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EXPLICIT ESTIMATES FOR THE SUMMATORY FUNCTION OF $\Lambda(n)/n$ FROM THE ONE OF $\Lambda(n)$

OLIVIER RAMARÉ

ABSTRACT. We prove that the error term $\sum_{n \leq x} \Lambda(n)/n - \log x + \gamma$ differs from $(\psi(x) - x)/x$ by a well controlled function. We deduce very precise numerical results from this formula.

1. INTRODUCTION

We define classically

$$\psi(x) = \sum_{n \leq x} \Lambda(n), \quad \tilde{\psi}(x) = \sum_{n \leq x} \Lambda(n)/n.$$

There has been a good amount of work to find explicit asymptotics for $\psi(x)$, see for instance [15], [16], [17], [18] and [13]. The quantity $\tilde{\psi}(x)$ has been much less studied though [16, Theorem 6] gives an estimate. There has been an attempt in a more general setting in [10] and recent attention has been turned to the Mertens product, as in [2]. The problem here is that one would really want to deduce such an estimate from the ones concerning $\psi(x)$, but such a method is missing. The aim of this paper is to provide a fairly simple roundabout, see Theorem 1.1 below.

Let us note that the prime number Theorem in the form $\psi(x) = (1 + o(1))x$ is “elementarily” equivalent to

$$(1.1) \quad \tilde{\psi}(x) = \log x - \gamma + o(1).$$

So in a sense, we are concerned with a quantitative version of this equivalence. A simple integration by parts is *not* enough, as it loses a log-factor. In effect, an estimate of the form $|\psi(x) - x|/x \leq 0.01$ for x large enough transfers into something like $|\tilde{\psi}(x) - \log x + \gamma| \leq 0.01 \log x$ which is of no interest. The Landau equivalence Theorem can however be made explicit, but forbid a saving better than $1/\sqrt{\log x}$ in a rough form ; allowing a saving of any power of $\log x$ is already theoretically not obvious, see [9] for instance. Here is a conjecture.

Conjecture (Strong form of Landau equivalence Theorem, I).

There exist two positive constants c_1 and c_2 such that

$$\left| \tilde{\psi}(x) - \log x + \gamma \right| \leq c_1 \max_{x/c_2 < y \leq c_2 x} \frac{|\psi(y) - y|}{y} + c_2 x^{-1/4}.$$

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Such a conjecture holds (almost trivially) true under the Riemann Hypothesis. The result of [3] indicates that such an inequality does not hold in the case of Beurling generalized integers. Indeed they show that the condition $\psi_{\mathcal{P}}(x) \sim x$ does not ensure that $\tilde{\psi}_{\mathcal{P}}(x) - \log x$ has a limit, with obvious notations.

Let us end this introduction with a remark: in [7], the authors exhibit, under the Riemann Hypothesis, a pseudo-periodical function that (essentially) takes the value $(\tilde{\psi}(e^{-y}) + y) e^{y/2}$ when $y < 0$ and $(\psi(e^y) - e^y) e^{-y/2}$ when $y > 0$. This means that the values of ψ and of $\tilde{\psi}$ may share a much more profound link than proposed in the above conjecture.

We are not able to prove our conjecture, but show in Lemma 2.2 that

$$\tilde{\psi}(x) - \log x + \gamma - \frac{\psi(x) - x}{x}$$

is a well-controlled function. Here are some consequences of this formula.

Theorem 1.1. *For $x \geq 8950$, we have*

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} + \mathcal{O}^*\left(\frac{1}{2\sqrt{x}}\right) + \mathcal{O}^*(1.75 \cdot 10^{-12}).$$

Furthermore when $\log x \geq 9270$, we have (with $R = 5.69693$)

$$\begin{aligned} \tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} + \mathcal{O}^*\left(\frac{1}{2\sqrt{x}}\right) \\ + \mathcal{O}^*\left(\frac{1 + 2\sqrt{(\log x)/R}}{2\pi} \exp\left(-2\sqrt{(\log x)/R}\right)\right). \end{aligned}$$

Corollary. *We have for $x > 1$,*

$$\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*(1.833/\log^2 x).$$

Furthermore, for $1 \leq x \leq 10^{10}$, we have $\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*(1.31/\sqrt{x})$. For $x \geq 23$, we have $\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*(0.0067/\log x)$.

As a comparison, [16, Theorem 6] proposes an inequality similar to the last one above, but with $1/2 = 0.5$ instead of 0.0067 . No error term with a saving of $1/\log^2 x$ is proposed.

