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A remark on the geometry of spaces of functions with prime frequencies.

P. Lefèvre, E. Matheron, O. Ramaré*

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Abstract

We prove that the space of continuous functions whose frequencies are products of r powers of primes numbers, contains some complemented copies of ℓ^1 hence is far from being isomorphic to the whole space of continuous functions, or to the disc algebra. Actually, our results are more general and may be written in some other frameworks, like the space L^1 .

Key Words: Prime numbers, continuous functions, disk algebra, thin sets.

AMS classification: 42A55, 43A46, 46B03.

Introduction

It is now a well known topic in analysis to study the geometry (from the point of view of Banach spaces) of the classical spaces of functions, with lacunary spectrum, and try to compare the “thinness” properties of a subset $\Lambda \subseteq \mathbb{Z}$ with the Banach space properties of the space X_Λ , where X is a Banach space of integrable functions on \mathbb{T} and X_Λ is the subspace of X consisting of the $f \in X$ whose spectrum lies in Λ : $\hat{f}(\gamma) = 0$ if $\gamma \notin \Lambda$. We refer to the monographs [12] or [6] to know more on the various notions of thin sets of integers.

This wide topic has been extensively studied: see for instance [1], [6], [9] or more recently [3].

We shall concentrate our attention on sets $\Lambda \subset \mathbb{Z}$ closely related to the set \mathcal{P} of prime numbers. Considering the set of prime numbers as “thin” is a question of point of view. On one hand, it is a small set since it has zero density in the set of integers. On the other hand, it contains arbitrarily large arithmetic progressions [2] which is forbidden for the most classical thin sets in harmonic analysis. The set \mathcal{P} shares some equirepartition properties with the set of all integers (see [11] for instance).

There is a nice result obtained by Lust-Piquard [8]: she showed that the space $C_{\mathcal{P}}(\mathbb{T})$ contains subspaces isomorphic to c_0 . It is well known that the same phenomenon occurs for the space $C(\mathbb{T})$ or the disc algebra $A(\mathbb{D}) = C_{\mathbb{N}}(\mathbb{T})$.

*preprint

As a consequence, \mathcal{P} may appear as a “not very thin” subset of \mathbb{N} . Actually, it was proved more recently in [7] that there are some rather thin sets Λ (p -Sidon sets for instance) such that $C_\Lambda(\mathbb{T})$ contains a subspace isomorphic to c_0 . A first natural question concerning prime numbers is whether the space $C_{\mathcal{P}}(\mathbb{T})$ contains not only some but many subspaces isomorphic to c_0 , i.e. satisfies the property (V) of Pełczyński (see the definition below). Note that very thin sets like Sidon sets (or Rosenthal sets) are far from having this property: when Λ is a Sidon set, $C_\Lambda(\mathbb{T})$ is isomorphic to ℓ^1 , hence cannot contain a subspace isomorphic to c_0 .

In an opposite direction, it is proved in [4] that the set \mathcal{P} of prime numbers is a Lust-Piquard set (= a totally ergodic set), hence sharing this property with “very thin sets” (like Sidon sets, or more generally Rosenthal sets). Let us recall that neither \mathbb{Z} or \mathbb{N} have this property.

A natural question then occurs: is the geometry of the space $C_{\mathcal{P}}(\mathbb{T})$ very different from the one of $C(\mathbb{T})$ or of the disc algebra? We shall give a positive answer, proving in particular that $C_{\mathcal{P}}(\mathbb{T})$ is isomorphic neither to $C(\mathbb{T})$ nor to the disc algebra.

Actually, writing $\mathcal{P}_r = \left\{ \prod_{1 \leq i \leq r} p_i^{\alpha_i} \mid p_i \in \mathcal{P}, \alpha_i \in \mathbb{N} \right\}$, our main result is the following:

Theorem.

Let $\Lambda \subset \mathbb{Z}$ such that $\mathcal{P} \subset \Lambda \subset \mathcal{P}_r \cup (-\mathcal{P}_r)$.

Then

1. $C_\Lambda(\mathbb{T})$ contains a complemented copy of ℓ^1 .
2. $L_\Lambda^1(\mathbb{T})$ and $L_{\mathbb{Z} \cup \Lambda}^1(\mathbb{T})$ contain a complemented copy of ℓ^2 .

As an immediate corollary, we have:

Corollary.

Let $\Lambda \subset \mathbb{Z}$ such that $\mathcal{P} \subset \Lambda \subset \mathcal{P}_r \cup (-\mathcal{P}_r)$.

Then

1. $C_\Lambda(\mathbb{T})$ does not have property (V) of Pełczyński.
2. $L_\Lambda^1(\mathbb{T})$ and $L_{\mathbb{Z} \cup \Lambda}^1(\mathbb{T})$ do not verify the Grothendieck Theorem.

In particular, $C_\Lambda(\mathbb{T})$ is isomorphic neither to $C(\mathbb{T})$ nor to the disc algebra. Neither $L_\Lambda^1(\mathbb{T})$ nor $L_{\mathbb{Z} \cup \Lambda}^1(\mathbb{T})$ is isomorphic to $L^1(\mathbb{T})$.

Recall that a Banach space X has the *property (V) of Pełczyński* if, for every non weakly compact (bounded) operator $T : X \rightarrow Y$, there is a subspace X_0 of X , isomorphic to c_0 , such that T is an isomorphism from X_0 onto $T(X_0)$ (i.e. T stabilizes a copy of c_0). This property is shared by $C(\mathbb{T})$ and $A(\mathbb{D})$.

We shall say that a Banach space X verifies the *Grothendieck Theorem* if every (bounded) operator from X to a Hilbert space is absolutely summing. This property is shared by $L^1(\mathbb{T})$. See [10] to know more on the subject.

