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# Some elementary explicit bounds for two mollifications of the Moebius function

O. Ramaré

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## Abstract

We prove that the sum  $\sum_{\substack{d \leq x, \\ (d,r)=1}} \mu(d)/d^{1+\varepsilon}$  is bounded by  $1 + \varepsilon$ , uniformly in  $x \geq 1$ ,  $r$  and  $\varepsilon > 0$ . We prove a similar estimate for the quantity  $\sum_{\substack{d \leq x, \\ (d,r)=1}} \mu(d) \log(x/d)/d^{1+\varepsilon}$ . When  $\varepsilon = 0$ ,  $r$  varies between 1 and a hundred, and  $x$  is below a million, this sum is non-negative and this raises the question as to whether it is non-negative *for every*  $x$ .

## 1 Introduction and results

Our first result is the following:

**Theorem 1.1.** *When  $r \geq 1$  and  $\varepsilon \geq 0$ , we have*

$$\left| \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \right| \leq 1 + \varepsilon.$$

This Lemma generalizes the estimate of [5, Lemme 10.2] which corresponds to the case  $\varepsilon = 0$ . This generalization is *not* straightforward at all and requires a change of proof. The case  $\varepsilon = 0$  and  $r = 1$  is classical. The parameter  $\varepsilon$  that is being introduced induces some flexibility useful when applying Rankin's method (devised in [8]). As it turns out, we can do somewhat better concerning the lower bound, and we prove that

$$-\frac{11}{15}(1 + \varepsilon) \leq \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}.$$

We ran computations covering the range  $1 \leq x \leq 10^6$  and  $1 \leq r \leq 100$  with  $\varepsilon = 0$ ; we found that the lowest lower bound was met at  $x = 13$  and  $r = 1$ . This raises the following question:

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**Question 1.** *It is true that*

$$\sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d} \geq -2323/30030 \quad ?$$

See section 2 for a very preliminary result in this direction.

We proceed by proving the following more involved form:

**Theorem 1.2.** *When  $r \geq 1$  and  $1.38 \geq \varepsilon \geq 0$ , we have*

$$\left| \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \log \frac{x}{d} \right| \leq 1.4 + 4.7\varepsilon + 3.3\varepsilon^2 + (1 + \varepsilon) \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} x^\varepsilon$$

where

$$\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} = \prod_{p|r} \frac{p^{1+\varepsilon}}{p^{1+\varepsilon} - 1}. \quad (1)$$

The dependence in  $r$  is optimal as seen by taking for  $r$  the product of every primes not more than  $\sqrt{x}$ . The proof is again unbalanced with respect to the upper and the lower bound, and we prove a somewhat better lower bound:

$$-(1.434 + 4.992\varepsilon + 3.558\varepsilon^2) \leq \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \log \frac{x}{d}.$$

I expect the factor  $x^\varepsilon$  in the upper bound to be a blemish; however, the (limited) numerical verifications we ran suggest that the factor  $r^{1+\varepsilon}/\phi_{1+\varepsilon}(r)$  *cannot* be omitted even if the condition  $r \leq x$  is added (this condition often appears in practice). It should be added that it is not difficult to prove that

$$\sum_{d \leq x} \frac{\mu(d)}{d} \log \frac{x}{d} \sim 1 \quad (x \rightarrow \infty)$$

which means that one cannot expect an arbitrary small constant in the right hand side of the inequality given in Theorem 1.2. We have checked that

$$0 \leq \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d} \log \frac{x}{d} \leq \frac{r}{\phi(r)} + 0.007 \quad (x \leq 10^6, 1 \leq r \leq 100)$$

(where  $x$  is a real number and not especially an integer) and all these maxima were in fact very close to  $r/\phi(r)$ . These computations raise two questions:

**Question 2.** *Is it true that*

$$\sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d} \log \frac{x}{d} \geq 0, \quad (x \geq 1, r \geq 1) \quad ?$$

**Question 3.** *Is it true that*

$$\sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d} \log \frac{x}{d} \leq \frac{r}{\phi(r)} + 1, \quad (x \geq 1, r \geq 1) \quad ?$$

In both these questions,  $x$  is only assumed to be a positive real number. On recalling what happens in the case of Turán's conjecture on the summatory function of the Liouville function divided by its argument, see [2], we believe that the answer to the first question is no. The sum is however less likely to be very erratic because of the smoothing factor, a factor that is absent in Turán's problem. In direction of these conjecture, we note the following formula

$$\int_1^\infty \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d} \log \frac{x}{d} \frac{dx}{x^{s+1}} = \frac{r^{1+s}}{\phi_{1+s}(r)} \frac{1}{s^2 \zeta(1+s)}$$

from which we easily deduce (on taking  $s = \varepsilon > 0$  and letting  $\varepsilon$  go to infinity) that

$$\limsup_x \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d} \log \frac{x}{d} \geq \frac{r}{\phi(r)}.$$

We discuss some related points in the last section.

