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# Tame behaviour of the mean value of multiplicative functions and some inequalities relating values of Dirichlet series <sup>\*†</sup>

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## Abstract

We show that the average of a bounded multiplicative function varies locally in a regular manner. Our precise result improves on a similar one by Elliott. We use two novel ingredients: a better smoothing device, and a bilinear inequality for values of Dirichlet series. This last inequality leads to a refinement of a Theorem of Barrucand & Louboutin on lower bounds of  $L(1, \chi)$ .

## 1 Introduction

In 1996, Hildebrand discovered in [10] that mean values of bounded multiplicative functions vary in a mild manner over a large range. This early proof used the large sieve inequality. Shortly afterward Elliott modified completely the approach in [5] and, using Halász method, managed to improve considerably the error term. Our aim here is to refine the latter estimate.

**Theorem 1.1** *Let  $g$  be a complex valued multiplicative function whose modulus is not more than 1. There exists a real number  $\tau$  of modulus  $\leq (\text{Log } x)^{1/8}$  such that*

$$\sum_{n \leq x} g(n) = w^{1+i\tau} \sum_{n \leq x/w} g(n) + \mathcal{O}\left(x \cdot \kappa^{\frac{-2}{9+\sqrt{81-32\xi}}} (\text{Log } \kappa)^{3/2}\right)$$

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uniformly in  $g$  and  $w_0$  such that  $1 \leq w \leq w_0 \leq x$ , and where

$$(1) \quad \kappa = \frac{2 \operatorname{Log} x}{\operatorname{Log} 2w_0}, \quad \xi \operatorname{Log} \operatorname{Log} x = \operatorname{Log} \kappa.$$

The exponent in the above Theorem is larger than  $1/9$  which is already twice better than the one in [5]. This saving comes from a better handling of the smoothing device, for which we appeal to [13] (see also [12] for some comments). Furthermore when  $w_0$  is not more than a power of  $\operatorname{Log} x$ , our exponent becomes  $1/8$ ; this improvement stems from a better inequality between values of Dirichlet series of completely multiplicative functions on the border of their half-plane of absolute convergence. The outcome of our discussion on such inequality is twofold: first, bilinear inequalities for the logarithm of such Dirichlet series seem more appropriate than linear ones; and second, we prove that if such a series has a *large* value at one point on a vertical line, then this point is isolated, except for its immediate neighbours. In passing, we shall recover and extend a result from [1] concerning lower bounds of  $L$ -functions at 1:

**Theorem 1.2** *Let  $\chi$  be a primitive Dirichlet character of conductor  $\mathfrak{f}$  and let  $\mathfrak{f}'$  be the conductor of  $\chi^2$ . Let us assume that  $\mathfrak{f}' \neq 1$ . Then we have*

$$(2) \quad |L(1, \chi)| \gg 1 / ((\operatorname{Log} \mathfrak{f})^{\sqrt{(1+\xi)/2}} \operatorname{Log} \operatorname{Log}(10\mathfrak{f}/\mathfrak{f}'))$$

with  $\xi = (\operatorname{Log} \operatorname{Log} \mathfrak{f}') / \operatorname{Log} \operatorname{Log} \mathfrak{f}$ .

The results in [1] correspond to the extreme case  $\xi = 0$  and otherwise to the exponent  $(3+\xi)/4$ . This exponent is larger than ours, though it is asymptotic to it when  $\xi$  narrows to 1.

The optimal exponent of  $\kappa$  in Theorem 1.1 is 1. Indeed, let us consider for the case when  $g = f_y$ , the characteristic function of those integers with no prime factors below some  $y = x^u$  for a fixed  $u < 1$ . We have the classical estimate

$$(3) \quad (1/x) \sum_{n \leq x} f_y(n) = \omega(u) + \mathcal{O}(1/\operatorname{Log}^2 y)$$

where  $\omega$  is the Buchstab function. This implies that

$$(1/x) \sum_{n \leq x} f_y(n) - (w/x) \sum_{n \leq x/w} f_y(n) \sim \omega'(u) \frac{\operatorname{Log} w}{u \operatorname{Log} x}$$

when  $(\operatorname{Log} w) / \operatorname{Log} x$  goes to zero and provided we choose  $u$  such that  $\omega'(u) \neq 0$ ; this is easily achieved. This example shows also what we mean by "uniformly in  $g$ ": our function may depend on  $x$ .

We can refine somewhat the previous approach when  $g$  is real-valued.

**Theorem 1.3** *Let  $g$  be a real valued multiplicative function whose modulus is not more than 1. We have*

$$\sum_{n \leq x} g(n) = w \sum_{n \leq x/w} g(n) + \mathcal{O} \left( x \cdot \kappa^{\frac{-1/4}{1+\sqrt{1-\xi/2}}} (\text{Log } \kappa)^{5/4} \right)$$

*uniformly in  $g$  and  $w_0$  such that  $1 \leq w \leq w_0 \leq x$ , and where  $\kappa$  and  $\xi$  are defined in (1).*

This time, the exponent of  $\kappa$  lies between  $(2 - \sqrt{2})/4 = 1/6.828\dots$  when  $\xi = 1$  and  $1/8$  when  $\xi = 0$ .

Concerning Theorems 1.1 and 1.3, let us specify that we handle the case of bounded multiplicative functions by noticing that their Dirichlet series compares to the one of a completely multiplicative one up to multiplication by a third series absolutely convergent in a larger half-plane.

We show at the end of this paper that  $\tau$  can in fact be taken not more than the saving in Theorem 1.1, namely  $\kappa^{\frac{2}{9+\sqrt{81-32\xi}}} (\text{Log } \kappa)^{-3/2}$ .

Hildebrand applied his idea to the Moebius function, see [11], and obtained an elementary proof of the prime number theorem. By using Theorem 1.1, one can for instance improve the Burgess inequality, see [3], on the initial interval  $[1, y]$ : we first extend the summation from ranging over  $[1, y]$  to range over  $[1, wy]$  for a suitable  $w$  and we prove that the resulting sum is indeed smaller than the trivial bound. This general principle can be applied to numerous situations. Hildebrand already used in [10] this idea and Elliott gave some other applications in [6] which are thus directly improved upon by our results.

