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Fast and precise computation of some Euler products

S. Ettahri, O. Ramaré and L. Surel

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Abstract

`File LoeschianConstant-09.tex` A003136 (Voir A301430)

1 Introduction

At the beginning of our query lie two constants, namely

$$\alpha_3 = \frac{1}{2^{1/2}3^{1/4}} \prod_{p\equiv 2[3]} \left(1-\frac{1}{p^2}\right)^{-1/2} \tag{1}$$

and

$$\alpha_1 = \frac{3^{1/4}\sqrt{\pi}}{2^{5/4}} \frac{\log(2+\sqrt{3})^{1/4}}{\Gamma(1/4)} \prod_{p\equiv 5,7,11[12]} \left(1-\frac{1}{p^2}\right)^{-1/2}. \tag{2}$$ {eq:12}

Both occur in number theory as densities. The number of integers n of the shape $n = x^2 - xy + y^2$, where x and y are integers (these are the so-called Loeschian numbers, see sequence A003136 of the [2]), is given by

$$N(x) = \alpha_3 \frac{x(1+o(1))}{\sqrt{\log x}}. \tag{3}$$ {eq:1}

This accounts for our interest in the first constant. The second one occurs because the number of Loeschian numbers that are also sums of two squares (see sequence A301430 of the [2]) is given by

$$N'(x) = \alpha_1 \frac{x(1+o(1))}{\sqrt{\log x}}.$$

The question we address here is devising a fast manner to compute the intervening Euler products. From sequence A301429 of the [2], we know that $\alpha_3 = 0.638909\ldots$ but we would like (much!) more digits. Similarly the constant $\alpha_1 = 0.30231614235\ldots$. We prove here that

Theorem 1.1. *We have*

$$\alpha_3 = 0.63890\,94054\,45343\,88225\,49426\,74928\,24509\,37549\,75508\,02912$$
$$33454\,21692\,36570\,80763\,10027\,64965\,82468\,97179\,11252\,86643\cdots$$

[2] O. F. Inc., 2019, *The On-Line Encyclopedia of Integer Sequence.*

and

$$\alpha_1 = 0.30231\,61425\,69799\,15540\,44179\,05922\,52433\,94675\,42586\,71306$$
$$13715\,35597\,72574\,35560\,81500\,06128\,08597\,69222\,39895\,08293\cdots$$

As a matter of fact, our method is more general and allows one to compute Euler products of the shape

$$\prod_{p\in\mathcal{A} \bmod q}(1-p^{-s})$$

for any s with $\Re s > 1$ and some subsets $\mathcal{A}$ of $(\mathbb{Z}/q\mathbb{Z})^\times$. The history of such formulae for scientific computations starts with D. Shanks in [7, equation (15)]. D. Shanks's approach has been put in a general context by P. Moree & D. Osburn in [3, equation (3.2)]. As a matter of fact, an accurate value of α_3 already follows from this paper, but not the one for α_1. The formulae we prove have a wider reach, though they fail to exhaust the problem. A subset $\mathcal{A}$ of $(\mathbb{Z}/q\mathbb{Z})^\times$ is said to be a *lattice-invariant* class is all its elements generate the same subgroup (see Definition 3.1 below). Here is a consequence of our approach.

Theorem 1.2. *Let q be some modulus and $\mathcal{A}$ be a* lattice-invariant *class of $(\mathbb{Z}/q\mathbb{Z})^\times$. For every $s > 1$, the product*

$$\prod_{p\ mod\ q\in\mathcal{A}}(1-p^{-s})$$

can be computed in double-exponential time.

This theorem applies in particular to $\prod_{p\equiv 1[q]}(1-p^{-s})$ and to $\prod_{p\equiv -1[q]}(1-p^{-s})$ and this is enough to compute α_1 and α_3. We end this paper with numerical examples.

The material of this paper has been used to write the script

`LatticeInvariantEulerProducts.sage`

which we shorten below in `LIEP.sage` and which can be found on the second author website. We give some details about this script when developing the proof below.

Precise statement of the main result

Let $q > 1$ be a modulus. We set G_0 be a subgroup of $G = (\mathbb{Z}/q\mathbb{Z})^\times$ and $G_0^\perp$ be the subgroup of characters that take the value 1 on G_0. Let $s > 1$ be a real number.

Remark 1.3. We assume that s is a real number while in fact, any complex value with real part > 1 would do. We do so because our prsentation relies on logarithms and this allows us to use the confortable notation of linear analys. If we skip this step and stick to the multiplicative setting, all our estimates still hold true.

[7] D. Shanks, 1964, "The second-order term in the asymptotic expansion of $B(x)$".
[3] P. Moree and R. Osburn, 2006, "Two-dimensional lattices with few distances".

Let $P \geqslant 2$ be a parameter. We shall compute directly what happens for the primes $< P$. We define accordingly

{eq:11}
$$L_P(s,\chi) = \prod_{p \geqslant P} (1 - \chi(p)/p^s). \tag{4}$$

We define, for any positive integer t:

{defgammaGzerot}
$$\gamma_s(G_0, t) = \log \prod_{\chi \in G_0^\perp} L_P(ts, \chi). \tag{5}$$

The parameter P has disappeared from our notation and the reader may stick with $P = 2$. When s is a real number, the number $\prod_{\chi \in G_0^\perp} L_P(ts, \chi)$ is indeed a positive real number, essentially because, when χ belongs to $G_0^\perp$, so does $\overline{\chi}$.