### Notation

We use here the notation  $h = \mathcal{O}^*(k)$  to mean that  $|h| \leq k$ . We denote by  $\tau(m)$  the number of (positive) divisors of  $m$ , and by  $(a, b)$  the gcd of  $a$  and  $b$ . For  $\varepsilon \geq 0$  and  $r \geq 1$  any natural squarefree number, we define two functions. The first one is alternatively defined by

$$f_{r,\varepsilon}(n) = \sum_{\substack{\ell|n, \\ (\ell,r)=1}} \frac{\mu(\ell)}{\ell^\varepsilon} \tau(n/\ell) \quad (2)$$

or, in multiplicative form, by:

$$f_{r,\varepsilon}(n) = \prod_{\substack{p^\nu || n, \\ p \nmid r}} \left( \nu + 1 - \frac{\nu}{p^\varepsilon} \right) \prod_{\substack{p^\nu || n, \\ p | r}} (\nu + 1). \quad (3)$$

We easily determine its Dirichlet series:  $\sum_{n \geq 1} f_{r,\varepsilon}(n)/n^s = \zeta(s)^2 / \zeta(s + \varepsilon)$ . We shall further write

$$f_{r,\varepsilon}(n) = \mathbb{1} \star g_{r,\varepsilon}(n) \quad (4)$$

where the function  $g_{r,\varepsilon}$  has the essential property of being non-negative and is being defined by:

$$g_{r,\varepsilon}(n) = \sum_{\substack{\ell|n, \\ (\ell,r)=1}} \frac{\mu(\ell)}{\ell^\varepsilon} \geq 0. \quad (5)$$

### Thanks

Sincere thanks are due to the careful referee who has checked our computations and indeed has rooted out several mistakes.

## 2 Verifying Theorem 1.1 for small values

We study what happens for small values here. The proof is pedestrian and painful, but I have not seen any way to avoid it, or to present it in a more general frame.

We study the following quantity:

$$m_0(r, x) = \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}. \quad (6)$$

**Lemma 2.1.** *When  $x < 10$  and  $\varepsilon \geq 0$ , we have  $-1/30 \leq m_0(r, x) \leq 1$ .*

*Proof.* The sum we consider reads

$$1 - \frac{h(2)}{2^{1+\varepsilon}} - \frac{h(3)}{3^{1+\varepsilon}} - \frac{h(5)}{5^{1+\varepsilon}} + \frac{h(6)}{6^{1+\varepsilon}} - \frac{h(7)}{7^{1+\varepsilon}}$$

where  $h$  is the characteristic function of the integers  $\leq x$  that are coprime with  $r$ . The minimum is clearly

$$1 - \frac{1}{2^{1+\varepsilon}} - \frac{1}{3^{1+\varepsilon}} - \frac{1}{5^{1+\varepsilon}}$$

which is minimal when  $\varepsilon = 0$ . This is the  $-1/30$ . The maximum contains the summand 1. If the summand  $1/6^{1+\varepsilon}$  is present, then so is the summand  $-1/2^{1+\varepsilon}$ . This concludes the proof.  $\square$

## 3 Auxiliaries

**Lemma 3.1.** *When  $\varepsilon \geq 0$ , we have*

$$\sum_{h \leq H} h^\varepsilon = \frac{H^{1+\varepsilon}}{1+\varepsilon} + \mathcal{O}^*(H^\varepsilon).$$

*This is also  $\leq H^{1+\varepsilon}$ . When  $H$  is an integer, we have  $\sum_{h \leq H} h^\varepsilon \geq \frac{H^{1+\varepsilon}}{1+\varepsilon}$ .*

*Proof.* Indeed, when  $\varepsilon > 0$ , a summation by parts gives us directly

$$\begin{aligned}\sum_{h \leq H} h^\varepsilon &= \sum_{h \leq H} \varepsilon \int_0^h dt/t^{1-\varepsilon} = \varepsilon \int_0^H \sum_{t < h \leq H} 1 dt/t^{1-\varepsilon} \\ &= \varepsilon \int_0^H (H-t) dt/t^{1-\varepsilon} + \mathcal{O}^*(H^\varepsilon).\end{aligned}$$

