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Comparing $L(s, \chi)$ with its truncated Euler product and generalization

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Abstract

We show that any L -function attached to a non-exceptionnal Hecke Grossencharakter Ξ may be approximated by a truncated Euler product when s lies near the line $\Re s = 1$. This leads to some refined bounds on $L(s, \Xi)$.

1 Introduction and results

For $L(1, \chi)$, see [8], [16] and [1].

We first need to fix some terminology. We select a number field \mathbb{K}/\mathbb{Q} be a number field of degree d and discriminant Δ . We denote its norm by N , as a shortcut to $N_{\mathbb{K}/\mathbb{Q}}$. We shall consider Hecke Grossencharakter Ξ with (finite) ideal \mathfrak{f} , of norm q , and associated with some finite set of infinite places. The conductor \mathfrak{f} being fixed, the main Theorem of [5] tells us there exists an absolute constant $C > 0$ such that no L -function $L(s, \Xi)$ has a zero ρ in the region

$$\Re \rho \geq 1 - \frac{C}{\text{Log max}(q\Delta, q\Delta|\Im s|)} \quad (1)$$

except at most one such character; this potential exception is real valued and may have at most one real zero β in this region. We refer to this hypothetical character as the exceptional character and term the remaining ones as being non-exceptional. See also [11]. In the case of Dirichlet characters, i.e. $\mathbb{K} = \mathbb{Q}$, we know from [13] that we may take $C = 1/6.3958$.

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Theorem 1 *Let Ξ be a non-exceptional Hecke Grossencharacter with (finite) conductor \mathfrak{f} of norm $q > 1$. We have*

$$L(s, \Xi) \asymp \prod_{\mathbf{N} \mathfrak{p} \leq q\Delta|s|} (1 - \Xi(\mathfrak{p})/\mathbf{N} \mathfrak{p}^s)^{-1}$$

when $1 \geq (\Re s - 1) \text{Log}(q\Delta(2 + |s|)) \geq -C/2$, the constant C being the one from (1).

The restriction to non-exceptional characters can be dispensed with if we assume $|\Im s| \geq 1/\text{Log}(q\Delta)$. Under the Riemann hypothesis for the implied L -function, we can restrict the above product to $p \leq \text{Log Log}(q\Delta|s|)$. As trivial consequences, we find via (a generalization of) Mertens theorems (see (9) below) that, under these conditions

$$\frac{q/\phi(q)}{\text{Log}(q\Delta|s|)} \ll |L(s, \Xi)| \ll \frac{\phi(q)}{q} \text{Log}(q\Delta|s|). \quad (2)$$

The upper bound is classical in the case of Dirichlet characters but improves considerably in the general case on the one given in Theorem 5 of [17], albeit being less explicit. The factor $q/\phi(q)$ in the lower one appears to be novel, even in the case of Dirichlet characters. For instance, it supersedes the one of Corollary 2 of [11] by the factor $q/\phi(q)$ and by the fact that it is valid for any non-exceptional character. From a historical viewpoint, [14] shows that $|L(1, \chi)| \gg 1/\text{Log}^5 q$ for non-real characters, and improves this in $|L(1, \chi)| \gg 1/\text{Log} q$ in [15]. The proof is somewhat more delicate than expected.

Note also (by again invoking Mertens' theorems) that we can restrict the product to $p \leq (q|s|)^a$ for any positive a .

Granville & Soundararajan investigated in [6] (see also [7]) the distribution of values of $L(1, \chi)$ (χ being a Dirichlet character) via an approximation by an Euler product and in particular, they show in their Proposition 1 that the Euler product may be truncated to $p \leq \text{Log} q$ for all but $\mathcal{O}(q^{1-2/\text{Log Log} q})$ characters. Note however that they aim at an exact approximation of $L(1, \Xi)$ while we only seek to recover its order of magnitude.

Our main ingredient is the following Lemma of independent interest.

Lemma 1 *Under the conditions above, $|L'/L(s, \Xi)| \ll \text{Log}(q\Delta(2 + |s|))$.*

In this Lemma also, the restriction to non-exceptional characters can be dispensed with if we assume $|\Im s| \geq 1/\text{Log}(q\Delta)$. The inequality $-\Re L'/L(s, \Xi) \leq c \text{Log}(q\Delta(2 + |s|))$ when $\Re s > 1$ is a classical element of the proof of the zero-free region for $L(\cdot, \Xi)$ (see [4, chapter 14] for instance); by using his local

method, Landau shows in [15, page 30] that $\Re L'/L(s, \Xi) \leq c \text{Log}(q\Delta(2+|s|))$. The above Lemma shows that much more is true and that only invoking a one-sided bound for the real part does not lead to any improvement.

Under the Riemann hypothesis for $L(\cdot, \Xi)$, the upper bound becomes $\text{Log Log}(q\Delta(2+|s|))$.

