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Olivier Ramaré

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From explicit estimates for the primes to explicit estimates for the Moebius function

Olivier Ramaré,
CNRS, Laboratoire Paul Painlevé,
Université Lille 1,
59 655 Villeneuve d'Ascq, France
Email: ramare@math.univ-lille1.fr

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Abstract

We prove two explicit estimates respectively slightly stronger than $|\sum_{d \leq D} \mu(d)|/D \leq 0.013/\text{Log } D$ for every $D \geq 1\,100\,000$ and than $|\sum_{d \leq D} \mu(d)/d| \leq 0.026/\text{Log } D$ for every $D \geq 61\,000$.

1 Introduction

There is a long litterature concerning explicit estimates for the summatory function of the Moebius function, and we cite for instance [21], [1], [4], [3], [6], [7], [10], [11]. The paper [5] proposes a very usefull annotated bibliography covering relevant items up to 1983. It has been known since the beginning of the 20th century at least (see for instance [13]) that showing that $M(x) = \sum_{n \leq x} \mu(n)$ is $o(x)$ is equivalent to showing that the Tchebychef function $\psi(x) = \sum_{n \leq x} \Lambda(n)$ is asymptotic to x . We have good explicit estimates for $\psi(x) - x$, see for instance [19], [22] and [9]. This is due to the fact that we can use analytic tools in this problem since the residues at the poles of the Dirichlet generating series (namely here $-\zeta'(s)/\zeta(s)$) are known. However this situation has no counterpart in the Moebius function case. It would thus be highly valuable to deduce estimates for $M(x)$ from estimates for $\psi(x) - x$, but a precise quantitative link is missing. I proposed some years back the following conjecture:

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Conjecture (Strong form of Landau's equivalence Theorem, II).

There exist positive constants c_1 and c_2 such that

$$|M(x)|/x \leq c_1 \max_{c_2x < y \leq x/c_2} |\psi(y) - y|/y + c_1x^{-1/4}.$$

Such a conjecture is trivially true under the Riemann Hypothesis. In this respect, we note that [23] proves that in case of the Beurling's generalized integers, one can have $M_{\mathcal{P}}(x) = o(x)$ without having $\psi(x) \sim x$. This reference has been kindly shown to me by Harold Diamond whom I warmly thank here.

We are not able to prove such a strong estimate, but we are still able to derive estimate for $M(x)$ from estimates for $\psi(x) - x$. Our process can be seen as a generalization of the initial idea of [21] also used in [10]. We describe it in the section 3, after a combinatorial preparation. Here is our main Theorem.

Theorem 1.1. *For $D \geq 1\,078\,853$, we have*

$$\left| \sum_{d \leq D} \mu(d) \right| \leq \frac{0.0130 \operatorname{Log} D - 0.118}{(\operatorname{Log} D)^2} D.$$

The last result of this shape is from [10] and has 0.10917 (starting from $D = 695$) instead of 0.0130.

On following an idea of [11] which we recall in the last section, we deduce from the above the following estimate.

Corollary 1.2. *For $D \geq 60\,298$, we have*

$$\left| \sum_{d \leq D} \mu(d)/d \right| \leq \frac{0.0260 \operatorname{Log} D - 0.118}{(\operatorname{Log} D)^2}.$$

The last result of this shape is from [11] and has 0.2185 (starting from $x = 33$) instead of 0.0260. Here are two results that are simpler to remember:

Corollary 1.3. *For $D \geq 60\,200$, we have*

$$\left| \sum_{d \leq D} \mu(d)/d \right| \leq \frac{\operatorname{Log} D - 4}{40(\operatorname{Log} D)^2}.$$

If we replace the -4 by 0 , the resulting bound is valid from $24\,270$ onward.

Corollary 1.4. *For $D \geq 50\,000$, we have*

$$\left| \sum_{d \leq D} \mu(d)/d \right| \leq \frac{3 \operatorname{Log} D - 10}{100(\operatorname{Log} D)^2}.$$

If we replace the -10 by 0 , the resulting bound is valid from $11\,815$ onward.

We will meet another problem in between, which is to relate quantitatively the error term $\psi(x) - x$ with the error term concerning the approximation of $\tilde{\psi}(x) = \sum_{n \leq x} \Lambda(n)/n$ by $\text{Log } x - \gamma$. This problem is surprisingly difficult but [16] offers a good enough solution.

I thank Harald Helfgott for interesting discussions that pushed me into pulling this note out of its drawer and François Dress for giving me the preprint [11]. This paper was done in majority when I was enjoying the hospitality of the Mathematical Sciences Institute in Chennai, and I thank this institution and my hosts Ramachandran Balasubramanian, Anirban Mukhopadhyay and Sanoli Gun for this opportunity to work in peace and comfort.