We proceed by continuity to cover the case  $\varepsilon = 0$ . When  $H$  is an integer, a comparison to an integral gives the result.  $\square$

**Lemma 3.2.** *For  $L > 1$ , we have*

$$\sum_{n \leq L} f_{r,\varepsilon}(n) \leq L \sum_{\ell \leq L} g_{r,\varepsilon}(\ell)/\ell. \quad (7)$$

*Proof.* We recall (4) and write, since  $g_{r,\varepsilon} \geq 0$

$$\sum_{n \leq L} f_{r,\varepsilon}(n) = \sum_{km \leq L} g_{r,\varepsilon}(m) \leq L \sum_{m \leq L} g_{r,\varepsilon}(m)/m.$$

The Lemma follows readily.  $\square$

**Lemma 3.3.** *For every integer  $n$  and any  $\varepsilon \geq 0$ , we have*

$$g_{1,\varepsilon}(\ell) \leq \sum_{mn=\ell} g_{1,\varepsilon/2}(m)g_{1,\varepsilon/2}(n).$$

*Proof.* We check that, when  $\alpha \geq 1$  is an integer and  $p$  a prime number,

$$\begin{aligned}g_{1,\varepsilon}(p^\alpha) &= 1 - \frac{1}{p^\varepsilon} = 1 - \frac{1}{p^{\varepsilon/2}} + \frac{1}{p^{\varepsilon/2}} \left(1 - \frac{1}{p^{\varepsilon/2}}\right) \\ &\leq g_{1,\varepsilon/2}(p^\alpha)g_{1,\varepsilon/2}(1) + g_{1,\varepsilon/2}(1)g_{1,\varepsilon/2}(p^\alpha) \\ &\leq \sum_{0 \leq \beta \leq \alpha} g_{1,\varepsilon/2}(p^{\alpha-\beta})g_{1,\varepsilon/2}(p^\beta).\end{aligned}$$

We conclude by invoking the multiplicativity of  $g_{1,\varepsilon/2}$ .  $\square$

**Lemma 3.4.** *We have when  $L \geq 7.2$ ,*

$$\sum_{p \leq L} \frac{\log p}{p-1} \leq \log L.$$

*Proof.* We cite [9, (2.8)]:

$$\sum_{p \leq L} \frac{\log p}{p} \leq \log L - \gamma - \sum_{p \geq 2} \frac{\log p}{p(p-1)} + \frac{1}{2 \log L}, \quad (L \geq 319)$$

from which we deduce, for  $L \geq 319$ ,

$$\sum_{p \leq L} \frac{\log p}{p-1} \leq \log L - \gamma + \frac{1}{2 \log L}.$$

A simple GP script shows that

$$\sum_{p \leq L} \frac{\log p}{p-1} \leq \log L$$

when  $1000 \geq L \geq 7.2$ , and the reader will conclude readily.  $\square$

**Lemma 3.5.** *We have, when  $L \geq 1$  and  $\varepsilon \geq 0$ ,*

$$\sum_{\ell \leq L} g_{1,\varepsilon}(\ell)/\ell \leq L^\varepsilon. \quad (8)$$

*Proof.* Verifying the stated inequality for  $1 \leq L < 8$  is (tedious but) easy, hence we can now assume that  $L \geq 8$ . We readily find that the sum in question is not more than

$$T = \prod_{p \leq L} \frac{1 - p^{-1-\varepsilon}}{1 - p^{-1}} = \exp \sum_{p \leq L} \log \left( 1 + \frac{1 - p^{-\varepsilon}}{p-1} \right).$$

We apply  $\log(1+x) \leq x$  for non-negative  $x$  and  $1 - p^{-\varepsilon} \leq \varepsilon \log p$  to get, when  $L \geq 8$ ,

$$T \leq \exp \varepsilon \sum_{p \leq L} \frac{\log p}{p-1} \leq L^\varepsilon$$

by invoking Lemma 3.4.  $\square$

**Lemma 3.6.** *We have, when  $L \geq 1$ ,  $r \geq 1$  and  $\varepsilon \geq 0$ ,*

$$\sum_{\ell \leq L} g_{r,\varepsilon}(\ell)/\ell \leq \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} L^\varepsilon. \quad (9)$$