We denote the set of *lattice-invariant* clases by $G^\sharp$ and the set of cyclic subgroups by $\mathscr{G}$. Both sets are in an obvious one-to-one correspondence. We consider the vector

$$\Gamma_s(t) = (\gamma_s(G_0, t))_{G_0 \in \mathscr{G}}. \tag{6}$$ {defGammaoft}

The vector $\Gamma_s(t)$ has cyclic subgroups as indices for its rows. The rather formal expression is computed by the function `GetFormalExpGamma` of the script `LIEP.sage` from the values of the Hurwitz zeta function; by "rather formal", we mean that we have not yet introduced the notion of interval arithmetic and that we did not take the logarithm of the value yet; the numerical computations will deliver complex numbers with a negligible imaginary part, which we have to convert to a real number before applying the logarithm function. We next define

$$V_s(t) = \left(\log \prod_{\substack{p + q\mathbb{Z} \in b, \\ p \geqslant P}} \left(1 - p^{-s}\right) \right)_{b \in G^\sharp}. \tag{7}$$ {defVsoft}

The vector $V_s(t)$ has classes as indices for its rows. We control the size of our vectors with the norm

$$\|W\| = \max_i |W_i| \tag{8}$$ {eq:8}

when W is the vector of coordinates W_i. We define the matrix M_1^{-1} by

$$M_1^{-1}\big|_{i=b, j=K} = \begin{cases} \mu(|\langle b \rangle / K|)/|G/K| & \text{when } K \subset \langle b \rangle, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$ {defL}

It is unusual to define a matrix by its inverse. In the natural course of the proof, a matrix M_1 will occur, whose inverse is the one above; it is computed in Proposition 4.1. The reader will readily check that there are no circularity in our definitions. Let us recall that the *exponent* of G is the maximal order of an element in G and is denoted by $\exp G$. To each divisor $d > 1$ of $\exp G$, we associate the matrix N_d whose columns and rows are indexed by cyclic subgroups of G and whose entries are given by

$$N_d\big|_{i=B_0, j=B_1} = \begin{cases} \mu(|B_0/K|) & \text{if } \exists K / |B_1/K| = d \text{ and } K \subset B_0 \cap B_1, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$ {defNd}

Such a subgroup K if it exists is unique: it is determined by B_1 and d as there are only one subgroup of index d in the cyclic group B_1. Here is our main theorem.

{precise}

Theorem 1.4. *For any integer $r \geqslant 2$, we have*

$$\left\| V_s(1) - \sum_{0 \leqslant v \leqslant r-1} (-1)^v \sum_{d_1 \cdots d_v \leqslant 2^r} \frac{N_{d_1}}{d_1} \cdots \frac{N_{d_v}}{d_v} M_1^{-1} \Gamma_s(d_1 \dots d_v) \right\| \leqslant 2r \left(\frac{|G^\sharp| d(\exp G)}{P} \right)^r \frac{1 + 2^{-r} s^{-1} P}{P^{s 2^r}} \quad (11)$$ {fineq}

where $d_1, \dots, d_r$ are all divisors of $\exp G$ excluding 1.

When $v = 0$, we use $d_1 \dots d_v = 1$ and $N_{d_1} \cdots N_{d_v} = \mathrm{Id}$.

2 A general mechanism

Let us start by presenting the mechanism of Shanks is a somewhat general setting.

{shanks}

Lemma 2.1. *Let $\mathcal{P}$ be a set of prime numbers and let f be a function from $\mathcal{P}$ to $\{\pm 1\}$. For every s with $\Re s > 1$, we have*

$$\prod_{\substack{p \in \mathcal{P}, \\ f(p) = -1}} (1 - p^{-s})^2 = \frac{\prod_{p \in \mathcal{P}} (1 - p^{-s})}{\prod_{p \in \mathcal{P}} (1 - f(p) p^{-s})} \prod_{\substack{p \in \mathcal{P}, \\ f(p) = -1}} (1 - p^{-2s}).$$

Proof. The proof is straighforward. We simply write

$$\prod_{\substack{p \in \mathcal{P}, \\ f(p) = -1}} \frac{(1 - p^{-s})^2}{1 - p^{-2s}} = \prod_{\substack{p \in \mathcal{P}, \\ f(p) = -1}} \frac{1 - p^{-s}}{1 + p^{-s}} = \prod_{\substack{p \in \mathcal{P}, \\ f(p) = -1}} \frac{1 - p^{-s}}{1 - f(p) p^{-s}}$$
$$= \prod_{p \in \mathcal{P}} \frac{1 - p^{-s}}{1 - f(p) p^{-s}}$$

as required. □

Shanks's method is efficient to deal with product of primes belonging to a coset modulo a quadratic character. We generalize it as follows.

{dede}

Lemma 2.2. *Let $q > 1$ be a modulus. We set G_0 be a subgroup of $G = (\mathbb{Z}/q\mathbb{Z})^\times$ and $G_0^\perp$ be the subgroup of characters that take the value 1 on G_0. For any integer b, we define $\langle b \rangle$ to the the subgroup generated by b modulo q. We have*

$$\prod_{\chi \in G_0^\perp} L_P(s, \chi) = \prod_{G_0 \subset K \subset G} \prod_{\substack{p \geqslant P, \\ \langle p \rangle G_0 = K}} \left(1 - p^{-|K/G_0| s}\right)^{-|G/K|}$$

and, for any element $a \notin G_0$ of order 2, we have

$$\prod_{\chi \in G_0^\perp} L_P(s, \chi)^{\chi(a)} = \prod_{\substack{G_0 \subset K \subset G, \\ a \in K}} \prod_{\substack{p \geqslant P, \\ \langle p \rangle G_0 = K}} \left(\frac{(1 - p^{|K/G_0| s/2})^2}{1 - p^{-|K/G_0| s}} \right)^{-|G/K|}$$

where $\hat{G}$ is the set of characters of G.

Case $G_0 = \{1\}$ of the first identity is classical in Dedekind zeta function theory, and can be found in [6, Proposition 13] in a rephrased form. Case $a \neq 1$ will not be required for the general theory. It may however lead quickly to the required result.