Notation

We define the shortcuts $R(x) = \psi(x) - x$ and $r(x) = \tilde{\psi}(x) - \text{Log } x + \gamma$, where we recall that

$$(1.1) \quad \tilde{\psi}(x) = \sum_{n \leq x} \Lambda(n)/n.$$

We shall use square-brackets to denote the integer part and parenthesis to denote the fractionnal part, so that $D = [D] + \{D\}$. But since this notation is used seldomly we shall also use square brackets in their usual function.

2 A combinatorial tool

We prove a formal identity in this section. Let F be a function and $Z = -F'/F$ the opposite of its logarithmic derivative. We look at

$$F[1/F]^{(k)} = P_k.$$

It is immediate to compute the first values and we find that

$$(2.1) \quad P_0 = F, \quad P_1 = Z, \quad P_2 = Z' + Z^2, \quad P_3 = Z'' + 3ZZ' + Z^3.$$

In general, the following recursion formula holds

$$(2.2) \quad P_k = F(P_{k-1}/F)' = P'_{k-1} + ZP_{k-1}.$$

Here is the result this leads to:

Theorem 2.1. *We have*

$$F[1/F]^{(k)} = \sum_{\sum_{i \geq 1} ik_i = k} \frac{k!}{k_1!k_2! \dots (1!)^{k_1} (2!)^{k_2} \dots} \prod_{k_i} Z^{(i-1)k_i}.$$

We can prove it by using the recursion formula given above. We present now a different line. Let us expand $1/F(s+X)$ in Taylor series around $X = 0$.

$$\frac{1}{F(s+X)} = \sum_{k \geq 0} [1/F(s)]^{(k)} \frac{X^k}{k!}.$$

We do the same for $-F'(s+X)/F(s+X)$ getting:

$$\frac{-F'(s+X)}{F(s+X)} = \sum_{k \geq 0} [Z(s)]^{(k)} \frac{X^k}{k!}.$$

Integrating formally this expression, we get

$$-\text{Log}(F(s+X)/F(s)) = \sum_{k \geq 1} [Z(s)]^{(k-1)} \frac{X^k}{k!}$$

where the constant term is chosen so that the constant term is indeed 0. We then apply the exponential formula

$$\exp\left(\sum_{k \geq 1} x_k X^k / k!\right) = \sum_{m \geq 0} Y_m(x_1, x_2, \dots) \frac{X^m}{m!}$$

where the $Y_m(x_1, x_2, \dots)$ are the complete exponential Bell polynomials whose expression yields the Theorem above.

3 The general argument

Let us specialize $F = \zeta$ in Theorem 2.1. The left hand side therein has a simple pole in $s = 1$ with a residue being the k -th Taylor coefficient of $1/\zeta(s)$ around $s = 1$, coefficient that we are to multiply by $k!$. Let us call \mathfrak{R}_k this residue. By a routine argument, we get

$$(3.1) \quad \sum_{\ell \leq L} \mathbb{1} \star (\mu \text{Log}^k)(\ell) = \mathfrak{R}_k L + o(L).$$

Note that, thanks to Theorem 2.1, the error term is quantified in terms of the error term in the approximations of both $\psi(x) - x$ and $\tilde{\psi}(x) - \text{Log } x + \gamma$. Getting to this error term in fact requires using a good enough error term for both these quantities (see for instance [12]). We then continue

$$(3.2) \quad \sum_{\ell \leq L} \mu(\ell) \text{Log}^k \ell = \sum_{d \leq L} \mu(d) \left(\mathfrak{R}_k \frac{L}{d} + o(L/d) \right)$$

which ensures us that $\sum_{\ell \leq L} \mu(\ell) \text{Log}^k \ell$ is $o(L \text{Log } L)$.

Case $k = 2$ is most enlightening. In this case, our method consist in writing

$$(3.3) \quad \sum_{\ell \leq L} \mu(\ell) \text{Log}^2 \ell = \sum_{d\ell \leq L} \mu(\ell) (\Lambda \star \Lambda(d) - \Lambda(d) \text{Log} d).$$

As it turns out, the main term of the summatory function of ΛLog (namely $L \text{Log} L$) cancels the one of $\Lambda \star \Lambda$. This requires the prime number Theorem. In deriving the prime number theorem from Selberg's formula $\mu \star \text{Log}^2 = \Lambda \text{Log} + \Lambda \star \Lambda$, it is a well known difficulty to show that both summands indeed contribute and this is another show-up of the parity principle. We modify (3.3) as follows:

$$(3.4) \quad 2\gamma + \sum_{\ell \leq L} \mu(\ell) \text{Log}^2 \ell = \sum_{d\ell \leq L} \mu(\ell) (\Lambda \star \Lambda(d) - \Lambda(d) \text{Log} d + 2\gamma).$$

Case $k = 1$ is classical, but it is interesting to note that this is the starting point of [21].