## 2 Frame of the proof

We follow closely [5], which in turn is a modification of a paper of Halász. We consider

$$(4) \quad S_0(g, y) = y^{-1} \sum_{n \leq y} g(n)$$

and

$$(5) \quad S(g, y, \beta) = y^{-1} \sum_{n \geq 1} \text{Log } n g(n) e^{-(n/y)^\beta}.$$

In order to compare both, let us first note the following Lemma whose proof is delayed until next section:

**Lemma 2.1** *We have for  $\beta \geq 1$ :*

$$\sum_{n \leq y} (1 - e^{-(n/y)^\beta}) \operatorname{Log} n + \sum_{n > y} e^{-(n/y)^\beta} \operatorname{Log} n \ll y \frac{\operatorname{Log} y}{\beta} + y + \beta \operatorname{Log} y.$$

We use furthermore the estimate  $\sum_{n \leq y} \operatorname{Log}(y/n) \ll y$  and get

$$(6) \quad S_0(g, y) \operatorname{Log} y = S(g, y, \beta) + \mathcal{O}(\beta^{-1} \operatorname{Log} y + 1 + y^{-1} \beta \operatorname{Log} y).$$

Halasz in [8] and Elliott in [5] use a different smoothing while Ruzsa uses none in [15]. The removal of the smoothing these first authors employ is responsible in their proof for halving the exponent of the saving by 2. If we were to not introduce any smoothing at all like in [15], the error term would have an additional  $\operatorname{Log} \operatorname{Log} x$  (possibly  $\operatorname{Log} \operatorname{Log} \operatorname{Log} x$  after some trimming).

Halász method relies on expressing the average in consideration in terms of an integral of the derivative of the Dirichlet series

$$(7) \quad D(g, s) = \sum_{n \geq 1} g(n)/n^s.$$

We use here:

$$(8) \quad S(g, y, \beta) = \frac{-1}{2i\pi\beta} \int_{\alpha-i\infty}^{\alpha+i\infty} D'(g, s) \Gamma(s/\beta) y^{s-1} ds$$

with

$$(9) \quad \alpha = 1 + (\operatorname{Log} x)^{-1}, \quad 1 \leq y \leq x, \quad \beta \geq 2.$$

The parameter  $\beta$  will tend to infinity at a slower pace than  $\operatorname{Log} x$  and will have to be tuned properly (this is done in (35)).

Once  $\alpha$  is fixed, we construct a real number  $\tau$  not too large such that:

1.  $D(g, s + i\tau)$  is appreciably smaller than  $\zeta(\alpha)$  for all  $s$  neither too close nor too far from 0. More precisely, we ensure that

$$(10) \quad \mu = (\alpha - 1) \max_{K(\alpha-1) \leq |t| \leq \beta^3} |D(g, s + i\tau)|$$

is small enough;

2. if  $\tau \neq 0$ , we need  $D(g, \alpha + i\tau)$  to be close enough to  $\zeta(\alpha)$ ; more precisely, we ensure that

$$\sum_{p \leq x} (1 - \Re g(p) p^{-i\tau})/p$$

is small enough.

Having such a  $\tau$ , we consider the multiplicative function

$$(11) \quad g_\tau(n) = n^{-i\tau} g(n)$$

which satisfies  $D(g_\tau, s) = D(g, s + i\tau)$ .

We follow Halász method as in [5] and apply (8) to  $g_\tau$ . By using the exponential decay of the  $\Gamma$ -function in vertical strips, we readily shorten this integral to  $|\Im s| \leq \beta^3$ . As in [5], the main work is then to study the medium range  $K(\alpha - 1) \leq |\Re s| \leq \beta^3$ . On the remaining range, we exchange  $x/w$  by  $x$  with an error term  $\mathcal{O}(K \text{Log } w_0)$  as in [5], and we present in section 10 an improvement of it in case  $g$  is real-valued. We thus reach

$$(12) \quad S(g_\tau, x/w, \beta) = S(g_\tau, x, \beta) + \mathcal{O}(\text{error}) + \mathcal{O}\left(K \frac{\text{Log } w_0}{\text{Log } x} \text{Log } x\right).$$

At this level we choose  $\beta$  and  $K$  optimally and derive an estimation of  $S_0(g_\tau, x/w) - S_0(g_\tau, x)$  by dividing the resultant error term by  $\text{Log } x$ ; we should in fact add a  $\mathcal{O}((\text{Log } w_0)/\text{Log } x)$  but this can be incorporated in the main error term simply by assuming that  $K \geq 2$ . The proof of the Theorem in case  $\tau = 0$  is complete at this level, and in particular when  $g$  takes only real values.

We remove  $\tau$  in case it is  $\neq 0$  by appealing to Lemma 8.6 from [15]:

**Lemma 2.2** *Let  $b$  be a real number and set  $L = \sum_{p \leq x} |1 - g(p)|/p$ . We have*

$$\sum_{n \leq x} g(n) n^{ib} = \frac{x^{ib}}{1 + ib} \sum_{n \leq x} g(n) + \mathcal{O}\left(x (e^L / \text{Log } x)^{1/2} \text{Log}^2(|b| + \text{Log } x)\right).$$

As a matter of fact, Ruzsa stated this Lemma for unimodular multiplicative functions  $g$ , that is those who verify  $|g(n)| = 1$ , but it is easily seen that the proof applies also in our somewhat more general context. It is used in this case by [5].

Let us stress again that none of the implied constants in the  $\mathcal{O}$ -symbols depend on  $g$  and we shall indeed use Lemma 2.2 on  $g_\tau$ , with  $b$  being  $\tau$ .

### 3 Proof of Lemma 2.1

This section is routine, but we need to handle completely the dependence in the parameter  $\beta$ . We simply use  $\sum_{n \leq t} \text{Log } n = t \text{Log } t - t + \mathcal{O}(\text{Log}(2t))$  and

an integration by parts:

$$\begin{aligned} \sum_{n \leq y} \text{Log } n e^{-(n/y)^\beta} &= \sum_{n \leq y} \text{Log } n \beta \int_n^y \left(\frac{t}{y}\right)^{\beta-1} e^{-(t/y)^\beta} \frac{dt}{y} + e^{-1} \sum_{n \leq y} \text{Log } n \\ &= \beta \int_1^y (t \text{Log } t - t + \mathcal{O}(\text{Log}(2t))) \left(\frac{t}{y}\right)^{\beta-1} e^{-(t/y)^\beta} \frac{dt}{y} + e^{-1} y \text{Log } y + \mathcal{O}(y). \end{aligned}$$

We call  $V$  the left-hand side. The above estimation simplifies into

$$\begin{aligned} V &= \beta \int_1^y (t \text{Log } t - t) (t/y)^{\beta-1} e^{-(t/y)^\beta} dt/y + e^{-1} y \text{Log } y + \mathcal{O}(y) \\ &= y\beta \int_{1/y}^1 (\text{Log } u + \text{Log } y - 1) u^\beta e^{-u^\beta} du + e^{-1} y \text{Log } y + \mathcal{O}(y) \\ &= y \text{Log } y \beta \int_{1/y}^1 u^\beta e^{-u^\beta} du + e^{-1} y \text{Log } y + \mathcal{O}(y) \end{aligned}$$