Notation. We use the classical counting function

$$(1.2) \quad N(T) = \sum_{\substack{\rho \\ 0 < \gamma \leq T}} 1,$$

where $\rho = \beta + i\gamma$ is a zero of the Riemann zeta-function. Furthermore, by $f(x) = \mathcal{O}^*(g(x))$ we mean $|f(x)| \leq g(x)$.

The computations required have been done via Pari/GP, see [12].

Thanks. Thanks are due to the referee for his/her careful reading that has helped improve these results.

2. AN EXPLICIT FORMULA

We will need [14, Lemma 4]:

Lemma 2.1. *Let g be a continuously differentiable function on $[a, b]$ with $2 \leq a \leq b < +\infty$. We have*

$$\begin{aligned} \int_a^b \psi(t)g(t)dt &= \int_a^b tg(t)dt - \sum_{\rho} \int_a^b \frac{t^{\rho}}{\rho}g(t)dt \\ &\quad + \int_a^b (\log 2\pi - \frac{1}{2} \log(1 - t^{-2}))g(t)dt. \end{aligned}$$

Here is our main formula.

Lemma 2.2. *We have, for $x \geq 1$:*

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} + \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho-1)} + \frac{B(x)}{x}.$$

where the sum is over the zeroes ρ of the Riemann zeta function that lie in the critical strip $0 < \Im s < 1$ (the so-called non trivial zeroes) and $B(x)$ is the bounded function given by

$$B(x) = \frac{1}{2} + \log 2\pi - \frac{x-1}{2} \log(1 - x^{-1}).$$

The main feature of the Lemma is that the sum over the zeroes is uniformly convergent, a feature not shared by the explicit formulae for $\psi(x)$ or for $\tilde{\psi}(x)$. In fact, the main difficulty is carried by the term $(\psi(x) - x)/x$.

Proof. We simply proceed by integration by parts:

$$\begin{aligned} \tilde{\psi}(x) &= \int_1^x \psi(t) \frac{dt}{t^2} + \frac{\psi(x)}{x} \\ &= \log x - \gamma + \int_x^{\infty} (\psi(t) - t) \frac{dt}{t^2} + \frac{\psi(x) - x}{x}. \end{aligned}$$

Note that the existence of the integral requires a strong enough form of the equivalence between $\psi(t)$ and t . Next we apply the explicit formula given in Lemma 2.1 and get

$$\begin{aligned} \int_x^Y (\psi(t) - t) \frac{dt}{t^2} &= - \sum_{\rho} \int_x^Y \frac{t^{\rho-2}}{\rho} dt + \int_x^Y (\log 2\pi - \frac{1}{2} \log(1 - t^{-2})) \frac{dt}{t^2} \\ &= - \sum_{\rho} \frac{Y^{\rho-1} - x^{\rho-1}}{\rho(\rho-1)} + \int_x^Y (\log 2\pi - \frac{1}{2} \log(1 - t^{-2})) \frac{dt}{t^2}. \end{aligned}$$

Since (1.1) is known to hold, and $\sum_{\rho} 1/|\rho(\rho-1)|$ is convergent, we can send Y to infinity and get

$$\int_x^Y (\psi(t) - t) \frac{dt}{t^2} = \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho-1)} + \int_x^{\infty} (\log 2\pi - \frac{1}{2} \log(1 - t^{-2})) \frac{dt}{t^2}.$$

□

3. KNOWN BOUNDS ON $\psi(x)$

In [13], we find that

$$(3.1) \quad |\psi(x) - x| \leq \sqrt{x} \quad (8 \leq x \leq 10^{10}).$$

If we change this \sqrt{x} by $\sqrt{2x}$, this is valid from $x = 1$ onwards. Furthermore

$$(3.2) \quad |\psi(x) - x| \leq 0.8 \sqrt{x} \quad (1500 \leq x \leq 10^{10}).$$

By [4, Théorème 1.3] improving on [18, Theorem 7], we have

$$(3.3) \quad |\psi(x) - x| \leq 0.0065x/\log x \quad (x \geq \exp(22))$$

We readily extend this estimate to $x \geq 3\,430\,190$ by using (3.1), and then to $x \geq 1\,514\,928$ by direct inspection.

We quote [4, Théorème 1.4] and in [5, Theorem 5.2]

$$(3.4) \quad |\vartheta(x) - x| \leq 3.965x/\log^2 x \quad (x > 2)$$

$$(3.5) \quad |\vartheta(x) - x| \leq 515x/\log^3 x \quad (x > 2)$$

In fact [5, Theorem 5.2] proposes the constant 21 instead of 515 in this inequality, but this preprint has not been published. We will not use this bound but take this opportunity to record this fact.