4 Some known estimates and straightforward consequences

Lemma 4.1 ([18]). $\max_{t \geq 1} \psi(t)/t = \psi(113)/113 \leq 1.04$.

Concerning small values, we quote from [17] the following result

$$(4.1) \quad |\psi(x) - x| \leq \sqrt{x} \quad (8 \leq x \leq 10^{10}).$$

If we change this \sqrt{x} by $\sqrt{2x}$, this is valid from $x = 1$ onwards. Furthermore

$$(4.2) \quad |\psi(x) - x| \leq 0.8 \sqrt{x} \quad (1\,500 \leq x \leq 10^{10}).$$

Lemma 4.2.

$$|\psi(x) - x| \leq 0.0065x / \text{Log} x \quad (x \geq 1\,514\,928).$$

Proof. By [8, Théorème 1.3] improving on [22, Theorem 7], we have

$$(4.3) \quad |\psi(x) - x| \leq 0.0065x / \text{Log} x \quad (x \geq \exp(22)).$$

We readily extend this estimate to $x \geq 3\,430\,190$ by using (4.2). We then use the function `WalkPsi` from the script `IntR.gp` (with the proper `model` function). \square

Lemma 4.3. For $x \geq 7\,105\,266$, we have

$$|\psi(x) - x|/x \leq 0.000\,213.$$

Proof. We start with the estimate from [20, (4.1)]

$$(4.4) \quad |\psi(x) - x|/x \leq 0.000\,213 \quad (x \geq 10^{10}).$$

We extend it to $x \geq 14\,500\,000$ by using (4.2). We complete the proof by using the following Pari/Gp script (see [15]):

```
{CalculerLambdas(Taille)=
  my(pk, Lambdas);
  Lambdas = vector(Taille);
  forprime(p = 2, Taille,
    pk = p;
    while(pk <= Taille, Lambdas[pk] = p; pk*=p));
  return(Lambdas);}
```

```
{model(n)=n}
```

```
{WalkPsi(zmin, zmax)=
  my(res = 0.0, mo, maxi, psiaux = 0.0, Lambdas);
  Lambdas = CalculerLambdas(zmax);
  for(y = 2, zmin,
    if(Lambdas[y]!=0, psiaux += log(Lambdas[y]),));
  maxi = abs(psiaux-zmin)/model(zmin);
  for(y = zmin+1, zmax,
    mo = 1/model(y);
    maxi = max(maxi, abs(psiaux-y)*mo);
    if(Lambdas[y]!=0, psiaux += log(Lambdas[y]),));
    maxi = max(maxi, abs(psiaux-y)*mo));
  print("|psi(x)-x|/model(x) <= ", maxi, " pour ",
    zmin, " <= x <= ", zmax);
  return(maxi);}
```

□

Lemma 4.4. *For $x \geq 59\,843$, we have*

$$|\psi(x) - x|/x \leq 0.0025.$$

Proof. The preceding Lemma proves it for $x \geq 7\,105\,266$. On using (4.2), we extend it to $x \geq 102\,500$. We complete the proof by using the same script as in the proof of Lemma 4.3. □

Lemma 4.5. *For $x \geq 32\,054$, we have*

$$|\psi(x) - x|/x \leq 0.003.$$

Proof. The preceding Lemma proves it for $x \geq 7\,105\,266$. On using (4.2), we extend it to $x \geq 102\,500$. We complete the proof by using the same script as in the proof of Lemma 4.3. □

We quote from [16] the following Lemma.

Lemma 4.6. *When $x \geq 23$, we have*

$$\tilde{\psi}(x) = \text{Log } x - \gamma + \mathcal{O}^*\left(\frac{0.0067}{\text{Log } x}\right).$$

Let us turn our attention to the summatory function of the Moebius function. In [6], we find the bound

$$(4.5) \quad |M(x)| \leq 0.571\sqrt{x} \quad (33 \leq x \leq 10^{12})$$

In [7], we find

$$(4.6) \quad |M(x)| \leq x/2360 \quad (x \geq 617\,973)$$

(see also [4]) which [2] (published also in [3]) improves in

$$(4.7) \quad |M(x)| \leq x/4345 \quad (x \geq 2\,160\,535).$$

Bounds for squarefree numbers

Lemma 4.7. *We have for $D \geq 1$*

$$\sum_{d \leq D} \mu^2(d) = \frac{6}{\pi^2}D + \mathcal{O}^*(0.7\sqrt{D}).$$

For $D \geq 10$, we can replace 0.7 by 0.5.