We continue by noticing that

$$\begin{aligned} \beta \int_{1/y}^1 u^\beta e^{-u^\beta} du &= \beta \int_0^1 u^\beta e^{-u^\beta} du + \mathcal{O}(\beta y^{-1}) \\ &= \int_0^1 v^{1/\beta} e^{-v} dv + \mathcal{O}(\beta y^{-1}) \\ &= \int_0^1 e^{-v} dv + \mathcal{O}\left(\beta y^{-1} + \int_0^1 (1 - v^{1/\beta}) dv\right) \\ &= 1 - e^{-1} + \mathcal{O}(\beta y^{-1} + \beta^{-1}). \end{aligned}$$

Let us now consider

$$\begin{aligned} W &= \sum_{n \geq 1} \text{Log } n e^{-(n/y)^\beta} \\ &= \sum_{n \geq 1} \text{Log } n \beta \int_n^\infty \left(\frac{t}{y}\right)^{\beta-1} e^{-(t/y)^\beta} \frac{dt}{y} \\ &= \beta \int_1^\infty (t \text{Log } t - t + \mathcal{O}(\text{Log}(2t))) \left(\frac{t}{y}\right)^{\beta-1} e^{-(t/y)^\beta} \frac{dt}{y}. \end{aligned}$$

We use yet another integration by parts to get

$$\begin{aligned} \beta \int_1^\infty \text{Log}(2t) \left(\frac{t}{y}\right)^{\beta-1} e^{-(t/y)^\beta} \frac{dt}{y} &= \text{Log } 2 + \int_1^\infty e^{-(t/y)^\beta} \frac{dt}{t} \\ &\leq \text{Log } 2 + \int_{1/y}^\infty e^{-u^\beta} \frac{du}{u} \leq \text{Log } 2 + \int_{1/y}^1 \frac{du}{u} + \int_1^\infty e^{-u^\beta} \frac{du}{u} \\ &\leq \text{Log}(2y) + \beta^{-1}. \end{aligned}$$

We continue as follows

$$\begin{aligned}
W &= y\beta \int_{1/y}^{\infty} (\operatorname{Log} u - 1 + \operatorname{Log} y) u^\beta e^{-u^\beta} du + \mathcal{O}(1) \\
&= y\beta \int_0^{\infty} (\operatorname{Log} u - 1 + \operatorname{Log} y) u^\beta e^{-u^\beta} du + \mathcal{O}(1 + \beta \operatorname{Log} y) \\
&= y(\operatorname{Log} y - 1)\Gamma(1 + \beta^{-1}) + y\beta^{-1}\Gamma'(1 + \beta^{-1}) + \mathcal{O}(1 + \beta \operatorname{Log} y)
\end{aligned}$$

and we conclude easily.

## 4 Linear inequalities for values of Dirichlet polynomials on difference sets

Difference sets are especially atuned with Dirichlet polynomials and appear in some important places in the theory of large values of Dirichlet polynomials, as in [13] and in [9].

Let  $D(s) = \sum_{n \geq 1} \psi_n n^{-s}$  be a Dirichlet series with non-negative coefficients and admitting an abscissa of absolute convergence not more than 1. The examples we have in mind are  $\operatorname{Log} \zeta(s)$  with  $\psi_n = \Lambda(n)/\operatorname{Log} n$  and  $-(\zeta'/\zeta)(s)$  with  $\psi_n = \Lambda(n)$ . We restrict our attention to series having non-negative coefficients for simplicity. The object of this part is to discuss properties of the shape

$$(13) \quad \forall (\varphi_a)_a \in \mathbb{R}^A, \forall \sigma > 1, \quad UD(\sigma) + \sum_{1 \leq a \neq b \leq A} u_{a,b} D(\sigma + i(\varphi_a - \varphi_b)) \geq 0$$

where  $U$  and the  $u_{a,b}$ 's are real numbers,  $u_{b,a} = u_{a,b}$  and  $A$  is a given integer. The set of such  $(U, (u_{a,b}))$  is obviously a cone and we would like to know the generators of its extremal half-lines. Clearly such a set depends on  $D$  (note that  $D$  can be 0), so we restrict our attention to universal  $(U, (u_{a,b}))$ , i.e. such that (13) holds for all  $D$ . In particular, it is enough to take polynomials instead of series, from which we deduce that it is enough to verify it for the monomials  $n^{-s}$ ; changing the  $\varphi_a$ 's, we infer that it is enough to have

$$(14) \quad \forall (\varphi_a)_a \in \mathbb{R}^A, \quad U + \sum_{1 \leq a \neq b \leq A} u_{a,b} e^{i(\varphi_a - \varphi_b)} \geq 0.$$

Another way of writing (14) is

$$(15) \quad \forall (z_a)_a \in \mathbb{U}^A, \quad U + \Re \sum_{1 \leq a \neq b \leq A} u_{a,b} z_a \bar{z}_b \geq 0,$$



where  $\mathbb{U}$  is the unit circle. The quantity  $U + \Re \sum_{1 \leq a \neq b \leq A} u_{a,b} z_a \bar{z}_b$  is an harmonic function as a function of  $z_1$ , which is non negative on the unit circle. By using a Poisson integral, we see that the positivity extends to  $|z_1| \leq 1$ . We treat similarly the other variables and get

$$(16) \quad \forall (z_a)_a \in \mathbb{D}^A, \quad U + \sum_{1 \leq a \neq b \leq A} u_{a,b} z_a \bar{z}_b \geq 0,$$

where  $\mathbb{D}$  is the unit disk. Set

$$(17) \quad H((u_{a,b})) = \min_{(z_a)_a \in \mathbb{D}^A} \sum_{1 \leq a \neq b \leq A} u_{a,b} z_a \bar{z}_b$$

which is finite since  $\mathbb{D}$  is compact and is clearly the largest value possible for  $-U$ . Note that it shows by taking  $z_1 = z_2 = \dots = z_A = 0$  that  $U \geq 0$ . The above discussion shows that any set  $\{u_{a,b}, 1 \leq a < b \leq A\}$  gives rise to an inequality like (14).

A classical case comes from

$$(18) \quad \left| \sum_{a=1}^A u_a e^{i\varphi_a} \right|^2 = \sum_{a=1}^A u_a^2 + \sum_{a \neq b} u_a u_b \cos(\varphi_a - \varphi_b)$$

This is used to prove that

**Lemma 4.1** *Let  $(u_a) \in \mathbb{R}^A$  and  $(\varphi_a) \in \mathbb{R}^A$ . Then*

$$0 \leq \sum_{a=1}^A u_a^2 + \sum_{a \neq b} u_a u_b e^{i(\varphi_a - \varphi_b)}.$$

The reader will readily show that the corresponding  $H$  is indeed  $-\sum_{a=1}^A u_a^2$ .