We go from ϑ to ψ by using [17, Theorem 6]

$$(3.6) \quad 0 \leq \psi(x) - \vartheta(x) \leq 1.0012\sqrt{x} + 3x^{1/3} \quad (x > 0).$$

Lemma 3.1. *For $x \geq 7\,105\,266$, we have*

$$|\psi(x) - x|/x \leq 0.000\,213.$$

Proof. We start with the estimate from [17, (4.1)]

$$(3.7) \quad |\psi(x) - x|/x \leq 0.000\,213 \quad (x \geq 10^{10}).$$

We extend it to $x \geq 14\,500\,000$ by using (3.1). We conclude by direct inspection. \square

Lemma 3.2. *We have for $x > 1$*

$$|\psi(x) - x| \leq 1.830x/\log^2 x, \quad |\psi(x) - x| \leq 516x/\log^3 x.$$

Proof. Indeed, we readily find that

$$\begin{aligned} |\psi(x) - x| \frac{(\log x)^2}{x} &\leq |\psi(x) - \vartheta(x)| \frac{(\log x)^2}{x} + |\vartheta(x) - x| \frac{(\log x)^2}{x} \\ &\leq \min\left(\frac{1.0012(\log x)^2}{\sqrt{x}} + \frac{3(\log x)^2}{x^{2/3}} + \frac{515}{\log x}, 0.0065 \log x\right) \end{aligned}$$

which is not more than 1.830, on using the first estimate for $x \geq \exp(281.5)$ and the second one for the smaller values. We prove the second estimate in the same way

$$\begin{aligned} |\psi(x) - x| \frac{(\log x)^3}{x} &\leq |\psi(x) - \vartheta(x)| \frac{(\log x)^3}{x} + |\vartheta(x) - x| \frac{(\log x)^3}{x} \\ &\leq \min\left(\frac{1.0012(\log x)^3}{\sqrt{x}} + \frac{3(\log x)^3}{x^{2/3}} + 515, 0.0065 \log^2 x\right) \end{aligned}$$

which is not more than 516, on using again the first estimate for $x \geq \exp(281.5)$ and the second one for the smaller values. For x lower, we first use

$$|\psi(x) - x| \leq (\log^2 x / (1.830\sqrt{x})) 1.830x / \log^2 x$$

which extends our bound till $x \geq 55$. A very primitif GP script shows that

$$|\psi(x) - x| \leq 1.417x / \log^2 x, \quad (1 \leq x \leq 10^5).$$

We proceed similarly for the bound with $\log^3 x$. □

4. LEMMAS ON THE ZEROES

We quote from [14]:

Lemma 4.1. *If T is a real number $\geq 10^3$ then*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + \mathcal{O}^*(0.67 \log \frac{T}{2\pi}).$$

This is a version of Theorem 19 of [15], relying on [1].

Lemma 4.2. *We have, when $T \geq 10^3$*

$$\sum_{\gamma \geq T_0} 1/\gamma^2 \leq \frac{\log(T/(2\pi))}{2\pi T} + 0.67 \frac{2\log(T/(2\pi)) + (1/2)}{T^2}.$$

Proof. We call S the sum to evaluated and we simply use integration by parts:

$$\begin{aligned} S &= 2 \int_T^\infty \frac{N(t) - N(T)}{t^3} dt \\ &\leq \frac{2}{(2\pi)^2} \int_{T/(2\pi)}^\infty \frac{u \log u - u + \frac{7}{8} + 0.67 \log u}{u^3} du \\ &\quad - \frac{\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} - 0.67 \log \frac{T}{2\pi}}{T^2} \\ &\leq \frac{\log(T/(2\pi))}{2\pi T} + 0.67 \frac{2\log(T/(2\pi)) + (1/2)}{T^2}. \end{aligned}$$

The Lemma follows readily. □

Lemma 4.3. *We have $\sum_\rho 1/|\rho(\rho - 1)| \leq 0.047$, where ρ ranges over all non trivial zeroes of ζ .*

In particular, we do not impose $\Im \rho > 0$. We prove this Lemma by using the file of the first 10^5 zeroes provided by Odlyzko [11].

We in fact used zeroes only up to height 10 000 and ran the computations using 28 digits precision on GP/Pari. Note that when $\Im \rho = 1/2$, we have $\rho(\rho - 1) = -|\rho|^2$. Truncation of the imaginary parts only increases the sum, while the high enough precision takes care of the machine error. The restricted sum is about 0.023 02 (with condition $\Im \rho > 0$). We next use Lemma 4.2 to handle the tail of the series. We finally double the value to remove the condition $\Im \rho > 0$, and round the value up.

