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# An unreliable single server retrial queue with collisions and transmission errors

Lamia Lakaour <sup>a\*</sup>, Djamil Aissani<sup>a</sup>, Karima Adel-Aissanou<sup>a</sup>,  
Kamel Barkaoui<sup>b</sup> and Sofiane Ziani<sup>a</sup>

<sup>a</sup>Research Unit LaMOS, Bejaia University, Bejaia 06000, Algeria.

<sup>b</sup>CEDRIC, Conservatoire National des Arts et Métiers, Paris, France.

## Abstract

The present paper deals with the performance evaluation of an  $M/M/1$  retrial queue with collisions, transmission errors and unreliable server. To the best of our knowledge, there are no works that have dealt with retrial queues by considering all the above-mentioned aspects (collisions, transmission errors and unreliable server). This queue can be used as a mathematical model of several computer systems and telecommunication networks. We apply the generating function method to derive the joint distribution of the server state and the orbit length in the steady state, and we obtain some performance measures. Finally, we provide numerical illustrations to show the effectiveness and the applicability of the model.

**Keywords:** Retrial queue; Collisions; Transmission errors; Unreliable server; Generating function.

## 1 Introduction

Wireless data networks, which are widely used nowadays, are subject to several problems such as collisions, transmission errors and server breakdowns. These motivate us to devise a stochastic model that takes into account all the aspects causing packets re-transmission to solve such problems. In this study, we consider a retrial queueing model which can be used to analyze the random access to the channel in wireless communications where the server corresponds to the channel radio and the RF module (Radio Frequency module), and the customers are the nodes packets. However, in the distributed coordination function (DCF) mode of IEEE 802.11 basic access scheme, the loss and the re-transmission of packets can be due to either collisions, transmission errors and breakdowns.

In wireless networks, collision takes place when at least two customers access the channel at the same time. In such a situation, the receivers can not decode any frame (packet) correctly. In this paper, we consider that the collision occurs when an arriving customer from outside the system namely a primary customer or a retrial customer (secondary customer) has access directly to the busy server. In this case, both the arriving customer and the customer in service will be moved to the orbit.

Usually, transmission errors are caused by an erroneous packet or by non-ideal channel conditions. Co-channel and adjacent channel interferences may also cause transmission errors (see Yeo and Agrawala (2003) and Yin (2007)). In our model, we consider that the packet is transmitted without transmission errors with probability  $\gamma$ .

The loss and re-transmission of a packet may be also due to the server failure. The failures could be caused by natural disasters, hardware failures, as well as from human actions such as cyber

attacks, which are becoming more and more sophisticated. Failures of radio channel are also due to the weakness of signal due to interference or fading, etc.

In the literature, we find works on queues with repeated attempts, some of them take into account collisions and others consider the server failures, but on the best of our knowledge, there is no work which takes into account transmission errors and the other factors together (retrials, collisions and server failures) and this is the object of the paper.

In the proposed model, an arriving customer repeats its attempts to enter the service if

1. the server is busy;
2. there is a transmission error;
3. there is a collision;
4. the server breaks down.

Queues with repeated attempts play a vital role in the performance analysis of many telecommunication processes, including local and wide area networks, switching systems, shared bus local area networks, digital cellular mobile networks, etc. The main feature of a retrial queue is to serve immediately the arriving customer if the server is free. However, if the server is busy, the arriving customer leaves the service area, joins a virtual group of the blocked customers, called orbit, and tries to get service again in a random order and at random time intervals (Dragieva (2013)).

Many retrial approaches have been proposed in the literature. The most usual one, described in the classical theory of retrial queues, is the so-called classical retrial policy where each customer (packet, call, etc.) in the orbit tries to get the service independently of the others. In this policy the intervals between successive repeated attempts are exponentially distributed with rate  $j\nu$ , when the orbit size is  $j$  (see Aissani (2008)). Another type of retrial policy, called constant retrial policy, is well known for modeling *ALOHA* protocol. According to this policy, the time between two successive repeated attempts is controlled by an electronic device. Thus, the retrial rate is  $(1 - \delta_{j,0})\eta$ , where  $\delta_{j,0}$  denotes Kronecker's delta and  $j$  the number of repeated customers (Fayolle (1986)). Artalejo and Gómez-Corral (1997) have proposed a linear retrial policy. This policy, which is also adopted in the present paper, combines both the last cited approaches (i.e., classical and constant retrial policies). Consequently, its rate is  $(1 - \delta_{j,0})\eta + j\nu$ .

In many situations involving data transmission from diverse sources there can be conflict for a limited number of channels or other facilities. Uncoordinated attempts by several sources to use a single server facility can result in 'Collision' leading to the loss of the transmission. Jonin (1982) and Falin and Sukharev (1985) have analyzed a type of retrial queueing system with collisions, called queue with double connections. For this queue, if an arriving customer finds the service facility busy, interrupts the service of the customer in service, both the arriving customer and the served customer will join the orbit and the server becomes free immediately. In Choi et al. (1992), the authors have analyzed a retrial queueing model with collisions and constant retrial rate arising from unslotted CSMA/CD (Carrier Sense Multiple Access with Collision Detection) protocol. They have derived the generating function of the limiting distribution of the orbit size when the server is idle. The service time of a customer consists of two consecutive phases. The collision occurs when the arriving customer finds the server busy with the first phase of a customer service time. Kim (2010) has considered a Markovian single server retrial queue with collisions and impatience. Kumar et al. (2010) have analyzed the feedback retrial queueing system with collisions and linear retrial policy. Wu

et al. (2011) have analyzed a retrial queueing system with preemptive resume and collisions. Kvach and Nazarov (2015) have considered the  $M/M/1/N$  retrial queue with collisions, while Jailaxmi et al. (2017) have conducted a performance analysis of an  $M/G/1$  retrial queue with general retrial time, modified M-vacations and collisions.

In Kim (2010); Kumar et al. (2010); Kvach and Nazarov (2015); Jailaxmi et al. (2017), if an arriving customer finds the server busy, the collision occurs between the arriving customer and the customer in service, resulting in the move of both customers to the retrial group. In all of the above mentioned works, the authors consider the collision as the only cause of unsuccessful transmission and re-transmission, which is not the case of the real-world communication systems. In this paper, other aspects causing an unsuccessful transmission and re-transmission are considered.

In many real-life situations the server is subject to unpredictable breakdowns. Therefore, queueing systems considering server breakdowns may provide a realistic representation of such situations. This type of systems can be encountered in communication networks, manufacturing and computer systems. Retrial queues dealing with servers failures and repairs have been introduced by Aissani (1988) and Kulkarni and Choi (1990). Several other works have been devoted to this field Aissani (1993, 1994); Anisimov and Atadzhanov (1992, 1994); Artalejo (1994); Sherman and Kharoufeh (2006); Falin (2010); Saggou et al. (2017). In Falin (2010), an  $M/G/1$  retrial queues with an unreliable server and general repair times have been considered. The author has found the joint distribution of the server state, the number of customers in the queue and the number of customers in the orbit in steady state when both service and repair times have a general distribution. Li et al. (2006) have considered a  $BMAP/G/1$  retrial queue with a server subject to breakdowns and repairs. They have assumed that the lifetime of the server is exponentially distributed while the repair time is general. In their work, the supplementary variable has been used. Li and Zhang (2017) have analyzed an  $M/G/1$  retrial  $G - queue$  with working breakdowns. They have supposed that during the breakdown period, the service still continues with a lower rate. In order to discuss the probability generating function of the number of customers in the retrial queue, the supplementary variable technique has been used. Sherman and Kharoufeh (2006) have analyzed an unreliable  $M/M/1$  retrial queue where the capacity of the orbit and the normal queue are considered infinite. They have provided stability condition as well as several stochastic decomposability results. Wu and Lian (2013) have analyzed an  $M/G/1$  retrial  $G - queue$  with priority under the Bernoulli vacation schedule subject to the server breakdowns and repairs using the matrix geometric method. The retrial  $G - queues$  with server breakdowns have been discussed in many papers such as Do (2011), Rakhee et al. (2013), Yang et al. (2013), Gao and Wang (2014), Bhagat and Jain (2016) and Chen et al. (2016). Several detailed reviews of retrial queues can be found in the bibliographies of Artalejo (1999, 2010), the survey papers of Yang and Templeton (1987), Aissani (1988), Falin (1990), Kulkarni and Liang (1997) and Kim and Kim (2016), the books of Falin and Templeton (1997) and Artalejo and Gómez-Corral (2008).

As stated above, several studies dealing with retrial queueing systems with collisions and retrial queues with unreliable servers have been considered in the literature. In these studies, the authors consider in their model of retrial queue only one event namely collisions or server breakdowns, which constitute their main draw back. In communication systems, retransmitting packets due to transmission errors caused by an erroneous packet, or non-ideal channel conditions is also one of the frequent factor in retransmitting data and not much work taking this aspect in their model of retrial is found in the literature. However, the consideration of retrial queueing systems taking into account all factors that cause the re-transmission of customers (collisions, transmission errors and server breakdowns) seems interesting. To the best of our knowledge, there are no works considering all these aspects together in the literature. To face these issues, an unreliable server retrial queueing

system with linear retrial rate, collisions of customers and transmission errors is investigated in this paper. To obtain some performance measures, the generating function method is applied. To assess our model, we give some numerical examples.

The next section presents the model description. Sect. 3 provides the steady-state probabilities of the orbit size and the server state, and gives the stability condition of the system. In Sect. 4, some performances are obtained using steady-state analysis, while in Sect. 5, numerical examples are given to illustrate the effectiveness of the theoretical results. Finally, the paper is concluded in Sect. 6.

## 2 Model description

let us consider an  $M/M/1$  retrial queue with collisions, transmission errors, unreliable server and linear retrial rate. Primary customers arrive from outside the system according to a Poisson process with rate  $\lambda$ . It is assumed that there is no waiting space and an arriving primary customer enters the service with probability  $p$  or enters and waits in the orbit of infinite size with probability  $(1-p)$  and tries to get served again. The definition of the probability "p" is motivated in modeling random access protocols in wireless communication networks to reduce collisions, i.e, Consider a communication line with slotted time, which is shared by several stations. The duration of the slot equals the transmission time of a single packet of data. If two or more stations are transmitting packets simultaneously, then a collision takes place, i.e. all packets are destroyed and must be retransmitted. If the stations involved in the conflict would try to retransmit destroyed packets in the nearest slot, then a collision occurs with certainty. To avoid this, each station independently of other stations, transmits the packet with probability  $p$  and delays actions with probability  $(1-p)$ . Each station introduces a random delay before next attempt to transmit the packet (see Artalejo and Falin (2002)).