Proof. We note that $\prod_{\chi \in G_0^\perp}(1-\chi(p)z)^{\chi(a)} = \prod_{\psi \in \hat{H}}(1-\psi(p)z)^{f(\psi)}$ when $\langle p \rangle = H$ and where

$$f(\psi) = \sum_{\substack{\chi \in G_0^\perp, \\ \chi|H=\psi}} \chi(a). \tag{12}$$ {eq:6}

The condition $\chi \in G_0^\perp$ can also be written as $\chi|G_0 = 1$, hence we can assume that $\psi|(H \cap G_0) = 1$. We write

$$\prod_{\chi \in G_0^\perp}(1-\chi(p)z)^{\chi(a)} = \prod_{\substack{\psi' \in \widehat{HG_0}, \\ \psi'|G_0=1}}(1-\psi(p)z)^{f'(\psi')}$$

where

$$f'(\psi') = \sum_{\substack{\chi \in G_0^\perp, \\ \chi|HG_0=\psi}} \chi(a). \tag{13}$$ {eq:76}

When a does not belong to HG_0, this sum vanishes; otherwise it equals $|G/(HG_0)|\psi'(a)$. The characters of HG_0 that vanish on G_0 are essentially the characters on the cyclic group $(HG_0)/G_0$. We thus have

$$\prod_{\substack{\psi' \in \widehat{HG_0}, \\ \psi'|G_0=1}}(1-\psi(p)z) = 1 - z^{|(HG_0)/G_0|}.$$

When $a^2 \equiv 1[q]$ and $a \notin G_0$, and since $(HG_0)/G_0$ is cyclic, of (even) order h say, the characters are given by $\chi(p^x) = e(cx/h)$ since p is a generator and where c ranges $\{0, \cdots, h-1\}$. We thus have, when $a \in H$,

$$\begin{aligned}
\prod_{\psi' \in \widehat{(HG_0)/G_0}}(1-\psi'(p)z)^{\psi'(a)} &= \prod_{c \bmod h}(1-e(c/h)z)^{e(c/2)} \\
&= \prod_{0 \leqslant d \leqslant (h-2)/2}(1-e(2d/h)z) \prod_{0 \leqslant d \leqslant (h-2)/2}(1-e((2d+1)/h)z)^{-1} \\
&= \frac{1-z^{h/2}}{1-(e(1/h)z)^{h/2}} = \frac{1-z^{h/2}}{1+z^{h/2}} = \frac{(1-z^{h/2})^2}{1-z^h}.
\end{aligned}$$

The reader will readily complete the proof by setting $K = HG_0$. □

Examples

Let us select for G_0 the kernel of a given quadratic character χ_1. The subgroup K can take only two values, G_0 or G. We thus get

$$L(s,\chi_1)L(s,\chi_0) = \prod_{\chi_1(p)=1}(1-p^{-s})^2 \prod_{\chi_1(p)=-1}(1-p^{-2s})$$

[6] J.-P. Serre, 1970, *Cours d'arithmétique.*

which gets converted into

$$L(s,\chi_1)L(s,\chi_0) = L(s,\chi_0)^2 \prod_{\chi_1(p)=-1} (1-p^{-s})^{-2} \prod_{\chi_1(p)=-1} (1-p^{-2s}) \tag{14}$$ {myshanks}

which is what follows from Lemma 2.1. In the same situation, let a be outside G_0. The only choice for K is $K = G$. We find that

$$L(s,\chi_0)/L(s,\chi_1) = \prod_{\chi_1(p)=-1} \frac{(1-p^{-s})^4}{(1-p^{-2s})^2}$$

which gets converted into

$$L(s,\chi_0)/L(s,\chi_1) = \prod_{\chi_1(p)=-1} \frac{(1-p^{-s})^2}{(1-p^{-2s})} = \prod_{\chi_1(p)=-1} \frac{1-p^{-s}}{1+p^{-s}}$$

which is rather trivial.

3 Products obtained

We want to compute Euler products of the shape

$$\prod_{p\in\mathcal{A} \bmod q} (1-1/p^s)$$

for $s > 1$ and some subset $\mathcal{A}$ of $(\mathbb{Z}/q\mathbb{Z})^\times$. Computing $L(s,\chi)$ is easier as it can be reduced to sums over integers is some arithmetic progressions. Equation (14) reduces the computations of $\prod_{p\in\mathcal{A} \bmod q}(1-1/p^s)$ to the one of $\prod_{p\in\mathcal{A} \bmod q}(1-1/p^{2s})$, and we can continue the process. We soon reach $\prod_{p\in\mathcal{A} \bmod q}(1-1/p^{2^N s})$ with a large enough N which can be approximated by $1+\mathcal{O}(2^{-2Ns})$. The object of this section is to devise a setting to understand which sums we relate together.

{m{ℭℓ} **Definition 3.1.** *Two elements g_1 and g_2 of the abelian group G are said to be* lattice-invariant *if and only if they generates the same group.*

The map between the cyclic subgroups of G and the lattice-invariant*-classes which, to a subgroup, associates the subset of its generators, is one-to-one.*

The function `GetLatticeInvariantClasses` of the script `LIEP.sage` gives the two lists: the one of the cyclic subgroups and the one of their generators, ordered similarly and in increasing size of the subgroup.

Any two elements of $(\mathbb{Z}/q\mathbb{Z})^\times$ equivalent according to it cannot be distinguished by using the formulae of Lemma 2.2. Conversely, the question is to know whether we are indeed able to distinguish each class.

We set $G = (\mathbb{Z}/q\mathbb{Z})^\times$. Let $G^\sharp$ be the set of all *lattice-invariant*-equivalence classes. To each class b, we attach the enumerable collection of symbols $(x_b^r)_{r\geqslant 1}$. We shall replace each of them according to

{eq:9}
$$x_b^r \mapsto \log \prod_{\substack{p+q\mathbb{Z}\in b,\\ p\geqslant P}} \big(1-p^{-s}\big). \tag{15}$$

We consider the module of finite formal combinations

$$\sum_{\substack{b\in G^\sharp,\\ r\geqslant 1}} \alpha_{b,r} x_b^r$$

with coefficients $\alpha_{b,r} \in \mathbb{Z}$ and indeterminates x_b^r. The superscript r is *not* a power. We consider the following special elements. Let $G_0 \subset K \subset G$ be two subgroups such that K/G_0 is cyclic. We define

$$g(G_0, K, t) = \sum_{b\in G^\sharp, bG_0=K} x_b^{t|K/G_0|}. \tag{16}$$ {eq:5}

With that, we find that

$$\gamma(G_0, t) = \sum_{G_0\subset K\subset G} |G/K| g(G_0, K, t). \tag{17}$$ {basicrel}

4 Iterating the formulae

The first identity of Lemma 2.2 gives us as many identities as there are subgroups G_0; we know by Definition 3.1 that the number of *lattice-invariant*-classes equals the one of cyclic subgroups. It turns out that it is enough to restrict our attention to cyclic subgroups G_0. Let $\mathscr{G}$ be the subset of such subgroups, which we order by inclusion. On recalling (6), we may then rewrite (17) in the form

$$\Gamma(t) = \sum_{d||G|} M_d V_s(dt) \tag{18}$$ {initeq}

where (this is the case $K = G_0$)

$$M_1\big|_{i=G_0,j=b} = \begin{cases} |G/K| & \text{if } b \subset G_0, \\ 0 & \text{otherwise,} \end{cases} \tag{19}$$ {eq:20}

and, where, when $d > 1$ (i.e. $G_0 \subsetneq K$), we have

$$M_d\big|_{i=G_0,j=b} = \begin{cases} |G/bG_0| & \text{if } |bG_0|/|G_0| = d, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$ {eq:19}

Equation (18) gives us a relation between $M_1 V_s(t)$ and $M_d V_s(dt)$ for several d's that are strictly larger than 1. Our roadmap is to invert the matrix M_1 and to iterate this formula. We compute explicitely M_1^{-1} by using some generalised Moebius inversion, which we first put in place.