This lemma is commonly employed to show that  $\zeta(1+it)$  cannot vanish if  $t \neq 0$ . One takes  $(u_1, u_2, u_3) = (1, 1, 1)$  and  $(\varphi_1, \varphi_2, \varphi_3) = (0, t \log n, -t \log n)$ . Lemma 2 from [5] or [4] (see (20) below) is a consequence of case  $A = 3$ ,  $(u_1, u_2, u_3) = (1, -1, -1)$  and  $(\varphi_1, \varphi_2, \varphi_3) = (0, \tau_1 \log n, \tau_2 \log n)$ .

We can use such inequalities for series that do not have positive coefficients by introducing a majorizing series as in the proof of Lemma 5.2 below.

## 5 Another linear inequality for values of Dirichlet polynomials

One can use (18) in a different fashion:

**Lemma 5.1** *Let  $(u_a) \in \mathbb{R}^A$  and  $(\varphi_a) \in \mathbb{R}^A$ . Let  $U$  be an upper bound for  $(\sum u_a^2)^{1/2}$ . Then*

$$\left| \sum_{a=1}^A u_a e^{i\varphi_a} \right| \leq U + \frac{1}{2U} \sum_{a \neq b} u_a u_b e^{i(\varphi_a - \varphi_b)}.$$

*Proof:* Notice first that the RHS is non-negative since it is equal to

$$U - \frac{\sum u_a^2}{2U} + \frac{1}{2U} \left| \sum u_a e^{i\varphi_a} \right|^2 \geq \frac{1}{2}U.$$

Then the RHS of (18) is easily seen to be not more than

$$\left( U + \frac{1}{2U} \sum_{a \neq b} u_a u_b \cos(\varphi_a - \varphi_b) \right)^2$$

hence the result. ◇◇◇

Lemma 5.1 is not a refinement of Lemma 4.1, due to the factor 1/2.

To derive an inequality on Dirichlet polynomials from such inequalities, the reader may simply adapt the proof of Lemma 5.2 that follows.

**Lemma 5.2** *Let  $h$  be a completely multiplicative function such that  $D(h, s)$  is absolutely convergent for  $\Re s > 1$ . Let  $h^\sharp$  be another completely multiplicative function that majorizes  $|h|$ . Let  $\sigma > 1$ ,  $(u_a) \in \mathbb{R}^A$ ,  $(t_a) \in \mathbb{R}^A$  and  $(\chi_a)$  be a family of Dirichlet characters. Let finally  $U$  be an upper bound for  $(\sum u_a^2)^{1/2}$ . Then  $\prod_{a=1}^A |D(h\chi_a, \sigma + it_a)|^{u_a}$  is not more than*

$$D(h^\sharp, \sigma)^U \left( \prod_{a \neq b} |D(h^\sharp \chi_a \bar{\chi}_b, \sigma + i(t_a - t_b))|^{u_a u_b} \right)^{1/2U}.$$

*Proof:* We simply use Lemma 5.1 over the quantity

$$\frac{1}{2} \sum_a u_a \{ \text{Log } D(h\chi_a, \sigma + it_a) + \text{Log } D(\bar{h}\chi_a, \sigma - it_a) \}$$

expanded in Dirichlet series:

$$\sum_{n \geq 2} \frac{\Lambda(n)}{n^\sigma \text{Log } n} \Re h(n) \sum_a \chi_a(n) u_a e^{-it_a \text{Log } n}.$$

◇◇◇

One can extend these inequalities from completely multiplicative to multiplicative functions as in [7] or [8]. This essentially relies on a hypothesis asserting that the values on powers of primes are small enough. If we assume like in this work that they are indeed bounded by one, we loose only a multiplicative constant on our upper bound.

While the theory of inequalities for values of Dirichlet series on difference sets runs rather smoothly, we have not been able to build a comfortable setting for inequalities similar to above one. On taking  $A = 2$ ,  $u_1 = u_2 = 1$ ,  $t_1 = t + \tau$  and  $t_2 = \tau$ , we reach

$$(19) \quad |D(h, \sigma + i\tau)D(h, \sigma + it + i\tau)| \leq D(h^\sharp, \sigma)^{\sqrt{2}} |D(h^\sharp, \sigma + it)|^{1/\sqrt{2}}$$

while inequality of Lemma 2 from [5] or [4] reads

$$(20) \quad |D(h, \sigma + i\tau)D(h, \sigma + it + i\tau)| \leq D(h^\sharp, \sigma)^{3/2} |D(h^\sharp, \sigma + it)|^{1/2}.$$

The latter is better when  $t$  is small due to the smaller sum  $\frac{3}{2} + \frac{1}{2} = 2 < \sqrt{2} + 1/\sqrt{2}$  while ours is better otherwise. On using this inequality, we would get our Theorem but with an exponent  $(2\sqrt{2} - 2)/(1 + 4\sqrt{2}) = 1/8.03 \dots$  when  $w$  is constant. We continue by deriving stronger inequalities, though they will lead only to a minor numerical improvement of this exponent. In a different context, Barrucand & Louboutin use (20) and (19) in [1], but for characters:

$$(21) \quad \begin{cases} |D(h\chi_1, \sigma)D(h\chi_2, \sigma)| \leq D(h^\sharp, \sigma)^{3/2} |D(h^\sharp\chi_1\bar{\chi}_2, \sigma)|^{1/2}, \\ |D(h\chi_1, \sigma)D(h\chi_2, \sigma)| \leq D(h^\sharp, \sigma)^{\sqrt{2}} |D(h^\sharp\chi_1\bar{\chi}_2, \sigma)|^{1/\sqrt{2}} \end{cases}$$

for  $\chi_2 = \bar{\chi}_1$  and  $h = h^\sharp = 1$ . The first of these is already stated in Lemma 2 of [4]. We dwell more on this aspect in section 7.

## 6 A bilinear inequality for values of a Dirichlet series

The proof of Lemma 5.1 goes by "completion of the square" and some precision is lost in the process. It can be recovered by working directly with squares.

Here is the line of approach. We start with  $\|f - \sum_i \xi_i \varphi_i\|^2 \geq 0$ , valid in any vector space equipped with a hermitian form  $\langle f|g \rangle$ , and expand it into

$$(22) \quad \|f\|^2 - \sum_i \xi_i \langle \varphi_i | f \rangle - \sum_i \bar{\xi}_i \langle f | \varphi_i \rangle + \sum_{i,j} \xi_i \bar{\xi}_j \langle \varphi_i | \varphi_j \rangle \geq 0.$$

The way to handle the last double sum will yield different results. We prove three of them, though we shall use only the last one in this paper. The first one is a now classical path to prove the large sieve inequality, see for instance [2].