Proof. [1] (see also [2]) proves that

$$\sum_{d \leq D} \mu^2(d) = \frac{6}{\pi^2}D + \mathcal{O}^*(0.1333\sqrt{D}) \quad (D \geq 1\,664)$$

and we use direct inspection using Pari/Gp to conclude. \square

Lemma 4.8. *Let $D/K \geq 1$. Let f be a non-negative non-decreasing C^1 function. We have*

$$\sum_{D/L < d \leq D/K} \mu^2(d)f(D/d) \leq 1.31f(L) + \frac{6D}{\pi^2} \int_K^L \frac{f(t)dt}{t^2} + 0.35\sqrt{D} \int_K^L \frac{f(t)dt}{t^{3/2}}.$$

Proof. We use a simple integration by parts to write

$$\begin{aligned} \sum_{D/L < d \leq D/K} \mu^2(d)f(D/d) &= \sum_{D/L < d \leq D/K} \mu^2(d) \left(f(K) + \int_K^{D/d} f'(t)dt \right) \\ &= \sum_{D/L < d \leq D/K} \mu^2(d)f(K) + \int_K^L \left(\sum_{D/L < d \leq D/t} \mu^2(d) \right) f'(t)dt. \end{aligned}$$

We then employ Lemma 4.7 to get the bound:

$$\frac{6D}{\pi^2 K} f(K) + \int_K^L \frac{6D}{\pi^2 t} f'(t) dt + 0.7 \sqrt{\frac{D}{K}} f(K) + 0.7 \int_K^L \sqrt{\frac{D}{t}} f'(t) dt$$

Two integrations by parts gives the expression

$$\frac{6}{\pi^2} f(L) + \int_K^L \frac{6D}{\pi^2 t^2} f(t) dt + 0.7 f(L) + 0.35 \sqrt{D} \int_K^L \frac{f(t) dt}{t^{3/2}}.$$

The Lemma follows readily. \square

5 A preliminary estimate on primes

Our aim here is to evaluate

$$(5.1) \quad R_4(D) = \sum_{d_1 \leq \sqrt{D}} \Lambda(d_1) R(D/d_1).$$

This remainder term is crucial in the final analysis and will be numerically one of the dominant terms.

Lemma 5.1. *When $D \geq 1$, and $\sqrt{D} \geq T \geq 1$, we have*

$$\sum_{d \leq T} \frac{\Lambda(d)}{d \operatorname{Log} \frac{D}{d}} \leq 1.04 \operatorname{Log} \frac{\operatorname{Log} D}{\operatorname{Log}(D/T)} + \frac{1.04}{\operatorname{Log} D}.$$

Proof. Let us define $f(t) = 1/(t \operatorname{Log} \frac{D}{t})$. We have by a classical summation by parts:

$$\begin{aligned} \sum_{d \leq T} \Lambda(d) f(d) &= \sum_{d \leq T} \Lambda(d) f(T) - \sum_{d \leq T} \Lambda(d) \int_d^T f'(t) dt \\ &\leq \frac{1.04}{\operatorname{Log}(D/T)} - 1.04 \int_1^T t f'(t) dt \\ &\leq \frac{1.04}{\operatorname{Log}(D/T)} - 1.04 [t f(t)]_1^T + 1.04 \int_1^T f(t) dt \\ &\leq \frac{1.04}{\operatorname{Log} D} + 1.04 \int_{D/T}^D \frac{dt}{t \operatorname{Log} t} \leq \frac{1.04}{\operatorname{Log} D} + 1.04 \operatorname{Log} \frac{\operatorname{Log} D}{\operatorname{Log}(D/T)} \end{aligned}$$

as required. \square

Lemma 5.2. *We have $|R_4(D)|/D \leq 0.0065$ when $D \geq 10^{10}$. When $D \geq 1\,300\,000\,000$, we have $|R_4(D)|/D \leq 0.0073$.*

The proof that follows is somewhat clumsy due to the fact that we have not been able to compute $R_4(D)$ for D up to 10^{10} . By inspecting the expression defining R_4 and the proof below, the reader will see one could try to get a better bound for

$$\sum_{D^{1/4} < d \leq \sqrt{D}} \Lambda(d) R(D/d).$$

Indeed one can compute the exact values of $R(D/d)$ and try to approximate them properly so as not to lose the sign changes in the expression. A proper model is even given by the explicit formula for $\psi(x)$. We have however tried to use the resulting polynomial, namely $x - \sum_{|\gamma| \leq G} x^{\frac{1}{2} + i\gamma} / (\frac{1}{2} + i\gamma)$ with $G = 20$, $G = 30$ and $G = 200$, but the approximation was very weak. It may be better to find directly a numerical fit for $R(x)$ on this limited range. It should be noted that the function $R(x)$ is highly erratic. Such a process would be important since the value 0.0065 that we get here decides for a large part of the final value in Theorem 1.1.

