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THE NEHARI MANIFOLD FOR A ψ -HILFER FRACTIONAL p -LAPLACIAN

J. VANTERLER DA C. SOUSA, JIABIN ZUO*, AND DONAL O'REGAN

ABSTRACT. In this paper, we discuss the existence and non-existence of weak solutions to the non-linear problem with a fractional p -Laplacian introduced by the ψ -Hilfer fractional operator, by combining the technique of Nehari manifolds and fibering maps. Also, we obtain some results on the ψ -Hilfer fractional operator and the Nehari manifold via the Euler functional.

1. INTRODUCTION

In this paper, we investigate the existence and non-existence of weak solutions to the nonlinear problem with a fractional p -Laplacian given by

$$(1.1) \quad \begin{cases} \mathbf{H}\mathbf{D}_T^{\alpha,\beta;\psi} \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^{p-2} \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right) = \lambda |\xi(x)|^{p-2} \xi(x) + b(x) |\xi(x)|^{q-1} \xi(x), \\ \mathbf{I}_{0+}^{\beta(\beta-1);\psi} \xi(0) = \mathbf{I}_T^{\beta(\beta-1);\psi} \xi(T) = 0, \end{cases}$$

where $\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi}(\cdot)$, $\mathbf{H}\mathbf{D}_T^{\alpha,\beta;\psi}(\cdot)$ are ψ -Hilfer fractional derivatives left-sided and right-sided of order $\frac{1}{p} < \alpha < 1$, type $0 \leq \beta \leq 1$, $1 < q < p - 1 < \infty$, $b \in L^\infty(\Omega)$ and $I_{0+}^{\beta(\beta-1);\psi}(\cdot)$ and $I_T^{\beta(\beta-1);\psi}(\cdot)$ are ψ -Riemann-Liouville fractional integrals left-sided and right-sided, for all $x \in \Omega = [0, T]$. We also have that $\xi \in \mathbb{H}_p^{\alpha,\beta;\psi}$ and $\lambda > 0$.

In the early 1960's, Nehari [1], introduced a method that became very useful in the calculus of variations and which is now called the Nehari manifold method. Nehari's original idea was to study a boundary value problem for certain second order nonlinear ordinary differential equations in an open interval (a, b) and show that the equation has a non-trivial solution that can be obtained through minimization problems with link. The fibering maps method is a good tool for solving elliptical differential problems [10, 13, 14, 15]. Relating the Nehari manifold with fibering maps was introduced by Drábek and Pohozaev (1997) [10] and discussed by Brown and Zhang (2003) [9] which related the Euler Lagrange functional to a real function.

In 2003, Figueiredo et al. [11], investigated the existence of solutions to an elliptic class of problems

$$\begin{cases} -\Delta u & = \lambda u^q + b(x) u^p, \text{ in } \Omega \\ u \geq 0, u \neq 0 & \text{in } \Omega \\ u & = 0 \end{cases}$$

where Ω is a bounded region of \mathbb{R}^n with smooth boundary, $0 < q < 1 < p < \frac{N+2}{N-2}$, $\lambda > 0$ and $a, b : \bar{\Omega} \rightarrow \mathbb{R}$ are regular functions that can change signs in Ω , using the Mountain Pass Theorem. In 2005, IL'yasov [13] and 2009 Wu [12], discussed the existence of solutions via the fibering approach and Nehari manifolds, respectively. In 1997 Drábek and Pohozaev [10] discussed positive solutions for p -Laplacian fibering maps, in 2003 Brown and Zhang [9], discussed the existence of solutions to the problem via a combination of fibering maps and Nehari manifolds.

In 2013, Nyamoradi et al. [14], investigated the existence and multiplicity of positive solutions to a quasilinear elliptic equation

$$(1.2) \quad \begin{cases} -\Delta_p u - \sum_{i=1}^k \mu_i \frac{|u|^{p-2} u}{|x-a_i|^p} & = f(x) |u|^{p^*-2} u + \lambda g(x) |u|^{q-2}, \text{ in } \Omega, \\ u & = 0, \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$ such that the points $a_i \in \Omega$, $i = 1, 2, \dots, k$ ($k \geq 2$), $0 \leq \mu_i < \bar{\mu} := \left(\frac{N-p}{p}\right)^p$, and $p^* := \frac{pN}{N-p}$ is the critical Sobolev exponent and $1 \leq q < p$,

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$\lambda > 0$ and f, g are continuous functions. With the problem (1.2) one can relate it to the following Hardy inequality

$$\int_{\Omega} \frac{|u|^p}{|x-a|^p} dx \leq \frac{1}{\mu} \int_{\Omega} |\nabla u|^p dx, \quad \forall a \in \mathbb{R}^N, \quad u \in C_0^\infty(\mathbb{R}^N).$$

Results on the existence and multiplicity of solutions for nonlinear boundary value problems of fractional differential equations in a fractional derivative space can be found in [16, 17, 18, 19, 33, 41]. Ledesma [20, 21, 22] and others discuss variational problems via fractional derivatives and integrals, with important techniques, such as the Mountain Pass Theorem, Nehari manifolds, critical point theory and fibering maps (see [17, 23, 24, 36, 37, 40, 42]). For other works see [25, 26, 27, 30, 31]. In 2020, Sousa et al. [2], discussed a new variational structure for the ψ -Hilfer fractional operator, justifying some fundamental properties in the variational context, in the ψ -fractional derivative spaces $\mathbb{H}_p^{\alpha, \beta; \psi}([0, T], \mathbb{R})$ and $\mathbb{H}_p^{2\alpha, \beta; \psi}([0, T], \mathbb{R})$, so one can discuss numerous boundary value problems in the fractional differential equation sense.

In 2018, Saoudi et al. [38], discussed the existence of solutions to the following nonlinear boundary value problem of the fractional differential equation:

$$\begin{cases} - {}_t\mathbb{D}_1^\alpha \left(|{}_0\mathbb{D}_t^\alpha (u(t))|^{p-2} {}_0\mathbb{D}_t^\alpha u(t) \right) & = f(t, u(t)) + \lambda g(t) |u(t)|^{q-2} u(t), \quad (t \in (0, 1)), \\ u(0) & = u(1) = 0, \end{cases}$$

where ${}_0\mathbb{D}_t^\alpha(\cdot)$, ${}_t\mathbb{D}_1^\alpha(\cdot)$ are Riemann-Liouville fractional derivatives left-sided and right-sided of order α with $\frac{1}{2} < \alpha < 1$, λ is a positive parameter, $2 < r < p < q$, $g \in C([0, 1])$, and $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, using a combination of the Nehari manifold and fibering maps.

Ghanmi and Zhang [44], investigated the existence of non-trivial solutions the following fractional boundary value problems

$$\begin{cases} {}_t\mathbb{D}_T^\alpha \left(|{}_0\mathbb{D}_t^\alpha (u(t))|^{p-2} {}_0\mathbb{D}_t^\alpha u(t) \right) & = \nabla W(t, u(t)) + \lambda g(t) |u(t)|^{q-2} u(t), \quad (t \in (0, T)), \\ u(0) & = u(T) = 0, \end{cases}$$

where ${}_0\mathbb{D}_t^\alpha(\cdot)$, ${}_t\mathbb{D}_T^\alpha(\cdot)$ are Riemann-Liouville fractional derivatives left-sided and right-sided α with $\frac{1}{p} < \alpha < 1$, $\nabla W(t, u)$ is the gradient of $W(t, u)$ at u and $W \in C([0, T] \times \mathbb{R}^n, \mathbb{R})$ is homogeneous of degree r , λ is a positive parameter, $g \in C([0, T])$, $1 < r < p < q$, using fibering maps and Nehari manifolds, for some positive constant λ_0 such that $0 < \lambda < \lambda_0$. Other works related to the existence of solutions for fractional boundary value problems can be found in [19, 25, 27, 36, 44, 42].

One of the main aims of this paper is to contribute new results to the theory of boundary value problem for fractional differential equations. The Euler functional associated with Eq.(1.1), for each $\xi \in \mathbb{H}_p^{\alpha, \beta; \psi}$, is given in the following form

$$\mathbf{E}_\lambda(\xi) = \frac{1}{p} \int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi(x) \right|^p dx - \frac{\lambda}{p} \int_0^T |\xi(x)|^p dx - \frac{1}{q+1} \int_0^T b(x) |\xi(x)|^{q+1} dx.$$

Motivated by the works above, this article aims to discuss the following results:

Theorem 1.1. *Suppose that there exists $\bar{\lambda} > 0$ such that for any $\lambda < \bar{\lambda}$, we have $\mathbf{L}_-(\lambda) \subseteq \mathbf{Y}_-$. Then,*

- (1) *There is a minimizing point for \mathbf{E}_λ in \mathbf{S}_λ^+ ,*
- (2) *There is a minimizing point for \mathbf{E}_λ in \mathbf{S}_λ^- ,*

proving that $\mathbf{L}_-(\lambda)$ is non-empty.

Theorem 1.2. *Suppose $\mathbf{L}_-(\lambda) \cap \mathbf{Y}_+ \neq \emptyset$. Then*

- (1) $\inf_{\xi \in \mathbf{S}_\lambda^+} \mathbf{E}_\lambda(\xi) = -\infty$, *when $\mathbf{S}_\lambda^+ \neq \emptyset$;*
- (2) $\inf_{\xi \in \mathbf{S}_\lambda^-} \mathbf{E}_\lambda(\xi) = 0$, *when $\mathbf{S}_\lambda^- \neq \emptyset$.*

Note $\mathbf{L}_-(\lambda)$, $\mathbf{Y}_-(\lambda)$, $\mathbf{Y}_+(\lambda)$, \mathbf{S}_λ^- and \mathbf{S}_λ^+ , will be presented in a later section.

In section 2 we present fundamental concepts of the ψ -Riemann-Liouville fractional integral and the ψ -Hilfer fractional derivative and some essential properties. We give the definition of the ψ -fractional derivative space and some new results are obtained, in particular, the integration by parts for the ψ -Hilfer fractional derivative. In section 3, we introduce the Nehari manifold and fibering maps and some results are proven involving the Euler functional in the ψ -fractional derivative space $\mathbb{H}_p^{\alpha, \beta; \psi}$. Finally, section 4 shows the existence and non-existence of weak solutions to the non-linear problem with a fractional p -Laplacian according to Eq.(1.1) in the sense of the ψ -Hilfer fractional derivative.

2. PRELIMINARIES AND THE ψ -FRACTIONAL DERIVATIVE SPACE $\mathbb{H}_p^{\alpha,\beta;\psi}([0, T], \mathbb{R})$.

In this section, we present the fundamental concepts of the ψ -Riemann-Liouville fractional integral and the ψ -Hilfer fractional derivative, as well as some fundamental results. We discuss a version of the integration by parts for the ψ -Hilfer fractional derivative and consider some particular cases. In addition, we present the variational structure that will define and prove the existence and non-existence of weak solutions to the fractional nonlinear problem p -Laplacian.

Let (a, b) ($-\infty \leq a < b \leq \infty$) be a finite or infinite interval of the real line \mathbb{R} and $\alpha > 0$. Also let $\psi(\cdot)$ be an increasing and positive monotone function on $[a, b]$, having a continuous derivative $\psi'(x) \neq 0$ on (a, b) . The left and right-sided fractional integrals of a function f with respect to another function ψ on $[a, b]$, are defined by [3, 4, 7]

$$(2.1) \quad \mathbf{I}_{a+}^{\alpha;\psi} \xi(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} \xi(t) dt$$

and

$$(2.2) \quad \mathbf{I}_{b-}^{\alpha;\psi} \xi(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} \xi(t) dt.$$

On the other hand, let $n - 1 < \alpha < n$, with $n \in \mathbb{N}$, $I = [a, b]$ is the interval such that $-\infty \leq a < b \leq \infty$ and there exist two functions $f, \psi \in C^n([a, b], \mathbb{R})$ such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The ψ -Hilfer fractional derivatives left-sided and right-sided $\mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi}(\cdot)$ ($\mathbf{H}\mathbf{D}_{b-}^{\alpha,\beta;\psi}(\cdot)$) function of order α and type $0 \leq \beta \leq 1$ are defined by [3, 4, 7]

$$(2.3) \quad \mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} \xi(x) = \mathbf{I}_{a+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathbf{I}_{a+}^{(1-\beta)(n-\alpha);\psi} \xi(x)$$

and

$$(2.4) \quad \mathbf{H}\mathbf{D}_{b-}^{\alpha,\beta;\psi} \xi(x) = \mathbf{I}_{b-}^{\beta(n-\alpha);\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathbf{I}_{b-}^{(1-\beta)(n-\alpha);\psi} \xi(x).$$

Choosing $\beta \rightarrow 0$, we have ψ -Riemann-Liouville fractional derivatives left-sided and right-sided, given by

$$(2.5) \quad D_{a+}^{\alpha;\psi} \xi(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathbf{I}_{a+}^{(n-\alpha);\psi} \xi(x)$$

and

$$(2.6) \quad D_{b-}^{\alpha;\psi} \xi(x) = \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathbf{I}_{b-}^{(n-\alpha);\psi} \xi(x).$$

The ψ -Hilfer fractional derivatives defined as above can be written in the following form

$$(2.7) \quad \mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} \xi(x) = \mathbf{I}_{a+}^{\gamma-\alpha;\psi} D_{a+}^{\gamma;\psi} \xi(x)$$

and

$$(2.8) \quad \mathbf{H}\mathbf{D}_{b-}^{\alpha,\beta;\psi} \xi(x) = \mathbf{I}_{b-}^{\gamma-\alpha;\psi} D_{b-}^{\gamma;\psi} \xi(x)$$

with $\gamma = \alpha + \beta(n - \alpha)$ and $\mathbf{I}_{a+}^{\gamma-\alpha;\psi}(\cdot)$, $\mathbf{I}_{b-}^{\gamma-\alpha;\psi}(\cdot)$, $D_{b-}^{\gamma;\psi}(\cdot)$ and $D_{a+}^{\gamma;\psi}(\cdot)$ as defined in Eq.(2.1), Eq.(2.2), Eq.(2.5) and Eq.(2.6).