The Moebius function associated to $\mathscr{G}$

We follow closely the exposition of Rota in [5]. On the algebra of functions f on couples (K, L) of points of $\mathscr{G}$ such that $K \subset L$ (the so-called *incidence algebra*, see [5, Section 3]), we define the convolution product

$$(f \star g)(K, L) = \sum_{K\subset H\subset L} f(K, H) g(H, L).$$

[5] G.-C. Rota, 1964, "On the foundations of combinatorial theory. I. Theory of Möbius functions".

We consider the $\mathscr{G}$-zeta function which is defined by

$$\zeta_{\mathscr{G}}(K,L) = \begin{cases} 1 & \text{when } K \subset L, \\ 0 & \text{otherwise.} \end{cases}$$

This function is shown to be invertible in the above algebra and its inverse is called the $\mathscr{G}$-Moebius function, denoted by $\mu_{\mathscr{G}}$. By definition, we have the two Moebius inversion formulas:

{eq:21}
$$\sum_{K \subset H \subset L} f(K,H) = g(K,L) \implies f(K,L) = \sum_{K \subset H \subset L} g(K,H)\mu_{\mathscr{G}}(H,L) \tag{21}$$

and

{eq:21b}
$$\sum_{K \subset H \subset L} f(H,L) = g(K,L) \implies f(K,L) = \sum_{K \subset H \subset L} \mu_{\mathscr{G}}(K,H)g(H,L). \tag{22}$$

We end this reminder with a formula giving the value of $\mu_{\mathscr{G}}(K,H)$.

Computing $\mu_{\mathscr{G}}(K,H)$

Let $C_p(K,H)$ be the number of chains of length p going from K to H, i.e. the number of $p+1$-uples $K = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_p = H$. Then (cf [5, Proposition 6])

{eq:22}
$$\mu_{\mathscr{G}}(K,H) = \sum_{p \geqslant 0} (-1)^p C_p(K,H). \tag{23}$$

Since the subgroups of a cyclic group are all cyclic, we only have to consider the chains in H/K. There is one and only one subgroup for each divisor of $|H/K|$, and any two such subgroups L_1 and L_2 are included according to whether their $|L_1||L_2|$ or not. This transfers the problem on a problem on integers. Let $c_p(n)$ be the number of $p+1$-divisibility chains between 1 and n. We have $c_0(n) = \mathbf{1}_{n=1}$ while $c_1(n) = \mathbf{1}_{n \geqslant 2}$ and $c_{p+1}(n) = (c_p \star c_1)(n)$. This proves that $c_p(n) = d_p^*(n)$, the number of p-tuples $(d_1, d_2, \ldots, d_p)$ of divisors of n that are such that $d_i \neq 1$ and $d_1 d_2 \cdots d_p = n$. We have

$$\sum_{n \geqslant 1} d_p^*(n)/n^s = (\zeta(s) - 1)^p$$

and thus the generating series of $\sum_{p \geqslant 0} (-1)^p d_p^*(n)$ is

$$\sum_{p \geqslant 0} (-1)^p (\zeta(s) - 1)^p = \frac{1}{1 + \zeta(s) - 1} = 1/\zeta(s).$$

We have proved that

{eq:23}
$$\mu_{\mathscr{G}}(K,H) = \mu(|H/K|). \tag{24}$$

[5] G.-C. Rota, 1964, "On the foundations of combinatorial theory. I. Theory of Möbius functions".

Inverting the matrix M_1

{compL}

Proposition 4.1. *The matrix M_1 is invertible and the coefficients of its inverse are given by*

$$M_1^{-1}\big|_{i=b,j=K} = \begin{cases} \mu(|\langle b\rangle/K|)/|G/K| & \text{when } K \subset \langle b\rangle, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We find that

$$M_1 V = (|G/K| \sum_{b\subset K} v_b)_K.$$

We replace b by the subroup $B = \langle b\rangle$ it generates. Inverting $f(K) = |G/K| \sum_{B\subset K} v_B$ is done with the Moebius function of $\mathscr{G}$. To do so, simply consider the more general function

$$F(H,K) = |G/K| \sum_{H\subset B\subset K} v^*(H,B) = |G/K|(v^* \star \zeta_{\mathscr{G}})(H,K)$$

where $v^*(H,B) = v_B$. This gets inverted in

$$v^*(H,B) = \sum_{H\subset K\subset B} F(H,K)|G/K|^{-1}\mu_{\mathscr{G}}(K,B)$$

which yield, by specializing $H = \{1\}$

$$v_B = \sum_{K\subset B} f(K)|G/K|^{-1}\mu_{\mathscr{G}}(K,B).$$

We could also have applied [5, Proposition 2 (**)]. This gives us

$$M_1^{-1}\big|_{i=B,j=K} = \begin{cases} \mu_{\mathscr{G}}(K,B)/|G/K| & \text{if } K \subset B, \\ 0 & \text{otherwise.} \end{cases}$$

Our proposition is proved. □

The function `GetM1Inverse` of the script `LIEP.sage` computes M_1^{-1}.

The recursion formula

We start from (18) and deduce that

$$V_s(t) = -\sum_{\substack{d\mid|G|,\\ d\neq 1}} M_1^{-1} M_d V_s(dt) + M_1^{-1}\Gamma(t). \tag{25}$$ {seceq}

We readily find that $N_d = dM_1^{-1}M_d$ is given by (10).