Proof. When $D \geq 1514928^2$, we have by Lemma 4.2 and Lemma 5.1:

$$|R_4(D)|/D \leq 0.0065 \sum_{d \leq \sqrt{D}} \frac{\Lambda(d)}{d \operatorname{Log}(D/d)} \leq 0.0065 \cdot \left(0.73 + \frac{1.04}{\operatorname{Log} D} \right).$$

This implies that $|R_4(D)|/D \leq 0.00499$ in the given range. When $10^{10} \leq D \leq 1514928^2$, we set $T = D/10^{10}$, we write

$$\begin{aligned} |R_4(D)|/D &\leq 0.000213 \sum_{d \leq T} \frac{\Lambda(d)}{d} + \frac{1}{D^{1/2}} \sum_{T < d \leq \sqrt{D}} \frac{\Lambda(d)}{\sqrt{d}} \\ &\leq 0.000213 \tilde{\psi}(T) \\ &\quad + \frac{1}{D^{1/2}} \left(\frac{\psi(\sqrt{D}) - \psi(T)}{D^{1/4}} + \frac{1}{2} \int_T^{\sqrt{D}} \frac{\psi(u) - \psi(T)}{u^{3/2}} du \right) \end{aligned}$$

i.e. on using $\psi(u) \leq u + \sqrt{u}$,

$$\begin{aligned} |R_4(D)|/D &\leq 0.000213 \tilde{\psi}(T) \\ &\quad + \frac{1}{D^{1/2}} \left(\frac{\psi(\sqrt{D})}{D^{1/4}} - \frac{\psi(T)}{T^{1/2}} + \frac{1}{2} \int_T^{\sqrt{D}} \frac{\psi(u)}{u^{3/2}} du \right) \\ &\leq 0.000213 \tilde{\psi}(T) \\ &\quad + \frac{1}{D^{1/2}} \left(\frac{\sqrt{D} + D^{1/4}}{D^{1/4}} - \frac{T - \sqrt{T}}{T^{1/2}} + D^{1/4} - \sqrt{T} + \operatorname{Log} \frac{\sqrt{D}}{T} \right) \end{aligned}$$

i.e. since $\tilde{\psi}(x) \leq \operatorname{Log} x$ when $x \geq 1$

$$\begin{aligned} |R_4(D)|/D &\leq 0.000213 \operatorname{Log} T \\ &\quad + \frac{1}{D^{1/2}} \left(2D^{1/4} - 2\sqrt{T} + 2 + \operatorname{Log} \frac{\sqrt{D}}{T} \right). \end{aligned}$$

We deduce that $|R_4(D)|/D \leq 0.0065$ when $D \geq 10^{10}$. When now $10^9 \leq D \leq 10^{10}$, we proceed as follows:

$$|R_4(D)|/D \leq \frac{1}{D^{1/2}} \left(\frac{\psi(1500)}{1500^{1/2}} + \frac{1}{2} \int_1^{1500} \frac{\psi(u)}{u^{3/2}} du \right) + \frac{0.8}{D^{1/2}} \left(\frac{\psi(\sqrt{D}) - \psi(1500)}{D^{1/4}} + \frac{1}{2} \int_{1500}^{\sqrt{D}} \frac{\psi(u) - \psi(1500)}{u^{3/2}} du \right).$$

$$\psi(1500) = 1509.27 + \mathcal{O}^*(0.01)$$

$$|R_4(D)|/D^{1/2} \leq (0.2 - 0.8) \frac{1509.3}{1500^{1/2}} + 0.642 + 0.8 \cdot 1.04 (2D^{1/4} - 1500^{1/2}).$$

The right hand side is not more than 0.0073 when $D \geq 1\,300\,000\,000$. \square

6 The relevant error term for the primes

The main actor of this section is the remainder term R_2^* defined by

$$(6.1) \quad \sum_{d \leq D} (\Lambda \star \Lambda(d) - \Lambda(d) \operatorname{Log} d) = -2[D]\gamma + R_2^*(D).$$

The object of this section is to derive explicit estimate for R_2^* from explicit estimates for the ψ . Most of the original work has been achieved already in the previous section, and we essentially put things in shape. Here is our result.

Lemma 6.1. *When $D \geq 1\,435\,319$, we have $|R_2^*(D)|/D \leq 0.0213$.*

We start by an expression for R_2^* .

Lemma 6.2.

$$|R_2^*(D)| \leq 2D|r(\sqrt{D})| + 2D^{1/2}R(\sqrt{D}) + R(\sqrt{D})^2 + R(D) \operatorname{Log} D + 1 + 2\gamma + 2R_4(D) + \left| \int_1^D R(t) \frac{dt}{t} \right|$$

where R_4 is defined in (5.1).

