perp(x_0 + t\sigma(x_0))\cdot\frac{d}{dt}\sigma(x_0 + t\sigma(x_0)) \\ &:= a(t) + sb(t), \end{aligned} \tag{5.11}$$

where, for a.e. $t \in (-R,R)$, we set

$$a(t) := \sigma(x_0)\cdot\sigma(x_0 + t\sigma(x_0)), \quad b(t) = -\sigma^\perp(x_0 + t\sigma(x_0))\cdot\frac{d}{dt}\sigma(x_0 + t\sigma(x_0)).$$

Since σ is Lipschitz in $\overline{B_{2R}(x_0)}$, there exists $K > 0$ such that

$$|\sigma(x_0 + t\sigma(x_0)) - \sigma(x_0)| \leq K|t| \quad \text{for all } t \in [-R,R]. \tag{5.12}$$

Thus, by (5.9) and (5.12), for a.e. $t \in (-R,R)$,

$$a(t) \geq 1 - K|t|, \quad |b(t)| \leq M, \tag{5.13}$$

and we can bound from below the right hand-side of (5.11) by

$$\det\nabla\Phi_{x_0}(t,s) \geq 1 - K|t| - M|s|.$$

Let $r > 0$ be small enough so that $1 - (K+M)r > \frac{1}{2}$, then we get that

$$\det\nabla\Phi_{x_0}(t,s) > 1/2 \quad \text{for a.e. } (t,s) \in Q_r. \tag{5.14}$$

Denoting by $U_{x_0} := \Phi_{x_0}(Q_r)$, we have so far established that $\Phi_{x_0} : Q_r \to U_{x_0}$ is a Lipschitz homeomorphism. With the help of (5.14), we now show that $\Phi_{x_0}^{-1}$ is Lipschitz in U_{x_0}. To that end, we prove in a manner similar to that in [22, Theorem 6.1] that $\Phi_{x_0}^{-1} \in W^{1,\infty}(U_{x_0};\mathbb{R}^2)$. Since we already know that $\Phi_{x_0}^{-1}$ is continuous in U_{x_0}, it enough to prove that $\Phi_{x_0}^{-1}$ has a weak gradient in $L^\infty(U_{x_0};\mathbb{M}^{2\times 2})$.

Let $\varphi \in \mathcal{C}_c^\infty(U_{x_0};\mathbb{R}^2)$ be an arbitrary test function. Using the area formula with the one-to-one Lipschitz function Φ_{x_0}, together with the chain rule formula $\nabla(\varphi\circ\Phi_{x_0}) = (\nabla\Phi_{x_0})^T\nabla\varphi\circ\Phi_{x_0}$, we get that for $i = 1,2$,

$$\begin{aligned} \int_{U_{x_0}} \Phi_{x_0}^{-1}(x)\,\partial_i\varphi(x)\,dx &= \int_{Q_r} \Phi_{x_0}^{-1}(\Phi_{x_0}(t,s))\,\partial_i\varphi(\Phi_{x_0}(t,s))\det\nabla\Phi_{x_0}(t,s)\,dt\,ds \\ &= \int_{Q_r} \big[\mathrm{cof}(\nabla\Phi_{x_0}(t,s))\nabla(\varphi\circ\Phi_{x_0})(t,s)\big]_i \begin{pmatrix} t \\ s \end{pmatrix} dt\,ds. \end{aligned}$$

Let $(\Phi_n)_{n\in\mathbb{N}}$ be an approximating sequence in $\mathcal{C}^\infty(\overline{Q_r};\mathbb{R}^2)$ such that $\Phi_n \to \Phi_{x_0}$ uniformly in $\overline{Q_r}$ and also in $W^{1,p}(Q_r;\mathbb{R}^2)$ for all $p<\infty$, then

$$\int_{U_{x_0}} \Phi_{x_0}^{-1}(x)\,\partial_i\varphi(x)\,dx = \lim_{n\to+\infty}\int_{Q_r}\big[\mathrm{cof}(\nabla\Phi_n(t,s))\nabla(\varphi\circ\Phi_n)(t,s)\big]_i\begin{pmatrix}t\\ s\end{pmatrix}dt\,ds.$$

Integrating by parts, using that $\mathrm{div}\big(\mathrm{cof}(\nabla\Phi_n)\big)=0$, as well as the area formula with the function Φ_{x_0} once more, we get

$$\begin{aligned}\int_{U_{x_0}} \Phi_{x_0}^{-1}(x)\,\partial_i\varphi(x)\,dx &= -\lim_{n\to+\infty}\int_{Q_r}\big(\mathrm{cof}(\nabla\Phi_n)\big)^{(i)}(\varphi\circ\Phi_n)\,dt\,ds\\ &= -\int_{Q_r}\big(\mathrm{cof}(\nabla\Phi_{x_0})\big)^{(i)}(\varphi\circ\Phi_{x_0})\,dt\,ds\\ &= -\int_{U_{x_0}}\frac{\big(\mathrm{cof}(\nabla\Phi_{x_0}(\Phi_{x_0}^{-1}))\big)^{(i)}}{\det\nabla\Phi_{x_0}(\Phi_{x_0}^{-1})}\varphi\,dx,\end{aligned}$$

where $A^{(i)}$ stands for the i-th row of the matrix $A\in\mathbb{M}^{2\times2}$. By definition of the weak gradient, (5.14) and since Φ_{x_0} is Lipschitz in $\overline{Q}_r$, we infer that

$$\nabla\Phi_{x_0}^{-1} = \frac{\big(\mathrm{cof}(\nabla\Phi_{x_0}(\Phi_{x_0}^{-1}))\big)^T}{\det\nabla\Phi_{x_0}(\Phi_{x_0}^{-1})}\in L^\infty(U_{x_0};\mathbb{M}^{2\times2}),$$

which completes the proof of the Proposition. □

The mapping Φ_{x_0} provides a convenient change of variables thanks to which we now deduce that the displacement is locally constant along characteristics.

Proposition 5.9. *The function $(t,s)\in Q_r\mapsto u\circ\Phi_{x_0}(t,s)\in BV(Q_r)$ only depends on t.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}^\infty(\Omega)\cap W^{1,1}(\Omega)$ be such that $u_n\to u$ in $L^1(\Omega)$ and $Du_n\rightharpoonup Du$ weakly* in $\mathcal{M}(\Omega;\mathbb{R}^2)$. Let us define the functions

$$v = u\circ\Phi_{x_0},\quad v_n = u_n\circ\Phi_{x_0}.$$

Note that $v_n\in W^{1,\infty}(Q_r)$. Because, up to a subsequence, $u_n\to u$ a.e. in Ω and because $\Phi_{x_0}^{-1}$, being Lipschitz, maps sets of zero Lebesgue measure into sets of zero Lebesgue measure, $v_n\to v$ a.e. in Q_r. The area formula (applied to the function Φ_{x_0}) together with (5.14) implies that,

$$\int_{Q_r}|v_n-v|\,dt\,ds = \int_{U_{x_0}}\frac{|u_n-u|}{\det\nabla\Phi_{x_0}(\Phi_{x_0}^{-1})}\,dx \le 2\int_{U_{x_0}}|u_n-u|\,dx\to0.$$

Hence $v_n\to v$ in $L^1(Q_r)$. Since $\Phi_{x_0}\in W^{1,\infty}(Q_r;\mathbb{R}^2)$ and $\nabla v_n = (\nabla\Phi_{x_0})^T\nabla u_n\circ\Phi_{x_0}$, the same change of variable argument yields in turn

$$\int_{Q_r}|\nabla v_n|\,dt\,ds \le \|\nabla\Phi_{x_0}\|_{L^\infty(Q_r)}\int_{Q_r}|\nabla u_n(\Phi_{x_0})|\,dt\,ds \le 2\|\nabla\Phi_{x_0}\|_{L^\infty(Q_r)}\int_{U_{x_0}}|\nabla u_n|\,dx\le C$$

for some constant $C>0$ independent of n. Hence $\nabla v_n\mathcal{L}^2\rightharpoonup Dv$ weakly* in $\mathcal{M}(Q_r;\mathbb{R}^2)$ and $v\in BV(Q_r)$.

Because Φ_{x_0} is Lipschitz in $\overline{Q_r}$, the function $(t,s)\in Q_r\mapsto g(t,s):=\det\nabla\Phi_{x_0}(t,s)=a(t)+sb(t)$ defined in (5.11) is in $L^\infty(Q_r)$ and it is affine with respect to s for a.e. $t\in(-r,r)$. It follows from an integration by parts that for all $\varphi\in\mathcal{C}_c^\infty(Q_r)$

$$\int_{Q_r}(\partial_s v_n)\,\varphi\,g\,dt\,ds = -\int_{Q_r}((\partial_s\varphi)\,v_n\,g+\varphi\,v_n\,\partial_s g)\,dt\,ds. \tag{5.15}$$

On the other hand, using that σ is constant on each characteristic line, *i.e.*, that $\sigma(x_0+t\sigma(x_0)) = \sigma(\Phi_{x_0}(t,s))$ for all $(t,s)\in Q_r$, we get that

$$\begin{aligned}\int_{Q_r}(\partial_s v_n)\,\varphi\,g\,dt\,ds &= \int_{Q_r}\nabla u_n(\Phi_{x_0}(t,s))\cdot\sigma^\perp(x_0+t\sigma(x_0))\,\varphi(t,s)\,g(t,s)\,dt\,ds\\ &= \int_{Q_r}\nabla u_n(\Phi_{x_0}(t,s))\cdot\sigma^\perp(\Phi_{x_0}(t,s))\,\varphi\circ\Phi_{x_0}^{-1}(\Phi_{x_0}(t,s))\,g(t,s)\,dt\,ds\\ &= \int_{U_{x_0}}\nabla u_n\cdot\sigma^\perp\varphi\circ\Phi_{x_0}^{-1}\,dx, \quad (5.16)\end{aligned}$$

where we used once more the area formula with the function Φ_{x_0} in the last equality. Since $\Phi_{x_0}^{-1}$ is continuous, σ is locally Lipschitz and $\varphi\in\mathcal{C}_c(Q_r)$, it follows that the function $\sigma^\perp\varphi\circ\Phi_{x_0}^{-1}$ belongs to $\mathcal{C}_c(U_{x_0})$. Gathering (5.15), (5.16), and passing to the limit leads to

$$-\int_{Q_r}((\partial_s\varphi)\,v\,g+\varphi\,v\,\partial_s g)\,dt\,ds = \int_{U_{x_0}}\varphi\circ\Phi_{x_0}^{-1}\sigma^\perp\cdot dDu.$$

Recalling Remark 4.5 together with the additive decomposition, we infer that $Du=\sigma+p=\sigma+\sigma|p|$. Thus, the right hand side of the previous equality vanishes and

$$\int_{Q_r}((\partial_s\varphi)\,v\,g+\varphi\,v\,\partial_s g)\,dt\,ds=0 \quad \text{for all } \varphi\in\mathcal{C}_c^\infty(Q_r). \quad (5.17)$$

Since $v\in BV(Q_r)$, it follows from slicing properties of BV-functions (see [2, Theorem 3.107]) that $s\mapsto v_t(s)=v(t,s)$ belongs to $BV((-r,r))$ for $\mathcal{L}^1$-a.e. $t\in(-r,r)$, and, using disintegration, that $D_s v=\mathcal{L}^1_t\otimes Dv_t$ in $\mathcal{M}(Q_r)$, *i.e.*,

$$\int_{Q_r}\varphi\,dD_s v=\int_{-r}^r\left(\int_{-r}^r\varphi(t,s)\,dDv_t(s)\right)ds \quad \text{for all } \varphi\in\mathcal{C}_c(Q_r).$$

Since $s\mapsto g(t,s)$ is affine, we can thus localize (5.17). For $\mathcal{L}^1$-a.e. $t\in(-r,r)$,

$$\int_{-r}^r g(t,s)\phi(s)\,dDv_t(s)=0 \quad \text{for all } \phi\in\mathcal{C}_c((-r,r)),$$

which means that

$$(a(t)+sb(t))Dv_t=0 \quad \text{in } \mathcal{M}((-r,r)) \quad \text{for } \mathcal{L}^1\text{-a.e. } t\in(-r,r).$$

Note that by our choice of r and (5.13) $a(t)\geq\frac{1}{2}$ for all $t\in(-r,r)$. If $b(t)=0$, we have $|a(t)+sb(t)|=|a(t)|\geq\frac{1}{2}$ for all $s\in(-r,r)$. On the other hand, if $b(t)\neq 0$, we get with (5.13) that

$$\left|\frac{a(t)}{b(t)}\right|\geq\frac{1}{2M}.$$

Since our choice of r ensures that $r<1/2M$, we conclude that $|a(t)+sb(t)|>0$ for $s\in(-r,r)$. We conclude that $Dv_t=0$ in $\mathcal{M}((-r,r))$ for $\mathcal{L}^1$-a.e. $t\in(-r,r)$, which proves that $D_s v=0$ in $\mathcal{M}(Q_r)$. □

We now proceed to the proof of Theorem 5.7.

Proof of Theorem 5.7. Since $\{U_x\}_{x\in\overline{\omega}}$ is an open covering of the compact set $\overline{\omega}$, we can extract a finite sub-covering. We can thus find finitely many points $x_1,\dots,x_N\in\overline{\omega}$ such that

$$\overline{\omega}\subset\bigcup_{i=1}^N U_{x_i}.$$

Let $Q_i = (-r_i, r_i)^2 = \Phi_{x_i}^{-1}(U_{x_i})$ be the corresponding open sets with $v_i := u \circ \Phi_{x_i} \in BV(Q_i)$. By Proposition 5.9, $D_s v_i = 0$ in $\mathcal{M}(Q_i)$, which means that the function

$$(t,s) \in Q_i \mapsto u\big(x_i + t\sigma(x_i) + s\sigma^\perp(x_i + t\sigma(x_i))\big)$$

is an s-independent $BV(Q_i)$-function. Using slicing properties of BV functions (see [2, Theorem 3.107]), there exists an $\mathcal{L}^1$-negligible set $N_i \subset (-r_i, r_i)$ such that for all $t \in (-r_i, r_i) \setminus N_i$,

$$s \in (-r_i, r_i) \mapsto u\big(x_i + t\sigma(x_i) + s\sigma^\perp(x_i + t\sigma(x_i))\big) \tag{5.18}$$

is constant. Set $Z_i := \Phi_{x_i}(N_i \times \{0\})$. Since $\mathcal{L}^1(N_i) = 0$ and Φ_{x_i} is Lipschitz in Q_r, $\mathcal{H}^1(Z_i) = 0$. Further, since Ω_p is bounded, there exists $T > 0$ large enough so that $\bigcup_{z\in Z_i} L_z \cap \Omega_p \subset \Phi_{x_i}(N_i \times (-T,T))$. Using again that $\mathcal{L}^1(N_i) = 0$, then $\mathcal{L}^2(N_i \times (-T,T)) = 0$ and because Φ_{x_i} is a Lipschitz function on $[-R,R] \times [-T,T]$, so $\mathcal{L}^2\left(\left(\bigcup_{z\in Z_i} L_z\right) \cap \Omega_p\right) = 0$.

From (5.18), it follows that, for all $x \in U_{x_i} \setminus \left(\bigcup_{z\in Z_i} L_z\right)$, the function

$$s \in (-r_i, r_i) \mapsto u(x + s\sigma^\perp(x))$$

is constant, that is u remains constant along $L_x \cap U_{x_i}$. In turn, $\{U_{x_i}\}_{1\le i\le N}$ is a finite covering of $\overline{\omega}$, so that u is constant on $L_x \cap \overline{\omega}$ for every $x \in \omega \setminus (\bigcup_{z\in Z_\omega} L_z)$ with $Z_\omega := \bigcup_{i=1}^N Z_i$ satisfying $\mathcal{H}^1(Z_\omega) = 0$ and $\mathcal{L}^2\left(\left(\bigcup_{z\in Z_\omega} L_z\right) \cap \Omega_p\right) = 0$. Finally, consider an exhaustion $\{\omega_k\}_{k\in\mathbb{N}}$ of open sets with $\omega_k = \{y \in \Omega_p : \operatorname{dist}(x, \partial\Omega_p) > \frac{1}{k}\}$ for all $k \ge 1$. Set $Z := \bigcup_{k\in\mathbb{N}} Z_{\omega_k}$ which satisfies $\mathcal{H}^1(Z) = 0$ and $\mathcal{L}^2\left(\left(\bigcup_{z\in Z} L_z\right) \cap \Omega_p\right) = 0$. In conclusion, for all $x \in \Omega_p \setminus \left(\bigcup_{z\in Z} L_z\right)$, the function u is constant on $L_x \cap \Omega_p$. □

6. Geometry of the solutions

In this section, we show that assumption (H) constrains the geometric structure of the plastic zone Ω_p, and of the solutions (σ, u) in that set. In particular, subsets $\mathbf{F}_{\bar z}$ of Ω_p which are bounded by characteristics intersecting at $\bar z \in \partial\Omega_p$ lead to boundary fans where the stress behaves like a vortex centered at the apex $\bar z$ of the fan, and the displacement is a monotone function of the angle (see Theorem 6.2). The complementary of those fans is made of either isolated characteristic lines (see Proposition 6.5), or convex sets $\mathbf{C}$ with one or two characteristics on the boundary (see Theorem 6.12). In any case, if the non characteristic boundary of either $\mathbf{F}_{\bar z}$, or $\mathbf{C}$ intersects the boundary of the domain Ω on a set of positive $\mathcal{H}^1$ measure, then the propagation of the prescribed Dirichlet boundary datum w through the characteristics inside $\mathbf{F}_{\bar z}$ or $\mathbf{C}$ provides a partial uniqueness property of the displacement (see Propositions 6.3 and 6.24). In the case of a connected component which is not a boundary fan, possible geometries are offered in Subsection 6.3. They consist of exterior fans (Proposition 6.26), constant zones (Proposition 6.28), or smooth one-parameter families of lines (see Paragraph 6.3.3). We conjecture that those are the only possibilities.