**Lemma 6.1 (Selberg)**  $\sum_i |\langle f|\varphi_i \rangle|^2 / \sum_j |\langle \varphi_i|\varphi_j \rangle| \leq \|f\|^2$ .

*Proof:* In (22), use

$$(23) \quad \sum_{i,j} \xi_i \bar{\xi}_j \langle \varphi_i|\varphi_j \rangle \leq \sum_{i,j} |\xi_i| |\xi_j| |\langle \varphi_i|\varphi_j \rangle| \leq \sum_{i,j} \frac{1}{2} (|\xi_i|^2 + |\xi_j|^2) |\langle \varphi_i|\varphi_j \rangle|.$$

The choice  $\xi_i = \langle f|\varphi_i \rangle / \sum_j |\langle \varphi_i|\varphi_j \rangle|$  yields the Lemma.  $\diamond \diamond \diamond$

**Lemma 6.2**  $\sum_i |\Re \langle f|\varphi_i \rangle|^2 / \sum_j |\Re \langle \varphi_i|\varphi_j \rangle| \leq \|f\|^2$ .

*Proof:* In (22), assume  $\xi_i$  to be real numbers. We can insert real part signs and proceed as before with the choice  $\xi_i = \Re \langle f|\varphi_i \rangle / \sum_j |\Re \langle \varphi_i|\varphi_j \rangle|$  to get the stated Lemma.  $\diamond \diamond \diamond$

In our application,  $\langle f|\varphi_i \rangle$  will be  $\text{Log } F(x_i)$ , so its real part will be  $\text{Log } |F(x_i)|$ . The previous Lemma majorizes the absolute value of this quantity, i.e. majorizes and minorizes  $|F(x_i)|$ . This precision is obtained because bounding  $|\Re \langle \varphi_i|\varphi_j \rangle|$  will also require some lower bounds. This is avoided in the next Lemma.

**Lemma 6.3** Assume  $\Re \langle f|\varphi_i \rangle \geq 0$  for all  $i$ . Then

$$\sum_i \frac{|\Re \langle f|\varphi_i \rangle|^2}{\sum_{j/\Re \langle \varphi_i|\varphi_j \rangle \geq 0} \Re \langle \varphi_i|\varphi_j \rangle} \leq \|f\|^2.$$

*Proof:* In (22), take nonnegative real  $\xi_i$ 's. Insert real parts, discard the  $\Re \langle \varphi_i|\varphi_j \rangle$  that are non positive and choose  $\xi_i = \Re \langle f|\varphi_i \rangle / \sum_{j/\Re \langle \varphi_i|\varphi_j \rangle \geq 0} \Re \langle \varphi_i|\varphi_j \rangle$ . The Lemma readily follows.  $\diamond \diamond \diamond$

**Lemma 6.4** Let  $D(s) = \sum_n \psi_n n^{-s}$  be a Dirichlet series absolutely convergent for  $\Re s > 1$ . We fix an upper bound  $\psi_n^\#$  of  $|\psi_n|$ . Let  $\sigma > 1$  and  $t$  be fixed real numbers. We put  $D^\#(s) = \sum_n \psi_n^\# n^{-s}$  and for  $\Delta$  and  $\delta \geq 0$  two real parameters we introduce

$$(24) \quad \Delta^+(\delta, \Delta) = \sup_{\delta \leq |t'| \leq 2\Delta} \Re D^\#(\sigma + it').$$

Let  $(u_a) \in \mathbb{R}^A$  be a family of points satisfying  $|u_a| \leq \Delta$  and  $|u_a - u_b| \geq \delta$  for  $a \neq b$ . Assume that for all  $a$ , we have  $\Re D(\sigma + iu_a + it) \geq 0$ . We have

$$\sum_{a=1}^A |\Re D(\sigma + iu_a + it)|^2 \leq D^\sharp(\sigma)(D^\sharp(\sigma) + (A-1)\Delta^+(\delta, \Delta)).$$

*Proof:* We simply apply Lemma 6.3 to the Hilbert space of sequences  $(r_n)$  such that  $\sum_{n \geq 1} |r_n|^2 n^{-\sigma} < \infty$  equipped with the hermitian product  $\langle r|s \rangle = \sum_{n \geq 1} r_n \overline{s_n} n^{-\sigma}$ .  $\diamond \diamond \diamond$

**Lemma 6.5** *Let  $0 \leq T_0 \leq T_1$  be two real parameters. Let  $h$  be a completely multiplicative function of modulus bounded by 1. There exists  $\tau \in [-T_1, T_1]$  such that*

$$\begin{aligned} \max_{T_0 \leq t \leq T_1} \frac{\max(0, \text{Log } |D(h, \alpha + i(t + \tau))|)^2}{\text{Log}^2 \zeta(\alpha)} \\ \leq 1 - \max_{T_0 \leq |t| \leq 2T_1} \frac{\text{Log}(\zeta(\alpha)/|\zeta(\alpha + it)|)}{2 \text{Log} \zeta(\alpha)} = B. \end{aligned}$$

Moreover, either  $\tau = 0$ , or  $|\tau| \geq T_0$  and

$$\sum_{p \geq 2} (1 - \Re h(p)p^{-i\tau})/p^\alpha \leq -(1 - \sqrt{B}) \text{Log}(\alpha - 1) + \mathcal{O}(1).$$

*Proof:* Either the right-hand side of our first inequality is  $\leq B$  for  $\tau = 0$  or there exists a  $t \in \pm[T_0, T_1]$ , which we call  $\tau$ , and such that

$$\text{Log}^2 |D(h, \alpha + i\tau)| > B \text{Log}^2 \zeta(\alpha).$$

Let then  $t$  such that  $T_0 \leq |t| \leq T_1$ . By Lemma 8.1, we have

$$\begin{aligned} \frac{\max(0, \text{Log } |D(h, \alpha + i(t + \tau))|)^2}{\text{Log}^2 \zeta(\alpha)} + \frac{\text{Log}^2 |D(h, \alpha + i\tau)|}{\text{Log}^2 \zeta(\alpha)} \\ \leq 1 + \max_{T_0 \leq |t| \leq 2T_1} \frac{\text{Log } |\zeta(\alpha + it)|}{\text{Log} \zeta(\alpha)} = 2B \end{aligned}$$

and we use our lower bound on  $\text{Log } |D(h, \alpha + i\tau)|$  to conclude.  $\diamond \diamond \diamond$