Proof. The proof is fully pedestrian. We have

$$\begin{aligned} \sum_{d \leq D} \Lambda(d) \operatorname{Log} d &= \psi(D) \operatorname{Log} D - \int_1^D \psi(t) dt/t \\ &= D \operatorname{Log} D - D + 1 + R(D) \operatorname{Log} D - \int_1^D R(t) dt/t. \end{aligned}$$

Concerning the other summand, Dirichlet hyperbola formula yields

$$\begin{aligned}
 \sum_{d_1 d_2 \leq D} \Lambda(d_1) \Lambda(d_2) &= 2 \sum_{d_1 \leq \sqrt{D}} \Lambda(d_1) \sum_{d_2 \leq D/d_1} \Lambda(d_2) - \psi(\sqrt{D})^2 \\
 &= 2D \sum_{d_1 \leq \sqrt{D}} \frac{\Lambda(d_1)}{d_1} - D \\
 &\quad - 2\sqrt{D}R(\sqrt{D}) - R(\sqrt{D})^2 + 2 \sum_{d_1 \leq \sqrt{D}} \Lambda(d_1) R(D/d_1) \\
 &= D \operatorname{Log} D - 2D\gamma - D \\
 &\quad + 2Dr(\sqrt{D}) - 2\sqrt{D}R(\sqrt{D}) - R(\sqrt{D})^2 + 2R_4(D).
 \end{aligned}$$

We reach $R_2^*(D) = R_3(D) - 1 + 2R_4(D) - R(D) \operatorname{Log} D + \int_1^D R(t) dt/t$, where

$$(6.2) \quad R_3(D) = 2Dr(\sqrt{D}) - 2\gamma\{D\} - 2\sqrt{D}R(\sqrt{D}) - R(\sqrt{D})^2.$$

The Lemma follows readily. \square

Lemma 6.3. *For the real number D verifying $3 \leq D \leq 110\,000\,000$, we have*

$$|R_2^*(D)| \leq 1.80\sqrt{D} \operatorname{Log} D.$$

When $110\,000\,000 \leq D \leq 1\,800\,000\,000$, we have

$$|R_2^*(D)| \leq 1.93\sqrt{D} \operatorname{Log} D.$$

We used a Pari/Gp script. The only non-obvious point is that we have precomputed the values of $\Lambda \star \Lambda - \Lambda \star \operatorname{Log}$ on intervals of length $2 \cdot 10^6$. On letting this script run longer (about twenty days), I would most probably be able to show that the bound $|R_2^*(D)| \leq 2\sqrt{D} \operatorname{Log} D$ holds when $D \leq 10^{10}$. This would improve a bit on the final result.

Lemma 6.4.

$$\int_1^{10^8} R(t) dt/t = -129.559 + \mathcal{O}^*(0.01).$$

See script `IntR.gp`.

Proof. We prove Lemma 6.1 here. Let us assume that $D \geq 1.3 \cdot 10^9$. We start with Lemma 6.2. We bound $r(\sqrt{D})$ via Lemma 4.6 (this requires $D \geq 23^2$), then $R(\sqrt{D})$ by Lemma 4.4 (this requires $D \geq 32054^2$), and $R(D) \operatorname{Log} D$ by using Lemma 4.2 (this requires $D \geq 1\,514\,928$). We bound R_4 by appealing to Lemma 5.2. We conclude by appealing to Lemma 4.3. All of that amounts to the bound:

$$\begin{aligned}
 |R_2^*(D)| \leq & \frac{4 \cdot 0.0067 D}{\operatorname{Log} D} + 0.006 D + (0.003)^2 D + 0.0065 D \\
 & + 0.0073 D + 132 + 0.000213 D - 0.000213 \cdot 10^8.
 \end{aligned}$$

We reach

$$(6.3) \quad |R_2^*(D)|/D \leq 0.0213$$

when $D \geq 1.3 \cdot 10^9$. Thanks to Lemma 6.3, we extend this bound to $D \geq 1\,435\,319$. \square

7 Estimating $M(D)$

We appeal to (3.4) and use Dirichlet hyperbola formula. We get in this manner our starting equation:

$$(7.1) \quad \sum_{d \leq D} \mu(d) \operatorname{Log}^2 d = 2\gamma + \sum_{d \leq D/K} \mu(d) R_2^*(D/d) \\ + \sum_{k \leq K} R_2^*(k) \sum_{D/(k+1) < d \leq D/k} \mu(d).$$