The inter-retrial time of each customer in the orbit is exponentially distributed and independent of the number of customers applying for the service. We assume that the repeated attempts follow the linear retrial policy; the probability of a repeated attempt from the orbit during the interval time  $(t, t + \Delta t)$  is  $(\eta(1 - \delta_{j,0}) + j\nu)\Delta t + o(\Delta t)$ , where  $j$  is the customers number at time  $t$  and  $\delta_{j,0}$  denotes Kronecker's delta. The customer in service could either end service successfully (without transmission error) and leaves the system with probability  $\gamma$  ( $0 < \gamma \leq 1$ ) or join the retrial group (due to transmission error) with probability  $(1 - \gamma)$ . If the server is busy, the arriving customer (primary or secondary) collides with the customer in service resulting in both being shifted to the orbit. We consider that the server is subject to random breakdowns only when a customer is in service and this according to a poisson process with rate  $\alpha$ . Customers whose service is interrupted by a breakdown have to leave the service zone and join the retrial queue. The repair times are considered exponentially distributed with repair rate  $\beta$ .

We assume that the service times are independent and follow an exponential distribution with mean  $1/\mu$ . The stochastic processes involved in the system are considered to be mutually independent.

The system is described by the bivariate process  $Y(t) = \{C(t), X(t); t \geq 0\}$ , where  $X(t)$  denotes the number of customers in the orbit at time  $t$  and  $C(t)$  denotes the server state defined as

$$C(t) = \begin{cases} 0, & \text{when the server is idle at time } t; \\ 1, & \text{when the server is busy at time } t; \\ 2, & \text{when the server breaks down and under repair at time } t. \end{cases}$$

It is clear that the process  $Y(t)$  is a continuous time Markov chain, irreducible, aperiodic and time-homogeneous with state space

$$S = \{(0, j), (1, j), (2, j') / j, j' \in \mathbb{Z}_+; j' \neq 0\}.$$

### 3 Analysis of the steady-state distribution

Let  $P_j = \lim_{t \rightarrow \infty} P\{C(t) = 0, X(t) = j\}$ ,  $Q_j = \lim_{t \rightarrow \infty} P\{C(t) = 1, X(t) = j\}$ ,  $j \geq 0$  and  $R_j = \lim_{t \rightarrow \infty} P\{C(t) = 2, X(t) = j\}$ ,  $j \geq 1$ , be the joint distributions of the server state and the number of customers in the orbit in the steady state.

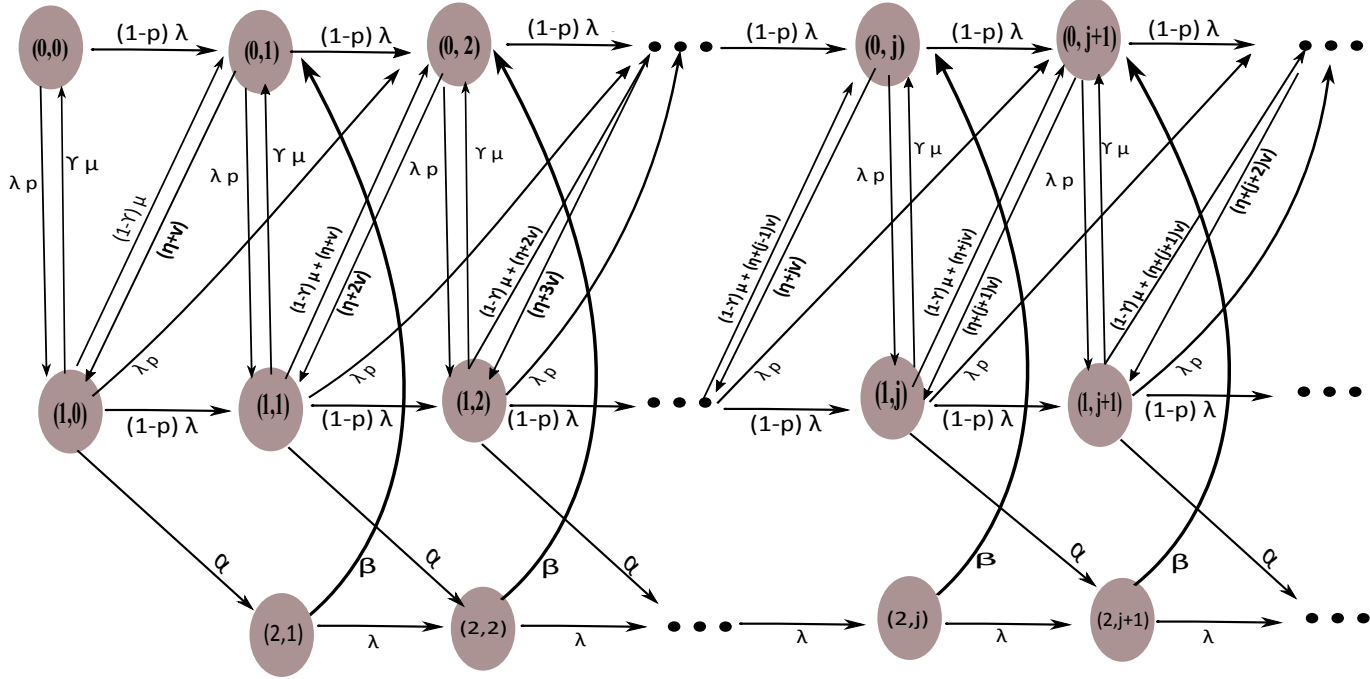


Figure 1: Transition diagram of an M/M/1 retrial queue with collisions, transmission errors, unreliable server and linear retrial policy.

In order to study the stability of the M/M/1 retrial queue with collisions, transmission errors, unreliable server and linear retrial policy, we consider the embedded Markov chain defined by  $\{X_n = X(t_n + 0), (n \geq 0)\}$ , with  $t_0 = 0$ , where  $t_n$  is the time when the server becomes idle for the  $n^{\text{th}}$  time due to the service completion of the  $n^{\text{th}}$  customer, collision of customers or reparation of the server, and  $X_n$  is the number of customers in the orbit just after  $t_n$ .

#### Proposition 1

The embedded Markov chain  $(X_n)_{n \geq 0}$  is ergodic if and only if

$$\rho = \frac{\lambda(2\beta + \alpha)}{\beta\gamma\mu} < 1, \quad (1)$$

where the parameters  $\lambda$ ,  $\beta$ ,  $\alpha$  and  $\gamma$  are as defined above.

#### Proof 1

To prove the sufficient condition, we use the criterion based on the theory of mean drift. The most important result of this theory is the Foster's criterion (see Pakes (1969) and Tweedie (1975)).

In our model the limiting mean drift is

$$\lim_{k \rightarrow \infty} \Delta_k = E[X_{n+1} - X_n | X_n = k] = \lambda(2\beta + \alpha) - \beta\gamma\mu, \forall k > j_0, j_0 \geq 0.$$

$(X_n)_{n \geq 0}$  is ergodic if  $\lim_{k \rightarrow \infty} \Delta_k < 0$ .

Hence  $\lambda(2\beta + \alpha) < \beta\gamma\mu$  if and only if  $\frac{\lambda(2\beta + \alpha)}{\beta\gamma\mu} = \rho < 1$ .

It can also be verified that this condition is necessary for ergodicity.  $\square$

Using the balance equations of the Markovian system illustrated in Fig. 1, we obtain

$$\lambda P_0 = \gamma\mu Q_0, \quad (2)$$

$$(\lambda + \eta + \nu)P_1 = (1-p)\lambda P_0 + (1-\gamma)\mu Q_0 + \gamma\mu Q_1 + \beta R_1, \quad (3)$$

$$\begin{aligned} (\lambda + \eta + j\nu)P_j &= (1-p)\lambda P_{j-1} + (\eta + (j-1)\nu)Q_{j-1} + (1-\gamma)\mu Q_{j-1} + \lambda p Q_{j-2} + \beta R_j \\ &\quad + \gamma\mu Q_j, \quad j = 2, 3, \dots, \end{aligned} \quad (4)$$

$$(\alpha + \lambda + \mu)Q_0 = \lambda p P_0 + (\eta + \nu)P_1, \quad (5)$$

$$(\alpha + \lambda + \mu + \eta + j\nu)Q_j = \lambda p P_j + (\eta + (j+1)\nu)P_{j+1} + (1-p)\lambda Q_{j-1}, \quad j = 1, 2, 3, \dots, \quad (6)$$

$$(\lambda + \beta)R_1 = \alpha Q_0, \quad (7)$$

$$(\lambda + \beta)R_j = \alpha Q_{j-1} + \lambda R_{j-1}, \quad j = 2, 3, \dots \quad (8)$$

To solve Eqs. (2)-(8), we use the following partial generating functions:

$$P(z) = \sum_{j=0}^{\infty} P_j z^j, \quad Q(z) = \sum_{j=0}^{\infty} Q_j z^j \quad \text{and} \quad R(z) = \sum_{j=1}^{\infty} R_j z^j, \quad z \in [0, 1], \quad (9)$$

where  $P(z)$  (respectively  $Q(z)$ ,  $R(z)$ ) is the PGF (Probability Generating Function) of the number of customers in the orbit when the server is idle (respectively busy, failing and under repair).

By multiplying both sides of Eqs. (2)-(8) by  $z^j$  and then making a summation, we obtain the following equations:

$$\begin{aligned} \nu z P'(z) + (\lambda + \eta - (1-p)\lambda z)P(z) - \eta P_0 &= (\gamma\mu + (1-\gamma)\mu z + \lambda p z^2 + \eta z)Q(z) \\ &\quad + \nu z^2 Q'(z) - \eta z Q_0 + \beta R(z), \end{aligned} \quad (10)$$

$$\nu P'(z) + (\lambda p + \frac{\eta}{z})P(z) - \frac{\eta}{z}P_0 = -\lambda(1-p)zQ(z) + \nu z Q'(z) - \eta Q_0 + (\lambda + \alpha + \mu + \eta)Q(z), \quad (11)$$

$$(\lambda(1-z) + \beta)R(z) = \alpha z Q(z). \quad (12)$$

Multiplying both sides of Eq. (11) by  $z$ , then subtracting the obtained equation from Eq. (10) and finally, summing with Eq. (12), we obtain

$$\lambda P(z) = (-\lambda z + \gamma\mu)Q(z) - \lambda R(z). \quad (13)$$

The following theorem gives the different probabilities of the server's state.