## 7 A detour towards $L$ -functions and a proof of Theorem 1.2

It is immediate to work out a version Lemma 6.4 with Dirichlet characters. In case of two characters and by applying this Lemma to the logarithm of the corresponding  $L$ -series multiplied by  $\epsilon = \pm 1$ , this reads

$$(25) \quad \max(0, \epsilon \operatorname{Log} |L(\sigma, \chi_1)|)^2 + \max(0, \epsilon \operatorname{Log} |L(\sigma, \chi_2)|)^2 \\ \leq \operatorname{Log} \zeta(\sigma) \left( \operatorname{Log} \zeta(\sigma) + \max(0, \operatorname{Log} |L(\sigma, \chi_1 \overline{\chi_2})|) \right).$$

We specialize it to  $\chi_2 = \overline{\chi_1}$  and get

$$2 \max(0, \epsilon \operatorname{Log} |L(\sigma, \chi)|)^2 \leq \operatorname{Log} \zeta(\sigma) \left( \operatorname{Log} \zeta(\sigma) + \max(0, \operatorname{Log} |L(\sigma, \chi^2)|) \right).$$

Let us assume that  $\chi^2$  has conductor  $f' \neq 1$  while  $\chi$  is primitive of conductor  $f$  and let  $\chi'$  the primitive character associated with  $\chi^2$ . For  $\sigma \geq 1$ , we have

$$|L(\sigma, \chi^2)| \leq |L(\sigma, \chi')| \frac{f/f'}{\phi(f/f')}$$

so that

$$\operatorname{Log} |L(\sigma, \chi^2)| \leq \operatorname{Log} \operatorname{Log} f' + \mathcal{O}(\operatorname{Log} \operatorname{Log} \operatorname{Log}(10f/f')).$$

Let us borrow the estimate (this is for instance a consequence of Lemma d of [14] used in equation (1) therein)

$$|L(1, \chi)| \gg |L(\sigma, \chi)|$$

for  $\sigma = 1 + 1/\operatorname{Log} f$  and let us assume that  $|L(1, \chi)| \leq 1$ . We set

$$(\operatorname{Log} f)^\xi = \operatorname{Log} f'$$

and Theorem 1.2 follows from our inequality with  $\epsilon = -1$ .

## 8 A bilinear inequality for values of real Dirichlet series

In case of Dirichlet series with real coefficients, we can look simultaneously at the contribution of  $\mathcal{D}(\sigma + it)$  and  $\mathcal{D}(\sigma - it)$  which are equal. In this case, it is slightly better to start the proof anew:

**Lemma 8.1** *Let  $\mathcal{D}(s) = \sum_n \psi_n n^{-s}$  be a Dirichlet series absolutely convergent for  $\Re s > 1$ . We assume that all  $\psi_n$  are real numbers and we fix an upper bound  $\psi_n^\#$  of  $|\psi_n|$ . Let  $\sigma > 1$  and  $t$  be fixed real numbers. We put  $\mathcal{D}^\#(s) = \sum_n \psi_n^\# n^{-s}$  and for  $\Delta$  and  $\delta > 0$  two real parameters we introduce*

$$(26) \quad \Delta^+(\delta, \Delta) = \sup_{\delta \leq |t'| \leq 2\Delta} \Re \mathcal{D}^\#(\sigma + it'), \quad W^+(\Delta) = \sup_{|2t-t'| \leq 2\Delta} \Re \mathcal{D}^\#(\sigma + it')$$

*Let  $(u_a) \in \mathbb{R}^A$  be a family of points satisfying  $|u_a| \leq \Delta$  and  $|u_a - u_b| \geq \delta$  for  $a \neq b$ . Assume that for all  $a$ , we have  $\Re \mathcal{D}(\sigma + iu_a + it) \geq 0$ . We have*

$$\sum_{a=1}^A |\Re \mathcal{D}(\sigma + iu_a + it)|^2 \leq \frac{1}{2} \mathcal{D}^\#(\sigma) (\mathcal{D}^\#(\sigma) + (A-1)\Delta^+(\delta, \Delta) + AW^+(\Delta)).$$

This yields for  $A = 1$ :

$$(27) \quad 2|\Re \mathcal{D}(\sigma + it)|^2 \leq \max\left(2, \mathcal{D}^\#(\sigma)(\mathcal{D}^\#(\sigma) + \Re \mathcal{D}^\#(\sigma + 2it))\right).$$

*Proof:* We have

$$\begin{aligned} \left| \sum_{a=1}^A w_a \Re \mathcal{D}(\sigma + it + iu_a) \right|^2 &= \left| \sum_{n \geq 1} \frac{\psi_n}{n^\sigma} \sum_{a=1}^A w_a \Re n^{-it-iu_a} \right|^2 \\ &\leq \mathcal{D}^\#(\sigma) \left\{ \sum_{n \geq 1} \frac{\psi_n^\#}{n^\sigma} \sum_{a,b} w_a w_b \Re n^{it+iu_a} \Re n^{it+iu_b} \right\} \end{aligned}$$

where the  $w_a$ 's are non-negative real numbers. This last expression reads also

$$\begin{aligned} \frac{1}{2} \mathcal{D}^\#(\sigma) \left( \sum_a w_a^2 \mathcal{D}^\#(\sigma) + \sum_{a \neq b} w_a w_b \Re \mathcal{D}^\#(\sigma + i(u_a - u_b)) \right. \\ \left. + \sum_{a,b} w_a w_b \Re \mathcal{D}^\#(\sigma + i(2t + u_a + u_b)) \right). \end{aligned}$$

We discard the  $\Re \mathcal{D}^\#(\sigma + iw)$  that are negative. Appealing to  $xy \leq (x^2 + y^2)/2$ , we get

$$\sum_{\substack{a \neq b, \\ \Re \mathcal{D}^\#(\sigma + i(u_a - u_b)) \geq 0}} w_a w_b \leq (A-1) \sum_a w_a^2$$

and a similar inequality for the part with  $\mathcal{D}^\#(\sigma + i(u_a + u_b))$ . We thus reach the upper bound

$$\frac{1}{2} \mathcal{D}^\#(\sigma) \sum_a w_a^2 \left( \mathcal{D}^\#(\sigma) + (A-1)\Delta^+(\delta, \Delta) + AW^+(\Delta) \right).$$

We simply take  $w_a = \Re \mathcal{D}(\sigma + it + iu_a)$ . ◇◇◇

## 9 On the Dirichlet series of a multiplicative function of modulus $\leq 1$

In this section,  $g$  is a multiplicative function of modulus  $\leq 1$  and we stress that none of the constants implied in the  $\ll$ -symbols depend on  $g$ . We first quote a result from section 10 of [15]:

**Lemma 9.1** *We write  $D(g, s) = D_0(g^*, s)J(s)$  with*

$$D_0(g^*, s) = \prod_{p \geq 2} \left(1 - \frac{g(p)}{p^s}\right)^{-1}$$

and where  $J(s)$  is holomorphic for  $\Re s > 1/2$  and verifies

$$1 \ll J(s) \ll 1, \quad J'(s) \ll 1 \quad (\Re s \geq 1).$$

The function  $g^*$  is completely multiplicative and still bounded in modulus by 1. We next quote Lemma 1 from [5] (see also (9.18 – 20) of [15] or pp 337-338 of [7]).