*Proof.* We use the notation  $d|r^\infty$  to say that each prime factor of  $d$  divides  $r$ . We write

$$\begin{aligned} \sum_{\ell \leq L} \frac{g_{r,\varepsilon}(\ell)}{\ell} &= \sum_{\substack{d|r^\infty, \\ d \leq L}} \sum_{\substack{\ell \leq L/d, \\ (\ell,r)=1}} \frac{g_{r,\varepsilon}(\ell)}{\ell d} \\ &\leq L^\varepsilon \sum_{d|r^\infty} \frac{1}{d^{1+\varepsilon}} = L^\varepsilon \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} \end{aligned}$$

by Lemma 3.5. The Lemma follows readily.  $\square$

**Lemma 3.7.**

$$\sum_{m \leq M} m^\varepsilon \tau(m) = \frac{M^{1+\varepsilon}}{1+\varepsilon} \left( \log M + 2\gamma - \frac{1}{1+\varepsilon} \right) + \mathcal{O}^* \left( 0.961(1+2\varepsilon)M^{\frac{1}{2}+\varepsilon} \right)$$

*Proof.* We recall part of [1, Theorem 1.1]:

$$\sum_{m \leq t} \tau(m) = t \log t + (2\gamma - 1)t + \mathcal{O}^*(0.961\sqrt{t}), \quad (t \geq 1).$$

Since  $(t \log t + (2\gamma - 1)t)/\sqrt{t}$  is seen to vary between  $-0.681$  and  $0.155$  when  $t$  varies between  $0$  and  $1$ , this estimate is also valid for  $t > 0$ . We use summation by parts and find that

$$\begin{aligned} \sum_{m \leq M} m^\varepsilon \tau(m) &= M^\varepsilon \sum_{m \leq M} \tau(m) - \varepsilon \int_0^M \sum_{m \leq t} \tau(m) dt / t^{1-\varepsilon} \\ &= M^{1+\varepsilon}(\log M + 2\gamma - 1) + \mathcal{O}^* \left( 0.961M^{\frac{1}{2}+\varepsilon} \right) \\ &\quad - \varepsilon \int_0^M (\log t + 2\gamma - 1)t^\varepsilon dt + \mathcal{O}^* \left( 0.961\varepsilon \int_0^M t^{\varepsilon-1/2} dt \right) \\ &= \frac{M^{1+\varepsilon}}{1+\varepsilon} \left( \log M + 2\gamma - \frac{1}{1+\varepsilon} \right) + \mathcal{O}^* \left( 0.961(1+2\varepsilon)M^{\frac{1}{2}+\varepsilon} \right). \end{aligned}$$

□

**Lemma 3.8.** *We have, when  $n \geq 2$ ,*

$$g_{r,\varepsilon}(n) \leq 1 - \frac{\mathbb{1}_{(n,r)=1}\mu^2(n)}{n^\varepsilon}.$$

*Proof.* Indeed, we verify that  $(1-a)(1-b) \leq (1-ab)$  when  $0 \leq a, b \leq 1$ . The Lemma readily follows by recursion on the number of prime factors of  $n$ . □

## 4 Some lemmas on squarefree numbers

Here is a Lemma from [4]:

**Lemma 4.1.** *We have, for  $D \geq 1664$*

$$\sum_{d \leq D} \mu^2(d) = \frac{6D}{\pi^2} + \mathcal{O}^*(0.1333\sqrt{D}).$$

*In particular, this is not more than  $0.62D$  when  $D \geq 1700$ .*



**Lemma 4.2.** *We have*

$$\sum_{d \leq x} \mu^2(d)/\sqrt{d} \leq 1.33 \sqrt{x}, \quad (x \geq 1).$$

If we are ready to assume larger, we would not save much since the best constant one can get is  $12/\pi^2 = 1.215 + \mathcal{O}^*(0.001)$ .

*Proof.* We use PARI/GP see [7] and the following script:

```
{check(borne) =
  my(res = 0.0, coef = 0);
  for(d = 1, borne,
    res += moebius(d)^2/sqrt(d);
    coef = max(coef, res/sqrt(d)));
  return(coef)}
```

It is then almost immediate to check our result when  $x \leq 10^7$ , despite the lack of refinement of the script proposed. For larger values, we use a summation by parts together with Lemma 4.1.  $\square$

**Lemma 4.3.** *We have*

$$\sum_{d \leq x} \mu^2(d) \leq \frac{11}{15} x, \quad (x \geq 9).$$

We note that  $11/15 = 0.7333\dots$  while the asymptotically best constant is rather lower, namely  $6/\pi^2 = 0.607927\dots$ . Reaching  $73/115 = 0.63478\dots$  already requires to take  $x \geq 75$ , and this means we would have to handle the possible divisibility by 21 primes in section 2. This is out of reach of the simple-minded method we have at our disposal.