### Theorem 1

For an M/M/1 retrial queue with collisions, transmission errors, unreliable server and linear retrial policy, under the stability condition  $\lambda(2\beta + \alpha) < \beta\gamma\mu$ , we have the following probabilities of the server

state:

(i) The probability that the sever is busy is

$$Q(1) = \frac{\lambda}{\gamma\mu}. \quad (14)$$

(ii) The probability that the server breaks down is

$$R(1) = \frac{\alpha\lambda}{\beta\gamma\mu}. \quad (15)$$

(iii) The probability that the server is idle is

$$P(1) = 1 - \frac{\lambda}{\gamma\mu} - \frac{\alpha\lambda}{\beta\gamma\mu}. \quad (16)$$

### **Proof 2**

(i) For  $z = 1$ , Eq. (13) becomes

$$\lambda (P(1) + Q(1) + R(1)) = \gamma\mu Q(1).$$

Under the stability condition (1), the limiting probability distribution exists, i.e.

$$P(1) + Q(1) + R(1) = 1,$$

then, we obtain the probability that the server is busy given in Eq.(14).

(ii) From Eq. (12), for  $z = 1$  and replacing  $Q(1)$  obtained previously, we get the probability that the server breaks down given in Eq.(15).

(iii) To obtain  $P(1)$ , we first put  $z = 1$  in Eq. (13) and then, we replace  $Q(1)$  and  $R(1)$  obtained previously.□

Now, let us consider the probability generating functions of the number of customers in the orbit when the server is busy, breaks down and under repair, and idle; and the probability generating functions of the number of customers in the orbit and in the system, which are given in the following theorem.

### **Theorem 2**

For an unreliable Markovian single server retrial queue with collisions, transmission errors and linear retrial policy, under the stability condition  $\rho < 1$ , we have:



(i) the probability generating function  $Q(z)$  of the orbit size when the server is busy is

$$\begin{aligned}
Q(z) &= z^{-\frac{\eta}{\nu}} \times \left[ \frac{\lambda}{\gamma\mu} \exp\left(\frac{(z^6-1)A}{6(\gamma\mu\nu(\beta+\lambda)^2)^7}\right) \exp\left(\frac{(z^5-1)B}{5(\gamma\mu\nu(\beta+\lambda)^2)^6}\right) \exp\left(\frac{(1-z^4)C}{4(\gamma\mu\nu(\beta+\lambda)^2)^5}\right) \right. \\
&\times \exp\left(\frac{(1-z^3)D}{3(\gamma\mu\nu(\beta+\lambda)^2)^4}\right) \exp\left(\frac{(1-z^2)E}{2(\gamma\mu\nu(\beta+\lambda)^2)^3}\right) \exp\left(\frac{(1-z)F}{(\gamma\mu\nu(\beta+\lambda)^2)^2}\right) \\
&- \frac{\lambda\eta}{\nu\gamma} P_0 \int_z^1 \exp\left(\frac{(z^6-u^6)A}{6(\gamma\mu\nu(\beta+\lambda)^2)^7}\right) \exp\left(\frac{(z^5-u^5)B}{5(\gamma\mu\nu(\beta+\lambda)^2)^6}\right) \exp\left(\frac{(u^4-z^4)C}{4(\gamma\mu\nu(\beta+\lambda)^2)^5}\right) \\
&\times \exp\left(\frac{(u^3-z^3)D}{3(\gamma\mu\nu(\beta+\lambda)^2)^4}\right) \exp\left(\frac{(u^2-z^2)E}{2(\gamma\mu\nu(\beta+\lambda)^2)^3}\right) \exp\left(\frac{(u-z)F}{(\gamma\mu\nu(\beta+\lambda)^2)^2}\right) u^{\frac{\eta}{\nu}-1} \\
&\times \left. \frac{(\gamma - \frac{\lambda}{\mu}u)(\beta + \lambda - \lambda u)}{-(2\lambda^2u - 2\lambda^2u^2 - \beta\gamma\mu + \alpha\lambda u + 2\beta\lambda u - \gamma\lambda\mu + \gamma\lambda\mu u)} du \right]; \tag{17}
\end{aligned}$$

(ii) the probability generating function  $R(z)$  of the orbit size when the server breaks down and under repair is

$$\begin{aligned}
R(z) &= \frac{\alpha z}{\lambda(1-z) + \beta} z^{-\frac{\eta}{\nu}} \times \left[ \frac{\lambda}{\gamma\mu} \exp\left(\frac{(z^6-1)A}{6(\gamma\mu\nu(\beta+\lambda)^2)^7}\right) \exp\left(\frac{(z^5-1)B}{5(\gamma\mu\nu(\beta+\lambda)^2)^6}\right) \exp\left(\frac{(1-z^4)C}{4(\gamma\mu\nu(\beta+\lambda)^2)^5}\right) \right. \\
&\times \exp\left(\frac{(1-z^3)D}{3(\gamma\mu\nu(\beta+\lambda)^2)^4}\right) \exp\left(\frac{(1-z^2)E}{2(\gamma\mu\nu(\beta+\lambda)^2)^3}\right) \exp\left(\frac{(1-z)F}{(\gamma\mu\nu(\beta+\lambda)^2)^2}\right) \\
&- \frac{\lambda\eta}{\nu\gamma} P_0 \int_z^1 \exp\left(\frac{(z^6-u^6)A}{6(\gamma\mu\nu(\beta+\lambda)^2)^7}\right) \exp\left(\frac{(z^5-u^5)B}{5(\gamma\mu\nu(\beta+\lambda)^2)^6}\right) \exp\left(\frac{(u^4-z^4)C}{4(\gamma\mu\nu(\beta+\lambda)^2)^5}\right) \\
&\times \exp\left(\frac{(u^3-z^3)D}{3(\gamma\mu\nu(\beta+\lambda)^2)^4}\right) \exp\left(\frac{(u^2-z^2)E}{2(\gamma\mu\nu(\beta+\lambda)^2)^3}\right) \exp\left(\frac{(u-z)F}{(\gamma\mu\nu(\beta+\lambda)^2)^2}\right) \\
&\times \left. u^{\frac{\eta}{\nu}-1} \frac{(\gamma - \frac{\lambda}{\mu}u)(\beta + \lambda - \lambda u)}{-(2\lambda^2u - 2\lambda^2u^2 - \beta\gamma\mu + \alpha\lambda u + 2\beta\lambda u - \gamma\lambda\mu + \gamma\lambda\mu u)} du \right]; \tag{18}
\end{aligned}$$

(iii) the probability generating function  $P(z)$  of the orbit size when the server is idle is

$$\begin{aligned}
P(z) &= \left(-z + \frac{\gamma\mu}{\lambda} - \frac{\alpha z}{\lambda(1-z) + \beta}\right) z^{-\frac{\eta}{\nu}} \left[ \frac{\lambda}{\gamma\mu} \exp\left(\frac{(z^6-1)A}{6(\gamma\mu\nu(\beta+\lambda)^2)^7}\right) \exp\left(\frac{(z^5-1)B}{5(\gamma\mu\nu(\beta+\lambda)^2)^6}\right) \right. \\
&\times \exp\left(\frac{(1-z^4)C}{4(\gamma\mu\nu(\beta+\lambda)^2)^5}\right) \exp\left(\frac{(1-z^3)D}{3(\gamma\mu\nu(\beta+\lambda)^2)^4}\right) \exp\left(\frac{(1-z^2)E}{2(\gamma\mu\nu(\beta+\lambda)^2)^3}\right) \exp\left(\frac{(1-z)F}{(\gamma\mu\nu(\beta+\lambda)^2)^2}\right) \\
&- \frac{\lambda\eta}{\nu\gamma} P_0 \int_z^1 \exp\left(\frac{(z^6-u^6)A}{6(\gamma\mu\nu(\beta+\lambda)^2)^7}\right) \exp\left(\frac{(z^5-u^5)B}{5(\gamma\mu\nu(\beta+\lambda)^2)^6}\right) \exp\left(\frac{(u^4-z^4)C}{4(\gamma\mu\nu(\beta+\lambda)^2)^5}\right) \\
&\times \exp\left(\frac{(u^3-z^3)D}{3(\gamma\mu\nu(\beta+\lambda)^2)^4}\right) \exp\left(\frac{(u^2-z^2)E}{2(\gamma\mu\nu(\beta+\lambda)^2)^3}\right) \exp\left(\frac{(u-z)F}{(\gamma\mu\nu(\beta+\lambda)^2)^2}\right) \\
&\times \left. u^{\frac{\eta}{\nu}-1} \frac{(\gamma - \frac{\lambda}{\mu}u)(\beta + \lambda - \lambda u)}{-(2\lambda^2u - 2\lambda^2u^2 - \beta\gamma\mu + \alpha\lambda u + 2\beta\lambda u - \gamma\lambda\mu + \gamma\lambda\mu u)} du \right]; \tag{19}
\end{aligned}$$

(iv) the probability generating function  $K(z)$  for the number of customers in the orbit is

$$\begin{aligned}
K(z) &= (-z + \frac{\gamma\mu}{\lambda} + 1)z^{-\frac{\eta}{\nu}} \left[ \frac{\lambda}{\gamma\mu} \exp\left(\frac{(z^6 - 1)A}{6(\gamma\mu\nu(\beta + \lambda)^2)^7}\right) \exp\left(\frac{(z^5 - 1)B}{5(\gamma\mu\nu(\beta + \lambda)^2)^6}\right) \exp\left(\frac{(1 - z^4)C}{4(\gamma\mu\nu(\beta + \lambda)^2)^5}\right) \right. \\
&\times \exp\left(\frac{(1 - z^3)D}{3(\gamma\mu\nu(\beta + \lambda)^2)^4}\right) \exp\left(\frac{(1 - z^2)E}{2(\gamma\mu\nu(\beta + \lambda)^2)^3}\right) \exp\left(\frac{(1 - z)F}{(\gamma\mu\nu(\beta + \lambda)^2)^2}\right) - \frac{\lambda\eta}{\nu\gamma} P_0 \int_z^1 \exp\left(\frac{(z^6 - u^6)A}{6(\gamma\mu\nu(\beta + \lambda)^2)^7}\right) \\
&\times \exp\left(\frac{(z^5 - u^5)B}{5(\gamma\mu\nu(\beta + \lambda)^2)^6}\right) \exp\left(\frac{(u^4 - z^4)C}{4(\gamma\mu\nu(\beta + \lambda)^2)^5}\right) \exp\left(\frac{(u^3 - z^3)D}{3(\gamma\mu\nu(\beta + \lambda)^2)^4}\right) \exp\left(\frac{(u^2 - z^2)E}{2(\gamma\mu\nu(\beta + \lambda)^2)^3}\right) \\
&\times \left. \exp\left(\frac{(u - z)F}{(\gamma\mu\nu(\beta + \lambda)^2)^2}\right) u^{\frac{\eta}{\nu} - 1} \frac{(\gamma - \frac{\lambda}{\mu}u)(\beta + \lambda - \lambda u)}{-(2\lambda^2u - 2\lambda^2u^2 - \beta\gamma\mu + \alpha\lambda u + 2\beta\lambda u - \gamma\lambda\mu + \gamma\lambda\mu u)} du \right]; \quad (20)
\end{aligned}$$