Theorem 2.1. [3, 4] *Let $n - 1 < \alpha < n$, $n \in \mathbb{N}$ and $0 \leq \beta \leq 1$. If $\xi \in C^n([a, b])$, then*

$$\mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} \xi(x) = D_{a+}^{n-\beta(n-\alpha);\psi} \left(\mathbf{I}_{a+}^{(n-\gamma);\psi} \xi(x) - \sum_{k=0}^{n-1} \frac{(\psi(x) - \psi(a))^k}{k!} D_{a+,k}^{\gamma;\psi} \xi(a) \right)$$

and

$$\mathbf{H}\mathbf{D}_{b-}^{\alpha,\beta;\psi} \xi(x) = D_{b-}^{n-\beta(n-\alpha);\psi} \left(\mathbf{I}_{b-}^{(n-\gamma);\psi} \xi(x) - \sum_{k=0}^{n-1} (-1)^k \frac{(\psi(b) - \psi(x))^k}{k!} D_{a+,k}^{\gamma;\psi} \xi(b) \right)$$

with $\gamma = \alpha + \beta(n - \alpha)$.

Lemma 2.2. [3, 4] *Let $\alpha > 0$ and $\delta > 0$. Then, we have the following semigroup property given by*

$$(2.9) \quad \mathbf{I}_{a+}^{\alpha;\psi} \mathbf{I}_{a+}^{\delta;\psi} \xi(x) = \mathbf{I}_{a+}^{\alpha+\delta;\psi} \xi(x)$$

and

$$(2.10) \quad \mathbf{I}_{b-}^{\alpha;\psi} \mathbf{I}_{b-}^{\delta;\psi} \xi(x) = \mathbf{I}_{b-}^{\alpha+\delta;\psi} \xi(x).$$

Theorem 2.3. [3, 4] *Let $\xi \in C^1([a, b], \mathbb{R})$, $\alpha > 0$ and $0 \leq \beta \leq 1$, and we have*

$$\mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} \mathbf{I}_{a+}^{\alpha;\psi} \xi(x) = \xi(x)$$

and

$$(2.11) \quad \mathbf{H}\mathbf{D}_{b-}^{\alpha,\beta;\psi} \mathbf{I}_{b-}^{\alpha;\psi} \xi(x) = \xi(x).$$

Theorem 2.4. [3, 4] *If $\xi \in C^n([a, b], \mathbb{R})$, $n - 1 < \alpha < n$ and $0 \leq \beta \leq 1$, then*

$$(2.12) \quad \mathbf{I}_{a+}^{\alpha;\psi} \mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} \xi(x) = \xi(x) - \sum_{k=1}^n \frac{(\psi(x) - \psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} \xi_{\psi}^{[n-k]} \mathbf{I}_{a+}^{(1-\beta)(n-\alpha);\psi} \xi(a)$$

and

$$(2.13) \quad \mathbf{I}_{b-}^{\alpha;\psi} \mathbf{H}\mathbf{D}_{b-}^{\alpha,\beta;\psi} \xi(x) = \xi(x) - \sum_{k=1}^n \frac{(\psi(b) - \psi(x))^{\gamma-k}}{\Gamma(\gamma-k+1)} \xi_{\psi}^{[n-k]} \mathbf{I}_{b-}^{(1-\beta)(n-\alpha);\psi} \xi(b),$$

where $\xi_{\psi}^{[n-k]}(x) := \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n-k} \xi(x)$.

Next, we will prove the integration by parts for the ψ -Riemann-Liouville fractional integral and the ψ -Hilfer fractional derivative.

The relation

$$(2.14) \quad \int_a^b \left(\mathbf{I}_{a+}^{\alpha;\psi} \xi(t) \right) \theta(t) dt = \int_a^b \xi(t) \psi'(t) \mathbf{I}_{b-}^{\alpha;\psi} \left(\frac{\theta(t)}{\psi'(t)} \right) dt$$

is valid.

One can prove Eq.(2.14) directly by interchanging the order of integration by the Dirichlet formula in the particular case Fubini theorem, i.e.,

$$\begin{aligned} \int_a^b \left(\mathbf{I}_{a+}^{\alpha;\psi} \xi(t) \right) \theta(t) dt &= \int_a^b \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \xi(s) ds \theta(t) dt \\ &= \int_a^b \frac{1}{\Gamma(\alpha)} \int_t^b \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \theta(t) dt \xi(s) ds \\ &= \int_a^b \xi(t) \psi'(t) \mathbf{I}_{b-}^{\alpha;\psi} \left(\frac{\theta(t)}{\psi'(t)} \right) dt. \end{aligned}$$

Theorem 2.5. *Let $\psi(\cdot)$ be an increasing and positive monotone function on $[a, b]$, having a continuous derivative $\psi'(\cdot) \neq 0$ on (a, b) . If $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$, then*

$$(2.15) \quad \int_a^b \left(\mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} \xi(t) \right) \theta(t) dt = \int_a^b \xi(t) \psi'(t) \mathbf{H}\mathbf{D}_{b-}^{\alpha,\beta;\psi} \left(\frac{\theta(t)}{\psi'(t)} \right) dt$$

for any $\xi \in AC^1$ and $\theta \in C^1$ satisfying the boundary conditions $\xi(a) = 0 = \xi(b)$.

Proof. In fact, using Eq.(2.8), Eq.(2.10) and Theorem 2.4 (Eq.(2.12)), we have

$$\begin{aligned} & \int_a^b \xi(t) \psi'(t) \mathbf{H}\mathbf{D}_{b-}^{\alpha,\beta;\psi} \left(\frac{\theta(t)}{\psi'(t)} \right) dt \\ &= \int_a^b \xi(t) \psi'(t) \mathbf{I}_{b-}^{1-\alpha;\psi} D_{b-}^{1;\psi} \left(\frac{\theta(t)}{\psi'(t)} \right) dt \\ &= \int_a^b \psi'(t) \left[\mathbf{I}_{a+}^{\alpha;\psi} \mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} \xi(t) + \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} d_j \right] \mathbf{I}_{b-}^{1-\alpha;\psi} D_{b-}^{1;\psi} \left(\frac{\theta(t)}{\psi'(t)} \right) dt \\ & \quad \left(\text{where } d_j = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right) \mathbf{I}_{a+}^{(1-\beta)(1-\alpha);\psi} \xi(a) \right) \end{aligned}$$

$$\begin{aligned}
&= \int_a^b \psi'(t) \mathbf{I}_{a+}^{\alpha;\psi} \mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} \xi(t) \mathbf{I}_{b-}^{1-\alpha;\psi} D_{b-}^{1;\psi} \left(\frac{\theta(t)}{\psi'(t)} \right) dt \\
&+ \frac{d_j}{\Gamma(\gamma)} \int_a^b \psi'(t) (\psi(t) - \psi(a))^{\gamma-1} \mathbf{I}_{b-}^{1-\gamma;\psi} D_{b-}^{1;\psi} \left(\frac{\theta(t)}{\psi'(t)} \right) dt \\
&= \int_a^b \mathbf{I}_{a+}^{\alpha;\psi} \mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} \xi(t) \mathbf{I}_{b-}^{-\alpha;\psi} \left(\frac{\theta(t)}{\psi'(t)} \right) dt \\
&= \int_a^b \left(\mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} \xi(t) \right) \theta(t) dt.
\end{aligned}$$

□

We now present some particular cases of the integration by parts of some fractional derivatives.

Theorem 2.6. *Let $0 < \alpha \leq 1$ and taking limit $\beta \rightarrow 1$ in Eq.(2.15), we have the integration by parts for ψ -Caputo fractional derivative, given by*

$$\int_a^b \left({}^{\mathbf{C}}\mathbf{D}_{a+}^{\alpha;\psi} \xi(t) \right) \theta(t) dt = \int_a^b \xi(t) \psi'(t) {}^{\mathbf{C}}\mathbf{D}_{b-}^{\alpha;\psi} \left(\frac{\theta(t)}{\psi'(t)} \right) dt$$

for any $\xi \in AC^1$ and $\theta \in C^1$ satisfying the boundary conditions $\xi(a) = 0 = \xi(b)$.

Theorem 2.7. *Let $0 < \alpha \leq 1$, $\psi(t) = t$ and taking limit $\beta \rightarrow 1$ in Eq.(2.15), we have the integration by parts for Caputo fractional derivative, given by*

$$\int_a^b \left({}^{\mathbf{C}}\mathbf{D}_{a+}^{\alpha} \xi(t) \right) \theta(t) dt = \int_a^b \xi(t) {}^{\mathbf{C}}\mathbf{D}_{b-}^{\alpha} \theta(t) dt$$

for any $\xi \in AC^1$ and $\theta \in C^1$ satisfying the boundary conditions $\xi(a) = 0 = \xi(b)$.

Theorem 2.8. *Let $0 < \alpha \leq 1$ and taking limit $\beta \rightarrow 0$ in Eq.(2.15), we have the integration by parts for ψ -Riemann-Liouville fractional derivative, given by*

$$\int_a^b \left(\mathbf{D}_{a+}^{\alpha;\psi} \xi(t) \right) \theta(t) dt = \int_a^b \xi(t) \psi'(t) \mathbf{D}_{b-}^{\alpha;\psi} \left(\frac{\theta(t)}{\psi'(t)} \right) dt$$

for any $\xi \in AC^1$ and $\theta \in C^1$ satisfying the boundary conditions $\xi(a) = 0 = \xi(b)$.

Theorem 2.9. *Let $0 < \alpha \leq 1$, $\psi(t) = t$ and taking limit $\beta \rightarrow 0$ in Eq.(2.15), we have the integration by parts for Riemann-Liouville fractional derivative, given by*

$$\int_a^b \left({}^{\mathbf{C}}\mathbf{D}_{a+}^{\alpha} \xi(t) \right) \theta(t) dt = \int_a^b \xi(t) {}^{\mathbf{C}}\mathbf{D}_{b-}^{\alpha} \theta(t) dt$$

for any $\xi \in AC^1$ and $\theta \in C^1$ satisfying the boundary conditions $\xi(a) = 0 = \xi(b)$.

Theorem 2.10. *Let $0 < \frac{1}{p} < \alpha < 1$ and $q = \frac{p}{p-1}$. Then for any $\xi \in L^p$, we have:*

- (1) $\mathbf{I}_{a+}^{\alpha;\psi}$ is Hölder continuous on $[a, b]$ with exponent $\alpha - \frac{1}{p} > 0$.
- (2) $\lim_{t \rightarrow a} \mathbf{I}_{a+}^{\alpha;\psi} \xi(t) = 0$.

Proof. The proof is a direct consequence of Proposition 7.3 [43]. □

Consequently $\mathbf{I}_{a+}^{\alpha;\psi} \xi(t)$ can be continuously extended by 0 in $t = a$. Finally, for any $\xi \in L^p$, we have $\mathbf{I}_{a+}^{\alpha;\psi} \xi \in C$. Moreover, the following inequality holds

$$\left\| \mathbf{I}_{a+}^{\alpha;\psi} \right\|_{L^p[0,T]} \leq \frac{(\psi(b) - \psi(a))}{\Gamma(\alpha) ((\alpha - 1)q + 1)^{\frac{1}{q}}} \|\xi\|_{L^p[0,T]}$$

for any $\xi \in L^p$.

Lemma 2.11. [2, 26] *Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$. For any $\xi \in L^p([0, T], \mathbb{R})$, we have*

$$\left\| \mathbf{I}_{a+}^{\alpha;\psi} \right\|_{L^p[0,T]} \leq \frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha + 1)} \|\xi\|_{L^p[0,T]}$$

for any $t \in [0, T]$.

Consider the Banach space $L^p([0, T], \mathbb{R})$ ($1 < p < \infty$) of functions on $[0, T]$ with values in \mathbb{R} with the norm

$$\|\xi\|_p = \left(\int_0^T |\xi(t)|^p dt \right)^{1/p}$$

and $L^\infty([0, T], \mathbb{R})$ is the Banach space of essentially bounded functions from $[0, T]$ into \mathbb{R} equipped with the norm

$$\|\xi\|_\infty = \text{ess sup}_{t \in [0, T]} |\xi(t)|.$$

Definition 2.12. [2, 26] Let $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ and $1 < p < \infty$, The ψ -fractional derivative space $\mathbb{H}_p^{\alpha, \beta; \psi} := \mathbb{H}_p^{\alpha, \beta; \psi}([0, T], \mathbb{R})$ is defined by the closure of $C_0^\infty([0, T], \mathbb{R})$, and is given by

$$(2.16) \quad \begin{aligned} \mathbb{H}_p^{\alpha, \beta; \psi} &= \left\{ \xi \in L^p([0, T], \mathbb{R}); \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi \in L^p([0, T], \mathbb{R}), \mathbf{I}_{0+}^{\beta(\beta-1)} \xi(0) = \mathbf{I}_T^{\beta(\beta-1)} \xi(T) = 0 \right\} \\ &= \overline{C_0^\infty([0, T], \mathbb{R})} \end{aligned}$$

with the following norm

$$(2.17) \quad \|\xi\|_{\mathbb{H}_p^{\alpha, \beta; \psi}} = \left(\|\xi\|_{L^p}^p + \left\| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi \right\|_{L^p}^p \right)^{1/p},$$

where $\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi}(\cdot)$ is the ψ -Hilfer fractional derivative with $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$.

Lemma 2.13. [2, 26] Let $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$ and $1 \leq p < \infty$. For any $\xi \in L^p([0, T], \mathbb{R})$ we have

$$\left\| \mathbf{I}_{0+}^{\alpha; \psi} \right\|_{L^p[0, T]} \leq \frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha + 1)} \|\xi\|_{L^p[0, T]},$$

for any $t \in [0, T]$.

Proposition 2.14. [2, 26] Let $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ and $1 < p < \infty$. For all $\xi \in \mathbb{H}_p^{\alpha, \beta; \psi}$, if $1 - \alpha \geq 1/p$ or $\alpha > 1/p$, we have

$$(2.18) \quad \|\xi\|_{L^p} \leq \frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha + 1)} \left\| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi \right\|_{L^p}.$$

Moreover, if $\alpha > 1/p$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$(2.19) \quad \|\xi\|_\infty \leq \frac{(\psi(T) - \psi(0))^{\alpha-1/p}}{\Gamma(\alpha) ((\alpha-1)q + 1)^{1/q}} \left\| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi \right\|_{L^p},$$

where $\|\xi\|_\infty = \sup_{t \in [0, T]} |\xi(t)|$.