**Lemma 9.2** *For  $1 < \sigma \leq 2$  and  $N \geq 0$ , we have*

$$\int_{|\tau| \geq N} \left| \frac{D'(g, s)}{sD(g, s)} \right|^2 d\tau \leq \frac{2\pi e^{42}}{(N+1)(\sigma-1)}$$

with  $s = \sigma + i\tau$ . Moreover, if  $\delta > 0$ ,

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{|D(g, s)|^{1+\delta}}{|s|^2} |ds| \ll \frac{1}{\delta(\sigma-1)^\delta}.$$

**Lemma 9.3 (Stirling)** *Suppose  $\sigma_0 \leq \sigma \leq \sigma_1$  and  $|t| \geq 1$ . Then*

$$|\Gamma(\sigma + it)| = \sqrt{2\pi} |t|^{\sigma-1/2} \exp(-\pi|t|/2) (1 + \mathcal{O}_{\sigma_0, \sigma_1}(1 + 1/|t|)).$$

**Lemma 9.4** *When  $\sigma > 1$  and  $\beta \geq 2$ , we have*

$$\begin{aligned} (\sigma - 1) \int_{|t| \geq T_0} |D'(g, s) \Gamma(s/\beta)| dt/\beta \\ \ll \frac{((\sigma - 1) \max_{\beta^3 \geq |t| \geq T_0} |D(g, s)|)^{\lambda/2}}{\sqrt{1-\lambda}} \text{Log } \beta + e^{-\beta^2} \end{aligned}$$

for any  $\lambda \in ]0, 1[$ .

We can of course replace  $D(g, s)$  by  $D(g^*, s)$  on the right-hand side, where  $g^*$  is defined in Lemma 9.1.



*Proof:* Let us first study the contribution of large  $|t|$ 's. We first note that

$$|\Gamma((\sigma + it)/\beta)/\beta| \ll e^{-|t|/\beta}/(\sigma + |t|) \quad (0 < \sigma \leq \beta).$$

This holds because, when  $|t| \leq \beta$ , we use the pole of  $\Gamma$  at  $s = 0$ , while for  $|t| \geq \beta$ , we use Lemma 9.3. We then proceed as in [15].

$$\int_{|t| \geq T_0} |D'(g, s)| |\Gamma((\sigma + it)/\beta)/\beta| dt \leq \sum_{k \in \mathbb{Z}} \frac{e^{-|k|/\beta}}{1 + |k|} \int_{\substack{\sigma + ik, \\ |s| \geq T_0}}^{\sigma + i(k+1)} |D'(g, s)| dt$$

Let us call  $I$  a typical set of integration (depending on  $k$ ). We use Cauchy's inequality and appeal to Lemma 9.2 concerning the part with  $D'(g, \cdot)/D(g, \cdot)$ . We write  $s = s' + ik$ , where  $s'$  ranges  $I'$  and note that  $D'(g, s) = D'(g_k, s')$  ( $g_k$  is defined in (11)). We get

$$\begin{aligned} \left( \int_{I'} |D'(g_k, s')| dt \right)^2 &= \left( \int_{I'} |D'/D(g_k, s')| |D(g_k, s')| dt \right)^2 \\ &\ll \int_{I'} |D'/D(g_k, s')|^2 dt \int_{I'} |D(g_k, s')|^2 dt \\ &\leq \int_{-\infty}^{\infty} |D'/D(g_k, s')|^2 \frac{dt}{|s'|^2} \int_{-\infty}^{\infty} |D(g_k, s')|^2 \frac{dt}{|s'|^2} \\ &\leq \frac{1}{\sigma - 1} \int_{-\infty}^{\infty} |D(g_k, s')|^2 \frac{dt}{|s'|^2}. \end{aligned}$$

When  $|k| \geq \beta^3$ , we bound the last integral simply by  $1/(\sigma - 1)$ . Otherwise, we need to save some more and we proceed as follows. For any  $\lambda \in [0, 1]$ , we have:

$$\begin{aligned} \int_{-\infty}^{\infty} \left| \frac{D(g_k, s')}{s'} \right|^2 dt &= \int_{-\infty}^{\infty} \frac{|D(g_k, s')|^\lambda |D(g_k, s')|^{2-\lambda}}{|s'|^2} dt \\ &\ll \max_{\beta^3 \geq |t| \geq T_0} |D(g, s)|^\lambda (\sigma - 1)^{\lambda-1} / (1 - \lambda) \end{aligned}$$

and the Lemma follows readily.  $\diamond \diamond \diamond$

**Lemma 9.5** *Assume  $g$  to be real-valued. When  $\sigma > 1$  and  $T_0 \in [\sigma - 1, 1]$ , we have*

$$|D(g, \sigma + iT_0)| \ll (\sigma - 1)^{-3/4} T_0^{-1/4}.$$

*Proof:* By Lemma 9.1, it is enough to consider the case where  $g$  is completely multiplicative. We set  $L = 1/(\sigma - 1)$ . In case  $|D(\sigma + iT_0)|$  is larger than 1, we get

$$2 \operatorname{Log}^2 |D(\sigma + iT_0)| \leq (\operatorname{Log} L + \mathcal{O}(1))(2 \operatorname{Log} L + \mathcal{O}(1) - \operatorname{Log} \sqrt{1 + (LT_0)^2})$$

so that

$$\operatorname{Log} |D(\sigma + iT_0)| \leq \operatorname{Log} L \left( 1 - \frac{\operatorname{Log} \sqrt{1 + (LT_0)^2}}{4 \operatorname{Log} L} \right) + \mathcal{O}(1)$$

as required. ◇◇◇

## 10 Continuity around $\Re s = 0$ when $g$ is real valued

The aim of this section is to study the difference

$$(28) \quad \Theta = \frac{-1}{2i\pi} \int_{-K(\alpha-1)}^{K(\alpha-1)} D'(g, s) \Gamma(s/\beta) (x^{s-1} - (x/w)^{s-1}) ds / \beta$$