(v) the probability generating function  $H(z)$  for the number of customers in the system is

$$\begin{aligned}
H(z) &= \frac{\gamma\mu}{\lambda} z^{-\frac{\eta}{\nu}} \left[ \frac{\lambda}{\gamma\mu} \exp\left(\frac{(z^6 - 1)A}{6(\gamma\mu\nu(\beta + \lambda)^2)^7}\right) \exp\left(\frac{(z^5 - 1)B}{5(\gamma\mu\nu(\beta + \lambda)^2)^6}\right) \exp\left(\frac{(1 - z^4)C}{4(\gamma\mu\nu(\beta + \lambda)^2)^5}\right) \right. \\
&\times \exp\left(\frac{(1 - z^3)D}{3(\gamma\mu\nu(\beta + \lambda)^2)^4}\right) \exp\left(\frac{(1 - z^2)E}{2(\gamma\mu\nu(\beta + \lambda)^2)^3}\right) \exp\left(\frac{(1 - z)F}{(\gamma\mu\nu(\beta + \lambda)^2)^2}\right) \\
&- \frac{\lambda\eta}{\nu\gamma} P_0 \int_z^1 \exp\left(\frac{(z^6 - u^6)A}{6(\gamma\mu\nu(\beta + \lambda)^2)^7}\right) \exp\left(\frac{(z^5 - u^5)B}{5(\gamma\mu\nu(\beta + \lambda)^2)^6}\right) \exp\left(\frac{(u^4 - z^4)C}{4(\gamma\mu\nu(\beta + \lambda)^2)^5}\right) \\
&\times \exp\left(\frac{(u^3 - z^3)D}{3(\gamma\mu\nu(\beta + \lambda)^2)^4}\right) \exp\left(\frac{(u^2 - z^2)E}{2(\gamma\mu\nu(\beta + \lambda)^2)^3}\right) \exp\left(\frac{(u - z)F}{(\gamma\mu\nu(\beta + \lambda)^2)^2}\right) \\
&\times \left. u^{\frac{\eta}{\nu} - 1} \frac{(\gamma - \frac{\lambda}{\mu}u)(\beta + \lambda - \lambda u)}{-(2\lambda^2u - 2\lambda^2u^2 - \beta\gamma\mu + \alpha\lambda u + 2\beta\lambda u - \gamma\lambda\mu + \gamma\lambda\mu u)} du \right]; \quad (21)
\end{aligned}$$

where

$$\begin{aligned}
P_0 &= \frac{\nu}{\mu\eta} \exp\left(\frac{-A}{6(\gamma\mu\nu(\beta + \lambda)^2)^7}\right) \exp\left(\frac{-B}{5(\gamma\mu\nu(\beta + \lambda)^2)^6}\right) \exp\left(\frac{C}{4(\gamma\mu\nu(\beta + \lambda)^2)^5}\right) \exp\left(\frac{D}{3(\gamma\mu\nu(\beta + \lambda)^2)^4}\right) \\
&\times \exp\left(\frac{E}{2(\gamma\mu\nu(\beta + \lambda)^2)^3}\right) \exp\left(\frac{F}{(\gamma\mu\nu(\beta + \lambda)^2)^2}\right) \left\{ \lim_{z \rightarrow 0^+} \int_z^1 \exp\left(\frac{-u^6 A}{6(\gamma\mu\nu(\beta + \lambda)^2)^7}\right) \exp\left(\frac{-u^5 B}{5(\gamma\mu\nu(\beta + \lambda)^2)^6}\right) \right. \\
&\times \exp\left(\frac{u^4 C}{4(\gamma\mu\nu(\beta + \lambda)^2)^5}\right) \exp\left(\frac{u^3 D}{3(\gamma\mu\nu(\beta + \lambda)^2)^4}\right) \exp\left(\frac{u^2 E}{2(\gamma\mu\nu(\beta + \lambda)^2)^3}\right) \exp\left(\frac{u F}{(\gamma\mu\nu(\beta + \lambda)^2)^2}\right) \\
&\times \left. u^{\frac{\eta}{\nu} - 1} \frac{(\gamma - \frac{\lambda}{\mu}u)(\beta + \lambda - \lambda u)}{-(2\lambda^2u - 2\lambda^2u^2 - \beta\gamma\mu + \alpha\lambda u + 2\beta\lambda u - \gamma\lambda\mu + \gamma\lambda\mu u)} du \right\}^{-1}; \quad (22)
\end{aligned}$$

$$\begin{aligned}
A &= \gamma\lambda^2(\beta + \lambda)^4\mu\nu^2(-\lambda^4(\beta + \lambda)^4\eta(\alpha + 2(\beta + \lambda + \gamma\mu))^6\nu^4 - \gamma^4\lambda^4(\beta + \lambda)^6\mu^4(-8\beta - 10\lambda - 4\eta + \alpha(-3 + 4p) - (2 + \gamma)\mu + \\
&4p(4(\beta + \lambda) + \gamma\mu) - 2\nu)v^4 + 5(\alpha + 2(\beta + \lambda + \gamma\mu))(\alpha\eta + 2\gamma\eta\mu + (\beta + \lambda)(\alpha + 2\eta + \mu - \gamma\rho\mu) + \alpha\nu + \beta\nu + \gamma\lambda^4(\beta + \lambda)^4\eta\mu(\alpha + \\
&4(\beta + \lambda) + \gamma\mu)(\alpha + 2(\beta + \lambda + \gamma\mu))^3\nu^4 + \lambda(\beta + \lambda + \nu)) - \gamma\lambda^4(\beta + \lambda)^4\mu(\alpha + 2(\beta + \lambda + \gamma\mu))^2\nu^4(6\gamma\eta\mu(\alpha + 4(\beta + \lambda) + \gamma\mu)^2 + (\alpha + \\
&2(\beta + \lambda + \gamma\mu))^2(-\alpha\beta + \beta^2 + 3\lambda^2 + (\alpha + 4\lambda)\eta + 4\beta(\lambda + \eta) - (\alpha + 4\beta)\lambda\rho - 2(\beta^2 + \lambda^2)p + \gamma\eta\mu + 2(\beta + \lambda)(\alpha + \mu - \gamma\rho\mu + \nu)) + \\
&4(\alpha + 2(\beta + \lambda + \gamma\mu))(2\gamma(\beta + \lambda)\eta\mu + (\alpha + 4(\beta + \lambda) + \gamma\mu)(\alpha\eta + 2\gamma\eta\mu + (\beta + \lambda)(\alpha + 2\eta + \mu - \gamma\rho\mu) + \alpha\nu + \beta\nu + \lambda(\beta + \lambda + \nu)))) - \\
&\gamma^3\lambda^4(\beta + \lambda)^4\mu^3\nu^4(4\gamma(\beta + \lambda)^2\eta\mu - (\beta + \lambda)^2(-1 + 2p)(\alpha + 2(\beta + \lambda + \gamma\mu))^2 + 4(\beta + \lambda)(\alpha + 4(\beta + \lambda) + \gamma\mu)(\alpha\eta + 2\gamma\eta\mu + (\beta + \\
&\lambda)(\alpha + 2\eta + \mu - \gamma\rho\mu) + \alpha\nu + \beta\nu + \lambda(\beta + \lambda + \nu)) + (\alpha + 4(\beta + \lambda) + \gamma\mu)^2(-\alpha\beta + \beta^2 + 3\lambda^2 + (\alpha + 4\lambda)\eta + 4\beta(\lambda + \eta) - (\alpha + 4\beta)\lambda\rho - \\
&2(\beta^2 + \lambda^2)p + \gamma\eta\mu + 2(\beta + \lambda)(\alpha + \mu - \gamma\rho\mu + \nu)) + 2(\beta + \lambda)(\alpha + 2(\beta + \lambda + \gamma\mu))((\alpha + 4(\beta + \lambda) + \gamma\mu)(\alpha + 2\beta + 3\lambda + 2\eta - \alpha\rho + \mu - \\
&p(4(\beta + \lambda) + \gamma\mu) + \nu) + 2(-\alpha\beta + \beta^2 + 3\lambda^2 + (\alpha + 4\lambda)\eta + 4\beta(\lambda + \eta) - (\alpha + 4\beta)\lambda\rho - 2(\beta^2 + \lambda^2)p + \gamma\eta\mu + 2(\beta + \lambda)(\alpha + \mu - \gamma\rho\mu + \\
&\nu)))) + \gamma^2\lambda^4(\beta + \lambda)^4\mu^2\nu^4(\gamma\eta\mu(\alpha + 4(\beta + \lambda) + \gamma\mu)^3 + (\beta + \lambda)(\alpha + 2(\beta + \lambda + \gamma\mu))^3(\alpha + 2\beta + 3\lambda + 2\eta - \alpha\rho + \mu - p(4(\beta + \lambda) + \\
&\gamma\mu) + \nu) + 3(\alpha + 4(\beta + \lambda) + \gamma\mu)(\alpha + 2(\beta + \lambda + \gamma\mu))(4\gamma(\beta + \lambda)\eta\mu + (\alpha + 4(\beta + \lambda) + \gamma\mu)(\alpha\eta + 2\gamma\eta\mu + (\beta + \lambda)(\alpha + 2\eta + \mu - \gamma\rho\mu) + \\
&\alpha\nu + \beta\nu + \lambda(\beta + \lambda + \nu))) + 3(\alpha + 2(\beta + \lambda + \gamma\mu))^2(2(\beta + \lambda)(\alpha\eta + 2\gamma\eta\mu + (\beta + \lambda)(\alpha + 2\eta + \mu - \gamma\rho\mu) + \alpha\nu + \beta\nu + \lambda(\beta + \lambda + \nu)) +
\end{aligned}$$

$$(\alpha + 4(\beta + \lambda) + \gamma\mu)(-\alpha\beta + \beta^2 + 3\lambda^2 + (\alpha + 4\lambda)\eta + 4\beta(\lambda + \eta) - (\alpha + 4\beta)\lambda p - 2(\beta^2 + \lambda^2)p + \gamma\eta\mu + 2(\beta + \lambda)(\alpha + \mu - \gamma p\mu + \nu)))));$$