Proof. Indeed we have

$$N_d\big|_{i=B_0,j=B_1} = d \sum_{\substack{K\subset B_0,\\ K\subset B_1,\\ |B_1/K|=d}} \mu(|B_0/K|)|G/K|^{-1}|G/B_1|.$$

This is exactly what we have written in (10). □

As a consequence, we see that only the d that divides the *exponent* of G. The function `GetNds` of the script `LIEP.sage` computes $(N_d)_d$.

$$V_s(t) = -\sum_{\substack{d \mid \exp G,\\ d\neq 1}} \frac{N_d}{d} V_s(dt) + M_1^{-1}\Gamma(t). \tag{26}$$

{trieq}

Unfolding the recursion

Let $z \geqslant 1$ and $r \geqslant 1$ be two parameters. We have

$$\begin{aligned} V_s(t) = (-1)^r &\sum_{d_1\cdots d_r \leqslant z} \frac{N_{d_1}}{d_1}\cdots\frac{N_{d_r}}{d_r} V_s(d_1\dots d_r t) \\ &+ \sum_{1\leqslant v\leqslant r} (-1)^v \sum_{\substack{d_1\cdots d_{v-1}\leqslant z,\\ d_1\cdots d_{v-1}d_v > z}} \frac{N_{d_1}}{d_1}\cdots\frac{N_{d_v}}{d_v} V_s(d_1\dots d_v t) \\ &+ \sum_{1\leqslant v\leqslant r-1} (-1)^v \sum_{d_1\cdots d_v\leqslant z} \frac{N_{d_1}}{d_1}\cdots\frac{N_{d_v}}{d_v} M_1^{-1}\Gamma(d_1\dots d_v t) + M_1^{-1}\Gamma(t) \end{aligned} \tag{27}$$

{recursion}

where $d_1, \dots, d_r$ are all divisors of $\exp G$ excluding 1. We can incorporate the last summand in the one before by considering as the value for $s = 0$.

Proof. Let us prove this formula by recursion. Case $r = 1$ is just (26). Let us see precisely what happens for $r = 2$. We start from

$$V_s(t) = -\sum_{\substack{d_1 \mid \exp G,\\ d_1\neq 1}} \frac{1}{d_1} N_{d_1} V_s(d_1 t) + M_1^{-1}\Gamma(t)$$

which we rewrite as

$$V_s(t) = -\sum_{\substack{d_1 \mid \exp G,\\ d_1\neq 1,\\ d_1\leqslant z}} \frac{N_{d_1}}{d_1} V_s(d_1 t) - \sum_{\substack{d_1 \mid \exp G,\\ d_1\neq 1,\\ d_1> z}} \frac{N_{d_1}}{d_1} V_s(d_1 t) + M_1^{-1}\Gamma(t).$$

We use again this equation on $V_s(d_1 t)$ when $d_1 \leqslant z$, and z/d_1 rather than z, getting

$$\begin{aligned} V_s(t) = &\sum_{\substack{d_1 \mid \exp G,\\ d_1\neq 1,\\ d_1\leqslant z}} \sum_{\substack{d_2 \mid \exp G,\\ d_2\neq 1,\\ d_1 d_2\leqslant z}} \frac{N_{d_1}}{d_1}\frac{N_{d_2}}{d_2} V_s(d_1 d_2 t) \\ &+ \sum_{\substack{d_1 \mid \exp G,\\ d_1\neq 1,\\ d_1\leqslant z}} \sum_{\substack{d_2 \mid \exp G,\\ d_2\neq 1,\\ d_1 d_2> z}} \frac{N_{d_1}}{d_1}\frac{N_{d_2}}{d_2} V_s(d_1 d_2 t) - \sum_{\substack{d_1 \mid \exp G,\\ d_1\neq 1,\\ d_1> z}} \frac{N_{d_1}}{d_1} V_s(d_1 t) \\ &- \sum_{\substack{d_1 \mid \exp G,\\ d_1\neq 1,\\ d_1\leqslant z}} \frac{N_{d_1}}{d_1} M_1^{-1}\Gamma(t) + M_1^{-1}\Gamma(t). \end{aligned}$$

To go from r to $r+1$, we select the divisors d_r that are such that $d_1 d_2 \cdots d_r \leqslant z$ and employ (26) on $V_s(d_1 \cdots d_r t)$. □

Lemma 4.2. *The coefficients of a product $N_{d_1}N_{d_2}\cdots N_{d_v}$ are at most (in absolute value) equal to $|G^\sharp|^{v-1}$, where $G^\sharp$ is the set of* lattice-invariant *classes (which is also the number of cyclic subgroups of G).*

End of the proof of Theorem 1.4

The formula (27) with $t=1$ contains most of our proof.

The number of possible d's is at most the number of divisors of $\exp G$ minus 1, so at most $d(\exp G)$. The coefficients of a typical product $N_{d_1}\cdots N_{d_v}$ are of size at most $|G^\sharp|^{v-1}$, we divide each coefficient by $d_1\cdots d_v$ which is at least z, and we have at most $d(\exp G)^v$ v-tuples $(d_1,\dots,d_v)$. As a consequence, each coordinate, says y, of the vector

$$\sum_{1\leqslant v\leqslant r}(-1)^v\sum_{\substack{d_1\cdots d_{v-1}\leqslant z,\\ d_1\cdots d_{v-1}d_v>z}}\frac{N_{d_1}}{d_1}\cdots\frac{N_{d_v}}{d_v}V_s(d_1\dots d_vt)$$

satisfies

$$|y|\leqslant r\,\frac{\big(|G^\sharp|d(\exp G)\big)^r}{z}\max_{D\geqslant 2^r}\|V_s(Dt)\|.$$

We deal similarly with the coordinates of the vector

$$(-1)^r\sum_{d_1\cdots d_r\leqslant z}\frac{N_{d_1}}{d_1}\cdots\frac{N_{d_r}}{d_r}V_s(d_1\dots d_rt)$$

except that the denominator $d_1\cdots d_r$ is not especially larger than z; we however select $z=2^r$ to ensure this condition. So, on combining both, we see that

$$\Big\|V_s(1)-\sum_{0\leqslant v\leqslant r-1}(-1)^v\sum_{d_1\cdots d_v\leqslant 2^r}\frac{N_{d_1}}{d_1}\cdots\frac{N_{d_v}}{d_v}M_1^{-1}\Gamma_s(d_1\dots d_v)\Big\|$$
$$\leqslant 2r\left(\frac{|G^\sharp|d(\exp G)}{2}\right)^r\max_{D\geqslant 2^r}\|V_s(D)\|. \quad (28)$$

To complete the proof, we simply need a bound for $\max_{D\geqslant 2^r}\|V_s(D)\|$ and such a bound is provided by the next lemma.