Generalization

Like many properties of Dirichlet L -functions, this one generalizes to a wide class of L -functions. We shall not describe such a general context but refer the reader to chapter 5 of [12]. We work under the conditions of their Theorem 5.10: $L(f, s)$ is an L -function of degree d such that the Rankin-Selberg convolutions $L(f \otimes f, s)$ and $L(f \otimes \bar{f}, s)$ exist, the latter having a simple pole at $s = 1$ while the former is entire if $f \neq \bar{f}$. We further suppose that $|\alpha_j(p)|^2 \leq p/2$ at the ramified primes. The notion of exceptional character is more complicated to define in a general context, since it requires a way of defining families of L -functions. Assuming that our candidate has no real zero in the classical zero-free region, we find that

$$L(f, s) \asymp \prod_{p \leq q(f, s)} (1 - \alpha_1(p)p^{-s})^{-1} \cdots (1 - \alpha_d(p)p^{-s})^{-1} \quad (3)$$

where the analytical conductor is defined there in equation (5.7).

Notations

We need some names for our variables, and the easiest path is to keep a fixed point $s_0 = \sigma_0 + it_0$, which will be s in the Theorem, and a running $s = \sigma + it$. We define

$$\mathcal{L} = \text{Log}(q\Delta(|s_0| + 2)). \quad (4)$$

The point $s_1 = \sigma_1 + it_0$ with $\sigma_1 = 1 + 1/\mathcal{L}$ will be of special interest.

2 Some material on primes in number fields

We can use the prime number Theorem for \mathbb{K}/\mathbb{Q} , but we prefer to sketch an elementary approach to the classical results we need. Such material is also contained in [18]. Assume we have an asymptotic estimate:

$$\sum_{N \alpha \leq X} 1 = c_0 X + \mathcal{O}(X/\text{Log}(2X)) \quad (5)$$

where \mathfrak{a} ranges the integral ideals of \mathbb{K} . Such an estimate is linked with the fact that the Dedekind zeta function $\zeta_{\mathbb{K}}$ of \mathbb{K} has a simple pole at $s = 1$. In particular c_0 is the residue of this function at $s = 1$. The results we seek also hold with the error term being simply $o(X)$, but our proof would require a modification. From this we deduce that

$$\sum_{N \mathfrak{a} \leq X} \text{Log } N \mathfrak{a} = c_0 X \text{Log } X + \mathcal{O}(X). \quad (6)$$

Writing $\zeta'_{\mathbb{K}}/\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}} \Lambda_{\mathbb{K}}(\mathfrak{a})/N \mathfrak{a}^s$ we find that $\sum_{\mathfrak{b}|\mathfrak{a}} \Lambda_{\mathbb{K}}(\mathfrak{b}) = \text{Log } N \mathfrak{a}$ and plugging this into (6), we get

$$c_0 X \text{Log } X + \mathcal{O}(X) = \sum_{N \mathfrak{b} \leq X} \Lambda_{\mathbb{K}}(\mathfrak{b}) \sum_{N \mathfrak{c} \leq X/N \mathfrak{b}} 1 = X \sum_{N \mathfrak{b} \leq X} \frac{\Lambda_{\mathbb{K}}(\mathfrak{b})}{N \mathfrak{b}}$$

by appealing to (5), from which we infer

$$\sum_{N \mathfrak{b} \leq X} \frac{\Lambda_{\mathbb{K}}(\mathfrak{b})}{N \mathfrak{b}} = \text{Log } X + \mathcal{O}(1). \quad (7)$$

Using the expression of $\zeta_{\mathbb{K}}$ as an Euler product, we find that $\Lambda_{\mathbb{K}}(\mathfrak{b})$ is zero except when \mathfrak{b} is a power of a prime \mathfrak{p} , at which point it takes the value $\text{Log } N \mathfrak{p}$. This finally leads us to the estimate

$$\sum_{N \mathfrak{p} \leq X} \frac{\text{Log } N \mathfrak{p}}{N \mathfrak{p}} = \text{Log } X + \mathcal{O}(1). \quad (8)$$

We infer $\sum_{N \mathfrak{p} \leq X} 1/N \mathfrak{p} = \text{Log } \text{Log } X + \mathcal{O}(1)$ and thus

$$\prod_{N \mathfrak{p} \leq X} (1 - 1/N \mathfrak{p}) \asymp 1/\text{Log } X \quad (9)$$

which is enough for our purpose. This is not what is referred to as Mertens' Theorem, since we do not have a proper asymptotic, but these estimates are enough for our purpose. We refer the reader to [3] for related material on explicit Mertens' Theorem in abelian number fields.