This equation is much more important than it looks since a bound for $R_2^*(k)$ that is $\ll k/(\operatorname{Log} k)^2$ shows that the second sum converges. A more usual treatment would consist in writing

$$\sum_{d \leq D} \mu(d) \operatorname{Log}^2 d = 2\gamma + \sum_{d \leq D/K} \mu(d) R_2^*(D/d) \\ + \sum_{k \leq K} (\Lambda \star \Lambda - \Lambda \operatorname{Log} + 2\gamma)(k) \sum_{D/K < d \leq D/k} \mu(d).$$

as in [21] for instance. However, when we bound $M(D/k) - M(D/(k+1))$ roughly by $D/(k(k+1))$ in (7.1), we get $D \sum_{k \leq K} |R_2^*(k)|/(k(k+1))$ which is expected to be $\mathcal{O}(D)$. On bounding $M(D/k) - M(D/K)$ by D/k in the second expression, we only get $D \sum_{k \leq K} |\Lambda \star \Lambda - \Lambda \operatorname{Log} - 2\gamma|(k)/k$ which is of size $D \operatorname{Log}^2 K$. Practically, if we want to use a bound of the shape $|M(x)| \leq x/4345$, we will loose the differentiating aspect and will bound $|M(D/k) - M(D/(k+1))|$ by $2D/(4345k)$ and not by $D/(4345k^2)$. It is thus better to use differentiation with respect to R_2^* when k is fairly small. It turns out that small is large enough! We write

$$(7.2) \quad \sum_{k \leq K} R_2^*(k) (M(D/k) - M(D/(k+1))) \\ = \sum_{k \leq K} (\Lambda \star \Lambda - \Lambda \operatorname{Log} + 2\gamma)(k) M(D/k) + R_2^*(K) M(D/K).$$

Lemma 7.1. *When $K = 462\,848$, we have*

$$\sum_{k \leq K} \frac{|\Lambda \star \Lambda - \Lambda \operatorname{Log} + 2\gamma|(k)}{k} + \frac{|R_2^*(K)|}{K} \leq 0.03739 \times 4345.$$

We can use the simple bound (6.3) and get, for $D/K \geq 2\,160\,535$

$$\left| \sum_{d \leq D} \mu(d) \operatorname{Log}^2 d \right| / D \leq \frac{2\gamma}{D} + 0.0213 \left(\frac{6}{\pi^2} \operatorname{Log} \frac{D}{K} + 1.166 \right) + 0.03739$$

$$\leq 0.0130 \operatorname{Log} D - 0.144$$

with $K = 462\,848$. Note that this lower bound of K has been chosen to verify

$$462\,848 \times 2\,160\,535 \leq 10^{12}.$$

Concerning the smaller values, we use summation by parts:

$$\sum_{d \leq D} \mu(d) \operatorname{Log}^2 d = \sum_{d \leq D} \mu(d) \operatorname{Log}^2 D - 2 \int_1^D \sum_{d \leq t} \mu(d) \frac{\operatorname{Log} t \, dt}{t}$$

which gives, when $33 \leq D \leq 10^{12}$,

$$\left| \sum_{d \leq D} \mu(d) \operatorname{Log}^2 d \right| \leq 0.571\sqrt{D} \operatorname{Log}^2 D + 2 \left| \int_1^{33} \sum_{d \leq t} \mu(d) \frac{\operatorname{Log} t \, dt}{t} \right|$$

$$+ 2 \cdot 0.571 \int_{33}^D \frac{\operatorname{Log} t \, dt}{\sqrt{t}}$$

$$\leq 0.571\sqrt{D} \operatorname{Log}^2 D + 2.284\sqrt{D} \operatorname{Log} D + 4.568\sqrt{D} - 43$$

and this is $\leq 0.0130 \operatorname{Log} D - 0.144$ when $D \geq 8\,613\,000$. We extend this bound to $D \geq 2\,161\,205$ by direct computations using Pari/Gp.

Let us state formally:

Lemma 7.2. *For $D \geq 2\,161\,205$, we have*

$$\left| \sum_{d \leq D} \mu(d) \operatorname{Log}^2 d \right| / D \leq 0.0130 \operatorname{Log} D - 0.144.$$

8 A general formula and proof of Theorem 1.1

Let $(f(n))$ be a sequence of complex numbers. We consider, for integer $k \geq 0$, the weighted summatory function

$$(8.1) \quad M_k(f, D) = \sum_{n \leq D} f(n) \operatorname{Log}^k n.$$

We want to derive information on $M_0(f, D)$ from information on $M_k(f, D)$. The traditional way to do that is in essence due to [14] and goes via a differential equation. It turns out that it is clearer and somewhat more precise to use the identity that follows.

Lemma 8.1. *We have, when $k \geq 0$, and for $D \geq D_0$,*

$$M_0(f, D) = \frac{M_k(f, D)}{\text{Log}^k D} + M_0(f, D_0) - \frac{M_k(f, D_0)}{\text{Log}^k D_0} - k \int_{D_0}^D \frac{M_k(f, t)}{t \text{Log}^{k+1} t} dt.$$

This formula in a special case is also used in [21] and [10].