*Proof.* We use PARI/GP see [7] and the following script:

```
{check(borneinf, bornesup) =
  my(res = 0.0, coef = 0);
  res = sum(d = 1, borneinf-1, moebius(d)^2);
  for(d = borneinf, bornesup,
    res += moebius(d)^2;
    coef = max(coef, res/d));
  return(coef)}
```

It is then almost immediate to check our result when  $x \leq 10^7$ , despite the lack of refinement of the script proposed. For larger values, the result is an immediate consequence of Lemma 4.1.  $\square$

## 5 Proof of Theorem 1.1

Lemma 2.1 establishes Theorem 1.1 when  $x < 10$ , so we may assume  $x \geq 10$ . We further may restrict our attention to integer values of  $x$ . We start with

$$S_0 = \sum_{n \leq x} n^\varepsilon g_{r,\varepsilon}(n) = \sum_{n \leq x} \sum_{\substack{d|n, \\ (d,r)=1}} \mu(d)(n/d)^\varepsilon.$$

Using the first expression yields  $0 \leq S_0$  as well as

$$S_0/x^\varepsilon \leq 1 + \sum_{2 \leq n \leq x} \left( g_{r,\varepsilon}(n) + \frac{\mathbb{1}_{(n,r)=1} \mu^2(n)}{n^\varepsilon} \right) - \sum_{\substack{2 \leq n \leq x, \\ (n,r)=1}} \frac{\mu^2(n)}{n^\varepsilon}$$

Each summand in the second sum is bounded above by 1 by Lemma 3.8. We get

$$0 \leq S_0/x^\varepsilon \leq x - \sum_{\substack{2 \leq n \leq x, \\ (n,r)=1}} \frac{\mu^2(n)}{n^\varepsilon}.$$

Let us write the second expression for  $S_0$ :

$$S_0 = \sum_{\substack{d \leq x, \\ (d,r)=1}} \mu(d) \sum_{m \leq x/d} m^\varepsilon.$$

We employ Lemma 3.1; we treat the case  $d = 1$  separately for the lower bound to reach

$$\begin{aligned} \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} - x^\varepsilon \sum_{\substack{2 \leq d \leq x, \\ (d,r)=1}} \mu^2(d) d^{-\varepsilon} &\leq S_0 \\ &\leq \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} + x^\varepsilon \sum_{\substack{d \leq x, \\ (d,r)=1}} \mu^2(d) d^{-\varepsilon}. \end{aligned}$$

The lower bound requires  $x$  to be an integer, but not the upper bound. We rewrite the above as

$$S_0 - x^\varepsilon \sum_{\substack{d \leq x, \\ (d,r)=1}} \mu^2(d) d^{-\varepsilon} \leq \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \leq S_0 + x^\varepsilon \sum_{\substack{2 \leq d \leq x, \\ (d,r)=1}} \mu^2(d) d^{-\varepsilon}.$$

By conjugating both estimates, we get,

$$-x^\varepsilon \sum_{\substack{d \leq x, \\ (d,r)=1}} \mu^2(d) d^{-\varepsilon} \leq \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \leq x^{1+\varepsilon}.$$

The right hand side is easily handled. We use Lemma 4.3 for the left hand side via, when  $x \geq 9$ :

$$\sum_{\substack{d \leq x, \\ (d,r)=1}} \mu^2(d)d^{-\varepsilon} \leq \sum_{d \leq x} \mu^2(d) \leq \frac{11}{15}x.$$

By conjugating both estimates, we get

$$-\frac{11}{15}(1 + \varepsilon) \leq \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \leq 1 + \varepsilon. \quad (x \geq 9) \quad (10)$$

Theorem 1.1 is proved.