We also know, thanks to [6], that the zeroes ρ in the critical strip and verifying $|\Im\rho| \leq 2.44 \cdot 10^{12} = T_0$ are all on the line $\Re\rho = 1/2$. We handle zeros with large imaginary part with the following Theorem from [8]

Lemma 4.4. *Every zero $\rho = \beta + i\gamma$ of ζ in the strip $0 < \beta < 1$ and $\gamma \geq 10$ verifies*

$$\beta \leq 1 - \varphi(\gamma) = 1 - 1/(R \log \gamma), \quad R = 5.696\,93.$$

5. PROOF OF THEOREM 1.1

We start with Lemma 2.2. Let us set

$$(5.1) \quad J(x) = \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho-1)}.$$

By considering the symmetry $\rho \mapsto 1 - \rho$, we get (remember that no zero of ζ lies on the segment $[0, 1]$)

$$J(x) = \sum_{\substack{\rho, \\ \Im\rho > 0}} \frac{x^{\rho-1} + x^{-\rho}}{\rho(\rho-1)}.$$

We are ready to majorize $J(x)$:

$$\begin{aligned} J(x) &\leq \sum_{|\gamma| \leq T_0} \frac{x^{-1/2}}{|\rho|^2} + \sum_{\gamma > T_0} \left(\frac{x^{-1/2}}{|\rho(\rho-1)|} + \frac{x^{-\varphi(\gamma)}}{|\rho(\rho-1)|} \right) \\ &\leq \frac{0.047}{\sqrt{x}} + \sum_{\gamma > T_0} \frac{x^{-\varphi(\gamma)}}{\gamma^2}. \end{aligned}$$

We first bound $x^{-\varphi(\gamma)}$ by 1 and get, by Lemma 4.2

$$J(x) \leq \frac{0.047}{\sqrt{x}} + \frac{\log(T_0/(2\pi))}{2\pi T_0} \left(1 + 1.36 \frac{2\pi}{T_0} \right) \leq \frac{0.047}{\sqrt{x}} + 1.75 \cdot 10^{-12}.$$

This proves the first part of Theorem 1.1. For large x , we can take advantage of the zero free region. We set $\varphi_2(\gamma) = x^{-\varphi(\gamma)}/\gamma^2$ and get

$$\begin{aligned} J(x) &\leq \frac{0.047}{\sqrt{x}} - \int_{T_0}^{\infty} (N(t) - N(T_0))\varphi_2'(t)dt \\ &\leq \frac{0.047}{\sqrt{x}} - \int_{T_0}^{\infty} (N^*(t) - N(T_0))\varphi_2'(t)dt - \int_{T_0}^{\infty} (N(t) - N^*(t))\varphi_2'(t)dt \\ &\leq \frac{0.047}{\sqrt{x}} + (N^*(T_0) - N(T_0))\varphi_2(T_0) + \int_{T_0}^{\infty} N^*(t)'\varphi_2(t)dt - \int_{T_0}^{\infty} (N(t) - N^*(t))\varphi_2'(t)dt \\ &\leq \frac{0.047}{\sqrt{x}} + 3 \cdot 10^{-24}x^{-\varphi(T_0)} + \int_{T_0}^{\infty} \frac{x^{-\varphi(t)} \log(t/(2\pi))dt}{2\pi t^2} \\ &\quad + \int_{T_0}^{\infty} \left| \frac{\log x}{2R} - \log^2 t \right| \frac{2x^{-\varphi(t)} \log(t/(2\pi))dt}{t^3 \log^2 t}. \end{aligned}$$

We now assume $\log x \geq 2R \log^2 T_0$ and infer the bound

$$\begin{aligned} J(x) &\leq \frac{0.047}{\sqrt{x}} + 6 \cdot 10^{-24} x^{-\varphi(T_0)} + \int_{T_0}^{\infty} \frac{x^{-\varphi(t)} \log(t/(2\pi)) dt}{2\pi t^2} + 0.67 \int_{T_0}^{\infty} \frac{x^{-\varphi(t)} dt}{t^3} \\ &\leq \frac{0.047}{\sqrt{x}} + 6 \cdot 10^{-24} x^{-\varphi(T_0)} + \int_{T_0}^{\infty} \frac{x^{-\varphi(t)} \log(t/6.25) dt}{2\pi t^2} \\ &\leq \frac{0.047}{\sqrt{x}} + \int_{T_0}^{\infty} \frac{x^{-\varphi(t)} \log t dt}{2\pi t^2}. \end{aligned}$$

$$I = \int_{T_0}^{\infty} \exp\left(-\frac{\log x}{R \log t} - \log t\right) \frac{\log t dt}{2\pi t} = \int_{\log T_0}^{\infty} \exp\left(-\frac{\log x}{Ru} - u\right) \frac{u du}{2\pi}.$$