The proof of the theorem relies on a gap property realized by products of powers of prime numbers:

Proposition.

Let $r \geq 1$. There is an infinite subset E_r of \mathcal{P} such that

$$\sup_{\lambda \in E_r} d(\lambda, \mathcal{P}_r \setminus \{\lambda\}) = +\infty$$

where $d(\lambda, A)$ denotes the distance: $d(\lambda, A) = \inf\{|\lambda - a|; a \in A\}$.

In other words, there are big symmetric holes in any Λ such that $\mathcal{P} \subset \Lambda \subset \mathcal{P}_r$. Even in the case $\Lambda = \mathcal{P}$, we did not find any reference of the preceding proposition. Of course, it is well known that there are big holes in \mathcal{P} , but the standard argument is to consider the set $\{n! + 2, \dots, n! + n\}$ with n arbitrarily large (which is obviously disjoint from the set of prime numbers), which provides one-sided holes in \mathcal{P} . The new point in the proposition is that we can even assume that these holes are symmetric. If we knew that there are infinitely many primorial numbers (resp. factorial numbers), then it would suffice to apply directly Prop.2 in [4] to prove the theorem.

Proof of the proposition.

For convenience, we shall define $\mathcal{P}_0 = \{1\}$.

We prove by induction on $r \in \mathbb{N}$ the assertion \mathcal{H}_r : “For every $a \geq 1$, there exists a prime number p such that $\{p - a, \dots, p - 1, p + 1, \dots, p + a\} \cap \mathcal{P}_r = \emptyset$.”

The property \mathcal{H}_0 is obvious.

Now assume that \mathcal{H}_{r-1} is true with $r \geq 1$. Given $a > 1$, choose a prime number $A > a + 1$ such that $\{A - a, \dots, A - 1, A + 1, \dots, A + a\} \cap \mathcal{P}_{r-1} = \emptyset$. Let

$$Q = \prod_{\substack{1 \leq k < 2A \\ k \neq A}} k^2 = \left(\frac{(2A - 1)!}{A} \right)^2.$$

Note that A and Q are coprime.

By the Dirichlet’s theorem on arithmetic progressions, we obtain a prime number $p > A$ such that $p \equiv A [Q]$.

We claim that this prime number p satisfies

$$d(p, \mathcal{P}_r \setminus \{p\}) \geq a.$$

The conclusion will then follow.

In order to prove the claim we shall actually prove that for every j in $\{1, \dots, A - 2\}$, neither $p - j$ nor $p + j$ belongs to \mathcal{P}_r .

Indeed, fix $j \in \{\pm 1, \dots, \pm a\}$ and assume that $p+j = \prod_{1 \leq i \leq r} q_i^{\alpha_i}$ where $q_i \in \mathcal{P}$, $\alpha_i \in \mathbb{N}$. Since $A+j$ divides Q and $p+j \equiv A+j [Q]$, the number $A+j$ divides $p+j$. Hence we can write $A+j = \prod_{1 \leq i \leq r} q_i^{\beta_i}$ where $\beta_i \leq \alpha_i$. If at least one of the β_i vanished, then $A+j$ would belong to \mathcal{P}_{r-1} , but this is false. Hence for every $1 \leq i \leq r$, we have $\beta_i \geq 1$. On the other hand, since $A < p$, there exists some $i_0 \in \{1 \dots, r\}$ such that $\beta_{i_0} < \alpha_{i_0}$. Now, consider $N = q_{i_0}(A+j)$. Since $\beta_{i_0} \geq 1$, we have q_{i_0} divides $A+j$, hence N divides $(A+j)^2$; but $(A+j)^2$ divides Q so that N divides Q . On the other hand N divides $p+j$ since $\beta_{i_0} + 1 \leq \alpha_{i_0}$. So N must divide $A+j$ which is false. The claim is proved and \mathcal{H}_r also. ■

Proof of the theorem.

It suffices to apply the preceding proposition and Prop.2 in [4]. Nevertheless, we repeat quickly the argument in our framework.

We shall produce a translation invariant projection which is nothing but a convolution with a Riesz product.

From the proposition we get by induction a sequence of prime numbers $(p_n)_{n \geq 1}$ which is a Hadamard sequence with ratio at least 3. We write $H = \{p_n \ n \geq 1\}$. The mesh

$$[H] = \left\{ \sum_{1 \leq n \leq m} \varepsilon_n p_n \mid \varepsilon_n \in \{-1, 0, 1\}; m \geq 1 \right\}$$

intersects $\mathcal{P}_r \cup (-\mathcal{P}_r)$ only on the $\pm p_n$'s.

Then we consider the Riesz product

$$R_N(x) = 2 \prod_{n=1}^N [1 + \cos(2\pi p_n x)].$$

A standard star-weak compactness argument in $M(\mathbb{T})$ provides a measure μ (bounded in norm by 2) whose Fourier coefficient are 1 on H and vanish on $\Lambda \setminus [H]$.

The convolution operator associated to μ is then a translation invariant projection from $C_\Lambda(\mathbb{T})$ onto $C_H(\mathbb{T})$, which is isomorphic to ℓ^1 (H is a Sidon set).

The same convolution operator also defines a projection from $L^1_\Lambda(\mathbb{T})$ onto $L^1_H(\mathbb{T})$, which is isomorphic to ℓ^2 (H is a $\Lambda(2)$ set).

At last, the same conclusion for the space $L^1_{\mathbb{Z}-\cup\Lambda}(\mathbb{T})$ holds, using Prop. 1.4 in [5] (see [?] for some details). ■

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