$$B = \gamma\lambda^5(\beta + \lambda)^7\mu\nu^5(-\eta(\alpha + 2(\beta + \lambda + \gamma\mu))^5 + \gamma^3(\beta + \lambda)\mu^3(-20\lambda^2 - 16\lambda\eta + \alpha^2(-1 + p) + 24\lambda^2p + 12\beta^2(-1 + 2p) - 8\lambda\mu - 5\gamma\lambda\mu - 4\gamma\eta\mu + 16\gamma\lambda p\mu - \gamma\mu^2 + \gamma^2p\mu^2 - 8\lambda\nu - \gamma\mu\nu - \alpha(4\eta + \beta(9 - 10p) + \mu + \gamma\mu - 2(6\lambda(-1 + p) + \gamma p\mu) + \nu) - 4b(4\eta + \lambda(8 - 12p) + (2 + \gamma - 4\gamma p)\mu + 2\nu)) + (\alpha + 2(\beta + \lambda + \gamma\mu))^3(4\gamma\eta\mu(\alpha + 4(\beta + \lambda) + \gamma\mu) + (\alpha + 2(\beta + \lambda + \gamma\mu))(\alpha\eta + 2\gamma\eta\mu + (\beta + \lambda)(\alpha + 2\eta + \mu - \gamma p\mu) + \alpha\nu + \beta\nu + \lambda(\beta + \lambda + \nu))) - \gamma\mu(\alpha + 2(\beta + \lambda + \gamma\mu))(3\gamma\eta\mu(\alpha + 4(\beta + \lambda) + \gamma\mu)^2 + (\alpha + 2(\beta + \lambda + \gamma\mu))^2(-\alpha\beta + \beta^2 + 3\lambda^2 + (\alpha + 4\lambda)\eta + 4\beta(\lambda + \eta) - (\alpha + 4\beta)\lambda p - 2(\beta^2 + \lambda^2)p + \gamma\eta\mu + 2(\beta + \lambda)(\alpha + \mu - \gamma p\mu + \nu))) + 3(\alpha + 2(\beta + \lambda + \gamma\mu))(2\gamma(\beta + \lambda)\eta\mu + (\alpha + 4(\beta + \lambda) + \gamma\mu)(\alpha\eta + 2\gamma\eta\mu + (\beta + \lambda)(\alpha + 2\eta + \mu - \gamma p\mu) + \alpha\nu + \beta\nu + \lambda(\beta + \lambda + \nu)))) + \gamma^2\mu^2(4\gamma(\beta + \lambda)\eta\mu(\alpha + 4(\beta + \lambda) + \gamma\mu) + (\beta + \lambda)(\alpha + 2(\beta + \lambda + \gamma\mu))^2(\alpha + 2\beta + 3\lambda + 2\eta - \alpha p + \mu - p(4(\beta + \lambda) + \gamma\mu) + \nu) + (\alpha + 4(\beta + \lambda) + \gamma\mu)^2(\alpha\eta + 2\gamma\eta\mu + (\beta + \lambda)(\alpha + 2\eta + \mu - \gamma p\mu) + \alpha\nu + \beta\nu + \lambda(\beta + \lambda + \nu))) + 2(\alpha + 2(\beta + \lambda + \gamma\mu))(2(\beta + \lambda)(\alpha\eta + 2\gamma\eta\mu + (\beta + \lambda)(\alpha + 2\eta + \mu - \gamma p\mu) + \alpha\nu + \beta\nu + \lambda(\beta + \lambda + \nu))) + (\alpha + 4(\beta + \lambda) + \gamma\mu)(-\alpha\beta + \beta^2 + 3\lambda^2 + (\alpha + 4\lambda)\eta + 4\beta(\lambda + \eta) - (\alpha + 4\beta)\lambda p - 2(\beta^2 + \lambda^2)p + \gamma\eta\mu + 2(\beta + \lambda)(\alpha + \mu - \gamma p\mu + \nu)))));$$

$$C = -\gamma\lambda^4(\beta + \lambda)^6\mu\nu^4(\alpha^4(\beta + \lambda + \nu) + 4(\beta + \lambda)^4(2\lambda - (-2 + \gamma)\mu + 2\nu) + \alpha^3(6\beta^2 + 7\lambda^2 + 4\gamma\mu\nu + \lambda(\mu + 2\gamma\mu + 7\nu) + \beta(13\lambda + \mu - \gamma(-3 + p)\mu + 7\nu)) + \alpha^2(12\beta^3 + 18\lambda^3 + 6\gamma^2\mu^2\nu + \gamma\lambda\mu((2 + \gamma)\mu + 14\nu) + \lambda^2(6\mu + 5\gamma\mu + 18\nu) + \beta^2(42\lambda + (6 + 7\gamma - 4\gamma p)\mu + 18\nu) + \beta(48\lambda^2 + 4\lambda((3 + 3\gamma - \gamma p)\mu + 9\nu) + \gamma\mu((2 + 4\gamma - 3\gamma p)\mu + 14\nu))) + \alpha(8\beta^4 + 20\lambda^4 + 4\gamma^3\mu^3\nu + 4\gamma\lambda^2\mu(\mu + 3\nu) + 4\lambda^3(3\mu + 5\nu) + 4\beta^3(11\lambda + 3\mu - \gamma p\mu + 5\nu) + \gamma^2\lambda\mu^2(\mu + 7\nu) + \beta^2(84\lambda^2 + \gamma\mu((4 + 3\gamma - 4\gamma p)\mu + 12\nu) + \lambda(36\mu - 8\gamma p\mu + 60\nu)) + \beta(68\lambda^3 + \gamma^2\mu^2(\mu - 3\gamma(-1 + p)\mu + 7\nu) + \gamma\lambda\mu((8 + 3\gamma - 4\gamma p)\mu + 24\nu) + \lambda^2(-4(-9 + \gamma p)\mu + 60\nu)))));$$

$$D = -\gamma\lambda^3(\beta + \lambda)^5\mu\nu^3(\alpha^3(\beta + \lambda + \nu) + 2(\beta + \lambda)^3(2\lambda - (-2 + \gamma)\mu + 2\nu) + \alpha^2(4\beta^2 + 5\lambda^2 + 3\gamma\mu\nu + \lambda(\mu + \gamma\mu + 5\nu) + \beta(9\lambda + \mu - \gamma(-2 + p)\mu + 5\nu)) + \alpha(4\beta^3 + 8\lambda^3 + 3\gamma^2\mu^2\nu + 4\lambda^2(\mu + 2\nu) + \gamma\lambda\mu(\mu + 5\nu) + \beta^2(16\lambda + (4 + \gamma - 2\gamma p)\mu + 8\nu) + \beta(20\lambda^2 + \lambda(8 + \gamma - 2\gamma p)\mu + 16\lambda\nu + \gamma\mu(\mu - 2\gamma(-1 + p)\mu + 5\nu)))));$$

$$E = \gamma\lambda^2(\beta + \lambda)^4\mu\nu^2(-\alpha^2(\beta + \lambda + \nu) - (\beta + \lambda)^2(2\lambda + 2\mu - \gamma\mu + 2\nu) - \alpha(2\beta^2 + 3\lambda^2 + \lambda\mu + 3\lambda\nu + 2\gamma\mu\nu + \beta(5\lambda + \mu + \gamma\mu - \gamma p\mu + 3\nu)))));$$

and

$$F = \gamma\lambda(\beta + \lambda)^3\mu\nu(-\alpha(\beta + \lambda + \nu) - (\beta + \lambda)(\lambda + \mu - \gamma p\mu + \nu)).$$

### Proof 3

(i) By differentiating Eq. (13) with respect to  $z$  and substituting into Eq. (10) and using Eq. (12), we obtain

$$\begin{aligned} Q'(z) &+ \frac{1}{\omega} \left[ \lambda^4(1 - 2p)z^4 + \lambda^3(\alpha(p - 1) - \mu + 4p\beta + 4p\lambda - 3\lambda - 2\beta + p\gamma\mu - 2\eta - \nu)z^3 \right. \\ &+ \lambda^2((\beta + \lambda)(2\alpha - 2\gamma p\mu + 2(\mu + \nu)) + 4\beta(\lambda + \eta) + \eta(\alpha + 4\lambda) - 2p(\beta^2 + \lambda^2) + \beta^2 + 3\lambda^2 - \alpha\beta p \\ &- \lambda p(\alpha + 4\beta) + \eta\gamma\mu)z^2 - \lambda(\beta + \lambda)((\beta + \lambda)(\alpha + 2\eta - \gamma p\mu + \mu) + \lambda(\lambda + \beta + \nu) + \beta\nu + \eta\alpha + \alpha\nu + 2\gamma\eta\mu)z \\ &\left. + \gamma\eta\mu(\beta + \lambda)^2 \right] Q(z) = \frac{\lambda\eta(\gamma - \frac{\lambda}{\mu}z)(\beta + \lambda - \lambda z)^2}{\gamma\omega} P_0, \end{aligned} \quad (23)$$

$$\text{where } \omega = -\nu z(\beta + \lambda(1 - z))((1 - z)(2\lambda^2z - \gamma\lambda\mu) - \beta\gamma\mu + \lambda z(\alpha + 2\beta)).$$

The equation given in Eq.(23) is a first order differential equation with second member. It has a singular point at  $z = 0$ , so the domain of solution is  $(0,1]$ .

Eq.(23) has the same form as the one below:

$$Q'(z) + \frac{az^4 + bz^3 + cz^2 + dz + e}{lz^4 + mz^3 + nz^2 + rz} Q(z) = g(z), z \in (0, 1], \quad (24)$$

where

$$\begin{aligned}
a &= \lambda^4(1 - 2p); \\
b &= \lambda^3(\alpha(p - 1) - \mu + 4p\beta + 4p\lambda - 3\lambda - 2\beta + p\gamma\mu - 2\eta - \nu); \\
c &= \lambda^2((\beta + \lambda)(2\alpha - 2\gamma p\mu + 2(\mu + \nu)) + 4\beta(\lambda + \eta) + \eta(\alpha + 4\lambda) - 2p(\beta^2 + \lambda^2) + \beta^2 + 3\lambda^2 - \alpha\beta - \lambda p(\alpha + 4\beta) + \eta\gamma\mu); \\
d &= -\lambda(\beta + \lambda)((\beta + \lambda)(\alpha + 2\eta - \gamma p\mu + \mu) + \lambda(\lambda + \beta + \nu) + \beta\nu + \eta\alpha + \alpha\nu + 2\gamma\eta\mu) \\
e &= \gamma\eta\mu(\beta + \lambda)^2; \\
l &= -2\lambda^3\nu; \\
m &= 4\beta\lambda^2\nu + 4\lambda^3\nu + \lambda^2\nu\gamma\mu + \lambda^2\alpha\nu; \\
n &= -2\lambda^2\nu\gamma\mu - 2\beta\lambda\nu\gamma\mu - 4\lambda^2\beta\nu - 2\lambda^3\nu - 2\beta^2\lambda\nu - \lambda^2\alpha\nu - \beta\lambda\alpha\nu; \\
r &= \nu\gamma\mu\lambda^2 + 2\lambda\beta\nu\gamma\mu + \beta^2\nu\gamma\mu;
\end{aligned}$$

$$g(z) = \frac{\lambda\eta(\gamma - \frac{\lambda}{\mu}z)(\beta + \lambda - \lambda z)^2}{\gamma\omega} P_0;$$

with

$$\omega = lz^4 + mz^3 + nz^2 + rz;$$

which can be written as

$$\omega = -\nu z(\beta + \lambda(1 - z))((1 - z)(2\lambda^2 z - \gamma\lambda\mu) - \beta\gamma\mu + \lambda z(\alpha + 2\beta)).$$