From the inequality (2.19), we also have

$$\|\xi\|_\infty \leq \frac{(\psi(T) - \psi(0))^{\alpha-1/p}}{\Gamma(\alpha) ((\alpha-1)q + 1)^{1/q}} \left\| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi \right\|_{\mathbb{H}_p^{\alpha, \beta; \psi}}$$

that $\mathbb{H}_p^{\alpha, \beta; \psi}$ is continuously injected into $C([0, T])$ for $\alpha > \frac{1}{p}$.

According to (2.18), we can consider $\mathbb{H}_p^{\alpha, \beta; \psi}$ with respect to the equivalent norm

$$(2.20) \quad \|\xi\| = \left\| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi \right\|_{L^p}.$$

Proposition 2.15. [2, 26] Let $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. For any $\xi \in \mathbb{H}_p^{\alpha, \beta; \psi}$, we have

$$\left\| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi \right\|_{L^p} \leq \frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha + 1)} \left\| \mathbf{H}\mathbf{D}_T^{\alpha, \beta; \psi} \left(\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi \right) \right\|_{L^p}.$$

Moreover, if $\alpha > 1/p$, $0 \leq \beta \leq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|\xi\|_\infty \leq \frac{(\psi(T) - \psi(0))^{\alpha-1/p} [\psi(T)]^\alpha}{\Gamma(\alpha) [(\alpha+1)q + 1]^{1/q} \Gamma(\alpha+1)} \left\| \mathbf{H}\mathbf{D}_T^{\alpha, \beta; \psi} \left(\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi \right) \right\|_{L^p}.$$

Proposition 2.16. [2, 26] *Let $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ and $1 < p < \infty$. Assume that $\alpha > 1/p$ and the sequence $\{\xi_k\}$ converges weakly to ξ in $\mathbb{H}_p^{\alpha,\beta;\psi}$ i.e., $\xi_k \rightharpoonup \xi$. Then $\xi_k \rightarrow \xi$ in $C([0, T], \mathbb{R})$, i.e., $\|\xi - \xi_k\|_\infty \rightarrow 0$ as $k \rightarrow \infty$.*

Proposition 2.17. [2, 26] *The space $\mathbb{H}_p^{\alpha,\beta;\psi}$ is compactly embedded in $C([0, T], \mathbb{R})$.*

Proposition 2.18. [2, 26] *Let $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ and $1 < p < \infty$. The fractional derivative space $\mathbb{H}_p^{\alpha,\beta;\psi}$ is a reflexive and separable Banach space.*

Choosing $p = 2$, in definition Eq.(2.16), we have the ψ -fractional derivative space $\mathbb{H}_2^{\alpha,\beta;\psi}$ is defined on $C_0^\infty([0, T], \mathbb{R})$ with respect to the norm

$$\|\xi\|_{\mathbb{H}_2^{\alpha,\beta;\psi}} = \left(\int_0^T |\xi(t)|^2 dt + \int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(t) \right|^2 dt \right)^{1/2}.$$

The space $\mathbb{H}_2^{\alpha,t;\psi}$ is a Hilbert space with the norm

$$\|\xi\|_{\mathbb{H}_2^{\alpha,\beta;\psi}} = \left(\int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(t) \right|^2 dt \right)^{1/2}$$

with $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$.

Theorem 2.19. [28] *Let E a normed vector space and $\{x_n\}_{n \in \mathbb{N}} \subset E$ be a sequence. Then, the following statements are valid:*

- (1) $x_n \rightharpoonup x \Leftrightarrow f(x_n) \rightarrow f(x)$ for all $f \in E^*$;
- (2) If $x_n \rightarrow x$, then $x_n \rightharpoonup x$;
- (3) If $x_n \rightharpoonup x$, then x_n is bounded and moreover $\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$.

Theorem 2.20. [29] *Let $\xi : M \rightarrow N$ be a differentiable map with $\dim M = m$, $\dim N = n$, $m \geq n$ and $p \in N$. Let $d\xi(x)$ be of class C^n , $\forall x \in M$, with $\xi(x) = p$. Then $\xi^{-1}(p)$ is a sub-variety of M , of dimension $m - n$.*

Now we state a compactness result for the integral space $\mathbb{H}_p^{\alpha,\beta;\psi}$ when $0 < \alpha \leq \frac{1}{p}$. This will be the principal key for our analysis in the sequel. For any $h \in \mathbb{R}$ and any $u \in L^p([0, T])$, we consider the translation of u by h defined by

$$(2.21) \quad \xi_n(u)(t) = \begin{cases} \xi(t+h) & , \quad t+h \in [0, T] \\ 0 & , \quad t+h \notin [0, T] \end{cases}.$$

Theorem 2.21. [28] (Frechet-Kolmogorov) *Let $\Omega \subset \mathbb{R}^n$ be bounded open and $1 \leq p < \infty$. A subset Y of $L^p(\Omega, \mathbb{R}^n)$ is relatively compact if and only if the following conditions are satisfied:*

- (1) Y is bounded, i.e., there exists a $c > 0$, such that $\|\xi\|_{L^p} \leq c$ for all $\xi \in Y$;
- (2) $f_n(\xi) \rightarrow \xi$ in $L^p(\Omega, \mathbb{R}^n)$ as $h \rightarrow 0$, uniformly for $\xi \in Y$, i.e.,

$$\sup_{\xi \in Y} \|f_n(\xi) - \xi\|_{L^p} \rightarrow 0$$

as $h \rightarrow 0$.

Theorem 2.22. *Let $\alpha \in (0, 1)$. Then the embedding $\mathbb{H}_p^{\alpha,\beta;\psi} \hookrightarrow L^p([0, T], \mathbb{R})$ is compact.*

Proof. From Proposition 2.14, the embedding $\mathbb{H}_p^{\alpha,\beta;\psi} \hookrightarrow L^p([0, T], \mathbb{R})$ is continuous, and so it is sufficient to prove that every bounded sequence in $\mathbb{H}_p^{\alpha,\beta;\psi}$ is pre-compact in $L^p([0, T], \mathbb{R})$. According to the Frechet-Kolmogorov theorem, it is sufficient to prove that

$$\sup_n \|f_n(\xi_n) - \xi_n\|_{L^p} \rightarrow 0$$

as $h \rightarrow 0$.

Now, using the Theorem 2.4, for any $u \in \mathbb{H}_p^{\alpha,\beta;\psi}$ and the fact that $\mathbf{I}_{0+}^{\alpha;\psi} \xi(0) = 0$, we have

$$\mathbf{I}_{0+}^{\alpha;\psi} \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(t) = \xi(t), \quad t \in [0, T].$$

Thus, for $h > 0$ and $t, t+h \in [0, T]$, we have

$$\|\xi_n(t+h) - \xi_n(t)\|_{L^p}^p = \int_0^T \left| \mathbf{I}_{0+}^{\alpha;\psi} \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(t+h) - \mathbf{I}_{0+}^{\alpha;\psi} \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(t) \right|^p dt$$

$$\begin{aligned}
&= \int_0^T \left| \begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t+h) - \psi(s))^{\alpha-1} \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_t^{t+h} \psi'(s) (\psi(t+h) - \psi(s))^{\alpha-1} \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(s) ds \\ &- \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(s) ds \end{aligned} \right|^p dt \\
&\leq \frac{2^{p-1}}{\Gamma(\alpha)^p} \int_0^T \left(\int_0^t |\psi'(s)| \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(s) \right| \left| (\psi(t+h) - \psi(s))^{\alpha-1} - (\psi(t) - \psi(s))^{\alpha-1} \right| ds \right)^p dt \\
&\quad + \frac{2^{p-1}}{\Gamma(\alpha)^p} \int_0^T \left(\int_t^{t+h} |\psi'(s) (\psi(t+h) - \psi(s))^{\alpha-1}| \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(s) \right| ds \right)^p dt.
\end{aligned}$$

Now, consider

$$\mathbf{A}_1 = \int_0^t |\psi'(s)| \left| (\psi(t+h) - \psi(s))^{\alpha-1} - (\psi(t) - \psi(s))^{\alpha-1} \right| \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(s) \right| ds$$

and

$$\mathbf{A}_2 = \int_t^{t+h} |\psi'(s) (\psi(t+h) - \psi(s))^{\alpha-1}| \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(s) \right| ds.$$

Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Now Hölder's inequality yields

$$\begin{aligned}
\mathbf{A}_1 &= \int_0^t |\psi'(s)|^{\frac{1}{p}} \left| (\psi(t+h) - \psi(s))^{\alpha-1} - (\psi(t) - \psi(s))^{\alpha-1} \right|^{\frac{1}{p}} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(s) \right| \times \\
&\quad |\psi'(s)|^{\frac{1}{q}} \left| (\psi(t+h) - \psi(s))^{\alpha-1} - (\psi(t) - \psi(s))^{\alpha-1} \right|^{\frac{1}{q}} ds \\
&\leq \frac{1}{\alpha^{\frac{1}{q}}} [(\psi(t+h) - \psi(s))^\alpha - ((\psi(t+h) - \psi(0))^\alpha - (\psi(t) - \psi(0))^\alpha)]^{\frac{1}{q}} \\
&\quad \times \left(\int_0^T |\psi'(s)| \left| (\psi(t+h) - \psi(s))^{\alpha-1} - (\psi(t) - \psi(s))^{\alpha-1} \right| \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(s) \right|^p ds \right)^{\frac{1}{p}}
\end{aligned}$$

(2.22)

and

$$\begin{aligned}
\mathbf{A}_2 &= \int_t^{t+h} |\psi'(s) (\psi(t+h) - \psi(s))^{\alpha-1}|^{\frac{1}{p}} \times \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(s) \right| |\psi'(s) (\psi(t+h) - \psi(s))^{\alpha-1}|^{\frac{1}{q}} ds \\
&\leq \left(\int_t^{t+h} |\psi'(s) (\psi(t+h) - \psi(s))^{\alpha-1}| \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(s) \right|^p ds \right)^{\frac{1}{p}} \left(\int_t^{t+h} |\psi'(s) (\psi(t+h) - \psi(s))^{\alpha-1}| ds \right)^{\frac{1}{q}},
\end{aligned}$$

(2.23)

respectively.

Performing the following variable change $s = t + \frac{h}{T}s$, it follows that $\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u_n(s) \rightarrow \mathbf{H}\mathbf{D}_{t+\frac{h}{T}s}^{\alpha,\beta;\psi} u_n(t + \frac{h}{T}s)$, yields

$$\int_t^{t+h} \psi'(s) (\psi(t+h) - \psi(s))^{\alpha-1} ds = \int_0^T \psi' \left(t + \frac{h}{T}s \right) \left(\psi(t+h) - \psi \left(\frac{Tt+hs}{T} \right) \right)^{\alpha-1} dt.$$

Thus, from (2.23), we have

$$\mathbf{A}_2 \leq \left(\int_t^{t+h} \left| \psi' \left(t + \frac{h}{T}s \right) \left(\psi(t+h) - \psi \left(\frac{Tt+hs}{T} \right) \right)^{\alpha-1} \right|^{\frac{1}{p}} \left| \mathbf{H}\mathbf{D}_{t+\frac{h}{T}s}^{\alpha,\beta;\psi} \xi_n \left(t + \frac{h}{T}s \right) \right|^p dt \right)^{\frac{1}{p}}$$

$$(2.24) \quad \times \left(\int_t^{t+h} \left| \psi' \left(t + \frac{h}{T} s \right) \left(\psi(t+h) - \psi \left(\frac{Tt+hs}{T} \right) \right)^{\alpha-1} \right| dt \right)^{\frac{1}{q}} dt.$$

Therefore, for every $n \in \mathbb{N}$, by (2.22) and (2.24) one has

$$(2.25) \quad \begin{aligned} \|\xi_n(t+h) - \xi_n(t)\|_{L^p}^p &\leq \frac{2^{p-1} \alpha^{-\frac{1}{q}}}{\Gamma(\alpha)^p} \int_0^T (\Psi^\alpha(t+h, t) - (\Psi^\alpha(t+h, 0) - \Psi^\alpha(t, 0)))^{\frac{p}{q}} \\ &\quad \times \left(\int_0^t |\psi'(s)| (\Psi^{\alpha-1}(t+h, s) - \Psi^{\alpha-1}(t, s)) \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_n(s) \right|^p ds \right) dt \\ &\quad + \frac{2^{p-1}}{\Gamma(\alpha)^p} \int_0^T \left(\int_0^T \left| \psi' \left(t + \frac{h}{T} s \right) \left(\psi(t+h) - \psi \left(\frac{Tt+hs}{T} \right) \right)^{\alpha-1} \right| \right. \\ &\quad \left. \left| \mathbf{H}\mathbf{D}_{t+\frac{h}{T}s}^{\alpha, \beta; \psi} \xi_n \left(t + \frac{h}{T} s \right) \right|^p dt \right) dt \\ &\quad \times \left(\int_0^T \psi' \left(t + \frac{h}{T} s \right) \left(\psi(t+h) - \psi \left(\frac{Tt+hs}{T} \right) \right)^{\alpha-1} dt \right)^{\frac{p}{q}} \\ &\leq \frac{2^{p-1}}{\Gamma(\alpha)^p} \frac{1}{\alpha^{\frac{p}{q}}} [\Psi^\alpha(T+h, T) - (\Psi^\alpha(T+h, 0) - \Psi^\alpha(T, 0))]^{\frac{p}{q}} \times \\ &\quad \int_0^T \int_s^T |\psi'(s)| (\Psi^{\alpha-1}(t+h, s) - \Psi^{\alpha-1}(t, s)) \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_n(s) \right|^p ds dt \\ &\quad + \frac{2^{p-1}}{\Gamma(\alpha)^p} (\psi(t+h) - \psi(t))^{\frac{\alpha p}{q}} \times \\ &\quad \int_0^T \left(\int_0^T \left| \psi' \left(t + \frac{h}{T} s \right) \left(\psi(t+h) - \psi \left(\frac{Tt+hs}{T} \right) \right)^{\alpha-1} \right| \right. \\ &\quad \left. \left| \mathbf{H}\mathbf{D}_{t+\frac{h}{T}s}^{\alpha, \beta; \psi} \xi_n \left(t + \frac{h}{T} s \right) \right|^p dt \right) dt \\ &\leq \frac{2^{p-1}}{\Gamma(\alpha)^p} \frac{1}{\alpha^{\frac{p}{q}}} [\Psi^\alpha(T+h, T) - (\Psi^\alpha(T+h, 0) - \Psi^\alpha(T, 0))]^{\frac{p}{q}} \\ &\quad \times \int_0^T \frac{|\psi'(s)|}{\psi'(t)} \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_n(s) \right|^p \frac{\Psi^\alpha(t+h, s) - (\Psi^\alpha(T+h, s) - \Psi^\alpha(T+h, s))}{\alpha} dt \\ &\quad + \frac{2^{p-1}}{\Gamma(\alpha)^p} \frac{1}{\alpha^p} (\psi(t+h) - \psi(t))^{\frac{\alpha p}{q} + \alpha} \left\| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_n(s) \right\|_{L^p}^p \\ &\leq \left\{ \begin{aligned} &\frac{2^{p-1}}{\Gamma(\alpha+1)^p} [\Psi^\alpha(T+h, T) - (\Psi^\alpha(T+h, 0) + \Psi^\alpha(T, 0))]^p \\ &+ \frac{2^{p-1}}{\Gamma(\alpha+1)^p} (\Psi^\alpha(T+h, T)) \end{aligned} \right\} \\ &\quad \times \left\| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_n(s) \right\|_{L^p}^p, \end{aligned}$$

where $\Psi^\alpha(t+h, t) := (\psi(t+h) - \psi(t))^\alpha$ and $\Psi^\alpha(t+h, 0) := (\psi(t+h) - \psi(0))^\alpha$.