Proof. Indeed, we have

$$k \int_{D_0}^D \frac{M_k(f, t)}{t \text{Log}^{k+1} t} dt = -\frac{M_k(f, D_0)}{\text{Log}^k D_0} + \sum_{n \leq D} f(n) \frac{\text{Log}^k n}{\text{Log}^k D} - \sum_{D_0 < n \leq D} f(n)$$

□

Proof. We proceed to the proof of Theorem 1.1. In the notation of Lemma 8.1, we have $M(D) = M_0(\mu, D)$. We have by Lemma 7.2 and with $D_0 = 2\,161\,205$:

$$\begin{aligned} |M(D)| &\leq \frac{0.0130 \text{Log } D - 0.144}{\text{Log}^2 D} D + M(D_0) - \frac{M_2(\mu, D_0)}{\text{Log}^2 D_0} \\ &\quad + 2 \int_{D_0}^D \frac{0.0130 \text{Log } t - 0.144}{\text{Log}^3 t} dt. \\ &\leq \frac{0.0130 \text{Log } D - 0.144}{\text{Log}^2 D} D - 3.48 + 2 \int_{D_0}^D \frac{0.0130 \text{Log } t - 0.144}{\text{Log}^3 t} dt. \\ &\leq \frac{0.0130 \text{Log } D - 0.118}{\text{Log}^2 D} D - 3.48 \\ &\quad - 0.0260 \frac{D_0}{\text{Log}^2 D_0} - \int_{D_0}^D \frac{0.236}{t \text{Log}^3 t} dt. \end{aligned}$$

(We used Pari/Gp to compute the quantity $M(D_0) - M_2(\mu, D_0)/\text{Log}^2 D_0$. We conclude by direct verification, again by relying on Pari/Gp. □

9 From M to m

We take the following Lemma from [11, (1.1)].

Lemma 9.1 (El Marraki). *We have*

$$|m(D)| \leq \frac{|M(D)|}{D} + \frac{1}{D} \int_1^D \frac{|M(t)| dt}{t} + \frac{\text{Log } D}{D}.$$

This Lemma may look trivial enough, but its teeth are hidden. Indeed, a usual summation by parts would bound $|m(D)|$ by an expression containing the integral of $|M(t)|/t^2$. An upper bound for $|M(t)|$ of the shape $ct/\text{Log } t$ would hence result in the useless bound $m(D) \ll \text{Log } \text{Log } D$.

Proof. We reproduce the proof, as it is short and the preprint we refer to is difficult to find. We have two equations, namely:

$$(9.1) \quad m(D) = \frac{M(D)}{D} + \int_1^D \frac{M(t)dt}{t^2}$$

and

$$(9.2) \quad \int_1^D \left[\frac{D}{t} \right] \frac{M(t)dt}{t} = \text{Log } D.$$

We deduce from the above that

$$m(D) = \frac{M(D)}{D} + \frac{1}{D} \int_1^D \left(\frac{D}{t} - \left[\frac{D}{t} \right] \right) \frac{M(t)dt}{t} + \frac{\text{Log } D}{D}.$$

The Lemma follows readily. □

Proof. We have, when $D \geq D_0 = 1\,078\,853$,

$$\begin{aligned} |m(D)| &\leq \frac{0.0130 \text{Log } D - 0.118}{(\text{Log } D)^2} + \frac{1}{D} \int_{D_0}^D \frac{0.0130 \text{Log } t - 0.118}{(\text{Log } t)^2} dt \\ &\quad + \frac{1}{D} \int_1^{D_0} \frac{|M(t)|dt}{t} + \frac{\text{Log } D}{D}, \\ &\leq \frac{0.0130 \text{Log } D - 0.118}{(\text{Log } D)^2} + \frac{1}{D} \int_{D_0}^D \frac{0.0130 dt}{\text{Log } t} \\ &\quad - \frac{1}{D} \int_{D_0}^D \frac{0.118 dt}{(\text{Log } t)^2} + \frac{301 + \text{Log } D}{D}. \end{aligned}$$

We continue by an integration by parts and some numerical computations:

$$\begin{aligned} |m(D)| &\leq \frac{0.0260 \text{Log } D - 0.118}{(\text{Log } D)^2} - \frac{0.105}{D} \int_{D_0}^D \frac{dt}{(\text{Log } t)^2} + \frac{-9795 + \text{Log } D}{D}, \\ &\leq \frac{0.0260 \text{Log } D - 0.118}{(\text{Log } D)^2} - \frac{1}{D} \int_{D_0}^D \frac{dt}{t} + \frac{-9795 + \text{Log } D}{D} \end{aligned}$$

This proves that $|m(D)|(\text{Log } D)^2 \leq 0.0260 \text{Log } D - 0.118$ as soon as $D \geq 1\,078\,853$. We extend this bound by direct inspection. □

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