6.1. **Boundary fans.** We set

$$\mathcal{F} := \Big\{\bar z \in \partial\Omega_p :\ \exists\, x, y \in \Omega_p \text{ with } y \neq x \text{ such that } L_x \cap L_y = \{\bar z\}\Big\}.$$

It is the set of all points on $\partial\Omega_p$ belonging to (at least) two distinct characteristic lines. The next result shows that any such point is the apex of a cone which is the unique intersection point of all other characteristic lines inside the cone.

Lemma 6.1. *Let $\bar z \in \mathcal{F}$ and $x,\ y \in \Omega_p$ with $y \neq x$ be such that $L_x \cap L_y = \{\bar z\}$. Then for all $z \in (\bar z + C(x - \bar z, y - \bar z)) \cap \Omega_p$,*

$$L_z \cap L_x = L_z \cap L_y = L_x \cap L_y = \{\bar z\}.$$

Proof. Since from hypothesis (H) the set Ω_p is assumed to be convex, there exist unique points $x' \in \partial\Omega_p \cap L_x$ and $y' \in \partial\Omega_p \cap L_y$ with $x' \neq \bar{z}$ and $y' \neq \bar{z}$ with the open segments $]\bar{z}, x'[$ and $]\bar{z}, y'[$ contained in Ω_p. Let us consider the (closed) triangle $T := (x', y', \bar{z}) \subset \overline{\Omega}_p$.

If $z \in \mathring{T}$, the line L_z passing through z must intersect at least one of the segments $[\bar{z}, x'[$ or $[\bar{z}, y'[$. Without loss of generality, we suppose that $[\bar{z}, x'[\cap L_z \neq \emptyset$ and let $\hat{z} \in [\bar{z}, x'[\cap L_z$. Let us assume that $\hat{z} \neq \bar{z}$. Since $L_z \cap L_x = \{\hat{z}\}$, we get from Proposition 5.4 that $\hat{z} \notin \Omega_p$ which contradicts the fact that $\hat{z} \in]\bar{z}, x'[\subset \Omega_p$. Thus $\hat{z} = \bar{z}$ and L_z passes through $\bar{z}$.

If $z \notin \mathring{T}$, take any point w on $]\bar{z}, z[\cap \mathring{T}$. Then, according to the previous argument L_w passes through the point $\bar{z}$, thus $[\bar{z}, w] \subset L_w$ and $z \in L_w$. Therefore, Proposition 5.3 ensures that $L_w = L_z$, hence L_z passes through $\bar{z}$ as well. □

For all $\bar{z} \in \mathcal{F}$, we consider the set of all points z in Ω_p such that the associated characteristic line L_z passes through the point $\bar{z}$, *i.e.*,

$$\mathbf{F}_{\bar{z}} := \mathrm{int}\{z \in \Omega_p : \ \bar{z} \in L_z\}.$$

In view of Lemma 6.1, it is a non empty open set of the form $\mathbf{F}_{\bar{z}} := (\bar{z} + C(x - \bar{z}, y - \bar{z})) \cap \Omega_p$, for some $x, y \in \overline{\Omega}_p$, and we call that open subset of Ω_p a *boundary fan* (see Figure 3). Note that it might be the case that $[\bar{z}, x] \subset \partial\Omega_p$ or $[\bar{z}, y] \subset \partial\Omega_p$. In addition, if $\mathbf{F}_{\bar{z}}$ and $\mathbf{F}_{\bar{z}'}$ are two distinct boundary fans, for some $\bar{z}$ and $\bar{z}' \in \mathcal{F}$ with $\bar{z} \neq \bar{z}'$, then Proposition 5.4 ensures that $\mathbf{F}_{\bar{z}} \cap \mathbf{F}_{\bar{z}'} = \emptyset$ so that there are at most countably many boundary fans. We define the following open subset of Ω_p

$$\mathscr{F} := \bigcup_{\bar{z} \in \mathcal{F}} \mathbf{F}_{\bar{z}}.$$

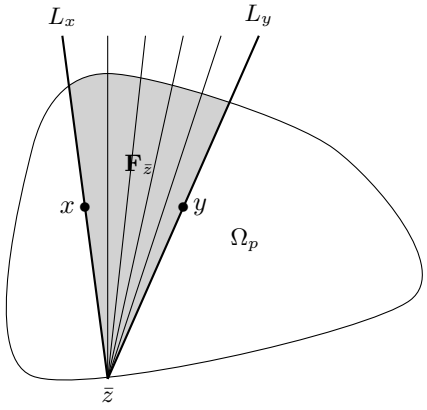

FIGURE 3. A boundary fan with apex $\bar{z}$

We now show a rigidity property of the Cauchy stress σ in $\mathscr{F}$: it is a vortex inside each boundary fan $\mathbf{F}_{\bar{z}}$. The corresponding displacement is constant along all characteristic lines. Note that the conclusion of Theorem 6.2 below is stronger than that of Theorem 5.7. Here we get a monotone function of the angle (which parameterizes the characteristics in the case of a fan).

Theorem 6.2. *Let $\bar{z} \in \mathcal{F}$. Then there exists $\alpha \in \{-1, 1\}$ such that*

$$\sigma(x) = \alpha \frac{(x - \bar{z})^{\perp}}{|x - \bar{z}|} \quad \text{for all } x \in \overline{\mathbf{F}}_{\bar{z}} \cap \Omega_p \setminus \{\bar{z}\}$$

and

$$u(x) = \alpha h\left(\frac{(x-\bar z)_2}{(x-\bar z)_1}\right) \text{ for all } x \in \mathbf{F}_{\bar z},$$

for some nondecreasing function $h : \mathbb{R} \to \mathbb{R}$.

Proof. Assume for simplicity that $\bar z = (0,0)$. We use the change of variables in polar coordinates $\Psi : (0,+\infty) \times (-\pi,\pi) \to \mathbb{R}^2 \setminus ((-\infty,0] \times \{0\})$ given by $\Psi(r,\theta) = (r\cos\theta, r\sin\theta)$. We set $e_r = (\cos\theta,\sin\theta)$, $e_\theta = (-\sin\theta,\cos\theta)$ and $\sigma_r = (\sigma\circ\Psi)\cdot e_r$, $\sigma_\theta = (\sigma\circ\Psi)\cdot e_\theta$ so that $\sigma\circ\Psi = \sigma_r e_r + \sigma_\theta e_\theta$. Since Ω_p is convex, so is $\mathbf{F}_{\bar z}$, thus

$$\mathbf{F}_{\bar z} = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 :\ r>0 \text{ and } \theta_0 < \theta < \theta_1\} \cap \Omega_p,$$

for some $-\frac{\pi}{2} \le \theta_0 < \theta_1 < \frac{\pi}{2}$. We define $\tilde F := \Psi^{-1}(\mathbf{F}_{\bar z}) = \{(r,\theta) \in \mathbb{R}^2 :\ (r\cos\theta, r\sin\theta) \in \mathbf{F}_{\bar z}\}$.

According to Proposition 5.3, for all $z \in \mathbf{F}_{\bar z}$, the vector field σ is constant along all characteristic lines L (which all pass through the origin), and, further, it is orthogonal to the direction of L. Therefore, for all $(r,\theta) \in \tilde F$,

$$\sigma(r\cos\theta, r\sin\theta) = f(\theta)e_\theta, \quad \sigma_\theta = f, \quad \sigma_r = 0$$

where $f : (\theta_0,\theta_1) \to \mathbb{R}$ is locally Lipschitz continuous. Using that $|\sigma| = 1$ together with the expression of the divergence in polar coordinates, we infer that $(\operatorname{div}\sigma)\circ\Psi = \frac{1}{r}D_r(r\sigma_r) + \frac{1}{r}D_\theta\sigma_\theta = 0$ in $\tilde F$, and thus

$$|f| = 1 \text{ in } (\theta_0,\theta_1) \text{ and } f' = 0 \text{ a.e. in } (\theta_0,\theta_1),$$

i.e., $f \equiv 1$ or $f \equiv -1$. Therefore, $\sigma\circ\Psi = \alpha e_\theta$, with $\alpha = 1$ or -1. Coming back to cartesian coordinates leads to the desired expression for σ in $\mathbf{F}_{\bar z}$, hence to $\overline{\mathbf{F}}_{\bar z} \cap \Omega_p \setminus \{\bar z\}$ by continuity.

Thanks to the additive decomposition $Du = \sigma + p$ in (3.2), and according to Remark 4.5,

$$Du = \sigma + p = \sigma(1+|p|) \quad \text{ in } \mathcal{M}(\Omega_p). \tag{6.1}$$

To simplify notation, we set $\mu := \mathcal{L}^2 + |p|$. Applying the curl to the previous equality and the fact that $\sigma \in \mathcal{C}^\infty(\mathbf{F}_{\bar z}; \mathbb{M}^{2\times 2}_{\rm sym})$ yields, in our notation,

$$0 = \operatorname{curl}(\sigma\mu) = -\sigma^\perp \cdot D\mu + \mu \operatorname{curl}\sigma \quad \text{ in } \mathcal{D}'(\Omega_p). \tag{6.2}$$

Since, for all $x \in \mathbf{F}_{\bar z} \setminus \{\bar z\}$, $\sigma(x) = \alpha\frac{x^\perp}{|x|}$, $\sigma^\perp(x) = -\alpha\frac{x}{|x|}$ and $\operatorname{curl}\sigma(x) = \frac{\alpha}{|x|}$, then

$$\frac{x}{|x|}\cdot D\mu + \frac{1}{|x|}\mu = 0 \quad \text{ in } \mathcal{D}'(\mathbf{F}_{\bar z}). \tag{6.3}$$

Let $\tilde\mu = \Psi^{-1}\#\mu \in \mathcal{M}(\tilde F)$ be the push-forward of μ by Ψ^{-1}. Then (6.3) is easily seen to imply that $D_r\tilde\mu = 0$ in $\mathcal{D}'(\tilde F)$ upon testing $D_r\tilde\mu$ by smooth functions of the form $\varphi\circ\Psi =: \tilde\varphi \in \mathcal{C}^\infty_c(\tilde F)$. This implies the existence of an orthoradial nonnegative measure $\eta \in \mathcal{M}((\theta_1,\theta_2))$ such that $\tilde\mu = \mathcal{L}^1_r \otimes \eta$, *i.e.*, for all $\varphi \in \mathcal{C}^\infty_c(\mathbf{F}_{\bar z})$,

$$\int_{\mathbf{F}_{\bar z}} \varphi\, d\mu = \int_{\tilde F} \tilde\varphi\, d\tilde\mu = \int_{\tilde F} \tilde\varphi(r,\theta)\, dr\, d\eta(\theta),$$

where, once again, $\varphi\circ\Psi = \tilde\varphi$. As a consequence, since $Du = \sigma\mu = \alpha\frac{x^\perp}{|x|}\mu$, we deduce that

$$\int_{\mathbf{F}_{\bar z}} \varphi\, dDu = \alpha\int_{\tilde F} \tilde\varphi(r,\theta)e_\theta\, dr\, d\eta(\theta). \tag{6.4}$$

But,

$$\begin{aligned}\int_{\mathbf{F}_{\bar z}} \varphi\, dDu &= -\int_{\mathbf{F}_{\bar z}} u\nabla\varphi\, dx\\ &= -\int_{\tilde F} u(r\cos\theta, r\sin\theta)\nabla\varphi(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta\\ &= -\int_{\tilde F} \tilde u\left[D_r\tilde\varphi\, e_r + \frac{1}{r}D_\theta\tilde\varphi\, e_\theta\right] r\, dr\, d\theta, \end{aligned} \tag{6.5}$$

where we set $\tilde u := u\circ\Psi$. Since $\tilde F$ does not contain the exceptional line $(-\infty, 0]\times\{0\}$, it follows that Ψ defines a $\mathcal{C}^\infty$-diffeomorphism between $\tilde F$ and $\mathbf{F}_{\bar z}$ so that $\tilde u \in BV(\tilde F)$. Thus, since $\partial_\theta e_\theta = -e_r$, (6.5) reads as

$$\int_{\mathbf{F}_{\bar z}} \varphi\, dDu = \langle (D_r(r\tilde u) - \tilde u)e_r + (D_\theta\tilde u)e_\theta, \tilde\varphi\rangle = \langle (D_r\tilde u)e_r + (D_\theta\tilde u)e_\theta, \tilde\varphi\rangle, \tag{6.6}$$

and gathering (6.4) and (6.6) yields

$$D_r\tilde u = 0, \quad D_\theta\tilde u = \alpha\mathcal{L}^1_r\otimes\eta \quad \text{ in } \mathcal{D}'(\tilde F).$$

By the first equation, there exists an orthoradial function $\tilde h \in BV((\theta_0,\theta_1))$ such that $\tilde u(r,\theta) = \alpha\tilde h(\theta)$ for a.e. $(r,\theta)\in\tilde F$. The second equation leads to $D\tilde h = \eta \geq 0$ which implies that $\tilde h$ is nondecreasing. Using that $\theta = \arctan\left(\frac{(x-\bar z)_2}{(x-\bar z)_1}\right)$ and setting $h = \tilde h\circ\arctan$, the result then follows coming back to cartesian coordinates. □

In the following result, we prove a partial uniqueness result for a fan for which a portion of its "top" boundary coincides with that of Ω, which of course may not happen.

Proposition 6.3. *Extend u by w outside Ω. Let $\mathbf{F}_{\bar z}$ be a fan centered at $\bar z\in\mathcal{F}$, and $C_{\bar z} := \bar z + C(x-\bar z, y-\bar z)$, for some x, $y\in\overline{\Omega}_p$, be the maximal open cone with vertex $\bar z$ such that $\mathbf{F}_{\bar z} = C_{\bar z}\cap\Omega_p$. Then, $u^+ = u^-$ $\mathcal{H}^1$-a.e. on $\partial\mathbf{F}_{\bar z}\cap C_{\bar z}$ and, in particular, $u = w$ $\mathcal{H}^1$-a.e. on $\partial\mathbf{F}_{\bar z}\cap C_{\bar z}\cap\partial\Omega$.*

Proof. Since $\partial\mathbf{F}_{\bar z}\cap C_{\bar z} = \partial\Omega_p\cap C_{\bar z}$, the set $\partial\mathbf{F}_{\bar z}\cap C_{\bar z}$ is open in the relative topology of $\partial\mathbf{F}_{\bar z}$ and Lipschitz. According to Remark 3.1, $\sigma\cdot\nu = \pm 1$ $\mathcal{H}^1$-a.e. on $\partial\mathbf{F}_{\bar z}\cap C_{\bar z}\cap J_u$. Using the explicit expression of σ on $\overline{\mathbf{F}}_{\bar z}\setminus\{\bar z\}$ given by Theorem 6.2, we deduce that $\sigma\cdot\nu$ coincides $\mathcal{H}^1$-a.e. with the usual scalar product of σ and ν on $\partial\mathbf{F}_{\bar z}\cap C_{\bar z}$. Therefore, $\nu = \pm\sigma$ $\mathcal{H}^1$-a.e. on $\partial\mathbf{F}_{\bar z}\cap C_{\bar z}\cap J_u$.

Assume that $\mathcal{H}^1(\partial\mathbf{F}_{\bar z}\cap C_{\bar z}\cap J_u) > 0$, since $\mathcal{H}^1$ almost every point of $\partial\mathbf{F}_{\bar z}\cap C_{\bar z}$ is a differentiability point of the boundary, we can find some $x_0\in\partial\mathbf{F}_{\bar z}\cap C_{\bar z}\cap J_u$ such that, up to a change of sign, $\nu(x_0) = \sigma(x_0)$. Let us consider the characteristic line $L_{x_0} = x_0 + \sigma^\perp(x_0)\mathbb{R}\subset C_{\bar z}$ which passes through the point $\bar z$ (because $x_0\in\overline{\mathbf{F}}_{\bar z}$), and let H be the closed half plane such that $\partial H = L_{x_0}$ which does not contain $\sigma(x_0)$ (so that $\sigma(x_0)$ is the outer unit normal to H). Since $\sigma(x_0)$ is also the unit outer normal to Ω_p at x_0, it results from the convexity of Ω_p that $\overline{\Omega}_p\subset H$. Note that since $x_0\in C_{\bar z}$ which is open, then the point x_0 does not belong to the boundary of the cone $C_{\bar z}$. Hence $C'_{\bar z} := H\cap C_{\bar z}$ is a cone with vertex $\bar z$ strictly contained in $C_{\bar z}$ which satisfies

$$\mathbf{F}_{\bar z} = C'_{\bar z}\cap\Omega_p,$$

which implies that either x or y does not belong to $\overline{\Omega}_p$, a contradiction. This argument proves that $\mathcal{H}^1(\partial\mathbf{F}_{\bar z}\cap C_{\bar z}\cap J_u) = 0$ and thus, that $u^+ = u^-$ $\mathcal{H}^1$-a.e. on $\partial\mathbf{F}_{\bar z}\cap C_{\bar z}$.