We note

$$T(z) = \frac{az^4 + bz^3 + cz^2 + dz + e}{lz^4 + mz^3 + nz^2 + rz},$$

so, we can write Eq.(24) as follows:

$$Q'(z) + T(z)Q(z) = g(z), z \in (0, 1]. \quad (25)$$

The solution of this equation is as follows

$$Q(z) = F(z) \exp\left(-\int_1^z T(x)dx\right), \quad (26)$$

where

$$F(z) = \int_1^z g(u) \exp\left(\int_1^u T(x)dx\right) du + k.$$

Substituting  $F(z)$  in Eq.(26), we get

$$Q(z) = \int_1^z g(u) \exp\left(\int_1^u T(x)dx\right) \exp\left(-\int_1^z T(x)dx\right) du + k \exp\left(-\int_1^z T(x)dx\right). \quad (27)$$

To determine the constant  $k$ , we put  $z = 1$  in Eq.(27) and we use the equation below corresponding to the probability to have a busy server

$$Q(1) = \frac{\lambda}{\gamma\mu},$$

we find

$$k = \frac{\lambda}{\gamma\mu},$$

then,

$$Q(z) = \int_1^z g(u) \exp\left(\int_1^u T(x)dx\right) \exp\left(-\int_1^z T(x)dx\right) du + \frac{\lambda}{\gamma\mu} \exp\left(-\int_1^z T(x)dx\right). \quad (28)$$

We used the power series of  $T(x)$  in the neighborhood of 0 to get  $\int T(x)dx$ . Then, we have

$$\begin{aligned}
\int T(x)dx &= \frac{e}{r} \log(|x|) - \frac{x^6}{6r^7} \left( (am + bl)r^5 + (-an^2 + (-2bm - 2cl)n - cm^2 - 2dlm - el^2)r^4 \right. \\
&\quad \left. + (bn^3 + (3cm + 3dl)n^2 + (3dm^2 + 6elm)n + em^3)r^3 + (-cn^4 + (-4dm - 4el)n^3 - 6em^2n^2)r^2 \right. \\
&\quad \left. + (dn^5 + 5emn^4)r - en^6 \right) - \frac{x^5}{5r^6} \left( (an + bm + cl)r^4 + (-bn^2 + (-2cm - 2dl)n - dm^2 - 2elm)r^3 \right. \\
&\quad \left. + (cn^3 + (3dm + 3el)n^2 + 3em^2n)r^2 + (-dn^4 - 4emn^3)r + en^5 \right) + \frac{x^4}{4r^5} \left( ar^4 + (-bn - cm - dl)r^3 \right. \\
&\quad \left. + (cn^2 + (2dm + 2el)n + em^2)r^2 + (-dn^3 - 3emn^2)r + en^4 \right) + \frac{x^3}{3r^4} (br^3 + (-cn - dm - el)r^2 \\
&\quad \left. + (dn^2 + 2emn)r - en^3 \right) + \frac{x^2}{2r^3} \left( cr^2 + (-dn - em)r + en^2 \right) + \frac{x}{r^2} (dr - en) + o(x^6) + \text{constant}. \quad (29)
\end{aligned}$$

We rewrite this integral as follows

$$\int T(x)dx = \frac{e}{r} \log(|x|) - \frac{x^6}{6r^7}A - \frac{x^5}{5r^6}B + \frac{x^4}{4r^5}C + \frac{x^3}{3r^4}D + \frac{x^2}{2r^3}E + \frac{x}{r^2}F + \text{constant}, \quad (30)$$

where

$$A = (am + bl)r^5 + (-an^2 + (-2bm - 2cl)n - cm^2 - 2dlm - el^2)r^4 + (bn^3 + (3cm + 3dl)n^2 + (3dm^2 + 6elm)n + em^3)r^3 + (-cn^4 + (-4dm - 4el)n^3 - 6em^2n^2)r^2 + (dn^5 + 5emn^4)r - en^6;$$

$$B = (an + bm + cl)r^4 + (-bn^2 + (-2cm - 2dl)n - dm^2 - 2elm)r^3 + (cn^3 + (3dm + 3el)n^2 + 3em^2n)r^2 + (-dn^4 - 4emn^3)r + en^5;$$

$$C = ar^4 + (-bn - cm - dl)r^3 + (cn^2 + (2dm + 2el)n + em^2)r^2 + (-dn^3 - 3emn^2)r + en^4;$$

$$D = br^3 + (-cn - dm - el)r^2 + (dn^2 + 2emn)r - en^3;$$

$$E = cr^2 + (-dn - em)r + en^2;$$

and

$$F = dr - en.$$

and we have

$$\begin{aligned} \exp\left(-\int_1^z T(x)dx\right) &= z^{-\frac{\epsilon}{r}} \exp\left(\frac{z^6-1}{6r^7}A\right) \exp\left(\frac{z^5-1}{5r^6}B\right) \exp\left(\frac{1-z^4}{4r^5}C\right) \exp\left(\frac{1-z^3}{3r^4}D\right) \\ &\times \exp\left(\frac{1-z^2}{2r^3}E\right) \exp\left(\frac{1-z}{r^2}F\right); \end{aligned} \quad (31)$$

$$\begin{aligned} \exp\left(\int_1^u T(x)dx\right) &= u^{\frac{\epsilon}{r}} \exp\left(\frac{1-u^6}{6r^7}A\right) \exp\left(\frac{1-u^5}{5r^6}B\right) \exp\left(\frac{u^4-1}{4r^5}C\right) \exp\left(\frac{u^3-1}{3r^4}D\right) \\ &\times \exp\left(\frac{u^2-1}{2r^3}E\right) \exp\left(\frac{u-1}{r^2}F\right). \end{aligned} \quad (32)$$

Substituting the equation corresponding to  $\exp\left(-\int_1^z T(x)dx\right)$ ,  $\exp\left(\int_1^u T(x)dx\right)$  and  $g(u)$  in Eq.(28), and after simplifications, we get the probability generating function  $Q(z)$  of the orbit size when the server is busy.

(ii) By substituting Eq.(17), obtained in (i), in Eq.(12), we obtain the probability generating function  $R(z)$  of the orbit size when the server breaks down and under repair.

(iii) By substituting Eq.(17) and Eq.(18), obtained in (i) and (ii) respectively, into (13), we obtain the probability generating function  $P(z)$  of the orbit size when the server is idle.

(iv) The probability generating function for the number of customers in the orbit denoted by  $K(z)$  is defined as  $K(z) = P(z) + Q(z) + R(z)$ . Substituting expressions for  $P(z)$ ,  $Q(z)$  and  $R(z)$ , we get  $K(z)$  given in Eq. (20).

(v) The probability generating function for the number of customers in the system, defined as  $H(z) = P(z) + R(z) + zQ(z)$ , is given in Eq. (21).

It can be noted that, as  $z \rightarrow 0_+$ ,  $Q(0) = Q_0 < \infty$  and  $z^{-\frac{1}{r}}$  diverges. Thus, from (17), we obtain the steady-state joint probability  $P_0$  that the server is idle and no customers are in the orbit.  $\square$

## 4 System characteristics

The following theorem summarizes some important characteristics of an unreliable single server retrial queue with collisions, transmission errors and linear retrial rate.

### Theorem 3

For a Markovian single server retrial queue with collisions, transmission errors, unreliable server and linear retrial rate, under the stability condition  $\lambda(2\beta + \alpha) < \beta\gamma\mu$ , we have the following performances:

(i) The mean number of customers  $L_q$  in the orbit is

$$\begin{aligned} L_q &= -\frac{\lambda}{\gamma\mu} - \frac{\eta}{v} + \frac{A}{(\gamma\mu\nu(\beta + \lambda)^2)^7} + \frac{B}{(\gamma\mu\nu(\beta + \lambda)^2)^6} - \frac{C}{(\gamma\mu\nu(\beta + \lambda)^2)^5} - \frac{D}{(\gamma\mu\nu(\beta + \lambda)^2)^4} \\ &- \frac{E}{(\gamma\mu\nu(\beta + \lambda)^2)^3} - \frac{F}{(\gamma\mu\nu(\beta + \lambda)^2)^2} - \frac{\mu\eta\beta}{v} \left( \frac{\gamma - \frac{\lambda}{\mu}}{\beta\gamma\mu - \alpha\lambda - 2\lambda\beta} \right) P_0. \end{aligned} \quad (33)$$

(ii) The mean number of customers  $L_s$  in the system is

$$L_s = -\frac{\eta}{v} + \frac{A}{(\gamma\mu\nu(\beta + \lambda)^2)^7} + \frac{B}{(\gamma\mu\nu(\beta + \lambda)^2)^6} - \frac{C}{(\gamma\mu\nu(\beta + \lambda)^2)^5} - \frac{D}{(\gamma\mu\nu(\beta + \lambda)^2)^4} - \frac{E}{(\gamma\mu\nu(\beta + \lambda)^2)^3} - \frac{F}{(\gamma\mu\nu(\beta + \lambda)^2)^2} - \frac{\mu\eta\beta}{v} \left( \frac{\gamma - \frac{\lambda}{\mu}}{\beta\gamma\mu - \alpha\lambda - 2\lambda\beta} \right) P_0. \quad (34)$$

(iii) The mean waiting time  $W_q$  in the orbit is

$$W_q = -\frac{1}{\gamma\mu} - \frac{\eta}{\lambda v} + \frac{A}{\lambda(\gamma\mu\nu(\beta + \lambda)^2)^7} + \frac{B}{\lambda(\gamma\mu\nu(\beta + \lambda)^2)^6} - \frac{C}{\lambda(\gamma\mu\nu(\beta + \lambda)^2)^5} - \frac{D}{\lambda(\gamma\mu\nu(\beta + \lambda)^2)^4} - \frac{E}{\lambda(\gamma\mu\nu(\beta + \lambda)^2)^3} - \frac{F}{\lambda(\gamma\mu\nu(\beta + \lambda)^2)^2} - \frac{\mu\eta\beta}{\lambda v} \left( \frac{\gamma - \frac{\lambda}{\mu}}{\beta\gamma\mu - \alpha\lambda - 2\lambda\beta} \right) P_0. \quad (35)$$