## 6 Proof of Theorem 1.2

The proof relies on two ways of writing the sum

$$S_1 = \sum_{n \leq x} n^\varepsilon f_{r,\varepsilon}(n) = \sum_{n \leq x} \sum_{\substack{d|n, \\ (d,r)=1}} \mu(d)(n/d)^\varepsilon \tau(n/d).$$

The first form shows that  $0 \leq S_1 \leq x^{1+2\varepsilon} r^{1+\varepsilon} / \phi_{1+\varepsilon}(r)$  by combining Lemma 3.2 together with Lemma 3.6. Let us write this sum differently:

$$S_1 = \sum_{\substack{d \leq x, \\ (d,r)=1}} \mu(d) \sum_{m \leq x/d} m^\varepsilon \tau(m)$$

and we use Lemma 3.7 to reach

$$S_1 = \frac{x^{1+\varepsilon}}{1 + \varepsilon} \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \left( \log \frac{x}{d} + 2\gamma - \frac{1}{1 + \varepsilon} \right) + \mathcal{O}^* \left( 0.961 \times 1.33 (1 + 2\varepsilon) x^{1+\varepsilon} \right)$$

since  $\sum_{d \leq x} \mu^2(d) / \sqrt{d} \leq 1.33\sqrt{x}$  by Lemma 4.2. We set

$$\alpha = 2\gamma - \frac{1}{1 + \varepsilon} \in [0, 1]. \quad (11)$$

All of that amounts to:

$$\begin{aligned} S_1 &= \frac{x^{1+\varepsilon}}{1 + \varepsilon} \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \left( \log \frac{x}{d} + \alpha \right) + \mathcal{O}^* (1.279(1 + 2\varepsilon)x^{1+\varepsilon}) \\ &= S_1^* + \alpha S_0 + \mathcal{O}^* (1.279(1 + 2\varepsilon)x^{1+\varepsilon}) \end{aligned}$$

say. We thus have

$$-1.279(1 + 2\varepsilon)x^{1+\varepsilon} \leq S_1^* + \alpha S_0 \leq 1.279(1 + 2\varepsilon)x^{1+\varepsilon} + x^{1+2\varepsilon} \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}.$$

We use (10) and Lemma 2.1, and reach

$$-1.279(1 + 2\varepsilon) - \alpha \leq x^{-1-\varepsilon} S_1^* \leq 1.279(1 + 2\varepsilon) + \frac{11}{15}\alpha + x^\varepsilon \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}.$$

We use  $\alpha \leq 2\gamma - 1 + \varepsilon$ . This gives

$$\begin{aligned} -1.434 - 4.992\varepsilon - 3.558\varepsilon^2 &\leq \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \log \frac{x}{d} \\ &\leq 1.393 + 4.684\varepsilon + 3.292\varepsilon^2 + (1 + \varepsilon) \frac{x^\varepsilon r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}. \end{aligned}$$

Since  $x^\varepsilon r^{1+\varepsilon} / \phi_{1+\varepsilon}(r) \geq 1$ , we check that the right hand side is larger than minus times the left hand side. Theorem 1.2 follows.

## 7 A generalization and a remark

It is not difficult to get along these lines the following Lemma:

**Lemma 7.1.** *When  $r \geq 1$  and  $k \geq 1$ , we have*

$$\sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \log^k \frac{x}{d} \ll_k \left( \frac{r}{\phi(r)} \right)^k (\log x)^{k-1}.$$

Such quantities appear for instance in [10] where cases  $k = 0$  and  $k = 1$  are used, while case  $k = 2$  is evaluated (there is a main term), but all with no coprimality conditions (i.e.  $r = 1$ ) and no  $\varepsilon$ . The reader will find in [3, Chapter 1] the evaluation of case  $k = 3$ ,  $r = 1$  and  $\varepsilon = 0$ . [6] also pertains to these quantities.

*Proof.* Indeed, we first prove that

$$\sum_{n \leq x} \sum_{\substack{d|n, \\ (d,r)=1}} \mu(d) (n/d)^\varepsilon \tau_{k+1}(n/d) \ll \left( \frac{r}{\phi(r)} \right)^k x (\log x)^{k-1}.$$

We then continue as in section 6. □

Here is a surprising elementary consequence.

**Lemma 7.2.** For any  $c > 0$ , we have

$$\sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d} - x^\varepsilon \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \ll_c \varepsilon \frac{r}{\phi(r)}$$

provided that  $0 \leq \varepsilon \leq c(\log x)^{-1}$ .

*Proof.* It is enough to consider

$$\int_0^\varepsilon \sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)x^\eta}{d^{1+\eta}} \log(x/d) d\eta \ll \varepsilon \frac{r}{\phi(r)}.$$

□

## References

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