Therefore the result follows from (2.25). \square

3. THE METHOD OF NEHARI MANIFOLDS AND PROPERTIES

In this section, we introduce some results about the Nehari manifold structure and fibering map analysis, related to Eq.(1.1). Now $\mathbf{E}_\lambda : \mathbb{H}_p^{\alpha, \beta; \psi} \rightarrow \mathbb{R}$, defined by

$$\mathbf{E}_\lambda(\xi) = \frac{1}{p} \int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi(x) \right|^p dx - \frac{\lambda}{p} \int_0^T |\xi(x)|^p dx - \frac{1}{q+1} \int_0^T b(x) |\xi(x)|^{q+1} dx$$

is the underlying functional associated to problem (1.1).

We say that $\xi \in \mathbb{H}_p^{\alpha, \beta; \psi}$ is a solution of the problem Eq.(1.1), if

$$\langle \mathbf{E}'_\lambda(\xi), \eta \rangle_{\mathbb{H}_p^{\alpha, \beta; \psi}} = \int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi(x) \right|^{p-2} \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi(x) \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \eta(x) dx$$

$$-\lambda \int_0^T |\xi(x)|^{p-2} \xi(x) \eta(x) dx - \int_0^T b(x) |\xi(x)|^{q-1} \xi(x) \eta(x) dx$$

for any $\eta \in \mathbb{H}_p^{\alpha, \beta; \psi}$ and the eigenvalue problem associated with the ψ -fractional p -Laplacian operator is given by

$$(3.1) \quad \begin{cases} \mathbf{H}\mathbb{D}_T^{\alpha, \beta; \psi} \left(\left| \mathbf{H}\mathbb{D}_{0+}^{\alpha, \beta; \psi} \xi(x) \right|^{p-2} \mathbf{H}\mathbb{D}_{0+}^{\alpha, \beta; \psi} \xi(x) \right) = \lambda |\xi(x)|^{p-2} \xi(x), \text{ for } x \in [0, T] \\ \mathbf{I}_{0+}^{\beta(\beta-1); \psi} \xi(0) = \mathbf{I}_T^{\beta(\beta-1); \psi} \xi(T) = 0. \end{cases}$$

Note that,

$$(3.2) \quad \lambda_1 = \min_{\xi \in \mathbb{H}_p^{\alpha, \beta; \psi}} \frac{\int_0^T \left| \mathbf{H}\mathbb{D}_{0+}^{\alpha, \beta; \psi} \xi(x) \right|^p dx}{\int_0^T |\xi(x)|^p dx}, \quad \xi \neq 0.$$

Lemma 3.1. *Let $0 < \alpha < 1$ and $0 \leq \beta \leq 1$.*

- (1) *The functional \mathbf{E}_λ is well defined on $\mathbb{H}_p^{\alpha, \beta; \psi}$;*
- (2) *The functional \mathbf{E}_λ is of class $C^1(\mathbb{H}_p^{\alpha, \beta; \psi}, \mathbb{R})$ and for all $\xi, \eta \in \mathbb{H}_p^{\alpha, \beta; \psi}$, we have*

$$(3.3) \quad \begin{aligned} \langle \mathbf{E}'_\lambda(\xi), \eta \rangle_{\mathbb{H}_p^{\alpha, \beta; \psi}} &= \int_0^T \left| \mathbb{D}_{0+}^{\alpha, \beta; \psi} \xi(x) \right|^{p-2} {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \xi(x) {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \eta(x) dx \\ &\quad - \lambda \int_0^T |\xi(x)|^{p-2} \xi(x) \eta(x) dx - \int_0^T b(x) |\xi(x)|^{q-1} \xi(x) \eta(x) dx. \end{aligned}$$

Proof. (1). From Hölder's inequality and (2.20), we have

$$\begin{aligned} \mathbf{E}_\lambda(\xi) &\leq \frac{1}{p} \int_0^T \left| {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \xi(x) \right|^p dx + \frac{\lambda}{p} \int_0^T |\xi(x)|^p dx + \frac{1}{q+1} \int_0^T b(x) |\xi(x)|^{q+1} dx \\ &\leq \frac{\left\| {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \xi \right\|^p}{p} + \frac{\lambda \|\xi\|_p^p}{p} + \frac{\|b\|_\infty \left\| |\xi|^{q+1} \right\|_{\frac{q}{q+1}}}{q+1} \\ &\leq c_1 \left\| {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \xi \right\|^p + c_2 \|\xi\|^p + c_3 \left\| |\xi|^{q+1} \right\| \end{aligned}$$

where c_2, c_3 , are the embedding constants. Therefore, \mathbf{E}_λ is well defined on $\mathbb{H}_p^{\alpha, \beta; \psi}$.

(2). Consider

$$\mathbf{H}(\xi) = \frac{1}{p} \left| {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \xi(x) \right|^p - \frac{\lambda}{p} |\xi(x)|^p - \frac{1}{q+1} b(x) |\xi(x)|^{q+1}.$$

Then, for all $\xi, \eta \in \mathbb{H}_p^{\alpha, \beta; \psi}$ and for almost every $t \in [0, T]$

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{\mathbf{H}(\xi(x) + s\eta(x)) - \mathbf{H}(\xi(x))}{s} &= \left| {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \xi(x) \right|^{p-2} {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \xi(x) {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \eta(x) \\ &\quad - \lambda |\xi(x)|^{p-2} \xi(x) \eta(x) - b(x) |\xi(x)|^{q-1} \xi(x) \eta(x). \end{aligned}$$

From the Lagrange mean value theorem, there exists a real number θ such that $|\theta| \leq s$ and

$$(3.4) \quad \begin{aligned} \frac{\mathbf{H}(\xi(x) + s\eta(x)) - \mathbf{H}(\xi(x))}{s} &= \left| {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} (\xi(x) + \theta\eta(x)) \right|^{p-2} {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} (\xi(x) + \theta\eta(x)) {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \eta(x) \\ &\quad - \lambda |(\xi(x) + \theta\eta(x))|^{p-2} (\xi(x) + \theta\eta(x)) \eta(x) \\ &\quad - b(x) |(\xi(x) + \theta\eta(x))|^{q-1} (\xi(x) + \theta\eta(x)) \eta(x) \\ &\leq \left| {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} (\xi(x) + \theta\eta(x)) \right|^{p-1} {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \eta(x) \\ &\quad + \lambda |(\xi(x) + \theta\eta(x))|^{p-1} \eta(x) + b(x) |(\xi(x) + \theta\eta(x))|^q \eta(x) \\ &\leq \left| {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \xi(x) \right|^{p-1} \left| {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \eta(x) \right| + \left| {}^H\mathbb{D}_{0+}^{\alpha, \beta; \psi} \eta(x) \right|^p \\ &\quad + \lambda |\xi(x)|^{p-1} |\eta(x)| + \lambda |\eta(x)|^p + b(x) |\xi(x)|^q |\eta(x)| + b(x) |\eta(x)|^{q+1}. \end{aligned}$$

Now Hölder's inequality yields

$$(3.5) \quad \int_0^T \left| {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^{p-1} \left| {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \eta(x) \right| dx \leq \left\| \left| {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \xi \right|^{p-1} \right\|_{\frac{p}{p-1}} \left\| {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \eta \right\|_p,$$

$$(3.6) \quad \int_0^T |\xi(x)|^{p-1} |\eta(x)| dx \leq \left\| |\xi|^{p-1} \right\|_{\frac{p}{p-1}} \|\eta\|_p,$$

and

$$(3.7) \quad \int_0^T |\xi(x)|^q |\eta(x)| dx \leq \|\xi\|_{\frac{q+1}{q}}^q \|\eta\|_{q+1}.$$

From (3.5), (3.6) and (3.7), one concludes that the last inequality in (3.4) is in $L^1(\Omega)$. Therefore, by the dominated convergence theorem, we get

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{\mathbf{E}_\lambda(\xi + s\eta) - \mathbf{E}_\lambda(\xi)}{s} &= \int_0^T \left| {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^{p-2} {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \xi(x) {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \eta(x) dx \\ &\quad - \lambda \int_0^T |\xi(x)|^{p-2} \xi(x) \eta(x) dx - \int_0^T b(t) |\xi(x)|^{q-1} \xi(x) \eta(x) dx. \end{aligned}$$

That is, \mathbf{E}_λ is Gâteaux differentiable. Finally, it is sufficient to prove that the Gâteaux derivative of \mathbf{E}_λ is continuous, and in this case we omit the details since it is similar to [44]. \square

Lemma 3.2. (1) Suppose that $\lambda < \lambda_1$. Then \mathbf{E}_λ is bounded from below in $\mathbb{H}_p^{\alpha,\beta;\psi}$.

(2) If $\lambda > \lambda_1$, then \mathbf{E}_λ is not bounded from below in $\mathbb{H}_p^{\alpha,\beta;\psi}$.

Proof. (1). Now (3.2), yields

$$\lambda_1 \int_0^T |\xi(x)|^p dx \leq \int_0^T \left| {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p dx,$$

and,

$$\int_0^T \left| {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p dx - \lambda \int_0^T |\xi(x)|^p dx \geq (\lambda_1 - \lambda) \int_0^T |\xi(x)|^p dx$$

implies that

$$\mathbf{E}_\lambda(\xi) \geq \frac{(\lambda_1 - \lambda)}{p} \int_0^T |\xi(x)|^p dx - \frac{1}{q+1} \int_0^T b(x) |\xi(x)|^{q+1} dx.$$

Now $b \in L^\infty$ so there exists a $\bar{b} \geq 0$ with

$$\mathbf{E}_\lambda(\xi) \geq \frac{(\lambda_1 - \lambda)}{p} \int_0^T |\xi(x)|^p dx - \frac{\bar{b}}{q+1} \int_0^T |\xi(x)|^{q+1} dx,$$

so, applying the Hölder inequality, note $1 < q < p-1$, $\frac{2}{p} < \frac{q+1}{p} < 1$ and $1 - \left(\frac{q+1}{p}\right) > 0$, we have

$$\mathbf{E}_\lambda(\xi) \geq \frac{1}{p} (\lambda_1 - \lambda) \int_0^T |\xi(x)|^p dx - \frac{\bar{b}}{q+1} T^{1-(q+1)/p} \left(\int_0^T |\xi(x)|^p dx \right)^{\frac{q+1}{p}}.$$

Therefore, \mathbf{E}_λ is bounded from below in $\mathbb{H}_p^{\alpha,\beta;\psi}$, when $\lambda < \lambda_1$.

(2). Let $\lambda > \lambda_1$. Let Ξ be the eigenfunction associated with the main eigenvalue λ_1 , that is, (Ξ, λ_1) is a solution to problem (3.1) and $\Xi \neq 0$, and then

$$(3.8) \quad \lambda_1 = \frac{\int_0^T \left| {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \Xi(x) \right|^p dx}{\int_0^T |\Xi(x)|^p dx}.$$

Performing the same procedure as in (1), and taking the limit in $\mathbf{E}_\lambda(t\Xi)$, with $t \rightarrow \infty$, yields

$$\lim_{t \rightarrow \infty} \mathbf{E}_\lambda(t\Xi) = \lim_{t \rightarrow \infty} |t|^p \left[\frac{\lambda_1}{p} \int_0^T |\Xi|^p dx - \frac{\lambda}{p} \int_0^T |\Xi|^p dx - \frac{1}{(q+1)t^{p-(q+1)}} \int_0^T |\Xi|^{q+1} dx \right]$$

$$= \lim_{t \rightarrow \infty} |t|^p \left[\frac{(\lambda_1 - \lambda)}{p} \int_0^T |\Xi|^p dx - \frac{1}{(q+1)t^{p-(q+1)}} \int_0^T |\Xi|^{q+1} dx \right].$$

Thus $\lim_{t \rightarrow \infty} \mathbf{E}_\lambda(t\Xi) = -\infty$, and therefore \mathbf{E}_λ is not bounded from below in $\mathbb{H}_p^{\alpha,\beta;\psi}$, when $\lambda > \lambda_1$. \square

Consider \mathbf{S}_λ , the subset of $\mathbb{H}_p^{\alpha,\beta;\psi}$, given by

$$\mathbf{S}_\lambda = \{ \xi \in \mathbb{H}_p^{\alpha,\beta;\psi}; \langle \mathbf{E}'_\lambda \xi, \xi \rangle = 0, \xi \neq 0 \},$$

called the Nehari manifold, where $\langle \cdot, \cdot \rangle$ denotes the usual duality. Note that if ξ is a critical point of the functional \mathbf{E}_λ , then $\mathbf{E}'_\lambda(\xi)\eta = 0$, for all $\eta \in \mathbb{H}_p^{\alpha,\beta;\psi}$.