Finally, using the boundary condition, we get that $u = w$ $\mathcal{H}^1$-a.e. on $\partial\mathbf{F}_{\bar z}\cap C_{\bar z}\cap\partial\Omega$. □

6.2. **Outside the fans.** The complementary set to the boundary fans, *i.e.*,

$$\mathscr{C} := \Omega_p \setminus \mathscr{F}$$

is a closed set in the relative topology of Ω_p.

6.2.1. *Topological structure of the complementary of the fans.* We first establish topological properties of the connected components of $\mathscr{C}$, which are closed in the relative topology of Ω_p.

Lemma 6.4. *Let* $\mathbf{C}$ *be a connected component of* $\mathscr{C}$*. Then* $\mathbf{C}$ *is convex and for all* $x \in \mathbf{C}$*,* $L_x \cap \Omega_p \subset \mathbf{C}$*.*

Proof. The proof is divided into three steps.

Step 1. We show that, for all $x \in \mathscr{C}$, $L_x \cap \Omega_p \subset \mathscr{C}$.

Let $x \in \mathscr{C}$, and consider the characteristic line L_x passing through x. Assume by contradiction that there exists $y \in L_x \cap \Omega_p$ such that $y \notin \mathscr{C}$. Then $L_x = L_y$ according to Remark 5.6 and, since $y \notin \mathscr{C}$, there exists $\bar{z} \in \mathcal{F}$ such that $y \in \mathbf{F}_{\bar{z}}$. Thus $x \in L_x \cap \Omega_p = L_y \cap \Omega_p \subset \mathbf{F}_{\bar{z}}$ which is impossible.

Step 2. We show that, for all $x \in \mathbf{C}$, $L_x \cap \Omega_p \subset \mathbf{C}$.

Consider the characteristic line L_x passing through x. We already know from the previous step that $L_x \cap \Omega_p \subset \mathscr{C}$. We assume by contradiction that there exists $y \in L_x \cap \Omega_p$ such that $y \notin \mathbf{C}$. Let C_y be the connected component of $\mathscr{C}$ which contains y. We distinguish two cases:

- If the segment $[x, y] \subset \mathscr{C}$, then the closed set $C' = \mathbf{C} \cup [x, y] \cup C_y$ is connected (because $[x, y] \cap \mathbf{C} \neq \emptyset$ and $[x, y] \cap C_y \neq \emptyset$), $C' \subset \mathscr{C}$ and x, $y \in C'$. Then C' is a connected subset of $\mathscr{C}$ which strictly contains $\mathbf{C}$, a contradiction.
- If there is $z \in [x, y]$ such that $z \notin \mathscr{C}$, then we can find $\bar{z} \in \mathcal{F}$ such that $z \in \mathbf{F}_{\bar{z}}$, which implies, by definition of a fan, that the line $L_x = L_z$ must pass through the vertex $\bar{z}$ of the fan $\mathbf{F}_{\bar{z}}$ and $L_x \cap \Omega_p \subset \mathbf{F}_{\bar{z}}$. In particular the point x belongs to $\mathbf{F}_{\bar{z}}$ which is again impossible since $x \in \mathscr{C}$.

Step 3. We show that $\mathbf{C}$ is convex.

Let x and $y \in \mathbf{C}$, and let us consider a point $z \in [x, y]$. Note that since Ω_p is convex, then $z \in \Omega_p$ and it makes sense to consider its associated characteristic line L_z. If, for all $z' \in L_z \cap \Omega_p$, $z' \notin \mathbf{C}$, it would then imply that $\mathbf{C} \subset H^+ \cup H^-$, where $H^\pm$ are both open half planes separated by the line L_z. Since both x and y belong to $\mathbf{C}$, the segment $[x, y]$ is not contained in L_z. Then, up to a permutation, $x \in H^-$ and $y \in H^+$, so that $\mathbf{C} \cap H^- \neq \emptyset$ and $\mathbf{C} \cap H^+ \neq \emptyset$. This last property contradicts the connectedness of $\mathbf{C}$. Therefore, there exists $z' \in L_z \cap \Omega_p$ such that $z' \in \mathbf{C}$, and, by Step 2, we have $L_z \cap \Omega_p = L_{z'} \cap \Omega_p \subset \mathbf{C}$ which implies that $z \in \mathbf{C}$. The proof of the lemma is complete. □

The following result shows that connected components of $\mathscr{C}$ with empty interior are actually characteristic lines.

Proposition 6.5. *Let* $\mathbf{C}$ *be a connected component of* $\mathscr{C}$ *be such that* $\mathring{\mathbf{C}} = \emptyset$*, then* $\mathbf{C} = L_x \cap \Omega_p$ *for some* $x \in \mathbf{C}$*.*

Proof. Since $\mathbf{C} \neq \emptyset$ there exists $x \in \mathbf{C}$ and, by Lemma 6.4, $L_x \cap \Omega_p \subset \mathbf{C}$. By convexity of Ω_p, the line L_x must intersects $\partial\Omega_p$ at two distinct points denoted by x' and x'', and thus, $]x', x''[= L_x \cap \Omega_p \subset \mathbf{C}$. Assume by contradiction that there is $y \in \mathbf{C} \setminus (L_x \cap \Omega_p)$. The same argument shows that $L_y \cap \Omega_p =]y', y''[\subset \mathbf{C}$. Since $\mathbf{C}$ is closed in Ω_p and convex, and since x', x'', y' and $y'' \in \mathbf{C}$ then $\overline{\mathbf{C}}$ contains the closed and convex hull of $\{x', x'', y', y''\}$ denoted by $\mathrm{conv}\{x', x'', y', y''\}$. Thus $\mathring{\mathbf{C}} \supset \mathrm{int}\,\mathrm{conv}\{x', x'', y', y''\} \neq \emptyset$ which is against the hypothesis, and finally $\mathbf{C} = L_x \cap \Omega_p$. □

Remark 6.6. Note that there might be uncountably many such characteristic lines corresponding to connected components of $\mathscr{C}$ with empty interior. ¶

We next focus on the (countably many) connected components $\mathbf{C}$ of $\mathscr{C}$ with non empty interior. Since σ is constant on each characteristic line inside $\mathbf{C}$, we can naturally extend σ to the part of boundary of $\mathbf{C}$ which is reached by a characteristic coming from the interior of $\mathbf{C}$, *i.e.*, $\partial\mathbf{C} \cap \bigcup_{z\in\mathbf{C}} L_z$. Specifically, we set

$$\sigma(x) = \sigma(z), \quad L_x := L_z \quad \text{if } x \in \partial\mathbf{C} \cap L_z,$$

in such a way that σ is constant on each $L_z \cap \overline{\mathbf{C}}$. The value of σ at x is unambiguous because there cannot be more than one characteristic line L_z coming from the interior of $\mathbf{C}$ and passing through x, lest x be the apex of a boundary fan contained in $\mathbf{C}$.

6.2.2. *Characteristic boundary points.* We introduce the set of characteristic boundary points which are the points in $\partial\mathbf{C}$ not crossed by a characteristic line coming from the interior of $\mathbf{C}$.

Definition 6.7. *Let $\mathbf{C}$ be a connected component of $\mathscr{C}$ be such that $\mathring{\mathbf{C}} \neq \emptyset$. We say that $x \in \partial\mathbf{C} \cap \partial\Omega_p$ is a characteristic boundary point of $\mathbf{C}$ if $x \notin L_z$ for all $z \in \mathbf{C}$. We denote by $\partial^c\mathbf{C}$ the set of all characteristic boundary points.*

The following result formalizes the idea that the set $\partial^c\mathbf{C}$ of all characteristic boundary points of $\mathbf{C}$ is made of points where the stress is normal to $\partial\mathbf{C}$, or, equivalently, where the characteristics are tangential to $\partial\mathbf{C}$. Note that $\sigma \in H^1_{\rm loc}(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$ so that σ is well defined on $\Omega \cap \partial\Omega_p$ as an element of $L^\infty(\Omega \cap \partial\Omega_p;\mathbb{R}^2)$ (see Remark 4.4). On the other hand, strictly speaking, σ might not be defined on $\partial\Omega \cap \partial\Omega_p$. However, since $\sigma \in H(\mathrm{div},\Omega) \cap L^\infty(\Omega;\mathbb{R}^2)$, the normal trace $\sigma\cdot\nu$ is well defined as an element of $H^{-1/2}(\partial\Omega;\mathbb{R}^2) \cap L^\infty(\partial\Omega;\mathbb{R}^2)$.

Proposition 6.8. *Let $\mathbf{C}$ be a connected component of $\mathscr{C}$ be such that $\mathring{\mathbf{C}} \neq \emptyset$ and $x \in \partial\mathbf{C} \cap \partial\Omega_p$.*

(i) *If $x \in \partial^c\mathbf{C}$ is an accumulation point of $(\partial\mathbf{C} \cap \partial\Omega_p) \setminus \partial^c\mathbf{C}$, denoting by $\Sigma(x)$ the set of all limits of sequences $(\sigma(x_n))_{n\in\mathbb{N}}$ where $(x_n)_{n\in\mathbb{N}} \subset \partial\mathbf{C} \cap \partial\Omega_p \setminus \partial^c\mathbf{C}$ is such that $x_n \to x$, then*

$$\Sigma(x) \subset \mathscr{N}_{\mathbf{C}}(x) \cup (-\mathscr{N}_{\mathbf{C}}(x)).$$

(ii) *Assume that the relative interior $(\partial\mathbf{C} \cap \partial\Omega_p)^\circ$ of $\partial\mathbf{C} \cap \partial\Omega_p$ is not empty and take x in that set. If x is not an accumulation point of $(\partial\mathbf{C} \cap \partial\Omega_p) \setminus \partial^c\mathbf{C}$, then $x \in \partial^c\mathbf{C}$ and there exists a maximal open set U_x containing x such that $S_x := (\partial\mathbf{C} \cap \partial\Omega_p) \cap U_x$ is an open line segment with $S_x \subset \partial^c\mathbf{C}$. In addition $\sigma\cdot\nu = 1$ or $\sigma\cdot\nu = -1$ $\mathcal{H}^1$-a.e. on S_x.*

Proof. Step 1. We first assume that $x \in \partial^c\mathbf{C}$ is an accumulation point of $(\partial\mathbf{C} \cap \partial\Omega_p) \setminus \partial^c\mathbf{C}$ and consider $\xi \in \Sigma(x)$. It means that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $(\partial\mathbf{C} \cap \partial\Omega_p) \setminus \partial^c\mathbf{C}$ such that $x_n \to x$ and $\sigma(x_n) \to \xi$. By convexity of $\mathbf{C}$, for each $n \in \mathbb{N}$, there exists another point $y_n \neq x_n$ in $\partial\mathbf{C} \cap L_{x_n}$. Up to a subsequence, we can assume that $y_n \to y \in \partial\mathbf{C}$. Let us distinguish two possibilities:

Case I. Assume first that $x \neq y$ and that there exists $\delta > 0$ such that for each $n \in \mathbb{N}$,

$$\max_{z\in[x_n,y_n]} \mathrm{dist}(z, \partial\mathbf{C} \cap \partial\Omega_p) \geq \delta > 0.$$

Therefore, there is $z_n \in [x_n, y_n]$ such that $\mathrm{dist}(z_n, \partial\mathbf{C} \cap \partial\Omega_p) \geq \delta$ and, by compactness, up to a further subsequence, we can assume that $z_n \to z$ for some $z \in \mathbf{C}$ with $\mathrm{dist}(z, \partial\mathbf{C} \cap \partial\Omega_p) \geq \delta$. Since σ is constant along L_{x_n}, we can write

$$x_n = z_n + \theta_n \sigma^\perp(z_n) \tag{6.7}$$

for some $\theta_n \in \mathbb{R}$. Using that $|\sigma(z_n)| = 1$, up to a further subsequence, we can also assume that $\theta_n \to \theta$ so that, passing to the limit in (6.7) and using the continuity of σ in Ω_p yields

$x = z + \theta\sigma^{\perp}(z) \in L_z$, which is against that $x \in \partial^c\mathbf{C}$. It shows that this first possibility never occurs.

Case II. We next suppose that $x \neq y$ and that, for a subsequence

$$\max_{z\in[x_n,y_n]} \operatorname{dist}(z, \partial\mathbf{C}\cap\partial\Omega_p) \to 0.$$

Since $x_n \to x$ and $y_n \to y$, then $[x_n, y_n] \to [x, y]$ in the sense of Hausdorff. Therefore, for all $z \in [x, y]$, there is $z_n \in [x_n, y_n]$ such that $z_n \to z$. In particular, $\operatorname{dist}(z_n, \partial\mathbf{C}\cap\partial\Omega_p) \to 0$ which leads to $z \in \partial\mathbf{C}\cap\partial\Omega_p$. This implies that $[x, y] \subset \partial\mathbf{C}\cap\partial\Omega_p$. Since $\sigma(x_n)$ is orthogonal to $[x_n, y_n]$ for each $n \in \mathbb{N}$, we deduce that ξ is orthogonal to $[x, y]$, hence ξ belongs to $\mathscr{N}_{\mathbf{C}}(x) \cup (-\mathscr{N}_{\mathbf{C}}(x))$.

Case III. We finally assume that $x = y$. Because of the convexity of $\mathbf{C}$, its boundary $\partial\mathbf{C}$ is Lipschitz continuous and thus, there exist $r > 0$ and a L-Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that, in a suitable coordinate system,

$$\begin{cases} \mathbf{C}\cap B_r(x) = \{(s_1, s_2) \in B_r(x) :\ s_2 < f(s_1)\}, \\ \partial\mathbf{C}\cap B_r(x) = \{(s_1, s_2) \in B_r(x) :\ s_2 = f(s_1)\}. \end{cases}$$

Since $\lim_n x_n = \lim_n y_n = x$, we can assume that n is large enough so that x_n and $y_n \in B_r(x)$, hence $x_n = (s_n, f(s_n))$ and $y_n = (t_n, f(t_n))$. Let H_n be a half plane such that $\partial H_n = L_{x_n}$ and H_n does not contain the portion of $\partial\mathbf{C}$ in between x_n and y_n. Let us denote by ξ_n be the unit exterior normal to H_n and let us fix $y \in \mathbf{C}$ (see Figure 4). Clearly, if $y \in \mathbf{C}\cap H_n$, then $\xi_n \cdot (y - x_n) \leq 0$.

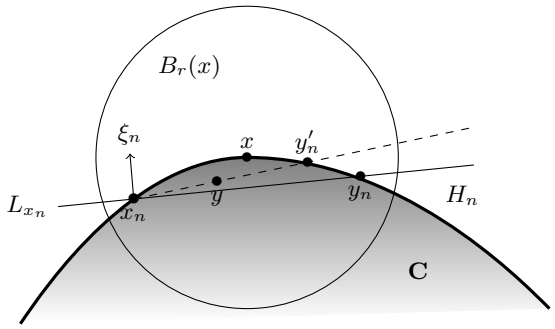

FIGURE 4. The case where $x \in \partial\mathbf{C}\cap\partial\Omega_p$ is an accumulation point of $(\partial\mathbf{C}\cap\partial\Omega_p)\setminus\partial^c\mathbf{C}$, and $x = \lim_n x_n = \lim_n y_n$ where $x_n \notin \partial^c\mathbf{C}$ and $x_n \neq y_n \in L_{x_n}\cap\partial\mathbf{C}$

On the other hand, if $y \in \mathbf{C}\setminus H_n$ we consider the point $y'_n \in \partial\mathbf{C}\cap(x_n + \mathbb{R}^*_+(y - x_n))$ which can be written as $y'_n = (t'_n, f(t'_n))$. Then, using the convexity of $\mathbf{C}$, we get that

$$\begin{aligned} \xi_n \cdot (y - x_n) &\leq \xi_n \cdot (y'_n - x_n) \leq |y'_n - x_n| = |(t'_n, f(t'_n)) - (s_n, f(s_n))| \\ &\leq \sqrt{1 + L^2}|t'_n - s_n| \leq \sqrt{1 + L^2}|x_n - y_n| \to 0. \end{aligned}$$

Thus, for all $y \in \mathbf{C}$, we have $\xi_n \cdot (y - x_n) \leq \sqrt{1 + L^2}|x_n - y_n|$. Denoting by $\tilde{\xi}$ any accumulation point of the sequence $(\xi_n)_{n\in\mathbb{N}}$, we infer that

$$\tilde{\xi} \cdot (y - x) \leq 0 \quad \text{for all } y \in \mathbf{C},$$

which means that $\tilde{\xi} \in \mathscr{N}_{\mathbf{C}}(x)$. Since $\sigma(x_n) = \pm\xi_n$, the previous argument shows that $\xi = \pm\tilde{\xi}$ and thus that $\Sigma(x) \subset \mathscr{N}_{\mathbf{C}}(x) \cup (-\mathscr{N}_{\mathbf{C}}(x))$.