3 Proof of Lemma 1

We start from Linnik's density lemma which the reader may find in [5, Lemma 7] or in [2, chapter 6] in case of Dirichlet characters. We define

$n(1 + it, r)$ to be the number of zeros ρ of $L(s, \chi)$ in the disc $|\rho - i - it| \leq r$. We have

$$\frac{L'}{L}(s, \Xi) = \frac{-\delta_{\Xi}}{s-1} + \sum_{|\rho-1-it_0| \leq 1/3} \frac{1}{s-\rho} + \mathcal{O}(\mathcal{L}) \quad (|s-1-it_0| \leq 1/4), \quad (10)$$

where δ_{Ξ} is 1 if Ξ is principal, and 0 otherwise. This is for instance Lemma 6 of [5]; In case of Dirichlet characters, this is (4) of chapter 16 of [4], and in a general context (5.28) of [12]. These two last proofs rely on a global representation of L'/L , while Fogel's one follows the local method of Landau. The latest refinements of this method may be found in [10] and [9].

One of the consequences of (10) is Linnik's density lemma:

$$n(1 + it, r) \ll r\mathcal{L} + 1. \quad (11)$$

Apply (10) to $s = \sigma + it$ with $\sigma \geq 1 - 2C/\mathcal{L}$ and to s_1 and subtract. For any zero ρ in the summation above, we have $|s - \rho| \geq |1 + it_0 - \rho|/2$ and thus

$$\begin{aligned} \left| \frac{L'}{L}(s, \Xi) - \frac{L'}{L}(s_1, \Xi) \right| &\leq \sum_{|\rho-1-it_0| \leq 1/3} \frac{4|\sigma - \sigma_1|}{|1 + it_0 - \rho|^2} + \mathcal{O}(\mathcal{L}) \\ &\leq |\sigma - \sigma_1| \sum_{0 \leq k \leq \text{Log } \mathcal{L}} \sum_{r=2^k \leq |\rho-1-it_0| \leq 2^{k+1}} \frac{4}{r^2} + \mathcal{O}(\mathcal{L}) \\ &\leq |\sigma - \sigma_1| \sum_{0 \leq k \leq \text{Log } \mathcal{L}} \frac{2n(1 + it_0, r)}{r^2} + \mathcal{O}(\mathcal{L}) \\ &\ll |\sigma - \sigma_1| \sum_{0 \leq k \leq \text{Log } \mathcal{L}} \left(\frac{\mathcal{L}}{r} + \frac{1}{r^2} \right) + \mathcal{O}(\mathcal{L}) \ll |\sigma - \sigma_1| \mathcal{L}^2 + \mathcal{L}. \end{aligned}$$

Notice furthermore that $|L'/L(s_1, \Xi)| \leq -\zeta'/\zeta(\sigma_1) \ll \mathcal{L}$, so that, when $\sigma \leq 1 + \mathcal{L}$, the above inequality reduces to

$$\left| \frac{L'}{L}(s, \Xi) \right| \ll \mathcal{L}. \quad (12)$$

This ends the proof in case of non-exceptional characters. In case of an exceptional character, we simply consider separately in (10) its contribution, namely $1/(s - \beta)$ which is again $\mathcal{O}(\mathcal{L})$. Under the Riemann hypothesis, we simply invoke Theorem 5.17 of [12].

4 Proof of the Theorem

Define $R = q\Delta|s_0|$. We check that

$$\left| \frac{L'}{L}(s, \Xi) + \sum_{\mathbf{N}\mathbf{p} \leq R} \frac{\Xi(\mathbf{p}) \text{Log } \mathbf{N}\mathbf{p}}{\mathbf{N}\mathbf{p}^s - \Xi(\mathbf{p})} \right| \ll \mathcal{L} + \text{Log } R \ll \mathcal{L} \quad (13)$$

when $s = \sigma + it_0$ and $1 \geq (\sigma - 1)\mathcal{L} \geq -C/2$. We integrate (12) between s_1 and s_0 and find that

$$|\text{Log } L_R(s_0, \Xi) - \text{Log } L_R(s_1, \Xi)| \ll 1 \quad (14)$$

with $L_R(s, \Xi) = \prod_{\mathbf{N}\mathbf{p} > R} (1 - \Xi(\mathbf{p})/\mathbf{N}\mathbf{p}^s)^{-1}$. Next we note that

$$\begin{aligned} |L_R(s_1, \Xi)| &\leq \prod_{\mathbf{N}\mathbf{p} > R} (1 - \mathbf{N}\mathbf{p}^{-\sigma_1})^{-1} \leq \exp \sum_{\mathbf{N}\mathbf{p} > R} \mathbf{N}\mathbf{p}^{-\sigma_1} \\ &\ll \exp \int_R^\infty \frac{dt}{t^{\sigma_1} \text{Log } t} = \exp \int_{R^{\sigma_1-1}}^\infty \frac{dv}{v^2 \text{Log } v} \end{aligned}$$

by setting $v = t^{\sigma_1-1}$, and where we have invoked (8). The last quantity is bounded since so is R^{σ_1-1} . Considering only real parts in (14), the Theorem readily follows.

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