Thus, we say that $u \in \mathbf{S}_\lambda$ if, and only if, it satisfies

$$(3.9) \quad \int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p dx - \lambda \int_0^T |\xi(x)|^p dx - \int_0^T b(x) |\xi(x)|^{q+1} dx = 0.$$

Proposition 3.3. *The set \mathbf{S}_λ is nonempty and is a sub-variety of $\mathbb{H}_p^{\alpha,\beta;\psi}$.*

Proof. Consider $\mathbf{T} : \mathbb{H}_p^{\alpha,\beta;\psi} \setminus \{0\} \rightarrow \mathbb{R}$, given by $\mathbf{T}(\xi) = \langle \mathbf{E}'_\lambda(\xi), \xi \rangle$.

Note,

$$\mathbf{T}(\xi) = \underbrace{\int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right|^p dx}_{J_1} - \lambda \underbrace{\int_0^T |\xi|^p dx}_{J_2} - \underbrace{\int_0^T b(x) |\xi|^{q+1} dx}_{J_3}.$$

Note that J_1, J_2, J_3 , are of class $C^1([0, T])$ and, therefore, it is observed that they are also of class $C^2([0, T])$, so we get that $\mathbf{T} \in C^1([0, T])$. Note that,

$$\begin{aligned} \mathbf{T}'(\xi)\eta &= p \int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right|^{p-2} \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \eta dx \\ &\quad - \lambda p \int_0^T |\xi|^{p-2} \xi \eta dx - (q+1) \int_0^T b(x) |\xi|^{q-1} \xi \eta dx. \end{aligned}$$

Now, consider $\eta \in \mathbb{H}_p^{\alpha,\beta;\psi}$ with $\eta \neq 0$. Without loss of generality assume

$$\int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \eta \right|^p dx - \lambda \int_0^T |\eta|^p dx > 0, \quad \int_0^T b(x) |\eta|^{q+1} dx > 0.$$

Consider the function $t \mapsto \mathbf{T}(t\eta)$, $t \in \mathbb{R}$ and $t > 0$. Then,

$$\mathbf{T}(t\eta) = t^p \int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \eta \right|^p dx - \lambda t^p \int_0^T |\eta|^p dx - t^{q+1} \int_0^T b(x) |\eta|^{q+1} dx.$$

Then as $1 < q < p-1 < \infty$, it follows that $\lim_{t \rightarrow \infty} \mathbf{T}(t\eta) = +\infty$ and $\lim_{t \rightarrow 0} \mathbf{T}(t\eta) = 0^-$. Thus, for some $\bar{t} > 0$, the function $\mathbf{T}(\bar{t}\eta)$ is equal to zero, that is, $\bar{t}\eta \in \mathbf{S}_\lambda$. Therefore $\mathbf{S}_\lambda \neq \emptyset$.

Now, we show that \mathbf{T} has no critical point in \mathbf{S}_λ . First, note that

$$(3.10) \quad \mathbf{T}'(\xi)\xi = p \int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right|^p dx - \lambda p \int_0^T |\xi|^p dx - (q+1) \int_0^T b(x) |\xi|^{q+1} dx.$$

For $\xi \in \mathbf{S}_\lambda$, with $\xi \neq 0$, we have

$$\mathbf{T}(\xi) = 0 \implies \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right|^p - \lambda |\xi|^p \right) dx - \int_0^T b(x) |\xi|^{q+1} dx = 0.$$

Thus,

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right|^p - \lambda |\xi|^p \right) dx = \int_0^T b(x) |\xi|^{q+1} dx.$$

Substituting in Eq.(3.10), gives

$$\mathbf{T}'(\xi)\xi = (p - (q+1)) \int_0^T b(x) |\xi|^{q+1} dx.$$

Therefore, as $1 < q < p-1$, we obtain that $\mathbf{T}'(\xi) \neq 0, \forall \xi \neq 0$ in \mathbf{S}_λ .

Let $\mathbf{M} = \mathbb{H}_p^{\alpha,\beta;\psi} \setminus \{0\}$. Note that 0 is a unique critical point in $\mathbf{T}^{-1}(0)$ and $0 \notin \mathbf{M}$, and note $\int_0^T b|\xi|^{q+1} dx > 0$. Therefore, 0 is a regular point of $\mathbf{T}|_{\mathbf{M}}$. From Theorem 2.20, it follows that $\mathbf{T}^{-1}|_{\mathbf{M}}(0)$ is a sub-variety of \mathbf{M} . Therefore, \mathbf{S}_λ is a C^1 sub-variety of $\mathbb{H}_p^{\alpha,\beta;\psi}$. \square

Let us consider real functions of positive variables as follows: $\Xi_\xi(t) : t \mapsto \mathbf{E}_\lambda(t\xi)$ ($t > 0$), and these functions are known as fibering maps for each $\xi \in \mathbb{H}_p^{\alpha,\beta;\psi}$, and we have

$$(3.11) \quad \Xi_\xi(t) = \frac{t^p}{p} \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right|^p - \lambda |\xi|^p \right) dx - \frac{t^{q+1}}{q+1} \int_0^T b(x) |\xi|^{q+1} dx,$$

$$(3.12) \quad \Xi'_\xi(t) = t^{p-1} \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx - t^q \int_0^T b(x) |\xi(x)|^{q+1} dx$$

and

$$(3.13) \quad \Xi''_\xi(t) = (p-1)t^{p-2} \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx - qt^{q-1} \int_0^T b(x) |\xi(x)|^{q+1} dx.$$

The following Lemma 3.4 considers the relationship between the Nehari manifold and fibering maps, which guarantees a correspondence between the critical points of $\Xi_\xi(t)$ and the elements of \mathbf{S}_λ .

Lemma 3.4. *Let Ξ_ξ be the operator defined in Eq.(3.11), with $t > 0$ and let $\xi \in \mathbb{H}_p^{\alpha,\beta;\psi}$. Then*

- (1) $\xi \in \mathbf{S}_\lambda$ if, and only if, $\Xi'_\xi(1) = 0$.
- (2) $t\xi \in \mathbf{S}_\lambda$ if, and only if, $\Xi'_\xi(t) = 0$.

Proof. (1). In fact, for $\xi \in \mathbf{S}_\lambda$ from Eq.(3.12), we have

$$\Xi'_\xi(1) = \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx - \int_0^T b(x) |\xi(x)|^{q+1} dx = \mathbf{E}'_\lambda(\xi)\xi = 0.$$

(2). We have

$$0 = \Xi'_\xi(t) = t^{p-1} \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx - t^q \int_0^T b(x) |\xi(x)|^{q+1} dx.$$

Multiplying both sides by t , so

$$0 = t^p \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx - t^{q+1} \int_0^T b(x) |\xi(x)|^{q+1} dx = \mathbf{E}'_\lambda(t\xi)t\xi.$$

For $t\xi \in \mathbf{S}_\lambda$, we have $0 = \mathbf{E}'_\lambda(t\xi)t\xi$, and then

$$0 = t^p \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx - t^{q+1} \int_0^T b(x) |\xi(x)|^{q+1} dx.$$

Dividing both sides by $t > 0$, yields

$$0 = t^{p-1} \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx - t^q \int_0^T b(x) |\xi(x)|^{q+1} dx = \Xi'_\xi(t).$$

\square

It is natural to divide the Nehari manifold into three distinct sets. Note that Eq.(3.12) and Eq.(3.13), yields

$$\Xi''_\xi(t) = t^{q-1}[(p-1) - q] \int_0^T b(x) |\xi(x)|^{q+1} dx$$

and as $t > 0$, $1 < q < p-1$, we have

$$\Xi''_\xi(t) > 0 \Leftrightarrow \int_0^T b(x) |\xi(x)|^{q+1} dx > 0,$$

$$\Xi''_\xi(t) < 0 \Leftrightarrow \int_0^T b(x) |\xi(x)|^{q+1} dx < 0$$

and

$$\Xi''_\xi(t) = 0 \Leftrightarrow \int_0^T b(x) |\xi(x)|^{q+1} dx = 0.$$

We can divide \mathbf{S}_λ into three sets:

$$\mathbf{S}_\lambda^+ = \left\{ \xi \in \mathbf{S}_\lambda; \int_0^T b(x)|\xi(x)|^{q+1} dx > 0 \right\},$$

$$\mathbf{S}_\lambda^- = \left\{ \xi \in \mathbf{S}_\lambda; \int_0^T b(x)|\xi(x)|^{q+1} dx < 0 \right\}$$

and

$$\mathbf{S}_\lambda^0 = \left\{ \xi \in \mathbf{S}_\lambda; \int_0^T b(x)|\xi(x)|^{q+1} dx = 0 \right\},$$

where \mathbf{S}_λ^+ , \mathbf{S}_λ^- and \mathbf{S}_λ^0 correspond to the points of local minimum, local maximum and inflection of Ξ_ξ , respectively.

The analysis of the behavior of Ξ_ξ is in relation to the sign of

$$\Theta(\xi) = \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx \quad \text{and}$$

$$\Upsilon(\xi) = \int_0^T b(x)|\xi(x)|^{q+1} dx.$$

Let $\xi \in \mathbb{H}_p^{\alpha,\beta;\psi}$ and $\Xi'_\xi(t) = 0$, and we conclude that:

- (1) If $\Theta(\xi)$ and $\Upsilon(\xi)$ have the same sign, then Ξ_ξ has a unique critical point,

$$t_\xi = \left(\frac{\int_0^T b(x)|\xi(x)|^{q+1} dx}{\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx} \right)^{\frac{1}{p-(1+q)}}.$$

This critical point is a minimum when $t_\xi \xi \in \mathbf{S}_\lambda^+$ if, and only if, $\Upsilon(\xi) > 0$. It is a maximum point when $t_\xi \xi \in \mathbf{S}_\lambda^-$ if, and only if, $\Upsilon(\xi) < 0$.

- (2) If $\Theta(\xi)$ and $\Upsilon(\xi)$ have opposite signs, then Ξ_ξ has no inflection points, so there are no multiples of ξ in \mathbf{S}_λ .

Therefore, we define (here $\|\cdot\| = \|\cdot\|_{\mathbb{H}_p^{\alpha,\beta;\psi}}$)

$$\mathbf{L}_+(\lambda) = \left\{ \xi \in \mathbb{H}_p^{\alpha,\beta;\psi}; \|\xi\| = 1 \text{ and } \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx > 0 \right\} \quad \text{and}$$

$$\mathbf{Y}_+ = \left\{ \xi \in \mathbb{H}_p^{\alpha,\beta;\psi}; \|\xi\| = 1 \text{ and } \int_0^T b(x)|\xi(x)|^{q+1} dx > 0 \right\}.$$

Similarly, we can define $\mathbf{L}_-(\lambda)$ and \mathbf{Y}_- when we replace “ > 0 ” by “ < 0 ” and $\mathbf{L}_0(\lambda)$ and \mathbf{Y}_0 when we replace “ > 0 ” by “ $= 0$ ”. We have the following results:

RESULTS I:

- (1) If $\xi \in \mathbf{L}_+(\lambda) \cap \mathbf{Y}_+$, then $t \mapsto \Xi_\xi(t)$ has a local minimum at $t = t_\xi$ and $t_\xi \xi \in \mathbf{S}_\lambda^+$.
- (2) If $\xi \in \mathbf{L}_-(\lambda) \cap \mathbf{Y}_-$, then $t \mapsto \Xi_\xi(t)$ has a local maximum at $t = t_\xi$ and $t_\xi \xi \in \mathbf{S}_\lambda^-$.
- (3) If $\xi \in \mathbf{L}_+(\lambda) \cap \mathbf{Y}_-$, then $t \mapsto \Xi_\xi(t)$ is strictly increasing and no multiple of ξ is in \mathbf{S}_λ .
- (4) If $\xi \in \mathbf{L}_-(\lambda) \cap \mathbf{Y}_+$, then $t \mapsto \Xi_\xi(t)$ is strictly decreasing and no multiple of ξ is in \mathbf{S}_λ .

The next result ensures that a critical point $\xi \in \mathbf{S}_\lambda$, of the functional \mathbf{E}_λ , restricted to the Nehari manifold, such that $\xi \notin \mathbf{S}_\lambda^0$ is really a critical point in $\mathbb{H}_p^{\alpha,\beta;\psi}$.

Lemma 3.5. *Suppose that ξ_0 is a local maximum point or a local minimum point for \mathbf{E}_λ in \mathbf{S}_λ , and that $\xi_0 \notin \mathbf{S}_\lambda^0$. Then ξ_0 is a critical point of \mathbf{E}_λ , and this means that $\mathbf{E}'_\lambda(\xi_0) = 0$ in $(\mathbb{H}_p^{\alpha,\beta;\psi})^*$.*

Proof. Let ξ_0 a local maximum or minimum point of \mathbf{E}_λ in \mathbf{S}_λ . By the Lagrange multiplier theorem, $\exists \mu \in \mathbb{R}$, such that

(1) $\mathbf{E}'_\lambda(\xi_0) = \mu r'_\lambda(\xi_0)$ in $(\mathbb{H}_p^{\alpha,\beta;\psi})^*$, where

$$r_\lambda(\xi_0) = \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_0(x) \right|^p - \lambda |\xi_0(x)|^p \right) dx - \int_0^T b(x) |\xi_0(x)|^{q+1} dx = \mathbf{E}'_\lambda(\xi_0) \xi_0 = 0.$$

Thus,

$$(2) \quad (3.14) \quad \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_0(x) \right|^p - \lambda |\xi_0(x)|^p \right) dx = \int_0^T b(x) |\xi_0(x)|^{q+1} dx.$$

Since $\xi_0 \in \mathbf{S}_\lambda$, (1) and Eq.(3.14), give

$$\begin{aligned} 0 &= \left\langle \mathbf{E}'_\lambda(\xi_0), \xi_0 \right\rangle_{\mathbb{H}_p^{\alpha,\beta;\psi}} \\ &= \mu \left\langle r'_\lambda(\xi_0), \xi_0 \right\rangle_{\mathbb{H}_p^{\alpha,\beta;\psi}} \\ &= p \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_0(x) \right|^p - \lambda |\xi_0(x)|^p \right) dx - (q+1) \int_0^T b(x) |\xi_0(x)|^{q+1} dx \\ &= (p-q-1) \int_0^T b(x) |\xi_0(x)|^{q+1} dx. \end{aligned}$$

Since $\xi_0 \notin \mathbf{S}_\lambda^0$, then $\Xi''_\xi(1) \neq 0$. Thus, $(r'_\lambda(\xi_0)\xi_0 \neq 0) \langle r'_\lambda(\xi_0), \xi_0 \rangle \neq 0$, giving us that $\mu = 0$, and $\mathbf{E}'_\lambda(\xi_0) = 0$. Therefore, ξ_0 is a critical point in $\mathbb{H}_p^{\alpha,\beta;\psi}$. \square

We now discuss the important role played by the condition $\mathbf{L}_-(\lambda) \subseteq \mathbf{Y}_-$, in determining the nature of the Nehari manifold.