Step 2. If x is not an accumulation point of $(\partial\mathbf{C} \cap \partial\Omega_p) \setminus \partial^c\mathbf{C}$, for $R > 0$ small enough, $\overline{B}_R(x)\cap(\partial\mathbf{C}\cap\partial\Omega_p)\setminus\{x\} \subset \partial^c\mathbf{C}$. By convexity of $\mathbf{C}$ and Ω_p and since $x \in (\partial\mathbf{C}\cap\partial\Omega_p)^\circ$, R can also be

chosen such that there exist two distinct points x' and x'' such that $(\partial\mathbf{C}\cap\partial\Omega_p)\cap\partial B_R(x)=\{x',x''\}$. Let C_R be the convex set delimited by the segment $[x',x'']$ and $B_R(x)\cap(\partial\mathbf{C}\cap\partial\Omega_p)$ (see Figure 5). If $\overline{B}_R(x)\cap(\partial\mathbf{C}\cap\partial\Omega_p)\neq[x',x'']$, then C_R has nonempty interior and, for every $y\in\mathring{C}_R$, the characteristic line L_y passing through y must intersect $\overline{B}_R(x)\cap(\partial\mathbf{C}\cap\partial\Omega_p)$. Since, except for the point x, this set is contained in $\partial^c\mathbf{C}$, the only possible intersection point is x. But then x would be the apex of a boundary fan contained in $\mathbf{C}$, which is impossible.

As a consequence, we must have that $\overline{B}_R(x)\cap(\partial\mathbf{C}\cap\partial\Omega_p)=[x',x'']$ is a closed line segment. Note that $x\in[x',x'']$ must belong to $\partial^c\mathbf{C}$. Indeed, if not let L_z be a characteristic line such that $x\in L_z$ for some $z\in\mathbf{C}$. Let us consider a triangle T with apexes x, $a\in L_z$ and $b\in]x,x'[$ in such a way that $\overline{T}\subset\mathring{\mathbf{C}}$ (see Figure 6). Then for any $y\in\mathring{T}$, the associated characteristic line L_y must pass through the point x (because $]x,a[\subset L_z$ and $]x,b[\subset]x,x'[\subset\partial^c\mathbf{C}$). Then x would be the apex of a boundary fan contained in $\mathbf{C}$ which is impossible (see Figure 6). We have thus established that the full segment $[x',x'']$ is contained in $\partial^c\mathbf{C}$. The maximal open set U_x is obtained by taking the union of all open balls $B_R(x)$ such that $B_R(x)\cap(\partial^c\mathbf{C}\cap\partial\Omega_p)$ is a segment.

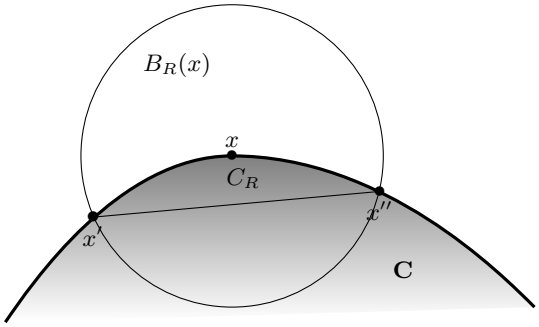

FIGURE 5. The case where $\mathring{C}_R\neq\emptyset$ leading to a contradiction

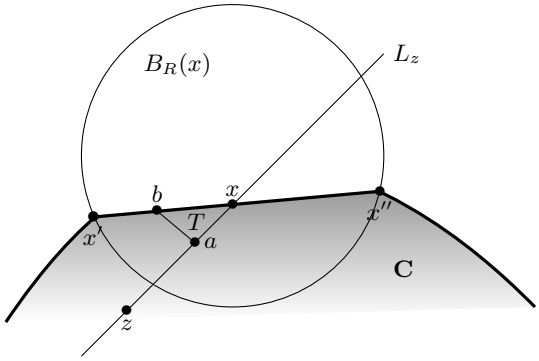

FIGURE 6. The case where $\mathring{C}_R=\emptyset$, hence $[x',x'']\subset\partial\mathbf{C}\cap\partial\Omega_p$

In order to show that $\sigma\cdot\nu=1$ or $\sigma\cdot\nu=-1$ $\mathcal{H}^1$-a.e. on S_x, we assume without loss of generality that $S_x=\{0\}\times(-1,1)$ and that $\nu=e_1$ in such a way that $\mathbf{C}\subset\mathbb{R}^-\times\mathbb{R}$. Since $S_x\subset\partial^c\mathbf{C}$, then no characteristic line from a point in $\mathbf{C}$ intersects S_x. Thus, if $\delta\in(0,1/2)$, $t\in(0,1)$ and $y\in(-1+\delta,1-\delta)$, then $(-t\varepsilon,y)\in\mathbf{C}$ for $\varepsilon>0$ small enough, and the characteristic line $L_{(-t\varepsilon,y)}$ passing through the point $(-t\varepsilon,y)$ intersects the vertical line $\{0\}\times\mathbb{R}$ at a point $(0,y_0^\varepsilon)$ with $|y_0^\varepsilon|\geq1$, or else $\sigma^\perp(-t\varepsilon,y)=\pm e_2$. Since $(-t\varepsilon,y)$ and $(0,y_0^\varepsilon)\in L_{(-t\varepsilon,y)}=(-t\varepsilon,y)+\mathbb{R}\sigma^\perp(-t\varepsilon,y)$, then

$$\sigma^\perp(-t\varepsilon,y)=\frac{\pm1}{\sqrt{t^2\varepsilon^2+(y_0^\varepsilon-y)^2}}\begin{pmatrix}t\varepsilon\\ y_0^\varepsilon-y\end{pmatrix}\ \text{ or }\ \pm e_2\to\pm e_2$$

as $\varepsilon\to0$.

Assume that there exist $y\neq z\in(-1+\delta,1-\delta)$ such that $\sigma^\perp(-t\varepsilon,y)\to e_2$ and $\sigma^\perp(-t\varepsilon,z)\to-e_2$ as $\varepsilon\to0$. Then, denoting by $(0,y_0)$ (resp. $(0,z_0)$) the intersection point of $L_{(-t\varepsilon,y)}$ (resp. $L_{(-t\varepsilon,z)}$) with the vertical line $\{0\}\times\mathbb{R}$, it means that $y_0\geq1$ and $z_0\leq-1$. We thus get that both characteristic lines $L_{(-t\varepsilon,y)}$ and $L_{(-t\varepsilon,z)}$ intersect at a point $x=(x_1,x_2)$ with $-1+\delta\leq x_2\leq1-\delta$ and $-\frac{t\varepsilon}{\delta}\leq x_1\leq0$ (see Figure 7).

Thus, for $\varepsilon>0$ small enough we infer that $x\in\mathbf{C}$ which is impossible according to Proposition 5.4. As a consequence,

$$\sigma((0,y)-\varepsilon te_1)\cdot e_1\to1\quad\text{ or }\quad\sigma((0,y)-\varepsilon te_1)\cdot e_1\to-1\tag{6.8}$$

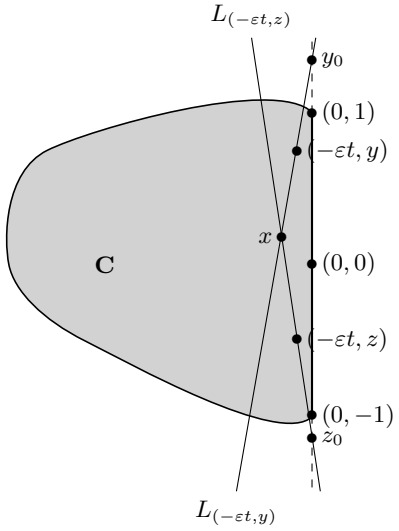

FIGURE 7.

for all $(t,y)\in(0,1)\times(-1+\delta,1-\delta)$. So there are only three possibilities: $\sigma(-t\varepsilon,y)=e_1$ for all $(t,y)\in(0,1)\times(-1+\delta,1-\delta)$; $\sigma(-t\varepsilon,y)=-e_1$ for all $(t,y)\in(0,1)\times(-1+\delta,1-\delta)$; or (6.8). Thus in any case, according to the dominated convergence theorem,

$$\lim_{\varepsilon\to 0}\int_0^1\int_{-1+\delta}^{1-\delta}|\sigma((0,y)-\varepsilon te_1)\cdot e_1-1|\,dy\,dt=0$$
$$\text{or}\quad \lim_{\varepsilon\to 0}\int_0^1\int_{-1+\delta}^{1-\delta}|\sigma((0,y)-\varepsilon te_1)\cdot e_1+1|\,dy\,dt=0. \quad (6.9)$$

On the other hand, using (2.1), we have for all $\varphi\in\mathcal{C}_c^\infty((-1,1))$,

$$\int_0^1\int_{-1}^1[\sigma((0,y)-\varepsilon te_1)\cdot e_1-(\sigma\cdot\nu)(0,y)]\varphi(y)\,dy\to 0. \quad (6.10)$$

Gathering (6.9) and (6.10) yields $\sigma\cdot\nu=1$ or $\sigma\cdot\nu=-1$ $\mathcal{H}^1$-a.e. in S_x. □

Remark 6.9. Assume that the relative interior $(\partial\mathbf{C}\cap\partial\Omega_p)^\circ$ is not empty and take x in the relative boundary $\partial\left[(\partial\mathbf{C}\cap\partial\Omega_p)^\circ\right]$ of that set. Then, in the setting of Step 2 in the above proof the segment S_x can be replaced by a segment with x as end point (and not one centered at x) under the only condition that x not be a "one sided accumulation point" of $(\partial\mathbf{C}\cap\partial\Omega_p)\setminus\partial^c\mathbf{C}$, that is that $\nexists x_n\in(\partial\mathbf{C}\cap\partial\Omega_p)\setminus\partial^c\mathbf{C}$ such that $x_n\to x$. To see that, it is enough to introduce a small ball B_R such that $\partial B_R\cap(\partial\mathbf{C}\cap\partial\Omega_p)=\{x,x'\}$ with $x'\in\partial^c\mathbf{C}$ and to take C_R in that step to be the convex set enclosed by $[x,x']$ and $B_R\cap(\partial\mathbf{C}\cap\partial\Omega_p)$. The rest of the proof is identical. ¶

6.2.3. *Topological structure of the boundary of the connected components.* We now establish a result concerning the topological structure of $\partial\mathbf{C}$ which can be split as the disjoint union of characteristic lines and continuous curves.

Proposition 6.10. *Let $\mathbf{C}$ be a connected component of $\mathscr{C}$ be such that $\mathring{\mathbf{C}}\neq\emptyset$ and $\mathbf{C}\neq\Omega_p$. Then, $\partial\mathbf{C}\cap\Omega_p\neq\emptyset$ and there exist an (at most) countable set $J\subset\mathbb{N}$ and, for all $j\in J$, distinct points*

$\hat{x}_j \in \Omega_p$ such that

$$\partial\mathbf{C} \cap \Omega_p = \bigcup_{j\in J} (L_{\hat{x}_j} \cap \Omega_p).$$

Moreover $\partial\mathbf{C} \cap \partial\Omega_p \neq \emptyset$ and there exist a (possibly uncountable) set I and, pairwise disjoint continuous curves $\{\Gamma_i\}_{i\in I}$ such that

$$\partial\mathbf{C} \cap \partial\Omega_p = \bigcup_{i\in I} \Gamma_i,$$

for some Lipschitz continuous mappings $\gamma_i : [0,1] \to \mathbb{R}^2$ with $\Gamma_i = \gamma_i([0,1])$.

Proof. Note that $\partial\mathbf{C} \cap \Omega_p \neq \emptyset$ otherwise $\Omega_p = \mathbf{C}$. If $x \in \partial\mathbf{C} \cap \Omega_p$, then there exists $r > 0$ such that $B_r(x) \subset \Omega_p$. Moreover, since $\mathbf{C}$ is closed in Ω_p, then $x \in \mathbf{C}$, hence $L_x \cap \Omega_p \subset \mathbf{C}$ by Lemma 6.4. If $L_x \cap \mathring{\mathbf{C}} \neq \emptyset$ then, by convexity of $\mathbf{C}$, L_x is not contained in the tangential cone to $\partial\mathbf{C}$ at x and thus $L_x \cap B_r(x) \setminus \mathbf{C} \neq \emptyset$, which is impossible according to Lemma 6.4. Thus $L_x \cap \mathring{\mathbf{C}} = \emptyset$ and $L_x \cap \Omega_p \subset \partial\mathbf{C}$. Since $\mathbf{C}$ is bounded and convex, its boundary is Lipschitz continuous (see Propositions 2.4.4 and 2.4.7 in [27]) and $\mathcal{H}^1(\partial\mathbf{C}) < +\infty$. Using that $L_x \cap \Omega_p$ is a nonempty open segment, $\mathcal{H}^1(L_x \cap \Omega_p) > 0$ and thus, there are at most countably many such characteristic lines. Therefore,

$$\partial\mathbf{C} \cap \Omega_p = \bigcup_{j\in J} (L_{\hat{x}_j} \cap \Omega_p),$$

for some nonempty (at most) countable set $J \subset \mathbb{N}$ and some distinct points $\hat{x}_j \in \Omega_p$, for all $j \in J$. Note that by convexity, for all $j \in J$, the set $\mathbf{C}$ is contained in one of the half planes delimited by $L_{\hat{x}_j}$ for each $j \in J$.

Remark that $\partial\mathbf{C} \cap \partial\Omega_p \neq \emptyset$ because, since there exists $x \in \mathbf{C} \neq \emptyset$, $L_x \cap \Omega_p \subset \mathbf{C}$ and L_x intersects $\partial\Omega_p$ at points which belong to $\partial\mathbf{C}$. Consider the connected components $\{\Gamma_i\}_{i\in I}$ of $\partial\mathbf{C} \cap \partial\Omega_p$. Note that for all $i \in I$, Γ_i is a closed, connected set with finite $\mathcal{H}^1$ measure, therefore, according to [16, Proposition C-30.1], we infer that Γ_i is arcwise connected and that there exists a Lipschitz continuous mapping $\gamma_i : [0,1] \to \mathbb{R}^2$ such that $\Gamma_i = \gamma_i([0,1])$ (with possibly $\gamma_i(0) = \gamma_i(1)$). □

The previous result motivates the following definition.

Definition 6.11. *Let $\mathbf{C}$ be a connected component of $\mathscr{C}$ be such that $\mathring{\mathbf{C}} \neq \emptyset$ and $\mathbf{C} \neq \Omega_p$. The set $\partial\mathbf{C} \cap \partial\Omega_p$ is called the exterior boundary of $\mathbf{C}$ and the set $\partial\mathbf{C} \cap \partial\Omega_p \setminus \partial^c\mathbf{C}$ is called the non-characteristic exterior boundary of $\mathbf{C}$.*

The next result shows that the topological structure of $\mathbf{C}$ is severely constrained. In fact, its boundary $\partial\mathbf{C}$ contains at most two characteristic lines and $\partial\mathbf{C} \cap \partial\Omega_p$ has at most two connected components (see Figures 8 and 9 for illustration).