Lemma 4.3. *Let $\mathcal{A}$ be a subset of the $G=(\mathbb{Z}/q\mathbb{Z})^\times$. Let $f>1$ be a real parameter. We have*

$$\Big|\log\prod_{\substack{p\in\mathcal{A},\\ p\geqslant P}}\big(1-p^{-f}\big)\Big|\leqslant\frac{1+Pf^{-1}}{P^f}.$$

Proof. We use

$$\log\prod_{\substack{p\in\mathcal{A},\\ p\geqslant P}}\big(1-p^{-f}\big)=-\sum_{\substack{p\in\mathcal{A},\\ p\geqslant P}}\sum_{k\geqslant 1}\frac{1}{kp^{kf}}$$

hence, by using a comparison to an integral, we find that

$$\Big|\log\prod_{\substack{p\in\mathcal{A},\\ p\geqslant P}}\big(1-p^{-f}\big)\Big|\leqslant\sum_{n\geqslant P}\frac{1}{n^f}\leqslant\frac{1}{P^f}+\int_P^\infty\frac{dt}{t^f}$$

□

5 Notes on the implementation

The parameter r is not very large, typically between 2 and 8. Since in (11), several products $d = d_1 \cdots d_v$ are equal, we store the computed values of $\Gamma_s(dt)$ in the dictionary `ComputedGammas` in the function `GetVs` of the script `LIEP.sage`. We do the same for the products $N_{d_1} \cdots N_{d_v} M_1^{-1}$ in the dictionary `ComputedProductNdsM1Inverse`. Since the list $[d_1, \cdots, d_v]$ cannot be a key for sur a dictionary, we replace it in the function `encode` by the string "d_1Ad_2A$\cdots d_v$A".

A final and very efficient time-saver is to be introduced: when ds is very large, we directly replace $L(ds, \chi)$ by an approximation of 1.

Concerning the general structure, the function `GetStructure` computes all the algebraical quantities that we need: the list of cyclic subgroups, the one of lattice-invariant classes, the exponent of our group, its character group, the set of invertible classes and, for each cyclic subgroup, the set of characters that are trivial on it.

Once the script is loaded via `load('LatticeInvariantEulerProducts.sage')`, a typical call will be

GetVs(12, 2, 100, 300)

to compute modulo 12 the possible constants with $s = 2$, asking for 100 decimal digits and using $P = 300$. The output is self explanatory. The number of decimal digits asked for is roughly handled and one may lose precision in between, but this is indicated at the end. A more precise treatment would first check the output and if the precision attained would not be enough, increase automatically this parameter. We prefer to let the users do that by themselves. For instance the call `GetVs(12, 2, 1010, 300)` produces only 628 accurate digits, but the call `GetVs(12, 2, 1500, 300)` produces after almost four times more time 1262 decimals digits. The digits presented when `WithLaTeX`= 1 are always accurate, so, in the first case, only 628 digits would be printed and 1262 in the second case.

There are two subsequent optional parameters `Verbose` and `WithLaTeX`. The first one may take the values 0, 1 and 2; when equal to 0, the function will simply do its job and return the list of the invariant classes and the one of the computed lower and upper values. When equal to 1, its default value, some information on the computation is given. At level 2, more informations is given, but that should not concern the casual user. When the parameter `Verbose` is at least 1 and `WithLaTeX` is 1, the values of the constants will be further presented in a format suitable for inclusion in a LaTeX-file. For instance, the call

GetVs(12, 2, 100, 300, 1, 1)

is the one used to prepare this document.

6 Some results

In this part, we exhibit some results for $s = 2$ and small q's. We decided to produce 100 decimal digits each time. Each computation took at most half a minute and we selected uniformly $P = 300$.

Modulo 3

$$\prod_{p\equiv 1[3]} (1-p^{-2})^{-1} = 1.03401\ 48754\ 14341\ 88053\ 90306\ 44413\ 04762\ 85789\ 65428\ 48909$$
$$98864\ 16825\ 03842\ 12222\ 45871\ 09635\ 80496\ 21707\ 98262\ 05962\ \cdots$$

$$\prod_{p\equiv 2[3]} (1-p^{-2})^{-1} = 1.41406\ 43908\ 92147\ 63756\ 55018\ 19079\ 82937\ 99076\ 95069\ 39316$$
$$21750\ 39924\ 96242\ 39281\ 06992\ 08849\ 94537\ 54858\ 50247\ 51141\ \cdots$$

Modulo 4

$$\prod_{p\equiv 1[4]} (1-p^{-2})^{-1} = 1.05618\ 21217\ 26816\ 14173\ 79307\ 65316\ 21989\ 05875\ 80425\ 46070$$
$$80120\ 04306\ 19830\ 27928\ 16062\ 22693\ 04895\ 12958\ 37291\ 59718\ \cdots$$

$$\prod_{p\equiv 3[4]} (1-p^{-2})^{-1} = 1.16807\ 55854\ 10514\ 28866\ 96967\ 37064\ 04040\ 13646\ 79021\ 45554$$
$$79928\ 40563\ 68111\ 38106\ 59377\ 71094\ 66904\ 07472\ 79588\ 48702\ \cdots$$

Modulo 5

$$\prod_{p\equiv 1[5]} (1-p^{-2})^{-1} = 1.01091\ 51606\ 01019\ 52260\ 49565\ 84289\ 51635\ 31275\ 34474\ 41032$$
$$78381\ 44787\ 08264\ 60620\ 03940\ 19317\ 65092\ 17713\ 39087\ 38209\ \cdots$$

$$\prod_{p\equiv 2,3[5]} (1-p^{-2})^{-1} = 1.55437\ 60727\ 20889\ 22081\ 75902\ 82565\ 55177\ 56056\ 30147\ 34257$$
$$40072\ 50077\ 94457\ 39239\ 00871\ 38641\ 44091\ 80733\ 87878\ 70683\ \cdots$$

$$\prod_{p\equiv 4[5]} (1-p^{-2})^{-1} = 1.00496\ 03239\ 22297\ 55899\ 37496\ 24810\ 25076\ 10886\ 10742\ 29796$$
$$54153\ 25350\ 41957\ 83218\ 23111\ 55915\ 45344\ 82860\ 11269\ 00178\ \cdots$$