We set

$$\frac{\log x}{Ru} + u = v$$

which gets solved in $(u^2 - uv + (\log x)/R = 0)$

$$2u = v \pm \sqrt{v^2 - 4(\log x)/R}.$$

We further get

$$\begin{aligned} 4u du &= \left(v \pm \sqrt{v^2 - 4(\log x)/R}\right) \left(1 \pm \frac{v}{\sqrt{v^2 - 4(\log x)/R}}\right) dv \\ &= \left(v \pm \sqrt{v^2 - 4(\log x)/R} \pm \frac{v^2}{\sqrt{v^2 - 4(\log x)/R}} + v\right) dv \\ &= \left(2v \pm \frac{2v^2 - 4(\log x)/R}{\sqrt{v^2 - 4(\log x)/R}}\right) dv \end{aligned}$$

so that I gets rewritten as

$$\begin{aligned} I &= \int_{2\sqrt{(\log x)/R}}^{\infty} e^{-v} \left(v + \frac{v^2 - 2(\log x)/R}{\sqrt{v^2 - 4(\log x)/R}}\right) \frac{dv}{4\pi} \\ &\quad + \int_{2\sqrt{(\log x)/R}}^{\frac{\log x}{R \log T_0} + \log T_0} e^{-v} \left(v - \frac{v^2 - 2(\log x)/R}{\sqrt{v^2 - 4(\log x)/R}}\right) \frac{dv}{4\pi} \end{aligned}$$

which yields

$$I \leq \int_{2\sqrt{(\log x)/R}}^{\infty} v e^{-v} \frac{dv}{2\pi} = \frac{1 + 2\sqrt{(\log x)/R}}{2\pi} \exp\left(-2\sqrt{(\log x)/R}\right).$$

It is then immediate to conclude the proof of Theorem 1.1.

6. PROOF OF THE COROLLARY

When $\log x \leq 2R \log^2 T_0$, but $x \geq 10^{10}$, we use Lemma 3.2 and get

$$\left|\tilde{\psi}(x) - \log x + \gamma\right| \log^2 x \leq 1.830 + \frac{\log^2 x}{2\sqrt{x}} + 1.68 \cdot 10^{-12} \log^2 x \leq 1.833.$$

When $8\,950 \leq x \leq 10^{10}$, we have

$$\left| \tilde{\psi}(x) - \log x + \gamma \right| \log^2 x \leq \frac{1.3 \log^2 x}{\sqrt{x}} + 1.68 \cdot 10^{-12} \log^2 x \leq 1.14.$$

When $\log x \geq 2R \log^2 T_0$, the bound becomes

$$1.830 + \frac{1 + 2\sqrt{(\log x)/R}}{2\pi} \exp\left(-2\sqrt{(\log x)/R}\right) \log^2 x \leq 1.832.$$

We complete the proof by direct inspection. For the limited range bound, we write

$$\left| \tilde{\psi}(x) - \log x + \gamma \right| \sqrt{x} \leq 1.3 + 1.68 \cdot 10^{-12} \sqrt{x} \leq 1.31$$

when $x \geq 8\,950$. We again conclude by direct inspection.

When $\log x \leq 2R \log^2 T_0$, but $x \geq 10^{10}$, we have

$$\left| \tilde{\psi}(x) - \log x + \gamma \right| \log x \leq 0.0065 + \frac{\log x}{2\sqrt{x}} + 1.68 \cdot 10^{-12} \log x \leq 0.0067.$$

When $8\,950 \leq x \leq 10^{10}$, we have

$$\left| \tilde{\psi}(x) - \log x + \gamma \right| \log x \leq \frac{1.3 \log x}{\sqrt{x}} + 1.68 \cdot 10^{-12} \log x \leq 0.0003.$$

When $\log x \geq 2R \log^2 T_0$, the bound becomes

$$0.0065 + \frac{1 + 2\sqrt{(\log x)/R}}{2\pi} \exp\left(-2\sqrt{(\log x)/R}\right) \log^2 x \leq 0.0066.$$

We complete the proof by direct inspection.

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