Theorem 6.12. *Let $\mathbf{C}$ be a connected component of $\mathscr{C}$ be such that $\mathring{\mathbf{C}} \neq \emptyset$ and $\mathbf{C} \neq \Omega_p$. Then $\partial\mathbf{C} \cap \partial\Omega_p$ has at most two connected components and*

i) *if $\partial\mathbf{C} \cap \partial\Omega_p$ has exactly two connected components, then $\partial^c\mathbf{C} = \emptyset$ and all characteristic lines that intersect $\partial\mathbf{C}$ must intersect both connected components.*
ii) *if $\partial\mathbf{C} \cap \partial\Omega_p$ is connected, then $\partial^c\mathbf{C}$ is either a point or a closed line segment contained in $\partial\mathbf{C} \cap \partial\Omega_p$, $\partial\mathbf{C} \cap \partial\Omega_p \setminus \partial^c\mathbf{C}$ has two connected components, and all characteristic lines that intersect $\partial\mathbf{C}$ must intersect both connected components.*

Remark 6.13. If $\mathbf{C} = \Omega_p$, arguments similar to those used in the proof of Theorem 6.12 would show that:

– $\partial^c\Omega_p$ has two connected components which are both either a point or a closed line segment;

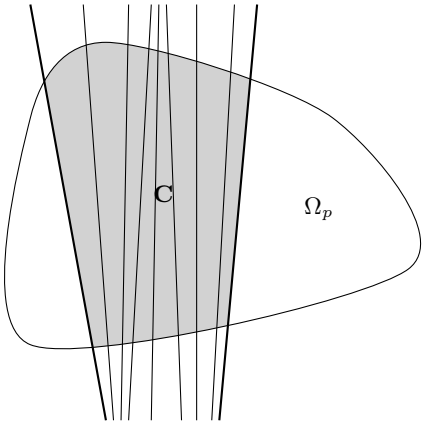

FIGURE 8. The case $\#(I) = 2$

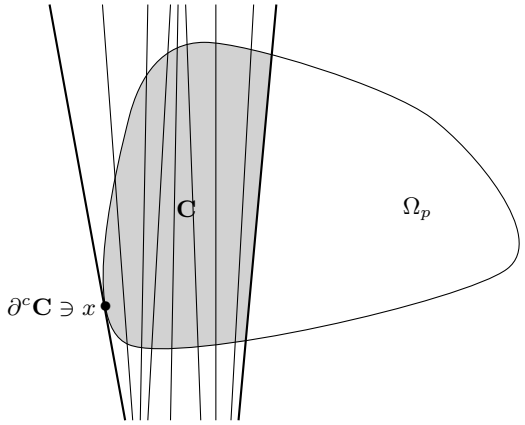

FIGURE 9. The case $\#(I) = 1$

– $\partial\Omega_p \setminus \partial^c\Omega_p$ has two connected components, and all characteristic lines that intersect $\partial\Omega_p$ must intersect both connected components. ¶

The proof of Theorem 6.12 relies on several technical results. We first show that there can be at most two boundary characteristic line segments.

Lemma 6.14. *$\#(J) = 1$ or 2 in the notation of Proposition 6.10.*

Proof. Since $\partial\mathbf{C} \cap \Omega_p \neq \emptyset$, it follows that $\#(J) \geq 1$.

Assume now that $\#(J) \geq 3$ and let L_1, L_2, L_3 be three distinct interior boundary characteristic lines, that is such that $L_i \cap \Omega_p \subset \partial\mathbf{C}$. Note that $L_i \cap \partial\Omega_p = \{x_i, x_i'\}$, where both points x_i and x_i' lie in $\partial\Omega_p \cap \partial\mathbf{C}$, so that $L_i \cap \Omega_p =]x_i, x_i'[$. Furthermore, because there can be no boundary or interior fans inside $\mathbf{C}$, the points x_1, x_1', x_2, x_2', x_3 and x_3' are pairwise distinct and $L_i \cap L_j \cap \overline{\mathbf{C}} = \emptyset$ for $i \neq j$. By Lemma 6.4, the middle points $y_i := (x_i + x_i')/2$ belong to $L_i \cap \Omega_p \cap \mathbf{C}$ and consequently, $L_{y_i} = L_i$. Moreover, for $i \neq j$, the closed segments $[y_i, y_j] \subset \Omega_p$ cannot be contained in a characteristic line L_x, for some $x \in \Omega_p$, otherwise L_x and L_i (resp. L_j) would intersect at y_i (resp. y_j), which is impossible in view of Proposition 5.4.

Now $\partial\mathbf{C} \setminus]x_1, x_1'[$ is a closed, connected set with finite $\mathcal{H}^1$ measure, therefore, according to [16, Proposition C-30.1], it is arcwise connected and there exists a Lipschitz continuous mapping $\gamma : [0,1] \to \mathbb{R}^2$ such that $\partial\mathbf{C} \setminus]x_1, x_1'[= \gamma([0,1])$ with $\gamma(0) = x_1$ and $\gamma(1) = x_1'$. Since $\partial\mathbf{C} = \gamma([0,1]) \cup]x_1, x_1'[$ with $]x_1, x_1'[\subset L_1$, it follows from Proposition 5.4 that L_2 and L_3 intersect $\partial\mathbf{C}$ in $\gamma((0,1))$. Let us renumber the L_i and exchange x_i with x_i' if necessary so that

$$x_2 = \gamma(s_2),\ x_3 = \gamma(s_3),\ x_3' = \gamma(s_3') \text{ with } 0 < s_2 < s_3 < s_3' < 1,$$

and consider $x_2' = \gamma(s_2')$ for some $0 < s_2' < 1$.

If $s_2' \in (s_3', 1)$, then the segment $]y_1, y_3[$ intersects L_2 at a point $y_2 \in \mathring{\mathbf{C}}$ (see Figure 10), which contradicts the fact that $L_2 \subset \partial\mathbf{C}$.

Now $s_2' \notin (s_3, s_3')$ because L_2 and L_3 cannot intersect in $\overline{\mathbf{C}}$, so the other possibility is that $s_2' \in (s_2, s_3)$ (see Figure 11). For any $t \in (0,1)$, define $y(t) := ty_3 + (1-t)y_1 \in]y_1, y_3[$. The intersection points of $L_{y(t)}$ with $\partial\Omega_p$, respectively denoted by $\gamma(s_t)$ and $\gamma(s_t')$, satisfy

$$s_t' \in (s_3', 1), \quad s_t \in (0, s_2) \cup (s_2', s_3).$$

Define

$$\underline{t} := \sup\{t \in [0,1] :\ L_{y(t)} \cap \gamma([0, s_2]) \neq \emptyset\}, \quad \bar{t} := \inf\{t \in [0,1] :\ L_{y(t)} \cap \gamma([s_2', s_3]) \neq \emptyset\}.$$

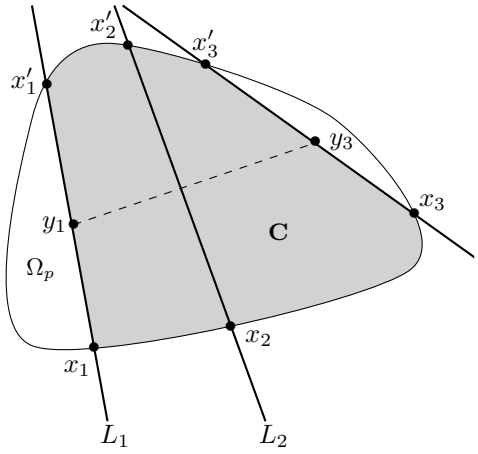

FIGURE 10. The first case: $s_2' \in (s_3', 1)$

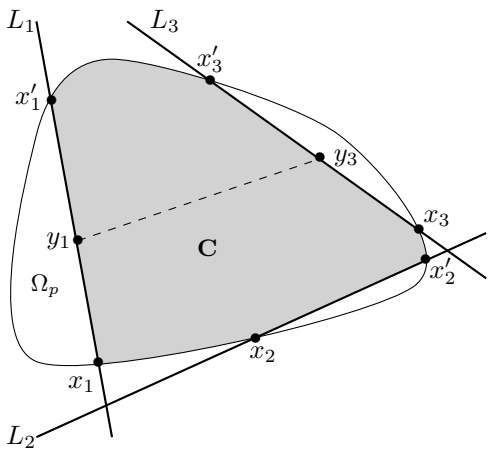

FIGURE 11. The second case: $s_2' \in (s_2, s_3)$

If, for all $y \in]y_1, y_3[$, the characteristic line L_y intersects $\gamma([s_2', s_3])$, then, by continuity of σ on $[y_1, y_3]$, we would have that $L_{y_1} \cap \gamma([s_2', s_3]) \neq \emptyset$ which is impossible since $L_{y_1} = L_1$ is disjoint from $\gamma([s_2', s_3])$. Therefore the set $\{t \in [0,1] : \ L_{y(t)} \cap \gamma([0, s_2]) \neq \emptyset\}$ is not empty and $\underline{t} > 0$. A similar argument also shows that $\underline{t} < 1$. Let $(t_n)_{n\in\mathbb{N}}$ be a maximizing sequence in $[0,1]$ such that $L_{y(t_n)} \cap \gamma([0, s_2]) \neq \emptyset$ for all $n \in \mathbb{N}$ and $t_n \to \underline{t}$. Since $\gamma(s_{t_n}) \in L_{y(t_n)} = y(t_n) + \mathbb{R}\sigma^\perp(y(t_n))$, there exists $\theta_n \in \mathbb{R}$ such that

$$\gamma(s_{t_n}) = y(t_n) + \theta_n \sigma^\perp(y(t_n)),$$

where $(\gamma(s_{t_n}))_{n\in\mathbb{N}}$ is a sequence in $\gamma([0, s_2])$ (hence bounded) and $(\theta_n)_{n\in\mathbb{N}}$ is a bounded sequence since $|\sigma^\perp(y(t_n))| = 1$ for all $n \in \mathbb{N}$. Therefore, up to a further subsequence $\gamma(s_{t_n}) \to \underline{x} \in \gamma([0, s_2])$ and $\theta_n \to \theta$. Thus, using that σ is continuous in $[y_1, y_3]$ (because $[y_1, y_3] \subset \Omega_p$), it follows that

$$\underline{x} = y(\underline{t}) + \theta\sigma^\perp(y(\underline{t})),$$

hence $\underline{x} \in L_{y(\underline{t})} \cap \gamma([0, s_2])$. Note that since $\underline{t} \in (0,1)$, then $\underline{y} := y(\underline{t}) \in]y_1, y_3[$. Moreover, $\underline{x} \neq \gamma(0) = x_1$ and $\underline{x} \neq \gamma(s_2) = x_2$ otherwise, $\underline{x}$ would be the apex of a boundary fan contained in $\mathbf{C}$, hence $\underline{x} \in L_{y(\underline{t})} \cap \gamma((0, s_2))$. A similar argument shows that $\bar{t} \in (0,1)$, $\bar{y} := y(\bar{t}) \in]y_1, y_3[$ and $L_{\bar{y}} \cap \gamma((s_2', s_3)) \neq \emptyset$.

Since $s_2 < s_2'$, then $\underline{t} < \bar{t}$, otherwise $L_{\underline{y}}$ and $L_{\bar{y}}$ would intersect inside $\mathring{\mathbf{C}}$ since $L_{\underline{y}}$ intersects $\gamma((s_1, s_2))$ and $L_{\bar{y}}$ intersects $\gamma((s_2', s_3))$. But this is impossible by Proposition 5.4.

Denote by $\underline{H}$ and $\overline{H}$ the open half-planes with boundary $L_{\underline{y}}$ and $L_{\bar{y}}$ that do not contain the points y_1 and y_3 respectively. The region $\mathbf{C}' := \mathbf{C} \cap \underline{H} \cap \overline{H}$ contains the characteristic lines $L_{y(t)}$ for all $t \in (\underline{t}, \bar{t})$. Such a line cannot intersect $L_{\underline{y}}$ and $L_{\bar{y}}$ by Proposition 5.4, and it cannot intersect the connected boundaries $\gamma([s_1, s_2]) \cup \gamma([s_2', s_3])$ by construction. The line $L_{y(t)}$ must therefore intersect $\gamma((s_2, s_2')) =]x_2, x_2'[\subset L_2$ (see Figure 11), which is impossible according, once again, to Proposition 5.4. □

Remark 6.15. Lemma 6.14 actually establishes, if $\mathbf{D}$ is any closed, convex subset of $\mathscr{C}$ with $\mathring{\mathbf{D}} \neq \emptyset$ and $\mathbf{D} \neq \Omega_p$ such that $\partial\mathbf{D} \cap \Omega_p$ is a countable union of disjoint characteristic line segments, that is such that

$$\partial\mathbf{D} \cap \Omega_p = \bigcup_{k\in K} (L_{\hat{x}_k} \cap \Omega_p),$$

for some countable set K and distinct points $\hat{x}_k \in \Omega_p$, then $\#(K) = 1$ or 2. ¶

Then, we show that $\partial\mathbf{C} \cap \partial\Omega_p$ has at most two connected components as well, that those cannot reduce to a single point, and that the extreme points lie on a characteristic line.

Lemma 6.16. *$\#(I) = \#(J) \in \{1, 2\}$ in the notation of Proposition 6.10. Moreover, for all $i \in I$, $\gamma_i(0) \neq \gamma_i(1)$. If $\#(J) = 1$ then $\gamma_1(0)$ and $\gamma_1(1)$ belong to L_1 while if $\#(J) = 2$ then $\gamma_1(0)$ and $\gamma_2(1)$ belong to one of the boundary characteristic lines, while $\gamma_1(1)$ and $\gamma_2(0)$ belong to the other one.*

Proof. Assume first that $\#(J) = 1$. Then the characteristic line $L_1 := L_{\hat{x}_1}$ intersects $\partial\Omega_p$ at two points, and thus determine a unique connected component $\Gamma = \partial\mathbf{C} \setminus L_1 = \partial\mathbf{C} \cap \partial\Omega_p$.

If $\#(J) = 2$, then the two characteristic lines $L_1 := L_{\hat{x}_1}$ and $L_2 := L_{\hat{x}_2}$ are distinct. They cannot intersect at $a \in \partial\Omega_p$ otherwise a would be the apex of a boundary fan contained in $\mathbf{C}$, which is not possible. They cannot intersect in $\mathring{\mathbf{C}}$ by virtue of Proposition 5.4. Thus they determine two disjoint connected components Γ_1 and Γ_2 for $\partial\mathbf{C} \setminus (L_1 \cup L_2) = \partial\mathbf{C} \cap \partial\Omega_p$.

The rest of the Lemma is a direct consequence of that geometry. □

We next show that, when $\#(J) = 2$, no characteristic line can intersect twice the same connected component of $\partial\mathbf{C} \cap \partial\Omega_p$.

Lemma 6.17. *If $\#(J) = 2$, then all characteristic lines L_x with $x \in \mathbf{C}$ intersect both Γ_1 and Γ_2.*

Proof. Assume by contradiction that there is a connected component of $\partial\mathbf{C} \cap \partial\Omega_p$, say Γ_1, and a characteristic line L that intersects Γ_1 at two distinct points, say $\bar{a} = \gamma_1(\bar{s})$ and $\bar{b} = \gamma_1(\bar{t})$ with $s < t$. First, $\bar{s} > 0$ and $\bar{t} < 1$, otherwise $\gamma_1(0)$ and/or $\gamma_1(1)$ would be the apex of a boundary fan contained in $\mathbf{C}$. Consider the closed hyperplane H bounded by L and containing both $\gamma(0)$ and $\gamma(1)$. Then $\mathbf{C} \cap H$ is a convex set in $\mathscr{C}$ which has three boundary characteristic line segments $L_1 \cap \Omega_p$, $L_2 \cap \Omega_p$ and $L \cap \Omega_p$, in contradiction with Remark 6.15. □

Provided that $\partial\mathbf{C} \cap \partial\Omega_p$ possesses two connected components, there are no characteristic boundary points.

Lemma 6.18. *Assume that $\#(J) = 2$, then $\partial^c\mathbf{C} = \emptyset$.*

Proof. Assume by contradiction that there is $x \in \partial^c\mathbf{C}$. Then, without loss of generality, $x \in \Gamma_1$ and, from Lemma 6.16, x cannot be an extreme point of Γ_1 so there exists $s \in (0, 1)$ such that $x = \gamma_1(s)$. Let us distinguish two cases:

Case I: If there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $\Gamma_1 \setminus \partial^c\mathbf{C}$ such that $x_n \to x$, then, according to Lemma 6.17, the characteristic line L_{x_n} intersects $\partial\mathbf{C}$ at another point y_n in Γ_2. There is $\delta > 0$ and a sequence of points $z_n \in \mathbf{C}$ with $\mathrm{dist}(z_n, \partial\mathbf{C} \cap \partial\Omega_p) \geq \delta$ such that $x_n = z_n + \theta_n\sigma^\perp(z_n)$ for some $\theta_n \in \mathbb{R}$. Then, up to a subsequence, $z_n \to z \in \mathbf{C}$, $\theta_n \to \theta$ so that $x = z + \theta\sigma^\perp(z) \in L_z$, a contradiction.

Case II: If there exists $R > 0$ such that $B_R(x) \cap \Gamma_1 \subset \partial^c\mathbf{C}$, according to Proposition 6.8 (ii), there exists a maximal open set U_x containing x such that $\Gamma_1 \cap U_x =]a, b[$ is a segment contained in $\partial^c\mathbf{C}$. Consider *e.g.* the point a and note that $a \neq \gamma_1(0)$ and $b \neq \gamma_1(1)$ otherwise, a (resp. b) would be the apex of a boundary fan according to Lemma 6.16. Indeed, one could consider a small open triangle T with vertices $a = \gamma_1(0)$ (resp. $b = \gamma_1(1)$), a point in $]a, b[$ and a point on $L_{\gamma_1(0)}$ (resp $L_{\gamma_1(1)}$) so that $\overline{T} \subset \mathring{\mathbf{C}}$. But then, any point $y \in \mathring{T}$ would be such that L_y contains $\gamma_1(0)$ (resp. $\gamma_1(1)$) which would therefore be a boundary fan contained in $\mathbf{C}$. Thus, $a = \gamma_1(s)$ for some $s \in (0, 1)$.