Modulo 7

$$\prod_{p\equiv 1[7]} (1-p^{-2})^{-1} = 1.00222\ 95338\ 19740\ 42627\ 18638\ 82440\ 54416\ 83422\ 51833\ 01832$$
$$79790\ 08537\ 95406\ 43291\ 50318\ 83415\ 39590\ 59797\ 47221\ 82235\ \cdots$$

$$\prod_{p\equiv 2,4[7]} (1-p^{-2})^{-1} = 1.34984\ 62584\ 24863\ 74985\ 86354\ 19845\ 81943\ 27190\ 25780\ 55506$$
$$37166\ 83604\ 83648\ 81410\ 33325\ 99543\ 93007\ 87688\ 11385\ 81977\ \cdots$$

$$\prod_{p\equiv 3,5[7]} (1-p^{-2})^{-1} = 1.18274\ 25972\ 09647\ 13108\ 58784\ 06326\ 06418\ 41454\ 09923\ 02135$$
$$98060\ 91576\ 08267\ 05876\ 38542\ 18908\ 26041\ 77900\ 01774\ 19544\ \cdots$$

$$\prod_{p\equiv 6[7]} (1-p^{-2})^{-1} = 1.00705\ 20326\ 03654\ 52429\ 40083\ 99736\ 08430\ 52698\ 73698\ 58258$$
$$78850\ 06093\ 35453\ 09488\ 18343\ 08755\ 41108\ 64586\ 07373\ 90204\ \cdots$$

Modulo 8

$$\prod_{p\equiv 1[8]} (1-p^{-2})^{-1} = 1.00483\,50650\,34191\,18655\,96551\,47589\,47548\,58058\,94319\,12665\,24335\,94429\,82900\,54799\,74898\,69404\,37759\,00902\,57396\,16131\cdots$$

$$\prod_{p\equiv 3[8]} (1-p^{-2})^{-1} = 1.13941\,87787\,53762\,35338\,45125\,46766\,30793\,81051\,30107\,10362\,61455\,18198\,30982\,41101\,68549\,80783\,85991\,69241\,91778\,51482\cdots$$

$$\prod_{p\equiv 5[8]} (1-p^{-2})^{-1} = 1.05109\,99867\,25307\,17314\,11652\,76738\,77788\,69209\,70907\,44411\,17294\,30277\,73004\,13438\,62260\,40465\,37645\,49362\,47222\,05107\cdots$$

$$\prod_{p\equiv 7[8]} (1-p^{-2})^{-1} = 1.02515\,03755\,35690\,45801\,58878\,05005\,52355\,11894\,33731\,56551\,18373\,75763\,76764\,33000\,02995\,14586\,30251\,80286\,39950\,93606\cdots$$

Modulo 9

$$\prod_{p\equiv 1[9]} (1-p^{-2})^{-1} = 1.00403\,38350\,51288\,79798\,24781\,19858\,91827\,40795\,64970\,36679\,36445\,41466\,37267\,39469\,92239\,23126\,66372\,93919\,69658\,67842\cdots$$

$$\prod_{p\equiv 4,7[9]} (1-p^{-2})^{-1} = 1.02986\,05881\,94067\,42193\,37664\,74782\,44214\,16682\,92769\,27452\,73207\,14817\,70307\,86916\,69892\,47036\,00965\,30253\,12905\,58265\cdots$$

$$\prod_{p\equiv 4,7[9]} (1-p^{-2})^{-1} = 1.40783\,70712\,90808\,40749\,73657\,22905\,08432\,39804\,43009\,44237\,40720\,14524\,97906\,23053\,87855\,38578\,70189\,74375\,81953\,76990\cdots$$

$$\prod_{p\equiv 8[9]} (1-p^{-2})^{-1} = 1.00442\,33235\,64563\,73391\,73441\,85781\,21579\,37352\,73638\,30955\,56280\,50892\,89398\,89075\,19700\,52740\,02286\,05779\,32565\,16286\cdots$$

Modulo 11

$$\prod_{p\equiv 1[11]} (1-p^{-2})^{-1} = 1.00232\,82408\,97736\,52733\,78057\,92469\,42582\,04345\,78064\,14205\,83982\,33031\,85907\,92992\,24880\,67695\,76044\,58944\,51196\,15620\cdots$$

$$\prod_{p\equiv 3,4,5,9[11]} (1-p^{-2})^{-1} = 1.17640\,19228\,83643\,95880\,33414\,33345\,34827\,96940\,97368\,30486\,16665\,34684\,14997\,26780\,56538\,00773\,03065\,73130\,57662\,69651\cdots$$

$$\prod_{p\equiv 2,6,7,8[11]} (1-p^{-2})^{-1} = 1.38240\,11442\,86238\,42027\,75000\,43769\,81637\,89094\,66912\,37304\,47620\,19574\,19754\,57218\,30125\,55995\,29700\,83532\,41364\,44524\cdots$$

$$\prod_{p\equiv 10[11]} (1-p^{-2})^{-1} = 1.00079\,37707\,14740\,00680\,22344\,89849\,48242\,62521\,98050\,34012\,64066\,25514\,76917\,65166\,82435\,21918\,91527\,18353\,68455\,84322\cdots$$