(iv) The mean waiting time  $W_s$  in the system is

$$W_s = -\frac{\eta}{\lambda v} + \frac{A}{\lambda(\gamma\mu\nu(\beta + \lambda)^2)^7} + \frac{B}{\lambda(\gamma\mu\nu(\beta + \lambda)^2)^6} - \frac{C}{\lambda(\gamma\mu\nu(\beta + \lambda)^2)^5} - \frac{D}{\lambda(\gamma\mu\nu(\beta + \lambda)^2)^4} - \frac{E}{\lambda(\gamma\mu\nu(\beta + \lambda)^2)^3} - \frac{F}{\lambda(\gamma\mu\nu(\beta + \lambda)^2)^2} - \frac{\mu\eta\beta}{\lambda v} \left( \frac{\gamma - \frac{\lambda}{\mu}}{\beta\gamma\mu - \alpha\lambda - 2\lambda\beta} \right) P_0. \quad (36)$$

(v) The probability  $R$  of having an empty system is

$$R = \frac{\nu(\lambda + \gamma\mu)}{\mu^2\eta\gamma} \exp\left(\frac{-A}{6(\gamma\mu\nu(\beta + \lambda)^2)^7}\right) \exp\left(\frac{-B}{5(\gamma\mu\nu(\beta + \lambda)^2)^6}\right) \exp\left(\frac{C}{4(\gamma\mu\nu(\beta + \lambda)^2)^5}\right) \exp\left(\frac{D}{3(\gamma\mu\nu(\beta + \lambda)^2)^4}\right) \times \exp\left(\frac{E}{2(\gamma\mu\nu(\beta + \lambda)^2)^3}\right) \exp\left(\frac{F}{(\gamma\mu\nu(\beta + \lambda)^2)^2}\right) \left\{ \lim_{z \rightarrow 0^+} \int_z^1 \exp\left(\frac{-u^6 A}{6(\gamma\mu\nu(\beta + \lambda)^2)^7}\right) \exp\left(\frac{-u^5 B}{5(\gamma\mu\nu(\beta + \lambda)^2)^6}\right) \times \exp\left(\frac{u^4 C}{4(\gamma\mu\nu(\beta + \lambda)^2)^5}\right) \exp\left(\frac{u^3 D}{3(\gamma\mu\nu(\beta + \lambda)^2)^4}\right) \exp\left(\frac{u^2 E}{2(\gamma\mu\nu(\beta + \lambda)^2)^3}\right) \exp\left(\frac{u F}{(\gamma\mu\nu(\beta + \lambda)^2)^2}\right) \times u^{\frac{\eta}{v}-1} \frac{(\gamma - \frac{\lambda}{\mu}u)(\beta + \lambda - \lambda u)}{-(2\lambda^2 u - 2\lambda^2 u^2 - \beta\gamma\mu + \alpha\lambda u + 2\beta\lambda u - \gamma\lambda\mu + \gamma\lambda\mu u)} du \right\}^{-1}. \quad (37)$$

(vi) The interrupted frequency  $N$  of the service caused by collisions is

$$N = (\lambda p + \eta) \frac{\lambda}{\gamma\mu} - \frac{\nu\lambda}{\gamma\mu Y} \left[ \lambda^4(1 - 2p) + \lambda^3(\alpha(p - 1) - \mu + 4p\beta + 4p\lambda - 3\lambda - 2\beta + p\gamma\mu - 2\eta - \nu) + \lambda^2((\beta + \lambda)(2\alpha - 2\gamma p\mu + 2(\mu + \nu)) + 4\beta(\lambda + \eta) + \eta(\alpha + 4\lambda) - 2p(\beta^2 + \lambda^2) + \beta^2 + 3\lambda^2 - \alpha\beta p - \lambda p(\alpha + 4\beta) + \eta\gamma\mu) - \lambda(\beta + \lambda)((\beta + \lambda)(\alpha + 2\eta - \gamma p\mu + \mu) + \lambda(\lambda + \beta + \nu) + \beta\nu + \eta\alpha + \alpha\nu + 2\gamma\eta\mu) + \gamma\eta\mu(\beta + \lambda)^2 \right] + \frac{\nu\beta^2\lambda\eta(\gamma - \frac{\lambda}{\mu})}{\gamma Y} P_0, \quad (38)$$

where  $Y = -\nu\beta(-\beta\gamma\mu + \lambda\alpha + 2\lambda\beta)$ .

(vii) The mean busy period  $E(L)$  of the system is

$$\begin{aligned}
E(L) &= \frac{1}{\lambda} \left( \frac{\mu\eta}{\nu} \exp\left(\frac{A}{6(\gamma\mu\nu(\beta+\lambda)^2)^7}\right) \exp\left(\frac{B}{5(\gamma\mu\nu(\beta+\lambda)^2)^6}\right) \exp\left(\frac{-C}{4(\gamma\mu\nu(\beta+\lambda)^2)^5}\right) \exp\left(\frac{-D}{3(\gamma\mu\nu(\beta+\lambda)^2)^4}\right) \right. \\
&\times \exp\left(\frac{-E}{2(\gamma\mu\nu(\beta+\lambda)^2)^3}\right) \exp\left(\frac{-F}{(\gamma\mu\nu(\beta+\lambda)^2)^2}\right) \left\{ \lim_{z \rightarrow 0^+} \int_z^1 \exp\left(\frac{-u^6 A}{6(\gamma\mu\nu(\beta+\lambda)^2)^7}\right) \exp\left(\frac{-u^5 B}{5(\gamma\mu\nu(\beta+\lambda)^2)^6}\right) \right. \\
&\times \exp\left(\frac{u^4 C}{4(\gamma\mu\nu(\beta+\lambda)^2)^5}\right) \exp\left(\frac{u^3 D}{3(\gamma\mu\nu(\beta+\lambda)^2)^4}\right) \exp\left(\frac{u^2 E}{2(\gamma\mu\nu(\beta+\lambda)^2)^3}\right) \exp\left(\frac{u F}{(\gamma\mu\nu(\beta+\lambda)^2)^2}\right) u^{\frac{7}{\nu}-1} \\
&\times \left. \frac{(\gamma - \frac{\lambda}{\mu}u)(\beta + \lambda - \lambda u)}{-(2\lambda^2 u - 2\lambda^2 u^2 - \beta\gamma\mu + \alpha\lambda u + 2\beta\lambda u - \gamma\lambda\mu + \gamma\lambda\mu u)} du \right\} - 1 \Bigg). \tag{39}
\end{aligned}$$

**Proof 4**

The mean number of customers  $L_q$  (respectively  $L_s$ ) in the orbit (respectively in the system) given in (i) (respectively in (ii)) under the steady-state condition are obtained by differentiating (20) (respectively (21)) with respect to  $z$  and evaluating at  $z = 1$ .

The mean waiting time  $W_q$  (respectively  $W_s$ ) in the orbit (respectively in the system) give in (iii) (respectively in (iv)) is related to the mean number of customers  $L_q$  (respectively  $L_s$ ) in the orbit (respectively in the system) by the Little formula (Little (1961)),  $L_q = \lambda W_q$  and  $L_s = \lambda W_s$ .

The probability  $R$  of having an empty system, given in (v), is obtained by summing the two probabilities  $P_0$  and  $Q_0$  given by (2) and (22).

(vi) The steady-state interrupted frequency  $N$  of the service caused by collisions given in Eq. (40), represents the sum of the interrupted frequency caused when the arriving customer from outside the system (primary customer) enters directly to the busy server and the interrupted frequency due to retrial of the customer from the orbit when the server is busy. Then

$$N = \lambda p Q(1) + \sum_{j=0}^{\infty} Q_j(\eta + j\nu), \tag{40}$$

which can be rewritten as

$$N = (\lambda p + \eta)Q(1) + \nu Q'(1). \tag{41}$$

By replacing  $Q(1)$  and  $Q'(1)$ , given respectively by (14) and (3), we obtain  $N$  given in Eq. (38).

(vii) The busy period of the system,  $L$ , starts with the arrival of a primary customer who finds the system empty and ends at the first departure epoch in which the system becomes empty again. From the regenerative processes theory, it is easy to get the following formula:

$$E(L) = \frac{1}{\lambda}(P_0^{-1} - 1).$$

Using (22), we obtain  $E(L)$  given in Eq.(39).□



## 5 Numerical illustrations

In this section, we give some numerical illustrations to show the effect of the variation and the impact of some parameters on the system performances.

In communication systems, it is interesting to determine the proportion of time that the channel is free and no packets need to be re-transmitted, i.e., no station ask for transmitting through this channel. Then, we first consider the variation of the steady state joint probability  $P_0$ , that the channel is idle and no packets in the orbit, according to different system parameters. In **Figs. 2-7**, we present the variation of the probability  $P_0$  against the retrial rate  $\nu$ , for three values of  $\gamma$  ( $\gamma = 0.3$ ,  $\gamma = 0.5$  and  $\gamma = 0.7$ ), where  $\gamma$  represents the probability that the packet is transmitted without error. Its values correspond to the channel conditions and packet state (erroneous packet or not).

As it is shown, the probability  $P_0$  increases when the retrial rate  $\nu$  increases for any value of  $\gamma$ . However,  $P_0$  increases considerably as the probability  $\gamma$  increases. When the probability  $\gamma$  becomes small, the number of packets that need re-transmission due to transmission errors become more important. This generates a large mass of packets in the orbit and consequently a low probability of having an empty orbit.

Really, the retrial rate is based on the stations number that try to transmit through the same channel. Then, it is very important to see the evolution of  $P_0$  for different retrial rate  $\eta$ . **Figs. 2-4** show the influence of  $\eta$  on the evolution of  $P_0$ . We notice that for a high retrial rate ( $\eta = 5$ ), values of  $P_0$  are more important, this can be explained by the fact that the higher rate  $\eta$  gives more chance to packets from the orbit to be transmitted.

In queueing systems with failures, the repair rate is one of the most important parameters. In telecommunication networks, it can interpret the ability of the system to resume service after an interruption. In **Figs. 5-7**, we show the impact of the repair rate  $\beta$  on  $P_0$ , while the other parameters are fixed. From these figures, we notice, as expected, that  $P_0$  increases with increasing the repair rate  $\beta$ , for any value of  $\gamma$ .

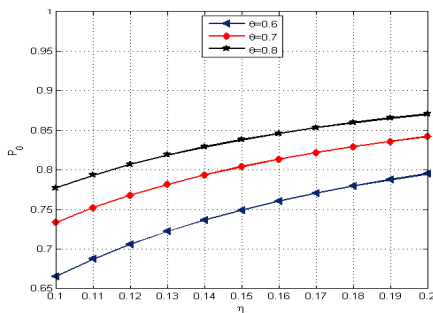


Figure 2:  $P_0$  as function of  $\nu$  for  $\lambda = 0.05$ ,  
 $\eta = 1.5$ ,  $\mu = 2$ ,  $\alpha = 10$ ,  $\beta = 10$ ,  $p = 0$ .