We claim that there exists a sequence $(s_n)_{n\in\mathbb{N}}$ in $(0,s)$ such that $x_n = \gamma_1(s_n) \in \Gamma_1 \setminus \partial^c\mathbf{C}$ for each $n \in \mathbb{N}$ and $x_n \to a$. Otherwise, according to Remark 6.9, the point a would be the right endpoint of a maximal open segment $]a', a[$ contained in $\Gamma_1 \cap \partial^c\mathbf{C}$ for some $a' = \gamma_1(s') \in \Gamma_1$ with $0 < s' < s$. Then two possibilities occur:

- Either both segments $]a', a[$ and $]a, b[$ are aligned so that, by Proposition 6.8 (ii) we get that the largest segment $]a', b[$ is contained in $\partial^c\mathbf{C}$, which is a contradiction with the maximality of $]a, b[$;
- Or the segments $]a', a[$ and $]a, b[$ are not aligned so that considering a triangle T with apexes a, $(a + a')/2$ and $(a + b)/2$, we would get that $\mathring{T} \subset \Omega_p$ (because $[a', a] \cup [a, b] \subset \partial\mathbf{C}$ and Ω_p is convex) and any point $z \in \mathring{T}$ would have an associated characteristic line L_z passing through a. This last case would lead to a boundary fan contained in $\mathbf{C}$ and centered at a, which is impossible.

According to the previous claim and Lemma 6.17, the characteristic line L_{x_n} intersects $\partial\mathbf{C}$ at another point $y_n \in \Gamma_2$. Since dist $(\Gamma_2, \Gamma_1) > 0$, there exist $\delta > 0$ and a sequence of points $(z_n)_{n\in\mathbb{N}}$ in $\mathbf{C}$ with $\mathrm{dist}(z_n, \partial\mathbf{C} \cap \partial\Omega_p) \geq \delta$ and such that $x_n = z_n + \theta_n\sigma^\perp(z_n)$ for some $\theta_n \in \mathbb{R}$. Then, up to a subsequence, $y_n \to y$, $z_n \to z \in \mathbf{C}$ and $\theta_n \to \theta$ so that, using the continuity of σ in Ω_p,

$$a = z + \theta\sigma^\perp(z) \in L_z,$$

hence $a \notin \partial^c\mathbf{C}$. Considering a small open triangle T with apexes a, $(a + b)/2$ and a point living on L_z in such a way that $T \subset \mathring{\mathbf{C}}$, then any point $w \in T$ has a characteristic line L_w which must pass through the point a, leading to a boundary fan contained in $\mathbf{C}$ and centered at a, a contradiction. □

Remark 6.19. As was the case for Remark 6.15 and with the same notation, Lemma 6.18 actually establishes that if $\#(K) = 2$, then $\partial^c\mathbf{D} = \emptyset$. ¶

We next focus on the case where $\partial\mathbf{C} \cap \partial\Omega_p$ is connected, and show that $\partial^c\mathbf{C}$ is connected set which separates $\partial\mathbf{C} \cap \partial\Omega_p$ into two connected components.

Lemma 6.20. *If $\#(J) = 1$, then the set $\partial^c\mathbf{C}$ is either a single point or a closed line segment and $\partial\mathbf{C} \cap \partial\Omega_p \setminus \partial^c\mathbf{C}$ has two connected components Γ_1' and Γ_2'. Further, all characteristic lines that intersect $\partial\mathbf{C} \setminus \partial^c\mathbf{C}$ must intersect both connected components Γ_1' and Γ_2'.*

Proof. Since $\#(I) = 1$, there exists a unique characteristic line $L = L_1$ such that $\mathbf{C} = \Omega_p \cap H$, where H is an closed hyperplane such that $\partial H = L$. Moreover,

$$\Gamma := \partial\mathbf{C} \cap \partial\Omega_p = \gamma([0, 1])$$

for some Lipschitz continuous mapping $\gamma : [0, 1] \to \mathbb{R}^2$.

Step 1. Let us first prove that $\partial^c\mathbf{C} \neq \emptyset$. To this aim, assume by contradiction that $\partial^c\mathbf{C} = \emptyset$.

We first consider the characteristic line passing through $\gamma(0)$ and $\gamma(1)$ and set $(s_0, t_0) = (0, 1)$. Assume that (s_n, t_n) are known so that $L_{\gamma(s_n)} \cap \overline{\Omega}_p = [\gamma(s_n), \gamma(t_n)]$. Let $V_n : [s_n, t_n] \to \mathbb{R}^+$ be the continuous function defined by

$$V_n(t) = \mathcal{H}^1(\gamma([s_n, t])) = \int_{s_n}^{t} |\dot\gamma(s)|\, ds \quad \text{for all } t \in [s_n, t_n]$$

which satisfies $V_n(s_n) = 0$ and $V_n(t_n) = \mathcal{H}^1(\gamma([s_n, t_n]))$. According to the intermediate valued Theorem, there exists $s_{n+1/2} \in (s_n, t_n)$ such that

$$V_n(s_{n+1/2}) = \frac{V_n(t_n)}{2} = \frac{\mathcal{H}^1(\gamma([s_n, t_n])}{2},$$

thus the curve $\gamma([s_n,t_n])$ is separated into two curves $\gamma([s_n,s_{n+1/2}])$ and $\gamma([s_{n+1/2},t_n])$ of equal length with

$$\mathcal{H}^1(\gamma([s_n,s_{n+1/2}])) = \mathcal{H}^1(\gamma([s_{n+1/2},t_n])) = \frac{1}{2}\mathcal{H}^1(\gamma([s_n,t_n])).$$

Since Ω_p is convex, the resulting triangle T_n with vertices $\gamma(s_n)$, $\gamma(t_n)$ and $\gamma(s_{n+1/2})$ satisfies $\mathring{T}_n \subset \Omega_p$. Moreover, $[\gamma(s_n),\gamma(s_{n+1/2})] \cup [\gamma(t_n),\gamma(s_{n+1/2})] \not\subset \partial\mathbf{C}$, otherwise it would also be lying in $\partial\mathbf{C} \cap \partial\Omega_p$ because that latter set has only one connected component by assumption. Then the characteristic line passing through $\gamma(s_{n+1/2})$ would necessarily intersect the open segment $]\gamma(s_n),\gamma(t_n)[\subset L_{\gamma(s_n)} \cap \Omega_p$, which is impossible by Proposition 5.4. We then define $t_{n+1/2}$ such that $\gamma(t_{n+1/2})$ is the other intersection point of $L_{\gamma(s_{n+1/2})}$ with $\partial\Omega_p$. Note that $t_{n+1/2} \in (s_n,t_n)$ otherwise $\gamma(s_n)$ (resp. $\gamma(t_n)$) would form a boundary fan contained in $\mathbf{C}$. Then if $s_{n+1/2} < t_{n+1/2}$, we define $(s_n,t_n) = (s_{n+1/2},t_{n+1/2})$, while if $s_{n+1/2} > t_{n+1/2}$, we define $(s_n,t_n) = (t_{n+1/2},s_{n+1/2})$ (see Figure 12). The sequence $(s_n)_{n\in\mathbb{N}}$ is increasing while $(t_n)_{n\in\mathbb{N}}$ is decreasing with $0 < s_n < s_{n+1} < t_{n+1} < t_n < 1$ for all $n \in \mathbb{N}$. Therefore, $s_n \to \bar{s}$ and $t_n \to \bar{t}$ for some $0 < \bar{s} \le \bar{t} < 1$, and the line segment $[s_n,t_n]$ converges in the sense of Hausdorff to $[\bar{s},\bar{t}]$. Since

$$\begin{aligned} |\gamma(t_{n+1}) - \gamma(s_{n+1})| \le \mathcal{H}^1(\gamma([s_{n+1},t_{n+1}])) &\le \mathcal{H}^1([\gamma(s_{n+1/2},t_n]) = \mathcal{H}^1(\gamma([s_n,s_{n+1/2}])) \\ &= \frac{\mathcal{H}^1(\gamma([s_n,t_n])}{2} \le \frac{\mathcal{H}^1(\gamma([0,1])}{2^{n+1}} \to 0, \end{aligned}$$

we deduce that $\gamma(\bar{s}) = \gamma(\bar{t}) =: x$. Since $x \notin \partial^c\mathbf{C}$ (because $\partial^c\mathbf{C}$ is empty), it lies on a characteristic line L_x which must intersect $[\gamma(s_n),\gamma(t_n)]$, a contradiction with Proposition 5.4.

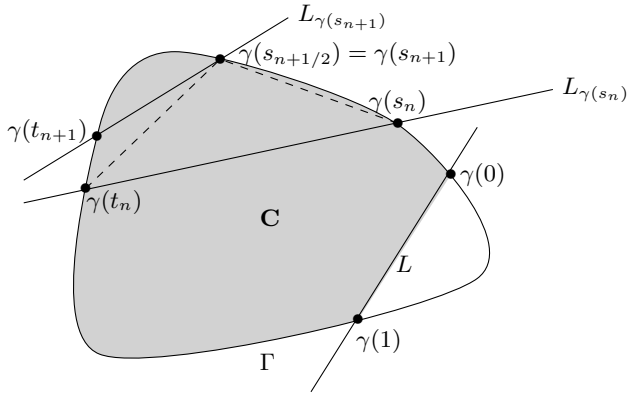

FIGURE 12. Construction of the sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$

Step 2. Assume now that $\partial^c\mathbf{C}$ is not connected and consider two distinct connected components S and S' with

$$t_0 := \inf\{t \in [0,1] :\ \gamma(t) \in S'\} > \sup\{s \in [0,1] :\ \gamma(s) \in S\} =: s_0.$$

Then there exists $\bar{s} \in [s_0,t_0]$ such that $x := \gamma(\bar{s}) \notin \partial^c\mathbf{C}$. Consider the characteristic line L_x passing through x. It must intersect Γ at some other point x' which cannot coincide with $\gamma(0)$ or $\gamma(1)$, otherwise it would form with L a boundary fan contained in $\mathbf{C}$ with apex at one of these points. Let H be the closed half-plane with L_x as boundary containing both $\gamma(0)$ and $\gamma(1)$. Then the region $\mathbf{C}' = \mathbf{C} \cap H$ would be a convex relatively closed subset of Ω_p such that, on the one hand, its boundary contains either S or S', and, on the other hand, $\partial\mathbf{C}' \cap \partial\Omega_p$ has at least two connected components. Hence $\partial^c\mathbf{C}' \neq \emptyset$ (since S and S' are contained in $\partial^c\mathbf{C}$, and thus in $\partial^c\mathbf{C}'$ as well),

which is in contradiction with Remark 6.19. So $\partial^c\mathbf{C}$ is connected and $\partial\mathbf{C}\cap\partial\Omega_p\setminus\partial^c\mathbf{C}$ has two connected components Γ_1' and Γ_2'.

Step 3. If $\partial^c\mathbf{C}$ is not reduced to a point, it must contain $\gamma((s',t'))$ for some $0<s'<t'<1$ so that its relative interior in $\partial\mathbf{C}$, denoted by $\mathrm{int}(\partial^c\mathbf{C})$, is not empty. Then, any point $x\in\mathrm{int}(\partial^c\mathbf{C})$ is not an accumulation point of $(\partial\mathbf{C}\cap\partial\Omega_p)\setminus\partial^c\mathbf{C}$ so that Proposition 6.8-(ii) ensures that $x\in]a_x,b_x[\subset\partial^c\mathbf{C}$ for some maximal open segment $]a_x,b_x[$ containing x. If $]a_x,b_x[\neq\mathrm{int}(\partial^c\mathbf{C})$, then a_x is not an accumulation of $(\partial\mathbf{C}\cap\partial\Omega_p)\setminus\partial^c\mathbf{C}$, and a new application of Proposition 6.8-(ii) shows that a_x is contained in another open segment $S\subset\partial^c\mathbf{C}$. Then, either $]a_x,b_x[$ and S are not aligned, which is impossible because a_x would be the apex of a boundary fan contained in $\mathbf{C}$ using an argument identical to that in Case II of the proof of Lemma 6.18; or $]a_x,b_x[$ and S are aligned so that $]a_x,b_x[\cup S\subset\partial^c\mathbf{C}$ is an open line segment containing x which is strictly larger that $]a_x,b_x[$, a contradiction to the maximality of $]a_x,b_x[$. Therefore we must have that $]a_x,b_x[=\mathrm{int}(\partial^c\mathbf{C})$, which shows that $\partial^c\mathbf{C}$ is a segment. If $a_x\notin\partial^c\mathbf{C}$, then there exists $z\in\mathbf{C}$ such that $a_x\in L_z$ and a_x would be the apex of a boundary fan contained in $\mathbf{C}$ by an argument identical to that used in Case II of the proof of Lemma 6.18. Therefore $a_x\in\partial^c\mathbf{C}$ and the same argument shows that $b_x\in\partial^c\mathbf{C}$. This proves that $\partial^c\mathbf{C}$ is a closed line segment.

Step 4. Assume that there is $z\in\mathbf{C}$ such that the characteristic line L_z intersects say Γ_1' at two distinct points $\gamma(s)$ and $\gamma(t)\in\Gamma_1'$. But then the same construction as in Step 1 would establish the existence of some $\check{s}\in[s,t]$ such that $\gamma(\check{s})\in\partial^c\mathbf{C}\cap\Gamma_1'$, which is impossible since $\Gamma_1'\subset\partial\mathbf{C}\setminus\partial^c\mathbf{C}$. Thus all characteristic lines that intersect Γ_1' must intersect Γ_2' as well. □

Remark 6.21. Remark that in Lemma 6.20, for any $x\in\Gamma_1'$, $y\in\Gamma_2'$, the line segment $[x,y]\not\subset\partial\mathbf{C}\cap\partial\Omega_p$. In other words, the boundary $\partial\mathbf{C}\cap\partial\Omega_p$ cannot be flat around $\partial^c\mathbf{C}$. Assuming otherwise, consider the Lipschitz parameterization γ of $\partial\mathbf{C}$ and assume, for example, that $x=\gamma(s)$, $y=\gamma(t)$ with $s<t$. Take L_x (resp. L_y) be the associated characteristic lines. Then, L_x must intersect Γ_2' at $\gamma(t')$ with $t'>t$ while L_y must intersect Γ_1' at $\gamma(s')$ with $s'<s$. But then, L_x and L_y must intersect in $\mathbf{C}$, which is impossible by Proposition 5.4. ¶

Remark 6.22. Theorem 6.12 can be rephrased as follows: the non-characteristic exterior boundary $\partial\mathbf{C}\cap\partial\Omega_p\setminus\partial^c\mathbf{C}$ has always exactly two connected components, denoted hereafter by Γ_1 and Γ_2 and all characteristic lines that intersect $\partial\mathbf{C}$ must intersect both Γ_1 and Γ_2. ¶

The particular geometrical structure of the connected components of $\mathscr{C}$ allows one to improve the continuity of the Cauchy stress σ up to the (non characteristic) boundary.

Theorem 6.23. *Let $\mathbf{C}$ be a connected component of $\mathscr{C}$ with nonempty interior. Then σ is continuous in $\overline{\mathbf{C}}\setminus\partial^c\mathbf{C}$.*

Proof. We already know that σ is locally Lipschitz continuous on $\mathbf{C}$. It thus remains to prove that σ is continuous on $\partial\mathbf{C}\cap\partial\Omega_p\setminus\partial^c\mathbf{C}$. According to Theorem 6.12 and Remark 6.22, we know that $\partial\mathbf{C}\cap\partial\Omega_p\setminus\partial^c\mathbf{C}$ has two connected components Γ_1 and Γ_2.

Let $x\in\partial\mathbf{C}\cap\partial\Omega_p\setminus\partial^c\mathbf{C}$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{\mathbf{C}}\setminus\partial^c\mathbf{C}$ such that $x_n\to x$. Without loss of generality, we can assume that $x\in\Gamma_1$. Since by Theorem 6.12 L_{x_n} must intersect both Γ_1 and Γ_2, there exists $a_n\in L_{x_n}\cap\Gamma_1$ and $b_n\in L_{x_n}\cap\Gamma_2$ which, up to a subsequence, satisfy $a_n\to a\in\Gamma_1$ and $b_n\to b\in\Gamma_2$. As a consequence, the closed segment $S_n:=L_{x_n}\cap\overline{\Omega}_p$ converges in the sense of Hausdorff to the segment $S=[a,b]$.