When $\lambda < \lambda_1$, we have

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx > 0,$$

for all $\xi \in \mathbb{H}_p^{\alpha,\beta;\psi}$. Thus

$$\mathbf{L}_+(\lambda) = \{ \xi \in \mathbb{H}_p^{\alpha,\beta;\psi}; \|\xi\| = 1 \}, \quad \mathbf{L}_-(\lambda) = \emptyset \quad \text{and} \quad \mathbf{L}_0(\lambda) = \emptyset.$$

When $\lambda = \lambda_1$, we get

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx \geq 0, \quad \forall \xi \in \mathbb{H}_p^{\alpha,\beta;\psi}.$$

Thus $\mathbf{L}_-(\lambda) = \emptyset$ and $\mathbf{L}_0(\lambda) = \{\Xi\}$, and when $\lambda > \lambda_1$, $\mathbf{L}_-(\lambda)$ becomes non-empty and becomes larger as λ increases.

Theorem 3.6. *Suppose that there exists $\bar{\lambda}$, such that for any $\lambda < \bar{\lambda}$, we have $\mathbf{L}_-(\lambda) \subseteq \mathbf{Y}_-$. Then, for all $\lambda < \bar{\lambda}$, we get*

- (1) $\mathbf{L}_0(\lambda) \subseteq \mathbf{Y}_-$ and, so, $\mathbf{L}_0(\lambda) \cap \mathbf{Y}_0 = \emptyset$,
- (2) \mathbf{S}_λ^+ is bounded,
- (3) $0 \notin \overline{\mathbf{S}_\lambda^-}$ and \mathbf{S}_λ^- is closed,
- (4) $\overline{\mathbf{S}_\lambda^+} \cap \mathbf{S}_\lambda^- = \emptyset$.

Proof. (1). Suppose, by contradiction, that $\mathbf{L}_0(\lambda) \not\subseteq \mathbf{Y}_-$. Thus, there exists $\xi \in \mathbf{L}_0(\lambda)$, such that $\xi \notin \mathbf{Y}_-$. This means that

$$\begin{aligned} \xi \in \mathbf{L}_0(\lambda) &\Rightarrow \xi \in \mathbb{H}_p^{\alpha,\beta;\psi}, \quad \|\xi\| = 1, \quad \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right|^p - \lambda |\xi(x)|^p \right) dx = 0 \quad \text{and} \\ \xi \notin \mathbf{Y}_- &\Rightarrow \int_0^T b(x) \left(\frac{|\xi|}{\|\xi\|} \right)^{q+1} dx \geq 0. \end{aligned}$$

If $\lambda < \mu < \bar{\lambda}$, then

$$0 = \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right|^p - \lambda |\xi|^p \right) dx > \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right|^p - \mu |\xi|^p \right) dx \Rightarrow \xi \in \mathbf{L}_-(\mu),$$

so that $\mathbf{L}_-(\mu) \not\subseteq \mathbf{Y}_-$, which gives us a contradiction to the theorem hypothesis. Now $\mathbf{L}_0(\lambda) \subseteq \mathbf{Y}_-$ and as $\mathbf{Y}_- \cap \mathbf{Y}_0 = \emptyset$, we get $\mathbf{L}_0(\lambda) \cap \mathbf{Y}_0 = \emptyset$.

(2). Suppose that \mathbf{S}_λ^+ is unbounded. Then there exists $\{\xi_n\} \in \mathbf{S}_\lambda^+$, such that

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n \right|^p - \lambda |\xi_n|^p \right) dx = \int_0^T b(x) |\xi_n|^{q+1} dx > 0$$

and $\|\xi_n\| \rightarrow \infty$, when $n \rightarrow \infty$.

Let $\eta_n = \frac{\xi_n}{\|\xi_n\|}$. Note $\{\eta_n\}$ is bounded so there exists a subsequence of $\{\eta_n\}$ (without loss of generality assume it is the whole sequence) with $\eta_n \rightharpoonup \eta_0$ in $\mathbb{H}_p^{\alpha,\beta;\psi}$. Now from Theorem 2.22 (without loss of generality assume it is the whole sequence) we have $\eta_n \rightarrow \eta_0$ in $L^p(\Omega)$ and in $L^{q+1}(\Omega)$, as $1 < q < p - 1$. Note also there exists a subsequence (without loss of generality assume it is the whole sequence) with η_n converging a.e. to η_0 . Since $\xi_n \in \mathbf{S}_\lambda^+$, we have

$$\int_0^T b(x) |\eta_n|^{q+1} dx = \frac{1}{\|\xi_n\|^{q+1}} \int_0^T b(x) |\xi_n|^{q+1} dx > 0.$$

Thus (note $b \in L^\infty$)

$$\int_0^T b(x) |\eta_0|^{q+1} dx \geq 0.$$

Furthermore, since $\xi_n \in \mathbf{S}_\lambda$, from Eq.(3.9), we have

$$(3.15) \quad \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n \right|^p - \lambda |\xi_n|^p \right) dx = \int_0^T b(x) |\xi_n|^{q+1} dx.$$

Divide Eq.(3.15) by $\|\xi_n\|^p$, and since $b(x) |\eta_n|^{q+1}$ is in $L^{q+1}([0, T])$ and $\|\xi_n\| \rightarrow \infty$, we have

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \eta_n \right|^p - \lambda |\eta_n|^p \right) dx = \int_0^T b(x) |\eta_n|^{q+1} \frac{1}{\|\xi_n\|^{p-(q+1)}} dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now

$$\lim_{n \rightarrow \infty} \int_0^T |\eta_n|^p dx = \int_0^T |\eta_0|^p dx$$

and

$$\int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \eta_0 \right|^p dx \leq \liminf_{n \rightarrow \infty} \int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \eta_n \right|^p dx.$$

Combining the above gives

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \eta_0 \right|^p - \lambda |\eta_0|^p \right) dx \leq \liminf_{n \rightarrow \infty} \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \eta_n \right|^p - \lambda |\eta_n|^p \right) dx = 0$$

and so, $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{L}_-(\lambda) \subseteq \Upsilon_-$ or $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{L}_0(\lambda) \subseteq \Upsilon_-$, which is impossible because $\int_0^T b(x) |\eta_0|^{q+1} dx \geq 0$.

Therefore, \mathbf{S}_λ^+ is bounded.

(3). Suppose that $0 \in \overline{\mathbf{S}_\lambda^-}$. Then, there exists a sequence $\{\xi_n\} \subseteq \mathbf{S}_\lambda^-$, such that

$$(3.16) \quad \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n \right|^p - \lambda |\xi_n|^p \right) dx = \int_0^T b(x) |\xi_n|^{q+1} dx < 0.$$

and $\|\xi_n\| \rightarrow 0$, when $n \rightarrow \infty$. Take $\eta_n = \frac{\xi_n}{\|\xi_n\|}$, and we have $\{\eta_n\}$ is bounded so there exists a subsequence of $\{\eta_n\}$ (without loss of generality assume it is the whole sequence) with $\eta_n \rightharpoonup \eta_0$ in $\mathbb{H}_p^{\alpha,\beta;\psi}$. Now from Theorem 2.22 (without loss of generality assume it is the whole sequence) we have $\eta_n \rightarrow \eta_0$ in $L^p(\Omega)$ and in $L^{q+1}(\Omega)$, as $1 < q < p - 1$. Note also there exists a subsequence (without loss of generality assume it is the whole sequence) with η_n converging a.e. to η_0 .

Since $\xi_n \in \mathbf{S}_\lambda^-$, we have

$$\int_0^T b(x) |\eta_n|^{q+1} dx = \frac{1}{\|\xi_n\|^{q+1}} \int_0^T b(x) |\xi_n|^{q+1} dx < 0.$$

Thus (note $b \in L^\infty$)

$$\int_0^T b(x) |\eta_0|^{q+1} dx \leq 0.$$

Divide Eq.(3.16) by $\|\xi_n\|^p$, and since $b(x)|\eta_n|^{q+1}$ is in $L^{q+1}([0, T])$ and $\|\xi_n\| \rightarrow 0$, we have

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \eta_n \right|^p - \lambda |\eta_n|^p \right) dx = \|\xi_n\|^{(q+1)-p} \int_0^T b(x) |\eta_n|^{q+1} dx \rightarrow -\infty \text{ as } n \rightarrow \infty$$

if $\int_0^T b(x) |\eta_0|^{q+1} dx < 0$. Thus

$$(3.17) \quad \int_0^T b(x) |\eta_0|^{q+1} dx = 0.$$

Now

$$\lim_{n \rightarrow \infty} \int_0^T |\eta_n|^p dx = \int_0^T \lim_{n \rightarrow \infty} |\eta_n|^p dx$$

and

$$\int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \eta_0 \right|^p dx \leq \liminf_{n \rightarrow \infty} \int_0^T \left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \eta_n \right|^p dx.$$

Combining the above gives

$$(3.18) \quad \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \eta_0 \right|^p - \lambda |\eta_0|^p \right) dx \leq \liminf_{n \rightarrow \infty} \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \eta_n \right|^p - \lambda |\eta_n|^p \right) dx \leq 0.$$

Hence, $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{L}_-(\lambda) \cap \Upsilon_0$ or $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{L}_0(\lambda) \cap \Upsilon_0$. In both cases, this is impossible. For the first one note that $\mathbf{L}_-(\lambda) \subseteq \overline{\Upsilon_-}$ and $\Upsilon_- \cap \Upsilon_0 = \emptyset$. For the second one note $\mathbf{L}_0(\lambda) \subseteq \Upsilon_-$ (see (1)) and $\Upsilon_- \cap \Upsilon_0 = \emptyset$. Therefore, $0 \notin \overline{\mathbf{S}_\lambda^-}$.

Next, we prove that $\overline{\mathbf{S}_\lambda^-}$ is closed. The idea here is to show that $\overline{\mathbf{S}_\lambda^-} \subseteq \mathbf{S}_\lambda^-$. Let $\xi \in \overline{\mathbf{S}_\lambda^-}$. Then there exists a sequence $\{\xi_n\} \subseteq \mathbf{S}_\lambda^-$ such that $\xi_n \rightarrow \xi$ in $\mathbb{H}^{\alpha, \beta; \psi}$. Also $\xi_n \rightarrow \xi$ in $L^p(\Omega)$ and in $L^{q+1}(\Omega)$, as $1 < q < p - 1$. Now

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_n \right|^p - \lambda |\xi_n|^p \right) dx = \int_0^T b(x) |\xi_n|^{q+1} dx < 0,$$

so taking the limit as $n \rightarrow \infty$ we have

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi \right|^p - \lambda |\xi|^p \right) dx = \int_0^T b(x) |\xi|^{q+1} dx \leq 0.$$

If $\int_0^T b(x) |\xi|^{q+1} dx = 0$ then both integrals are equal to 0, so $\frac{\xi}{\|\xi\|} \in \mathbf{L}_0(\lambda) \cap \Upsilon_0$, which contradicts (1). Hence both integrals must be negative and so $\xi \in \mathbf{S}_\lambda^-$. Thus $\overline{\mathbf{S}_\lambda^-} = \mathbf{S}_\lambda^-$, so \mathbf{S}_λ^- is closed.

(4). Suppose that there exists $\xi \in \overline{\mathbf{S}_\lambda^+} \cap \mathbf{S}_\lambda^-$. Note from (3) that $\xi \neq 0$. Since $\xi \in \mathbf{S}_\lambda^-$ we have

$$(3.19) \quad \int_0^T b(x) |\xi|^{q+1} dx < 0.$$

Also since $\xi \in \overline{\mathbf{S}_\lambda^+}$, we have

$$(3.20) \quad \int_0^T b(x) |\xi|^{q+1} dx \geq 0,$$

which is impossible. Thus, we conclude that $\overline{\mathbf{S}_\lambda^+} \cap \mathbf{S}_\lambda^- = \emptyset$. \square

Lemma 3.7. *Suppose that $\mathbf{L}_-(\lambda) \cap \Upsilon_+ \neq \emptyset$. Thus, there exists $k > 0$ such that, for any $\varepsilon > 0$, there exists $\xi_\varepsilon \in \mathbf{L}_+(\lambda) \cap \Upsilon_+$ such that*

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{a+}^{\alpha, \beta; \psi} \xi_\varepsilon(x) \right|^p - \lambda |\xi_\varepsilon(x)|^p \right) dx < \varepsilon$$

and

$$\int_0^T b(x) |\xi_\varepsilon(x)|^{q+1} dx > k.$$

Proof. Let $\xi \in \mathbf{L}_-(\lambda) \cap \mathbf{Y}_+$, so that

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx < 0$$

and

$$\int_0^T b(x) |\xi(x)|^{q+1} dx > 0.$$

We can choose $h \in \mathbb{H}_p^{\alpha,\beta;\psi}$ in such a way that $\int_0^T \left| \mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} h(x) \right|^p dx$ is arbitrarily large and so we can take h so that

$$\int_0^T b(x) |(\xi + th)(x)|^{q+1} dx > \frac{1}{2} \int_0^T b(x) |\xi(x)|^{q+1} dx, \quad \forall t \in [0, 1]$$

and

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{a+}^{\alpha,\beta;\psi} (\xi + th)(x) \right|^p - \lambda |(\xi + th)(x)|^p \right) dx > 0.$$

Consider $\xi_t := \frac{\xi + th}{\|\xi + th\|}$. So, for $t \in [0, 1]$, we have $\xi_t \in \mathbf{Y}_+$. In fact,

$$\int_0^T b(x) |\xi_t(x)|^{q+1} dx \geq \frac{1}{(\|\xi\| + \|h\|)^{q+1}} \frac{1}{2} \int_0^T b(x) |\xi(x)|^{q+1} dx.$$

Now, $\xi_0 \in \mathbf{L}_-(\lambda)$ and $\xi_1 \in \mathbf{L}_+(\lambda)$. Take

$$\phi(t) = \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_t(x) \right|^p - \lambda |\xi_t(x)|^p \right) dx, \quad \forall t \in [0, 1].$$

So, the function $\phi : [0, 1] \rightarrow \mathbb{R}$ is a continuous function such that $\phi(0) < 0$ and $\phi(1) > 0$ and so for any $\varepsilon > 0$ given, there are values of t such that ξ_t has the required properties. \square

4. EXISTENCE AND NONEXISTENCE OF WEAK SOLUTIONS

In this section, we will discuss the existence and non-existence of weak solutions to the nonlinear problem with a fractional p -Laplacian as given by Eq.(1.1), introduced via the ψ -Hilfer fractional derivative.