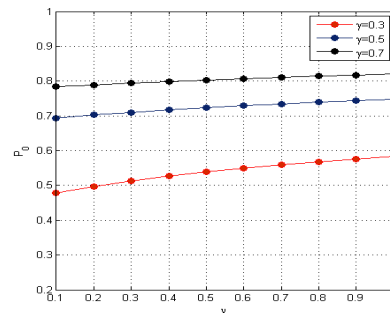


Figure 3:  $P_0$  as function of  $\nu$  for  $\lambda = 0.05$ ,  
 $\eta = 3$ ,  $\mu = 2$ ,  $\alpha = 10$ ,  $\beta = 10$ ,  $p = 0$ .

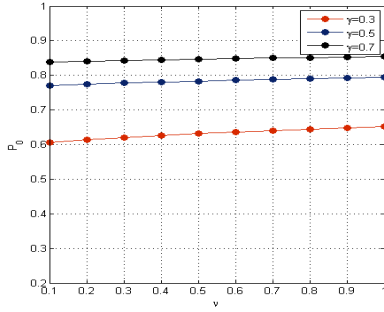


Figure 4:  $P_0$  as function of  $v$  for  $\lambda = 0.05$ ,  
 $\eta = 5, \mu = 2, \alpha = 10, \beta = 10, p = 0$ .

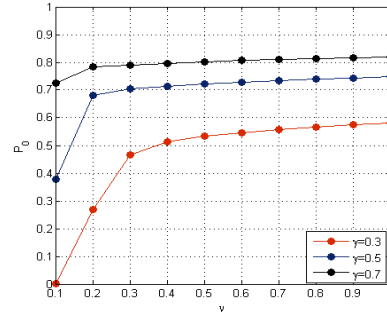


Figure 5:  $P_0$  as function of  $v$  for  $\lambda = 0.05$ ,  
 $p = 0, \mu = 2, \alpha = 5, \beta = 3, \eta = 2$ .

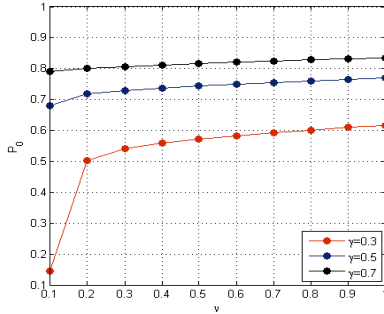


Figure 6:  $P_0$  as function of  $v$  for  $\lambda = 0.05$ ,  
 $p = 0, \mu = 2, \alpha = 5, \beta = 4, \eta = 2$ .

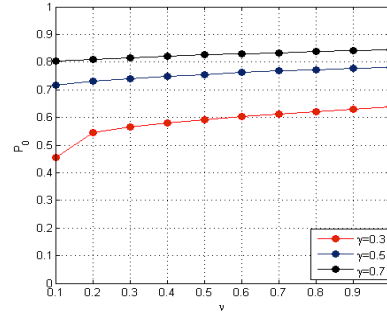


Figure 7:  $P_0$  as function of  $v$  for  $\lambda = 0.05$ ,  
 $p = 0, \mu = 2, \alpha = 5, \beta = 5, \eta = 2$ .

Recent communication protocols try to minimize collisions but cannot avoid them completely. Calculating the number of service interruptions due to collisions is very beneficial for evaluating these protocols. Then, we illustrate in **Fig. 8** how the steady-state interrupted frequency  $N$  of the service caused by collisions varies when the probability  $p$  increases, for two values of the arrival rate  $\lambda$  ( $\lambda = 0.05$  and  $\lambda = 0.2$ ). We observe that  $N$  increases according to the probability  $p$ . This effect is less visible when the arrival rate  $\lambda$  is small. Indeed, there is not a great influence on the frequency  $N$  because the service rate is high ( $\mu = 2$ ) compared to the arrival rate  $\lambda$  ( $\lambda = 0.05$ ). These conditions allow the channel to serve packets without having a lot of arrivals from outside the system and reduce considerably the collisions.

As shown in **Figs. 9-11**, the number of packets needing re-transmission ( $L_q$ ) is reduced when the retrial rate become large, this fact is also visible when increasing the probability  $p$ .

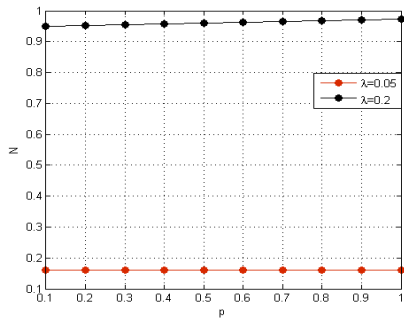


Figure 8:  $N$  versus  $p$  for  $\gamma = 0, 5$ ,  $\eta = 3$ ,  $\mu = 2$ ,  $\nu = 1.5$ ,  $\alpha = 1.5$  and  $\beta = 3$ .

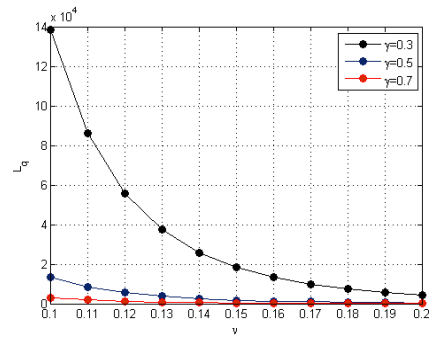


Figure 9:  $L_q$  versus  $v$  for  $p = 0$ ,  $\lambda = 0.1$ ,  $\eta = 1.5$ ,  $\mu = 10$ ,  $\alpha = 0.01$ ,  $\beta = 0.5$ .

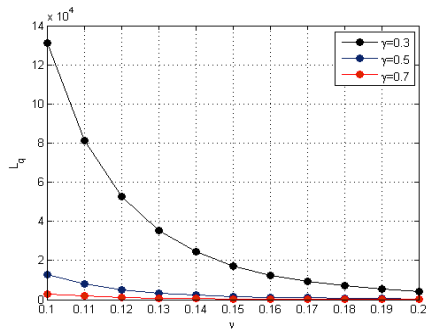


Figure 10:  $L_q$  versus  $v$  for  $p = 0.5$ ,  $\lambda = 0.1$ ,  $\eta = 1.5$ ,  $\mu = 10$ ,  $\alpha = 0.01$ ,  $\beta = 0.5$ .

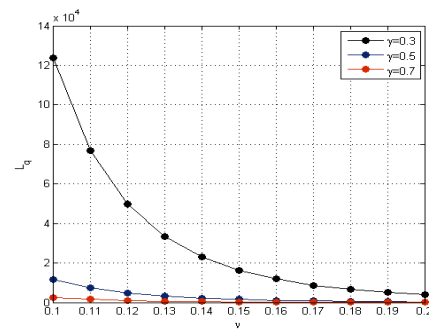


Figure 11:  $L_q$  versus  $v$  for  $p = 1$ ,  $\lambda = 0.1$ ,  $\eta = 1.5$ ,  $\mu = 10$ ,  $\alpha = 0.01$ ,  $\beta = 0.5$ .

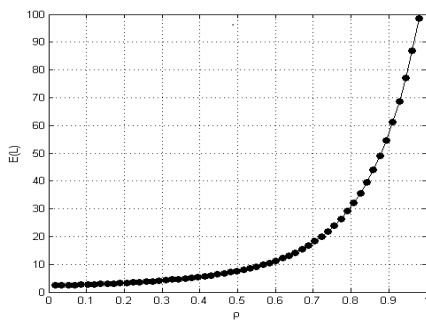


Figure 12:  $E(L)$  versus  $\rho$  for  $p = 0$ ,  $\eta = 3$ ,  $\nu = 1$ .

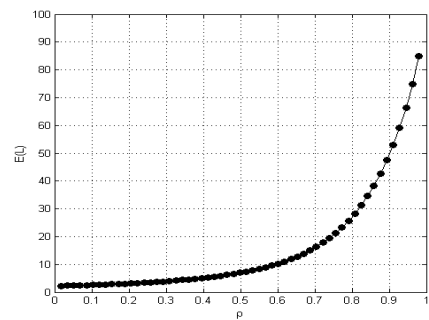


Figure 13:  $E(L)$  versus  $\rho$  for  $p = 0.5$ ,  $\eta = 3$ ,  $\nu = 1$ .

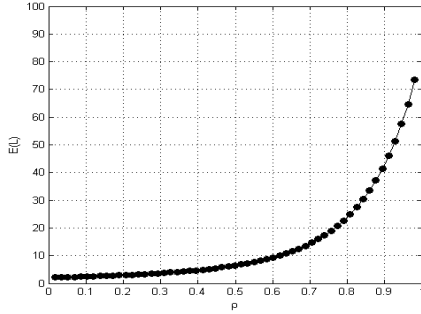


Figure 14:  $E(L)$  versus  $\rho$  for  $p = 1$ ,  
 $\eta = 3, v = 1$ .

**Figs. 12-14**, represent the variation of the mean busy period  $E(L)$  of the system according to  $\rho$  (the utilization factor) for three values of  $p$  ( $p=0, p=0.5, p=1$ ). We observe that  $E(L)$  increases according to  $\rho$ , this characteristic is reduced when primary packets join directly the channel ( $p = 1$ ) compared with the case where packets are transmitted with a probability  $p = 0$  or  $p = 0.5$ . This appears significantly in the case when the system approaches the saturation ( $\rho \simeq 1$ ).

Really, this is explained by the fact that when  $p = 1$ , packets have more chance to be transmitted without passing through the orbit and thus reduce the number of retrieval packets.

## 6 Conclusion

In this paper, we have investigated an unreliable Markovian single server retrieval queue with collisions, transmission errors and linear retrieval policy. This queueing model can be used for performance evaluation of channel random access in communication systems. To that end, we have used the Markov process theory to derive the steady-state equations. We have applied the generating function method to obtain the joint steady-state probability generating functions of the server state and the orbit length, and we have also computed essential characteristics. Finally, we have presented numerical illustrations for several system characteristics based on given numerical values for the system parameters. The obtained numerical results confirm the effectiveness of our analytical research. Several future works could be drawn from our results. First, It is possible to work on some system parameters like  $(\mu, p, \beta)$  to obtain a desired value of  $P_0$ . Then, one can develop an adaptive algorithm for the management of the probability  $p$  to according to a fixed  $E(L)$  value. It is also interesting to consider our result to emerge to cognitive radio management.

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