Case I: Assume first that there exists $\delta>0$ such that for all $n\in\mathbb{N}$,

$$\max_{z\in S_n}\mathrm{dist}(z,\partial\mathbf{C}\cap\partial\Omega_p)\geq\delta,$$

then there exists $z_n \in S_n$ such that $\operatorname{dist}(z_n, \partial\mathbf{C} \cap \partial\Omega_p) \geq \delta$ and, up to a subsequence, $z_n \to z$ for some $z \in S$ with $\operatorname{dist}(z, \partial\mathbf{C} \cap \partial\Omega_p) \geq \delta$. Since $x_n \in L_{z_n}$, there exists $\theta_n \in \mathbb{R}$ such that

$$x_n = z_n + \theta_n \sigma^{\perp}(z_n).$$

Note that, up to a further subsequence, $\theta_n \to \theta \in \mathbb{R}$ and thus, by continuity of σ in Ω_p, we have $x = z + \theta\sigma^{\perp}(z)$ which ensures that $x \in L_z$. Thus, using that σ is constant along characteristics and, once again, the continuity of σ in Ω_p, we get that

$$\sigma(x_n) = \sigma(z_n) \to \sigma(z) = \sigma(x).$$

Case II: Assume next that, up to a subsequence

$$\max_{z \in S_n} \operatorname{dist}(z, \partial\mathbf{C} \cap \partial\Omega_p) \to 0.$$

By Hausdorff convergence, for all $z \in S$, there exists a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in S_n$ and $z_n \to z$. Thus, $\operatorname{dist}(z_n, \partial\mathbf{C} \cap \partial\Omega_p) \to 0$ which ensures that $S \subset \partial\mathbf{C}$. Then $S = [a, b]$ is a line segment contained $\partial\mathbf{C} \cap \partial\Omega_p$ with $a \in \Gamma_1$ and $b \in \Gamma_2$, a contradiction with Remark 6.21. This second case is therefore impossible. □

The following result is analogous to Proposition 6.3. It gives a uniqueness result for the displacement in some portion of $\mathbf{C}$ whose boundary intersects the boundary of the domain Ω.

Proposition 6.24. *Let $\mathbf{C}$ be a connected component of $\mathscr{C}$ with nonempty interior. Extend u by w outside Ω. Then, $u^+ = u^-$ $\mathcal{H}^1$-a.e. on $\partial\mathbf{C} \cap \partial\Omega_p \setminus \partial^c\mathbf{C}$ and, in particular, $u = w$ $\mathcal{H}^1$-a.e. on $\partial\mathbf{C} \cap \partial\Omega_p \cap \partial\Omega \setminus \partial^c\mathbf{C}$.*

Proof. In the current setting, $\partial\mathbf{C} \cap \partial\Omega_p \setminus \partial^c\mathbf{C}$ has two connected components Γ_1 and Γ_2 which are $\mathcal{C}^1$ curves. According to Remark 3.1, $\sigma \cdot \nu = \pm 1$ $\mathcal{H}^1$-a.e. on $\partial\mathbf{C} \cap \partial\Omega_p \cap J_u \setminus \partial^c\mathbf{C}$. Theorem 6.23 ensures that σ is continuous in $\overline{\mathbf{C}} \setminus \partial^c\mathbf{C}$. We thus deduce that $\sigma \cdot \nu$ coincides $\mathcal{H}^1$-a.e. with the usual scalar product of σ and ν on $\partial\mathbf{C} \cap \partial\Omega_p \setminus \partial^c\mathbf{C}$. Therefore, $\nu = \pm\sigma$ $\mathcal{H}^1$-a.e. on $\partial\mathbf{C} \cap \partial\Omega_p \cap J_u \setminus \partial^c\mathbf{C}$ and by continuity of σ and ν on $\partial\mathbf{C} \cap \partial\Omega_p \setminus \partial^c\mathbf{C}$, it follows that $\nu = \pm\sigma$ everywhere on $\partial\mathbf{C} \cap \partial\Omega_p \cap J_u \setminus \partial^c\mathbf{C}$.

Assume that $\mathcal{H}^1(\partial\mathbf{C} \cap \partial\Omega_p \cap J_u \setminus \partial^c\mathbf{C}) > 0$, we can find a point $x_0 \in \partial\mathbf{C} \cap \partial\Omega_p \cap J_u \setminus \partial^c\mathbf{C}$ such that, up to a change of sign, $\nu(x_0) = \sigma(x_0)$. It thus follows that L_{x_0} is tangent to $\mathbf{C}$ at x_0 which is in contradiction with the fact that $x_0 \notin \partial^c\mathbf{C}$. Therefore, $\mathcal{H}^1(\partial\mathbf{C} \cap \partial\Omega_p \cap J_u \setminus \partial^c\mathbf{C}) = 0$ and thus, $u^+ = u^-$ $\mathcal{H}^1$-a.e. on $\partial\mathbf{C} \cap \partial\Omega_p \setminus \partial^c\mathbf{C}$.

Using the Dirichlet boundary condition, we get that $u = w$ $\mathcal{H}^1$-a.e. on $\partial\mathbf{C} \cap \partial\Omega_p \cap \partial\Omega \setminus \partial^c\mathbf{C}$. □

Remark 6.25. If Ω was a convex domain and if σ was such that $|\sigma| \equiv 1$ in Ω, then, $\Omega = \Omega_p$ would be the disjoint union of characteristic lines L_{x_λ} (see Proposition 6.5), of countably many boundary fans $\mathbf{F}_{\bar{z}_i}$, and of countably many connected components $\mathbf{C}_j$ (with non empty interior) such that every point in $\partial\mathbf{C}_j \cap \partial\Omega \setminus \partial^c\mathbf{C}_j$ is traversed by a characteristic line (see Theorem 6.12). Since all characteristic lines cut $\partial\Omega$ into two distinct points by convexity of Ω, then u is entirely determined by w in each fan $\mathbf{F}_{\bar{z}_j}$ according Proposition 6.3. Similarly, by virtue of Proposition 6.24, $u(x) = w(x)$ for $\mathcal{H}^1$-a.e. $x \in \partial\mathbf{C}_j \cap \partial\Omega \setminus \partial^c\mathbf{C}_j$.

Since it is generically not so that the values of w at both points of $L_x \cap \partial\Omega := \{a_x, b_x\}$ should match, u should experience a jump at some point $\bar{x} \in L_x \cap \Omega$ with $\sigma(\bar{x})$ as normal direction (see Remark 3.1) which is not possible since $\sigma(\bar{x})$ is normal to L_x. Therefore, in the scalar case with a Dirichlet boundary condition over the entire boundary, a convex body cannot be entirely plastified unless it is made only of boundary fans.

However, as seen in the example in the introduction, the situation is different in the case of mixed boundary conditions. ¶

6.3. **Miscellanea.**

6.3.1. *Exterior fans.* In contrast with the case of boundary fans, it might happen that two characteristic lines which intersect outside $\overline{\Omega}_p$ will not generate an exterior fan. In particular, there is no analogue to Lemma 6.1. We thus need to force this property by introducing the (possibly empty) set

$$\mathcal{F}^{\rm ext} := \Big\{ \bar z \in \mathbb{R}^2 \setminus \overline{\Omega}_p : \exists\, x, y \in \Omega_p \text{ with } y \neq x \text{ such that } L_x \cap L_y = \{\bar z\} \text{ and } \bar z \in L_z \text{ for all } z \in (\bar z + C(x-\bar z, y-\bar z)) \cap \Omega_p \Big\}.$$

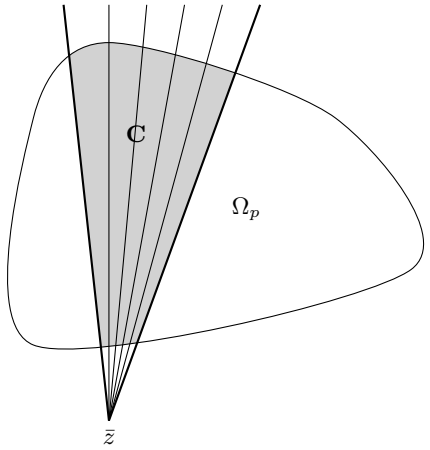

FIGURE 13. An exterior fan with apex $\bar z$ in the case $\#(I) = 2$

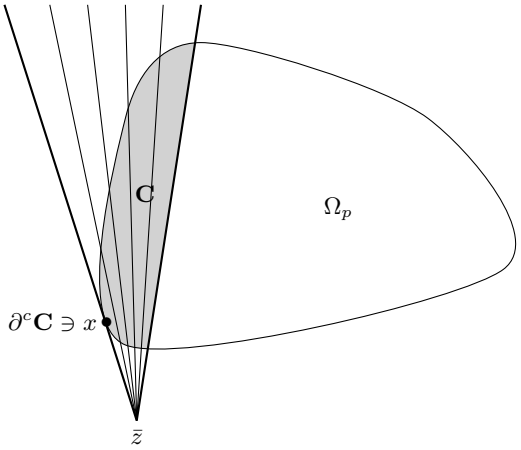

FIGURE 14. An exterior fan with apex $\bar z$ in the case $\#(I) = 1$

For all $\bar z \in \mathcal{F}^{\rm ext}$, there exists a maximal open set $(\bar z + C(x-\bar z, y-\bar z)) \cap \Omega_p$, for some x and $y \in \overline{\Omega}_p$, which is denoted by $\mathbf{F}_{\bar z}$, and which is called an *exterior fan*. Arguing exactly as in the proof of Theorem 6.2, we obtain the following rigidity result inside exterior fans.

Proposition 6.26. *If $\mathbf{C}$ is a connected component of $\mathscr{C}$ with nonempty interior such that $\mathbf{C} = \overline{\mathbf{F}}_{\bar z} \cap \Omega_p$, for some $\bar z \in \mathcal{F}^{\rm ext}$, then there exists $\alpha \in \{-1, 1\}$ such that*

$$\sigma(x) = \alpha \frac{(x-\bar z)^\perp}{|x-\bar z|} \quad \text{for all } x \in \overline{\mathbf{F}}_{\bar z} \cap \Omega_p$$

and

$$u(x) = \alpha h\left(\frac{(x-\bar z)_2}{(x-\bar z)_1}\right) \text{ for all } x \in \mathbf{F}_{\bar z},$$

for some nondecreasing function $h : \mathbb{R} \to \mathbb{R}$.

Remark 6.27. The analogue of Proposition 6.3 also holds true for exterior fans and there can be no jumps on $\partial \mathbf{F}_{\bar z} \cap C_{\bar z}$, where $C_{\bar z}$ is the maximal open cone such that $\mathbf{F}_{\bar z} = C_{\bar z} \cap \Omega_p$. Also, $u = w$ $\mathcal{H}^1$-a.e. on $\partial \mathbf{F}_{\bar z} \cap C_{\bar z} \cap \partial\Omega$. ¶

6.3.2. *Constant zones.* Constant zones have a special structure.

Proposition 6.28. *If $\mathbf{C}$ is a connected component of $\mathscr{C}$ with nonempty interior on which $\sigma = \bar\sigma$ is constant, then $u(x) = \bar\sigma \cdot x + v(\bar\sigma \cdot x)$ for all $x \in \mathring{\mathbf{C}}$ for some non decreasing function v, and, if $\partial\mathbf{C} \cap \partial\Omega_p$ has two connected components, then both characteristic lines L_1 and L_2 lying on $\partial\mathbf{C} \cap \Omega_p$ are parallel to $\mathbb{R}\bar\sigma^\perp$.*

Proof. Assume for simplicity that $\bar{\sigma} = e_1 = (1,0)$. Addressing the last part of the Proposition, if one of both characteristic lines, say L_1, lying on $\partial\mathbf{C} \cap \Omega_p$ is not parallel to $e_2 = (0,1)$, then there exists $x \in \mathring{\mathbf{C}}$ such that L_x intersects L_1 inside Ω_p which is absurd in view of Proposition 5.4.

Further, with the help of (6.2),

$$D_2|p| = \bar{\sigma}^{\perp} \cdot D|p| = 0 \quad \text{ in } \mathcal{D}'(\mathring{\mathbf{C}}),$$

which means that $|p|$ is independent of x_2. As a consequence, $|p| = \mu_{x_1} \otimes \mathcal{L}^1_{x_2}$ for some nonnegative measure $\mu = \mu_{x_1} \in \mathcal{M}(\mathbb{R})$, *i.e.*,

$$\int_{\mathring{\mathbf{C}}} \varphi \, d|p| = \int_{\mathring{\mathbf{C}}} \varphi(x_1, x_2) \, d\mu(x_1) \, dx_2 \quad \text{ for all } \varphi \in \mathcal{C}_c(\mathring{\mathbf{C}}).$$

This implies, in view of (6.1), that $Du = e_1(1 + \mu_{x_1} \otimes \mathcal{L}^1_{x_2})$ on $\mathring{\mathbf{C}}$. Hence $u(x) = x_1 + v(x_1)$ for a.e. $x \in \mathring{\mathbf{C}}$ for some $v \in BV(\mathbb{R})$ with $Dv = \mu \geq 0$. □

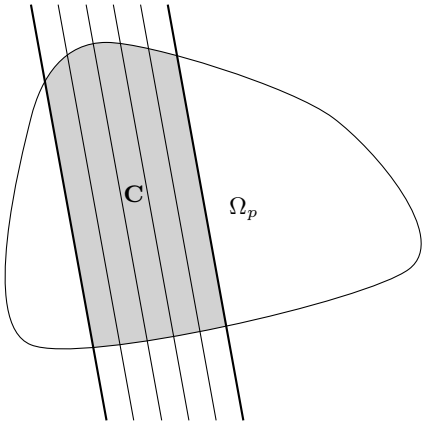

FIGURE 15. A constant zone in the case $\#(I) = 2$

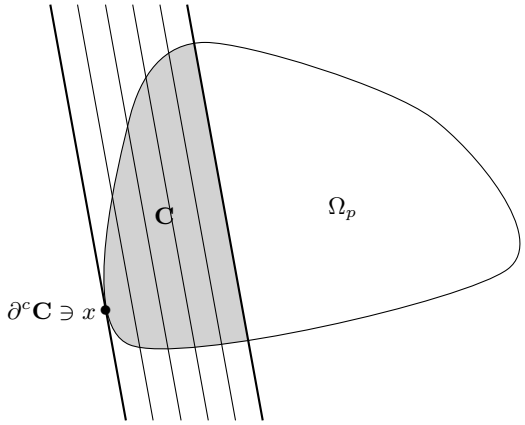

FIGURE 16. A constant zone in the case $\#(I) = 1$

6.3.3. *An example of a non-fan, non-constant structure.* Assume that a connected component $\mathbf{C}$ of Ω_p is such that $\mathbf{C} \subset (0,R)^2$, for some $R > 0$, and that the characteristic lines are given by the one-parameter family

$$L_t = \left\{ (x,y) \in \mathbb{R}^2 : \; y = \frac{x}{t} - t \right\}, \quad t > 0.$$

Note that this family of lines does not intersect at a single point so that it does not define an exterior fan (see Figure 17). However, we are going to construct an example of smooth solutions (u, σ, p) of

$$\begin{cases} \operatorname{div} \sigma = 0, \quad |\sigma| = 1 & \text{in } \mathbf{C}, \\ \nabla u = \sigma + p, \quad p = \sigma|p| & \text{in } \mathbf{C}, \end{cases}$$

such that u and σ are constant along $L_t \cap \mathbf{C}$ for all $t > 0$. Doing so, we will have satisfied all equations (3.2) on $\mathbf{C}$, ignoring the boundary condition on $\partial\Omega$ and recalling Remark 4.5.

Let us start by constructing a (unique) stress $\sigma : \mathbf{C} \to \mathbb{R}^2$. Indeed, since $\sigma(x,y)$ must be an unimodular vector orthogonal to L_t at $(x, y = \frac{x}{t} - t)$, then

$$\sigma(x,y) = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} -1 \\ t \end{pmatrix} \quad \text{ with } \quad t = \frac{-y + \sqrt{y^2 + 4x}}{2},$$

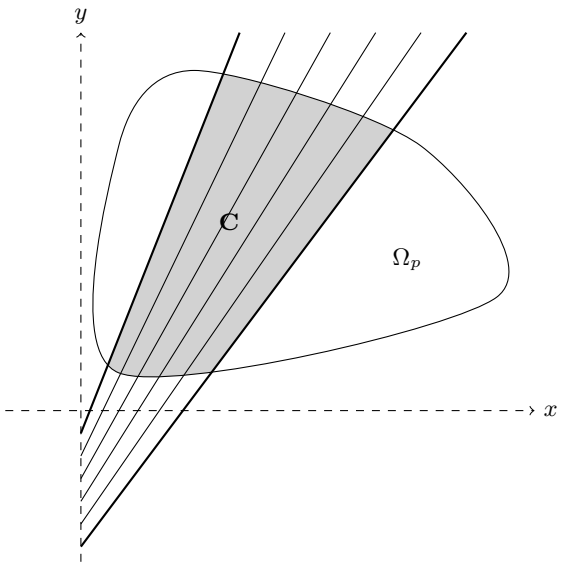

FIGURE 17. An example of a non-fan, non-constant structure

hence

$$\sigma(x,y) := \frac{2}{\sqrt{4+\left(\sqrt{y^2+4x}-y\right)^2}} \begin{pmatrix} -1 \\ \frac{\sqrt{y^2+4x}-y}{2} \end{pmatrix}.$$

Then $\sigma \in \mathcal{C}^\infty(\mathbf{C};\mathbb{R}^2)$ and, by construction, $|\sigma| = 1$ in $\mathbf{C}$. Further a lengthy computation would show that $\operatorname{div}\sigma = 0$ in $\mathbf{C}$.