Theorem 4.1. *Suppose that there exists $\bar{\lambda} > 0$, such that for any $\lambda < \bar{\lambda}$, we have $\mathbf{L}_-(\lambda) \subseteq \mathbf{Y}_-$. Then, for all $\lambda < \bar{\lambda}$, we have*

- (1) \mathbf{E}_λ is bounded from below in \mathbf{S}_λ^+ .
- (2) $\inf_{\xi \in \mathbf{S}_\lambda^-} \mathbf{E}_\lambda(\xi) > 0$, proving that \mathbf{S}_λ^- is nonempty.

Proof. (1). It is a direct consequence of bounded of \mathbf{S}_λ^+ .

(2). Notice that $\mathbf{E}_\lambda(\xi) \geq 0$, for all $\xi \in \mathbf{S}_\lambda^-$. In fact, if $\xi \in \mathbf{S}_\lambda^-$, then $\xi \in \mathbf{S}_\lambda$, so

$$\begin{aligned} \mathbf{E}_\lambda(\xi) &= \left(\frac{1}{p} - \frac{1}{q+1} \right) \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi \right|^p - \lambda |\xi|^p \right) dx \\ (4.1) \quad &= \left(\frac{1}{p} - \frac{1}{q+1} \right) \int_0^T b(x) |\xi|^{q+1} dx \geq 0. \end{aligned}$$

Suppose that $\inf_{\xi \in \mathbf{S}_\lambda^-} \mathbf{E}_\lambda(\xi) = 0$. Then, there exists $\{\xi_n\} \subseteq \mathbf{S}_\lambda^-$, so that $\lim_{n \rightarrow \infty} \mathbf{E}_\lambda(\xi_n) = 0$. From (4.1), we observe that

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n \right|^p - \lambda |\xi_n|^p \right) dx \rightarrow 0 \quad \text{and} \quad \int_0^T b(x) |\xi_n|^{q+1} dx \rightarrow 0, \quad \text{when } n \rightarrow \infty.$$

Let $\eta_n = \frac{\xi_n}{\|\xi_n\|}$. As we saw, $0 \notin \overline{\mathbf{S}_\lambda^-}$ (see (3) of Theorem 3.6), therefore there exists a $c > 0$ with $\|\xi_n\| > c$. Hence,

$$\lim_{n \rightarrow \infty} \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \eta_n \right|^p - \lambda |\eta_n|^p \right) dx = \lim_{n \rightarrow \infty} \frac{1}{\|\xi_n\|^p} \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n \right|^p - \lambda |\xi_n|^p \right) dx = 0,$$

and

$$\lim_{n \rightarrow \infty} \int_0^T b(x) |\eta_n|^{q+1} dx = \lim_{n \rightarrow \infty} \frac{1}{\|\xi_n\|^p} \int_0^T b(x) |\xi_n|^{q+1} dx = 0,$$

Also there exists a subsequence of $\{\eta_n\}$ (without loss of generality assume it is the whole sequence) with $\eta_n \rightharpoonup \eta_0$ in $\mathbb{H}_p^{\alpha, \beta; \psi}$ and as usual we may assume without loss of generality that

$$(4.2) \quad \begin{cases} \eta_n \rightharpoonup \eta_0 \text{ in } \mathbb{H}_p^{\alpha, \beta; \psi}; \\ \eta_n \rightarrow \eta_0 \text{ in } L^p(\Omega); \\ \eta_n \rightarrow \eta_0 \text{ in } L^{q+1}(\Omega); \\ \eta_n \rightarrow \eta_0 \text{ a.e. } \Omega. \end{cases}$$

From the dominated convergence theorem (note $b \in L^\infty$), we have

$$0 = \lim_{n \rightarrow \infty} \int_0^T b(x) |\eta_n|^{q+1} dx = \int_0^T b(x) |\eta_0|^{q+1} dx,$$

so $\frac{\eta_0}{\|\eta_0\|} \in \Upsilon_0$.

Now $\xi_n \subseteq S_\lambda^-$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^T |\eta_n|^p dx &= \int_0^T \lim_{n \rightarrow \infty} |\eta_n|^p dx, \\ \int_0^T |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \eta_0|^p dx &\leq \liminf_{n \rightarrow \infty} \int_0^T |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \eta_n|^p dx, \end{aligned}$$

so

$$(4.3) \quad \int_0^T \left(|\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \eta_0|^p - \lambda |\eta_0|^p \right) dx \leq \liminf_{n \rightarrow \infty} \int_0^T \left(|\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \eta_n|^p - \lambda |\eta_n|^p \right) dx \leq 0.$$

Hence, $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{L}_0(\lambda)$ or $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{L}_-(\lambda)$. However from above $\frac{\eta_0}{\|\eta_0\|} \in \Upsilon_0$, and this is a contradiction, because $\mathbf{L}_-(\lambda) \subseteq \Upsilon_-$, $\Upsilon_- \cap \Upsilon_0 = \emptyset$ and $\mathbf{L}_0(\lambda) \cap \Upsilon_0 = \emptyset$ (see (1) of Theorem 3.6). Therefore, $\inf_{\xi \in S_\lambda^-} \mathbf{E}_\lambda(\xi) > 0$. \square

Theorem 4.2. *Suppose that there exists $\bar{\lambda} > 0$, such that for any $\lambda < \bar{\lambda}$, we have $\mathbf{L}_-(\lambda) \subseteq \Upsilon_-(\lambda)$. Then, for all $\lambda < \bar{\lambda}$,*

- (1) *There is a minimizing point for \mathbf{E}_λ in \mathbf{S}_λ^+ ,*
- (2) *There is a minimizing point for \mathbf{E}_λ in \mathbf{S}_λ^- , proving that $\mathbf{L}_-(\lambda)$ is non-empty.*

Proof. (1). In view of Theorem 4.1, we know that \mathbf{E}_λ is bounded from below in \mathbf{S}_λ^+ . Thus, there exists a minimizing sequence $\{\xi_n\} \subseteq \mathbf{S}_\lambda^+$, such that

$$(4.4) \quad \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(\xi_n) = \inf_{\xi \in \mathbf{S}_\lambda^+} \mathbf{E}_\lambda(\xi).$$

Since \mathbf{S}_λ^+ is bounded (see (2) of Theorem 3.6) there exists a subsequence of $\{\xi_n\}$ (without loss of generality assume it is the whole sequence) such that

$$(4.5) \quad \begin{cases} \xi_n \rightharpoonup \xi_0 \text{ in } \mathbb{H}_p^{\alpha, \beta; \psi}; \\ \xi_n \rightarrow \xi_0 \text{ in } L^p(\Omega); \\ \xi_n \rightarrow \xi_0 \text{ in } L^{q+1}(\Omega); \\ \xi_n \rightarrow \xi_0 \text{ a.e. } \Omega. \end{cases}$$

Thus, we have

$$(4.6) \quad \lim_{n \rightarrow \infty} \int_0^T |\eta_n|^p dx = \int_0^T \lim_{n \rightarrow \infty} |\eta_n|^p dx$$

and

$$(4.7) \quad \int_0^T |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_0|^p dx \leq \liminf_{n \rightarrow \infty} \int_0^T |\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_n|^p dx.$$

Combining the above gives

$$\int_0^T \left(|\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_0|^p - \lambda |\xi_0|^p \right) dx \leq \liminf_{n \rightarrow \infty} \int_0^T \left(|\mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_n|^p - \lambda |\xi_n|^p \right) dx$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \inf \int_0^T b(x) |\xi_n|^{q+1} dx \\
(4.8) \quad &= \int_0^T b(x) |\xi_0|^{q+1} dx.
\end{aligned}$$

Also, from (4.5) and $\{\xi_n\} \subseteq \mathbf{S}_\lambda^+$, we have

$$\int_0^T b(x) |\xi_0|^{q+1} dx = \lim_{n \rightarrow \infty} \int_0^T b(x) |\xi_n|^{q+1} dx \geq 0,$$

so, $\frac{\xi_0}{\|\xi_0\|} \in \Upsilon_+ \cup \Upsilon_0$. Note that, if $\frac{\xi_0}{\|\xi_0\|} \in \Upsilon_0$, it follows from (4.8) that $\frac{\xi_0}{\|\xi_0\|} \in \mathbf{L}_0(\lambda) \cup \mathbf{L}_-(\lambda)$. First if $\frac{\xi_0}{\|\xi_0\|} \in \mathbf{L}_-(\lambda)$ then $\frac{\xi_0}{\|\xi_0\|} \in \mathbf{L}_-(\lambda) \subseteq \Upsilon_-$ so $\frac{\xi_0}{\|\xi_0\|} \in \Upsilon_-$, and so we have $\frac{\xi_0}{\|\xi_0\|} \in \Upsilon_- \cap \Upsilon_0$, which is impossible. Secondly if $\frac{\xi_0}{\|\xi_0\|} \in \mathbf{L}_0(\lambda)$ then $\frac{\xi_0}{\|\xi_0\|} \in \mathbf{L}_0(\lambda) \subseteq \Upsilon_-$ so $\frac{\xi_0}{\|\xi_0\|} \in \Upsilon_-$, and so we have $\frac{\xi_0}{\|\xi_0\|} \in \Upsilon_- \cap \Upsilon_0$, which is impossible. Therefore, $\frac{\xi_0}{\|\xi_0\|} \in \Upsilon_+$. Also as above (since $\Upsilon_- \cap \Upsilon_+ = \emptyset$) neither $\frac{\xi_0}{\|\xi_0\|} \in \mathbf{L}_-(\lambda)$ nor $\frac{\xi_0}{\|\xi_0\|} \in \mathbf{L}_0(\lambda)$ is possible, so $\frac{\xi_0}{\|\xi_0\|} \in \mathbf{L}_+(\lambda)$, and as a result $\frac{\xi_0}{\|\xi_0\|} \in \mathbf{L}_+(\lambda) \cap \Upsilon_+$.

From our previous (1) of **RESULTS I**, we obtain that the fibering map Ξ_{ξ_0} has a local minimum at t_{ξ_0} and $t_{\xi_0} \xi_0 \in \mathbf{S}_\lambda^+$, where

$$t_{\xi_0} = \left[\frac{\int_0^T b(x) |\xi_0|^{q+1} dx}{\int_0^T (|\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_0|^p - \lambda |\xi_0|^p) dx} \right]^{\frac{1}{p-(q+1)}} \geq 1.$$

Furthermore, we have

$$\begin{aligned}
\mathbf{E}_\lambda(\xi_0) &= \int_0^T (|\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_0|^p - \lambda |\xi_0|^p) dx - \int_0^T b(x) |\xi_0|^{q+1} dx \\
&\leq \lim_{n \rightarrow \infty} \inf \int_0^T (|\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n|^p - \lambda |\xi_n|^p) dx - \lim_{n \rightarrow \infty} \inf \int_0^T b(x) |\xi_n|^{q+1} dx \\
&\leq \lim_{n \rightarrow \infty} \inf \left(\int_0^T (|\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n|^p - \lambda |\xi_n|^p) dx - \int_0^T b(x) |\xi_n|^{q+1} dx \right) \\
&= \lim_{n \rightarrow \infty} \left(\int_0^T (|\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n|^p - \lambda |\xi_n|^p) dx - \int_0^T b(x) |\xi_n|^{q+1} dx \right) \\
(4.9) \quad &= \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(\xi_n).
\end{aligned}$$

Also from (4.8), we have the $\Xi'_{\xi_0}(1) \leq 0$. Now suppose $t_{\xi_0} > 1$. Then we have $\Xi'_{\xi_0}(1) < 0$. Since Ξ_{ξ_0} has a local minimum at t_{ξ_0} and $t_{\xi_0} \xi_0 \in \mathbf{S}_\lambda^+$ it follows that $\mathbf{E}_\lambda(t_{\xi_0} \xi_0) = \Xi_{\xi_0}(t_{\xi_0}) < \Xi_{\xi_0}(1) = \mathbf{E}_\lambda(\xi_0)$.

From (4.9) and (4.4) we have

$$\mathbf{E}_\lambda(t_{\xi_0} \xi_0) < \mathbf{E}_\lambda(\xi_0) \leq \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(\xi_n) = \inf_{\xi \in \mathbf{S}_\lambda^+} \mathbf{E}_\lambda(\xi),$$

which is impossible. Hence $t_{\xi_0} = 1$ so $\xi_0 = t_{\xi_0} \xi_0 \in \mathbf{S}_\lambda^+$. Thus ξ_0 is a minimizing point for \mathbf{E}_λ in \mathbf{S}_λ^+ ; note $\mathbf{E}_\lambda(\xi_0) \leq \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(\xi_n) = \inf_{\xi \in \mathbf{S}_\lambda^+} \mathbf{E}_\lambda(\xi)$ and since $\xi_0 \in \mathbf{S}_\lambda^+$ then $\mathbf{E}_\lambda(\xi_0) = \inf_{\xi \in \mathbf{S}_\lambda^+} \mathbf{E}_\lambda(\xi)$.