Modulo 12

$$\prod_{p\equiv 1[12]} (1-p^{-2})^{-1} = 1.00761\,32452\,14144\,96559\,45395\,42226\,07374\,36651\,26958\,68527\,45643\,01998\,09609\,77426\,73644\,88547\,67836\,44849\,33202\,14325\cdots$$

$$\prod_{p\equiv 5[12]} (1-p^{-2})^{-1} = 1.04820\,19053\,78977\,39250\,51304\,63281\,36174\,16812\,89837\,18149\,90001\,13571\,41001\,64738\,59094\,95187\,94893\,09617\,76977\,21642\cdots$$

$$\prod_{p\equiv 7[12]} (1-p^{-2})^{-1} = 1.02620\,21484\,34059\,19863\,35581\,85010\,26060\,28936\,85994\,49429\,66639\,85216\,99350\,23670\,85667\,49199\,08664\,69711\,36008\,63206\cdots$$

$$\prod_{p\equiv 11[12]} (1-p^{-2})^{-1} = 1.01177\,86384\,68011\,52312\,67190\,96158\,71334\,79259\,11961\,40490\,98220\,06644\,81223\,19559\,87175\,39371\,09183\,34305\,17862\,41782\cdots$$

Modulo 13

$$\prod_{p\equiv 1[13]} (1-p^{-2})^{-1} = 1.00065\,68661\,98289\,66605\,74722\,84730\,65103\,24737\,50365\,90184\,05159\,27092\,48866\,57922\,38005\,08700\,91710\,86513\,17616\,09196\cdots$$

$$\prod_{p\equiv 3,9[13]} (1-p^{-2})^{-1} = 1.12706\,12740\,31738\,17588\,59892\,83155\,15629\,26278\,07968\,11746\,82779\,67965\,70450\,70290\,07193\,03711\,22677\,99924\,94356\,98502\cdots$$

$$\prod_{p\equiv 5,8[13]} (1-p^{-2})^{-1} = 1.04384\,79529\,58163\,65554\,40607\,78295\,27290\,54540\,06610\,44072\,10832\,56178\,04912\,84183\,53532\,90965\,05901\,10590\,78096\,19725\cdots$$

$$\prod_{p\equiv 4,10[13]} (1-p^{-2})^{-1} = 1.00628\,51384\,49121\,21016\,97949\,85276\,52774\,02943\,27367\,61695\,62341\,94965\,66031\,00088\,80143\,94131\,68766\,62368\,89868\,86315\cdots$$

$$\prod_{p\equiv 2,6,7,11[5]} (1-p^{-2})^{-1} = 1.38005\,21668\,42131\,56025\,87780\,51917\,60220\,78300\,01261\,32132\,71452\,02278\,31109\,86101\,82228\,60818\,00993\,59554\,83705\,63152\cdots$$

$$\prod_{p\equiv 12[13]} (1-p^{-2})^{-1} = 1.00019\,47228\,43353\,67566\,71300\,08739\,85014\,82395\,69148\,48279\,40388\,11317\,05577\,95031\,73324\,84500\,62196\,13236\,95450\,44607\cdots$$

Modulo 15

$$\prod_{p\equiv 1[15]} (1-p^{-2})^{-1} = 1.00148\,97422\,73492\,93694\,01590\,74709\,94741\,86204\,16057\,84703\,39056\,96965\,27900\,69768\,31015\,93499\,57569\,20627\,49312\,46472\cdots$$

$$\prod_{p\equiv 4[15]} (1-p^{-2})^{-1} = 1.00317\,84702\,52335\,21553\,03385\,74605\,13341\,60876\,38887\,06373\,99726\,98300\,94166\,44951\,29890\,77512\,51898\,54937\,99553\,73817\cdots$$

$$\prod_{p\equiv 11[15]} (1-p^{-2})^{-1} = 1.00941\,13980\,00770\,01550\,89289\,87201\,77757\,07722\,90866\,70412\,16926\,38886\,93678\,57069\,72491\,86135\,73076\,88296\,92897\,11517\cdots$$

$$\prod_{p\equiv 14[15]} (1-p^{-2})^{-1} = 1.00177\,62085\,28901\,80614\,36146\,02808\,40796\,91353\,09274\,51496\,50189\,63697\,10472\,98862\,81970\,33771\,62313\,65210\,99539\,20779\cdots$$

$$\prod_{p\equiv 2,8[15]} (1-p^{-2})^{-1} = 1.34246\,04555\,75815\,20935\,38219\,61912\,33400\,63700\,39854\,01914\,89581\,22207\,92416\,73586\,96682\,97806\,80125\,69488\,81498\,50864\cdots$$

$$\prod_{p\equiv 7,13[15]} (1-p^{-2})^{-1} = 1.02920\,54527\,13850\,30813\,47218\,56152\,24186\,80527\,04739\,70305\,24517\,59610\,07321\,59017\,16749\,03353\,09603\,86166\,91723\,02744\cdots$$

Modulo 16

$$\prod_{p\equiv 1[16]} (1-p^{-2})^{-1} = 1.00378\,12963\,11174\,37713\,11351\,57040\,29912\,37432\,41715\,47512\,58825\,42256\,35367\,55019\,06798\,95994\,89800\,21007\,81973\,43643\cdots$$

$$\prod_{p\equiv 7[16]} (1-p^{-2})^{-1} = 1.02325\,48779\,01073\,97853\,55761\,58294\,87207\,77486\,14958\,00981\,48231\,42083\,87589\,50311\,26921\,49019\,48194\,65496\,71178\,11340\cdots$$

$$\prod_{p\equiv 9[16]} (1-p^{-2})^{-1} = 1.00104\,97998\,50572\,71636\,94193\,27915\,29477\,99415\,64430\,03051\,11770\,36575\,50177\,59023\,63659\,02494\,18529\,79978\,84276\,50040\cdots$$

$$\prod_{p\equiv 15[16]} (1-p^{-2})^{-1} = 1.00185\,24182\,16883\,23320\,82998\,60789\,73812\,61209\,24826\,80727\,98647\,97998\,43453\,46728\,16095\,21879\,11383\,12968\,15980\,55553\cdots$$

$$\prod_{p\equiv 3,11[16]} (1-p^{-2})^{-1} = 1.13941\,87768\,04159\,78365\,16336\,31967\,81864\,56473\,61983\,14189\,29444\,68291\,73201\,89509\,29669\,31553\,38592\,46601\,11751\,23656\cdots$$

$$\prod_{p\equiv 5,13[16]} (1-p^{-2})^{-1} = 1.05109\,99847\,01455\,13922\,37935\,04670\,04872\,53144\,42898\,28568\,56916\,17813\,30298\,25358\,17313\,32680\,61853\,91602\,73863\,21989\cdots$$

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