We now construct the displacement $u : \mathbf{C} \to \mathbb{R}$ as $u(x,y) := f(\sqrt{y^2+4x}-y)$ for all $(x,y) \in \mathbf{C}$, for some smooth decreasing function $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f'(t) \le -\frac{R+t}{\sqrt{4+t^2}} \quad \text{for all } t > 0. \tag{6.11}$$

Note that, by construction, u is constant along all lines L_t for all $t > 0$. We can immediately compute ∇u on $\mathbf{C}$. We get

$$\begin{aligned} \nabla u(x,y) &= -\frac{2}{\sqrt{y^2+4x}} \begin{pmatrix} -1 \\ \frac{\sqrt{y^2+4x}-y}{2} \end{pmatrix} f'(\sqrt{y^2+4x}-y) \\ &= -\left(\frac{4+\left(\sqrt{y^2+4x}-y\right)^2}{y^2+4x} \right)^{1/2} \sigma(x,y)\, f'(\sqrt{y^2+4x}-y). \end{aligned}$$

We then define the plastic strain $p : \mathbf{C} \to \mathbb{R}^2$ by

$$p(x,y) := \nabla u(x,y) - \sigma(x,y) = -\left(\frac{\sqrt{4+\left(\sqrt{y^2+4x}-y\right)^2}}{\sqrt{y^2+4x}} f'(\sqrt{y^2+4x}-y) + 1 \right) \sigma(x,y).$$

Note that, in view of (6.11),

$$-f'(\sqrt{y^2+4x}-y)\frac{\sqrt{4+\left(\sqrt{y^2+4x}-y\right)^2}}{\sqrt{y^2+4x}}-1\geq 0 \quad \text{for all } (x,y)\in\mathbf{C},$$

so that

$$|p(x,y)|=-\left(\frac{\sqrt{4+\left(\sqrt{y^2+4x}-y\right)^2}}{\sqrt{y^2+4x}}f'(\sqrt{y^2+4x}-y)+1\right) \quad \text{for all } (x,y)\in\mathbf{C},$$

and thus $p=\sigma|p|$. We have thus constructed smooth functions u and p satisfying $\nabla u=\sigma+p$ and $p=\sigma|p|$ in $\mathbf{C}$. Note that there are many choices of functions f satisfying (6.11), even considering only non-increasing BV-functions.

Appendix

Let $I\subset\mathbb{R}$ be a bounded open interval. If $x_0\in I$ and $\rho>0$, we denote by $I_{x_0,\rho}=(x_0-\rho,x_0+\rho)$. We recall that a function $f\in L^1(I)$ has *vanishing mean oscillation* in I, and we write $f\in \mathrm{VMO}(I)$, if

$$\frac{1}{\mathcal{L}^1(I\cap I_{x_0,\rho})}\int_{I\cap I_{x_0,\rho}}|f(x)-f_{x_0,\rho}|\,dx\to 0, \quad \text{as } \rho\to 0$$

uniformly with respect to $x_0\in I$, where $f_{x_0,\rho}:=\frac{1}{\mathcal{L}^1(I\cap I_{x_0,\rho})}\int_{I_{x_0,\rho}}f(y)\,dy$ is the average of f on $I\cap I_{x_0,\rho}$.

We first show a (well known) result: the fractional Sobolev space $H^{1/2}(I)$ is continuously embedded into $\mathrm{VMO}(I)$ (see [12, Example 2]).

Lemma A.1. *If $f\in H^{1/2}(I)$, then $f\in \mathrm{VMO}(I)$.*

Proof. Since $f\in H^{1/2}(I)$, then by definition

$$\iint_{I\times I}\frac{|f(x)-f(y)|^2}{|x-y|^2}\,dx\,dy<+\infty.$$

Therefore, by absolute continuity of the integral, for each $\varepsilon>0$ there exists $\delta>0$ such that for every Lebesgue measurable set $A\subset I\times I$ with $\mathcal{L}^2(A)\leq\delta$, then

$$\iint_{A}\frac{|f(x)-f(y)|^2}{|x-y|^2}\,dx\,dy\leq\varepsilon^2.$$

Let $x_0\in I$ and $\rho\leq\sqrt{\delta}/2$ so that $\mathcal{L}^2(I_{x_0,\rho}\times I_{x_0,\rho})=4\rho^2\leq\delta$. According to the triangle and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}\frac{1}{\mathcal{L}^1(I\cap I_{x_0,\rho})}\int_{I\cap I_{x_0,\rho}}|f(x)-f_{x_0,\rho}|\,dx &\leq \frac{1}{\mathcal{L}^1(I\cap I_{x_0,\rho})^2}\iint_{(I\cap I_{x_0,\rho})^2}|f(x)-f(y)|\,dx\,dy\\ &\leq \left(\frac{1}{\mathcal{L}^1(I\cap I_{x_0,\rho})^2}\iint_{(I\cap I_{x_0,\rho})^2}|f(x)-f(y)|^2\,dx\,dy\right)^{1/2}.\end{aligned}$$

If x and $y\in I\cap I_{x_0,\rho}$, then $|x-y|\leq\mathcal{L}^1(I\cap I_{x_0,\rho})$, hence

$$\frac{1}{\mathcal{L}^1(I\cap I_{x_0,\rho})}\int_{I_{x_0,\rho}}|f-f_{x_0,\rho}|\,dx\leq\left(\iint_{(I\cap I_{x_0,\rho})^2}\frac{|f(x)-f(y)|^2}{|x-y|^2}\,dx\,dy\right)^{1/2}\leq\varepsilon,$$

which proves that $f \in \mathrm{VMO}(I)$. □

We now show that a characteristic function which belongs to $H^{1/2}(I)$ must be constant.

Lemma A.2. *Let $\chi = \mathbf{1}_A$ for some Lebesgue measurable set $A \subset I$. If $\chi \in H^{1/2}(I)$ then either $\mathcal{L}^1(A) = 0$ or $\mathcal{L}^1(I \setminus A) = 0$.*

Proof. For $\mathcal{L}^1$-a.e. $y \in I \cap I_{x,\rho}$ we have that $\mathrm{dist}(\chi_{x,\rho}, \{0,1\}) \le |\chi_{x,\rho} - \chi(y)|$. Lemma A.1 implies that $\chi \in \mathrm{VMO}(I)$. Therefore, integrating over $I \cap I_{x,\rho}$, we get

$$\mathrm{dist}(\chi_{x,\rho}, \{0,1\}) \le \frac{1}{\mathcal{L}^1(I \cap I_{x,\rho})} \int_{I \cap I_{x,\rho}} |\chi - \chi_{x,\rho}|\, dy \to 0$$

as $\rho \to 0$, uniformly with respect to $x \in I$. In particular, there exists $\rho_0 > 0$ such that $\mathrm{dist}(\chi_{x,\rho}, \{0,1\}) < 1/3$ for all $0 < \rho < \rho_0$ and all $x \in I$. Let $\rho < \rho_0$. Since the function $x \in I \mapsto \chi_{x,\rho}$ is continuous, then, either $\chi_{x,\rho} = 0$ for all $x \in I$, or $\chi_{x,\rho} = 1$ for all $x \in I$. But, since $\chi_{x,\rho} \to \chi(x)$ as $\rho \to 0$ for every Lebesgue point $x \in I$ of χ, then, either $\chi = 0$ a.e. in I, or $\chi = 1$ a.e. in I. □

We next prove that a piecewise constant function on a Lipschitz graph cannot belong to $H^{1/2}$.

Lemma A.3. *Let $\Gamma = \Gamma^- \cup \Gamma^+$ where $\Gamma^\pm$ are segments in $\mathbb{R}^2$ such that $\overline{\Gamma}^+ \cap \overline{\Gamma}^-$ is a single point. Let $f : \Gamma \to \mathbb{R}$ be such that $f = \alpha$ on Γ^- and $f = \beta$ on Γ^+ where $\alpha \neq \beta$. Then $f \notin H^{1/2}(\Gamma)$.*

Proof. Without loss of generality, we suppose that $\Gamma \supset \{(x_1, x_2) \in \mathbb{R}^2 :\ -L < x_1 < L,\ x_2 = a|x_1|\}$ where $L > 0$, $a > 0$, $\Gamma^- = \{(x_1, x_2) \in \mathbb{R}^2 :\ -L < x_1 < 0,\ x_2 = -ax_1\}$ and $\Gamma^+ = \{(x_1, x_2) \in \mathbb{R}^2 :\ 0 < x_1 < L,\ x_2 = ax_1\}$. Let us show that

$$\iint_{\Gamma \times \Gamma} \frac{|f(x) - f(y)|^2}{|x - y|^2}\, d\mathcal{H}^1(x)\, d\mathcal{H}^1(y) = +\infty.$$

Indeed, using the definition of a curvilinear integral and a change of variables

$$\begin{aligned}
&\iint_{\Gamma \times \Gamma} \frac{|f(x) - f(y)|^2}{|x - y|^2}\, d\mathcal{H}^1(x)\, d\mathcal{H}^1(y) \\
&\qquad = 2|\alpha - \beta|^2 \iint_{\Gamma^- \times \Gamma^+} \frac{d\mathcal{H}^1(x)\, d\mathcal{H}^1(y)}{|x - y|^2} \\
&\qquad = 2(1 + a^2)|\alpha - \beta|^2 \iint_{(-L,0) \times (0,L)} \frac{dx_1\, dy_1}{(x_1 - y_1)^2 + a^2(x_1 + y_1)^2} \\
&\qquad = 2(1 + a^2)|\alpha - \beta|^2 \iint_{(0,L)^2} \frac{ds\, dt}{(s + t)^2 + a^2(s - t)^2}.
\end{aligned}$$

Since $(s + t)^2 + a^2(s - t)^2 \le 2(1 + a^2)(s^2 + t^2)$, we deduce that

$$\iint_{\Gamma \times \Gamma} \frac{|f(x) - f(y)|^2}{|x - y|^2}\, d\mathcal{H}^1(x)\, d\mathcal{H}^1(y) \ge |\alpha - \beta|^2 \iint_{(0,L)^2} \frac{ds\, dt}{s^2 + t^2} \ge |\alpha - \beta|^2 \iint_{D_L} \frac{ds\, dt}{s^2 + t^2},$$

where $D_L := (0, L)^2 \cap B_L(0)$ is the right upper quarter of disk. Using a change of variables in polar coordinates, we get that

$$\iint_{D_L} \frac{ds\, dt}{s^2 + t^2} = \frac{\pi}{2} \int_0^L \frac{dr}{r} = +\infty,$$

which completes the proof of the result. □

Acknowledgements

This research was supported by the National Science Fundation Grants DMS-1615839 and DMS/DMREF-1922371 and was completed in part while the first author was visiting the Courant Institute in April 2019 with the support of Grant DMS-1615839.

References

[1] L. Ambrosio: Transport equation and Cauchy problem for BV vector fields, *Inventiones Math.*, **158** (2004), 227–260.
[2] L. Ambrosio, N. Fusco, D. Pallara: *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, Oxford, 2000.
[3] G. Anzellotti: Pairings between measures and bounded functions and compensated compactness, *Ann. Mat. Pura Appl.*, **135**-4 (1983), 293–318.
[4] G. Anzellotti: On the existence of the rates of stress and displacements for Prandtl-Reuss plasticity, *Quart. Appl. Math.* **41** (1984), 181–208.
[5] G. Anzellotti: On the extremal stress and displacement in Hencky plasticity, *Duke Math. J.*, **51**-1 (1984), 133–147 .
[6] G. Anzellotti, S. Luckhaus: Dynamical evolution of elasto-perfectly plastic bodies, *Appl. Math. Optim.*, **15**-2 (1987), 121–140.
[7] J.-F. Babadjian, C. Mifsud: Hyperbolic structure for a simplified model of dynamical perfect plasticity, *Arch. Rational Mech. Anal.* **223**-2 (2017), 761–815.
[8] J.-F. Babadjian, M. G. Mora: Stress regularity in quasi-static perfect plasticity with a pressure dependent yield criterion, *J. Differential Eq.* **264**-8 (2018), 5109–5151.
[9] J.M. Ball: Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rat. Mech. Anal.*, **63**-4 (1977), 337–403.
[10] A. Bensoussan, J. Frehse: Asymptotic behaviour of Norton-Hoff's law in plasticity theory and H^1 regularity, *Boundary value problems for partial differential equations and applications, RMA Res. Notes Appl. Math.*, **29** (1993), 3–25.
[11] P. Bochard, P. Pegon: Kinetic selection principle for curl-free vector fields of unit norm, *Comm. Partial Diff. Eq.*, **42** (2017), 1375–1402.
[12] H. Brezis, L. Nirenberg: Degree theory and BMO. I. Compact manifolds without boundaries, *Selecta Math. (N.S.)*, **1**-2 (1995), 197–263.
[13] G.-Q. Chen, H. Frid: Divergence-measure fields and hyperbolic conservation laws, *Arch. Ration. Mech. Anal.*, **147**-2 (1999), 89–118.
[14] G. Crippa, C. De Lellis: Estimates for transport equations and regularity of the DiPerna-Lions flow, *J. Reine Angew. Math.*, **616** (2008), 15–46.
[15] G. Dal Maso, A. De Simone, M.G. Mora: Quasistatic evolution problems for linearly elastic perfectly plastic materials, *Arch. Rat. Mech. Anal.*, **181**-2 (2006), 237–291.
[16] G. David: *Singular sets of minimizers for the Mumford-Shah functional*, Progress in Mathematics (Volume 233), Birkhaüser Verlag, Basel, 2005.
[17] C. De Lellis, L. Székelyhidi Jr.: The Euler equations as a differential inclusion. *Annals Math.*, **170**-3 (2009), 1417–1436.
[18] A. Demyanov: Regularity of stresses in Prandtl-Reuss perfect plasticity, *Calc. Var. Partial Differential Equations*, **34**-1 (2009), 23–72.
[19] A. De Simone, R. V. Kohn, S. Müller, F. Otto: A compactness result in the gradient theory of phase transitions, *Proc. Roy. Soc. Edinburgh*, **131** (2001), 833–844.
[20] R. J. Di Perna, P. L. Lions: Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.* **98** (1989), 511–547.
[21] L.C. Evans, R.F. Gariepy: *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.
[22] I. Fonseca, W. Gangbo: *Degree theory in analysis and applications*, Oxford lecture series in mathematics and its applications 2, Clarendon Press, Oxford, 1995.
[23] G. A. Francfort, A. Giacomini: Small-strain heterogeneous elastoplasticity revisited, *Comm. Pure Appl. Math.*, **65**-9 (2012), 1185–1241.
[24] G.A. Francfort, A. Giacomini, J.-J. Marigo: The taming of plastic slips in von Mises elasto-plasticity, *Interfaces Free Bound.*, **17** (2015), 497–516.
[25] G.A. Francfort, A. Giacomini, J.-J. Marigo: A case study for uniqueness of elasto-plastic evolutions: The bi-axial test, *J. Math. Pures Appl.*, **105** (2016), 198–227.

[26] G.A. Francfort, A. Giacomini, J.-J. Marigo: The elasto-plastic exquisite corpse: A Suquet legacy, *J. Mech. Phys. Sol.*, **97** (2016), 125–139.
[27] A. Henrot, M. Pierre: *Shape variation and optimization. A geometrical analysis.* EMS Tracts in Mathematics, 28, European Mathematical Society (EMS), Zürich, 2018.
[28] R. Ignat: Two-dimensional unit-length vector fields of vanishing divergence, *J. Funct. Anal.*, **262**-8 (2012), 3465–3494.
[29] P.-E. Jabin, F. Otto, B. Perthame: Line-energy Ginzburg-Landau models: zero-energy states, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, **1**-1 (2002), 187–202.
[30] R. Kohn, R. Temam: Dual spaces of stresses and strains, with applications to Hencky plasticity, *Appl. Math. Optim.*, **10**-1 (1983), 1–35.
[31] J. Lubliner: *Plasticity Theory*, Macmillan Publishing Company, New York, 1990.
[32] A. Mielke: Evolution of rate-independent systems, In A. Dafermos and E. Feireisl, editors, *Evolutionary equations. Vol. II*, Handb. Differ. Equ., 461–559, Elsevier/North-Holland, Amsterdam, 2005.
[33] M.G. Mora: Relaxation of the Hencky model in perfect plasticity, *J. Math. Pures Appl.*, **9-106**-4 (2016), 725–743.
[34] F. Murat, C. Trombetti: A chain rule formula for the composition of a vector-valued function by a piecewise smooth function, *Boll. U. Mat. Ital., Serie 8*, **6-B**-3 (2003), 581–595.
[35] G.A. Seregin: On the differentiability of extremals of variational problems of the mechanics of ideally elasto-plastic media, (Russian) *Differentsial'nye Uravneniya*, **23**-11 (1987), 1981–1991, 2022.
[36] P.-M. Suquet: Sur les équations de la plasticité: existence et régularité des solutions, *J. Mécanique*, **20**-1 (1981), 3–39.
[37] R. Temam: *Problèmes Mathématiques en Plasticité*, Gauthier-Villars, Paris, 1983.

(J.-F. Babadjian) Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France.
E-mail address: jean-francois.babadjian@universite-paris-saclay.fr

(G.A. Francfort) Université Paris-Nord, LAGA, Avenue J.-B. Clément, 93430 - Villetaneuse, France & Courant Institute, New York University, 251 Mercer St., NY-NY10012, USA
E-mail address: gilles.francfort@univ-paris13.fr, gilles.francfort@cims.nyu.edu