(2). Let $\{\xi_n\}$ be a minimizing sequence for \mathbf{E}_λ in \mathbf{S}_λ^- and from Theorem 4.1 we have

$$(4.10) \quad \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(\xi_n) = \lim_{n \rightarrow \infty} \left(\int_0^T |\mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n|^p dx - \lambda \int_0^T |\xi_n|^p dx - \int_0^T b(x) |\xi_n|^{q+1} dx \right) = \inf_{\xi \in \mathbf{S}_\lambda^-} \mathbf{E}_\lambda(\xi) > 0.$$

Now we claim that $\{\xi_n\}$ is a bounded sequence. Suppose otherwise, so we may assume that $\|\xi_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Let $\eta_n = \frac{\xi_n}{\|\xi_n\|}$. Then there exists a subsequence $\{\eta_n\}$ (without loss of generality assume it is the

whole sequence), such that

$$(4.11) \quad \begin{cases} \eta_n \rightharpoonup \eta_0 \text{ in } \mathbb{H}_p^{\alpha,\beta;\psi}; \\ \eta_n \rightarrow \eta_0 \text{ in } L^p(\Omega); \\ \eta_n \rightarrow \eta_0 \text{ in } L^{q+1}(\Omega); \\ \eta_n \rightarrow \eta_0 \text{ a.e. } \Omega. \end{cases}$$

From (4.10) we have

$$\lim_{n \rightarrow \infty} \int_0^T \left(|\mathbf{HD}_{0+}^{\alpha,\beta;\psi} \eta_n|^p - \lambda |\eta_n|^p \right) dx = \lim_{n \rightarrow \infty} \int_0^T b(x) |\eta_n|^{q+1} dx \|\xi_n\|^{q+1-p} = 0.$$

From (4.11), we have

$$\int_0^T \left(|\mathbf{HD}_{0+}^{\alpha,\beta;\psi} \eta_0|^p - \lambda |\eta_0|^p \right) dx \leq \liminf_{n \rightarrow \infty} \int_0^T \left(|\mathbf{HD}_{0+}^{\alpha,\beta;\psi} \eta_n|^p - \lambda |\eta_n|^p \right) dx = 0.$$

Moreover, since $\xi_n \in \mathbf{S}_\lambda^-$, we have

$$\int_0^T b(x) |\eta_n|^{q+1} dx = \frac{1}{\|\xi_n\|^{q+1}} \int_0^T b(x) |\xi_n|^{q+1} dx < 0,$$

so,

$$\int_0^T b(x) |\eta_0|^{q+1} dx = \lim_{n \rightarrow \infty} \int_0^T b(x) |\eta_n|^{q+1} dx \leq 0.$$

Since $\xi_n \in \mathbf{S}_\lambda$, we have

$$\begin{aligned} \mathbf{E}_\lambda(\xi_n) &= \left(\frac{1}{p} - \frac{1}{q+1} \right) \int_0^T \left(|\mathbf{HD}_{0+}^{\alpha,\beta;\psi} \xi_n|^p - \lambda |\xi_n|^p \right) dx \\ &= \left(\frac{1}{p} - \frac{1}{q+1} \right) \int_0^T b(x) |\xi_n|^{q+1} dx, \end{aligned}$$

which implies

$$(4.12) \quad \int_0^T b(x) |\xi_n|^{q+1} dx = \left(\frac{1}{p} - \frac{1}{q+1} \right)^{-1} \mathbf{E}_\lambda(\xi_n).$$

Since, $\{\mathbf{E}_\lambda(\xi_n)\}$ is bounded, it follows that $\int_0^T b(x) |\xi_n|^{q+1} dx$ is bounded so

$$\int_0^T b(x) |\eta_0|^{q+1} dx = \lim_{n \rightarrow \infty} \int_0^T b(x) |\eta_n|^{q+1} dx = \lim_{n \rightarrow \infty} \frac{1}{\|\xi_n\|^{q+1}} \int_0^T b(x) |\xi_n|^{q+1} dx = 0.$$

Thus, $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{L}_-(\lambda) \cap \mathbf{Y}_0$ or $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{L}_0(\lambda) \cap \mathbf{Y}_0$. First if $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{L}_-(\lambda) \cap \mathbf{Y}_0$, then $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{L}_-(\lambda) \subset \mathbf{Y}_-$, so $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{Y}_-$, and so we have $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{Y}_- \cap \mathbf{Y}_0$, which is impossible. Secondly, if $\frac{\eta_0}{\|\eta_0\|} \in \mathbf{L}_0(\lambda) \cap \mathbf{Y}_0$, it is also impossible, because it contradicts (1) of Theorem 3.6.

Hence $\{\xi_n\}$ is bounded. Thus we may assume there exists a subsequence $\{\xi_n\}$ (without loss of generality assume it is the whole sequence) such that

$$(4.13) \quad \begin{cases} \xi_n \rightharpoonup \xi_0 \text{ in } \mathbb{H}_p^{\alpha,\beta;\psi}; \\ \xi_n \rightarrow \xi_0 \text{ in } L^p(\Omega); \\ \xi_n \rightarrow \xi_0 \text{ in } L^{q+1}(\Omega); \\ \xi_n \rightarrow \xi_0 \text{ a.e. } \Omega. \end{cases}$$

Now since $\lim_{n \rightarrow \infty} \mathbf{E}_\lambda(\xi_n) > 0$ and $1 < q < p - 1$, taking the limit as $n \rightarrow \infty$ on both sides of Eq.(4.12), we have

$$(4.14) \quad \int_0^T b(x) |\xi_0|^{q+1} dx = \lim_{n \rightarrow \infty} \int_0^T b(x) |\xi_n|^{q+1} dx = \left(\frac{1}{p} - \frac{1}{q+1} \right)^{-1} \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(\xi_n) < 0.$$

From (4.14) and (4.13), we get

$$\int_0^T \left(|\mathbf{HD}_{0+}^{\alpha,\beta;\psi} \xi_0|^p - \lambda |\xi_0|^p \right) dx \leq \liminf_{n \rightarrow \infty} \int_0^T \left(|\mathbf{HD}_{0+}^{\alpha,\beta;\psi} \xi_n|^p - \lambda |\xi_n|^p \right) dx$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \inf \int_0^T b(x) |\xi_n|^{q+1} dx \\
(4.15) \quad &= \int_0^T b(x) |\xi_0|^{q+1} dx < 0.
\end{aligned}$$

Thus, $\frac{\xi_0}{\|\xi_0\|} \in \mathbf{L}_-(\lambda) \cap \Upsilon_-$. Now Ξ_ξ has a local maximum at t_{ξ_0} (see (2) of **RESULTS I**), and $t_{\xi_0} \xi_0 \in \mathbf{S}_\lambda^-$, where

$$t_{\xi_0} = \left[\frac{\int_0^T b(x) |\xi_0|^{q+1} dx}{\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_0 \right|^p - \lambda |\xi_0|^p \right) dx} \right]^{\frac{1}{p-(q+1)}} \leq 1.$$

Also from (4.15), we have $\Xi'_{\xi_0}(1) \geq 0$. Now since $t_{\xi_0} \xi_n \rightarrow t_{\xi_0} \xi_0$ in $\mathbb{H}_p^{\alpha, \beta; \psi}$ we have

$$\mathbf{E}_\lambda(t_{\xi_0} \xi_0) \leq \liminf_{n \rightarrow \infty} \mathbf{E}_\lambda(t_{\xi_0} \xi_n).$$

Also since $t \rightarrow \mathbf{E}_\lambda(t \xi_n)$ reaches its maximum at $t = 1$ so from (4.10) we have

$$\mathbf{E}_\lambda(t_{\xi_0} \xi_0) \leq \liminf_{n \rightarrow \infty} \mathbf{E}_\lambda(t_{\xi_0} \xi_n) \leq \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(\xi_n) = \inf_{\xi \in \mathbf{S}_\lambda^-} \mathbf{E}_\lambda(\xi),$$

and so since $t_{\xi_0} \xi_0 \in \mathbf{S}_\lambda^-$ we must have

$$\mathbf{E}_\lambda(t_{\xi_0} \xi_0) = \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(\xi_n) = \inf_{\xi \in \mathbf{S}_\lambda^-} \mathbf{E}_\lambda(\xi) \text{ i.e. } \mathbf{E}_\lambda(t_{\xi_0} \xi_0) = \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(\xi_n).$$

Thus $\xi_n \rightarrow t_{\xi_0} \xi_0$ in $\mathbb{H}_p^{\alpha, \beta; \psi}$ and since $\xi_n \rightarrow \xi_0$ in $\mathbb{H}_p^{\alpha, \beta; \psi}$ we must have $t_{\xi_0} = 1$.

Hence $t_{\xi_0} = 1$ so $\xi_0 = t_{\xi_0} \xi_0 \in \mathbf{S}_\lambda^-$. Thus ξ_0 is a minimizing point for \mathbf{E}_λ in \mathbf{S}_λ^- ; note $\mathbf{E}_\lambda(\xi_0) \leq \lim_{n \rightarrow \infty} \inf \mathbf{E}_\lambda(\xi_n) \leq \lim_{n \rightarrow \infty} \mathbf{E}_\lambda(\xi_n) = \inf_{\xi \in \mathbf{S}_\lambda^-} \mathbf{E}_\lambda(\xi)$ and since $\xi_0 \in \mathbf{S}_\lambda^-$ then $\mathbf{E}_\lambda(\xi_0) = \inf_{\xi \in \mathbf{S}_\lambda^-} \mathbf{E}_\lambda(\xi)$. \square

The existence of the above minimizing point implies the existence of corresponding non-negative solutions for Eq.(1.1). Indeed, for example, ξ_0 is a minimizing point for \mathbf{E}_λ in \mathbf{S}_λ^- . Since $\mathbf{E}_\lambda(\xi) = \mathbf{E}_\lambda(|\xi|)$, we can assume that ξ_0 is not negative in $[0, T]$. Note that, \mathbf{S}_λ^- is closed, so ξ_0 is a local minimum for \mathbf{E}_λ in \mathbf{S}_λ^- .

Theorem 4.3. *Suppose $\mathbf{L}_-(\lambda) \cap \Upsilon_+ \neq \emptyset$. Then*

- (1) $\inf_{\xi \in \mathbf{S}_\lambda^+} \mathbf{E}_\lambda(\xi) = -\infty$, when $\mathbf{S}_\lambda^+ \neq \emptyset$;
- (2) $\inf_{\xi \in \mathbf{S}_\lambda^-} \mathbf{E}_\lambda(\xi) = 0$, when $\mathbf{S}_\lambda^- \neq \emptyset$.

Proof. (1). Suppose $\xi_0 \in \mathbf{L}_-(\lambda) \cap \Upsilon_+$. It follows from Lemma 3.7 that there exists $k > 0$ and a sequence $\{\xi_n\} \subseteq \mathbf{L}_+(\lambda) \cap \Upsilon_+$ such that

$$\int_0^T b(x) |\xi_n(x)|^{q+1} dx \geq k \text{ and } 0 < \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_n(x) \right|^p - \lambda |\xi_n(x)|^p \right) dx < \frac{1}{n}.$$

So for $t_n \xi_n \in \mathbf{S}_\lambda^+$, we have

$$\begin{aligned}
\mathbf{E}_\lambda(t_n \xi_n) &= \left(\frac{1}{p} - \frac{1}{q+1} \right) (t_n)^p \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_n(x) \right|^p - \lambda |\xi_n(x)|^p \right) dx \\
&= \left(\frac{1}{p} - \frac{1}{q+1} \right) \frac{\left(\int_0^T b(x) |\xi_n(x)|^{q+1} dx \right)^{\frac{p}{p-(q+1)}}}{\left(\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha, \beta; \psi} \xi_n(x) \right|^p - \lambda |\xi_n(x)|^p \right) dx \right)^{\frac{q+1}{p-(q+1)}}} \\
&\leq \left(\frac{1}{p} - \frac{1}{q+1} \right) n^{\frac{q+1}{p-(q+1)}} k^{\frac{p}{p-(q+1)}} \rightarrow -\infty
\end{aligned}$$

when $n \rightarrow \infty$.

Thus $\inf_{\xi \in \mathbf{S}_\lambda^+} \mathbf{E}_\lambda(\xi) = -\infty$.

(2). Let $\xi \in \mathbf{L}_-(\lambda) \cap \mathbf{Y}_+$. We can choose $h \in \mathbb{H}_p^{\alpha,\beta;\psi}$ such that

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} (\xi + th)(x) \right|^p - \lambda |(\xi + th)(x)|^p \right) dx < \frac{1}{2} \int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi(x) \right|^p - \lambda |\xi(x)|^p \right) dx$$

for all $t \in [0, 1]$ and

$$\int_0^T b(x) |(\xi + h)(x)|^{q+1} dx < 0.$$

Let $\xi_t := \frac{\xi + th}{\|\xi + th\|}$, then for $t \in [0, 1]$, we have $\xi_t \in \mathbf{L}_-(\lambda)$. Moreover $\xi_0 \in \mathbf{Y}_+$ and $\xi_1 \in \mathbf{Y}_-$. By the same argument as in the proof of (1), we can deduce that there exist $\delta > 0$ and a sequence $\{\xi_n\} \subset \mathbf{L}_-(\lambda) \cap \mathbf{Y}_-$ such that

$$\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n \right|^p - \lambda |\xi_n(x)|^p \right) dx < -\delta$$

and

$$\int_0^T b(x) |\xi_n(x)|^{q+1} dx < -\frac{1}{n}.$$

Thus for $t_n \xi_n \in \mathbf{S}_\lambda^-$, we have

$$\begin{aligned} \mathbf{E}_\lambda(t_n \xi_n) &= \left(\frac{1}{p} - \frac{1}{q+1} \right) \frac{\left(\int_0^T b(x) |\xi_n(x)|^{q+1} dx \right)^{\frac{p}{p-(q+1)}}}{\left(\int_0^T \left(\left| \mathbf{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \xi_n(x) \right|^p - \lambda |\xi_n(x)|^p \right) dx \right)^{\frac{q+1}{p-(q+1)}}} \\ &\leq \left(\frac{1}{p} - \frac{1}{q+1} \right) \left(-\frac{1}{n} \right)^{\frac{p}{p-(q+1)}} \left(\frac{1}{\delta} \right)^{\frac{q+1}{p-(q+1)}} \rightarrow 0, \end{aligned}$$

when $n \rightarrow \infty$. Hence $\mathbf{E}_\lambda(t_n \xi_n) \rightarrow 0$ as $n \rightarrow \infty$ and $\inf_{\xi \in \mathbf{S}_\lambda^-} \mathbf{E}_\lambda(u) = 0$. \square

5. CONCLUDING REMARKS

In this paper we obtain the existence and non-existence of weak solutions to the nonlinear problem with a fractional p -Laplacian introduced by the ψ -Hilfer fractional operator via the combination of Nehari manifold and fibering maps. The results investigated here were discussed in the ψ -fractional derivative space $\mathbb{H}_p^{\alpha,\beta;\psi}$. The results presented here will allow one to investigate other p -Laplacian problems in the fractional sense.

Authors' contributions

All authors' contributions to this manuscript are the same. All authors read and approved the final manuscript.

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