where  $s = \alpha + it$ . Since  $K(\alpha - 1) \leq 1 \leq \beta$ , the factor  $\Gamma(s/\beta)/\beta$  is essentially constant. We handle the  $w^{z-1} - 1$  in a standard fashion:

$$\frac{w^{z-1} - 1}{z - 1} = \int_0^{\operatorname{Log} w} e^{(z-1)u} du$$

so that, when  $w \geq 1$

$$|w^{s-1} - 1| \leq w^{\Re s - 1} |s - 1| \operatorname{Log} w.$$

As a consequence, we could majorize  $|w^{s-1} - 1|$  uniformly by  $(1 + K)(\alpha - 1)$  as in [5] but some more precision results in a much better outcome. First note that, with the shorthand  $T_0 = K(\alpha - 1)$ ,

$$\int_{-T_0}^{T_0} |D'/D(g, s)|^2 dt \ll 1/(\alpha - 1).$$

We handle the part with  $D(g, s)$  somewhat more carefully and write

$$\int_{-T_0}^{T_0} |D(g, s)|^2 |s - 1|^2 dt \leq \max_{|t| \leq T_0} (|D(g, s)|^\lambda |s - 1|^2) \int_{-T_0}^{T_0} |D(g, s)|^{2-\lambda} dt.$$

The reader may better understand what happens here by invoking Lemma 9.5. Our actual treatment is more precise. Since  $g$  is real-valued, we can apply (27) to  $g^*$  of Lemma 9.1 and get

$$(29) \quad \frac{2 \operatorname{Log}^2 |D(g^*, s)|}{\operatorname{Log}^2 \zeta(\alpha)} \leq 1 + \frac{\operatorname{Log} \left( \frac{1}{\sqrt{(\alpha-1)^2 + 4t^2}} + \mathcal{O}(1) \right)}{2 \operatorname{Log} \zeta(\alpha)}$$

with  $s = \alpha + it$ . We can majorize  $\lambda \operatorname{Log} |D(g^*, s)| + 2 \operatorname{Log} \sqrt{(\alpha - 1)^2 + t^2}$  by

$$\frac{\lambda \operatorname{Log} \zeta(\alpha)}{\sqrt{2}} \sqrt{1 + \frac{\operatorname{Log}\left(\frac{1}{\sqrt{(\alpha-1)^2+4t^2}} + \mathcal{O}(1)\right)}{2 \operatorname{Log} \zeta(\alpha)}} + 2 \operatorname{Log} \sqrt{(\alpha - 1)^2 + 4t^2}.$$

We can rewrite this upper bound as  $a\sqrt{1 + b \operatorname{Log}(u^{-1} + c)} + 2 \operatorname{Log} u$  whose derivative reads

$$\left( -\frac{ab}{4(1+cu)\sqrt{1+b \operatorname{Log}(u^{-1}+c)}} + 1 \right) \frac{2}{u} > 0.$$

We can thus majorize this quantity by its value in  $T_0$ . Majorizing that by the value over the interval  $[T_0, \beta^3]$ , we reach

$$(30) \quad \int_{-K(\alpha-1)}^{K(\alpha-1)} |D'(g, s)| |x^{s-1} - (x/w)^{s-1}| dt \ll K \frac{\mu^{\lambda/2}}{\sqrt{1-\lambda}} \operatorname{Log} w$$

where  $\mu$  is defined in (10).

## 11 The proof of Theorem 1.3

We first quantify the error term named *error* in (12). From (27) applied to  $g^*$  of Lemma 9.1, we get

$$(31) \quad \frac{\operatorname{Log}^2 |D(g^*, s)|}{\operatorname{Log}^2 \zeta(\alpha)} \leq 1 + \frac{\operatorname{Log}\left(\frac{1}{\sqrt{4K^2+1}} + c(\alpha-1) \operatorname{Log} \beta\right)}{2 \operatorname{Log} \zeta(\alpha)} = 1 - u$$

for a constant  $c$  and we feed this information in Lemma 9.4. We further use (30). The resulting error term is  $\mathcal{O}$  of:

$$\frac{(\alpha-1)^{\lambda(1-\sqrt{1-u})/2}}{\sqrt{1-\lambda}} \left( \operatorname{Log} \beta + K \frac{\operatorname{Log} w}{\operatorname{Log} x} \right) + \frac{1}{\beta} + \frac{1}{\operatorname{Log} x} + \frac{\beta \operatorname{Log}(x/w_0)}{(x/w_0) \operatorname{Log} x}.$$

To compare with [5], our way of handling the smoothing yields a  $\operatorname{Log} \beta$  instead of a  $\beta$ . To compare with [15], we would get a  $\operatorname{Log} \beta$  but only for  $\beta \gg \operatorname{Log} x \operatorname{Log} \operatorname{Log} x$ .

We take

$$(32) \quad K = \beta = \kappa \quad \text{and} \quad \lambda = 1 - (\operatorname{Log} \kappa)^{-1}.$$

We can majorize  $K$  by  $\sqrt{4K^2 + 1}$  and the latter by  $c_2 (\operatorname{Log} x)^{2u} \operatorname{Log} \kappa$  for some constant  $c_2$  since

$$2u \operatorname{Log} \operatorname{Log} x + \mathcal{O}(1) = \frac{1}{2} \operatorname{Log}(4K^2 + 1) - \operatorname{Log}(1 + c(\alpha-1)\sqrt{4K^2 + 1} \operatorname{Log} \beta).$$

Note next that

$$(\alpha-1)^{(1-\sqrt{1-u})/2} = (\alpha-1)^{\frac{u}{2(1+\sqrt{1-u})}} \ll \kappa^{\frac{1}{4(1+\sqrt{1-u})}} (\operatorname{Log} \kappa)^{1/4}.$$

## 12 The proof of Theorem 1.1

To handle the case of a complex valued  $g$ , we proceed as in [5] by replacing our appeal of (27) by Lemma 6.5. We first prove that

$$(33) \quad S_0(g_\tau, x) = S_0(g_\tau, x/w) + \mathcal{O}(x \cdot \varepsilon) \quad \text{with} \quad \varepsilon = (\text{Log } \kappa) / \kappa^{\frac{2}{9 + \sqrt{81 - 32\xi}}}.$$

*Proof:* We again quantify the error term named *error* in (12). From Lemma 6.5 applied to  $g_\tau^*$  of Lemma 9.1, we get

$$(34) \quad \frac{\text{Log}^2 |D(g_\tau^*, s)|}{\text{Log}^2 \zeta(\alpha)} \leq 1 + \frac{\text{Log}\left(\frac{1}{\sqrt{4K^2+1}} + c(\alpha-1)\text{Log } \beta\right)}{2\text{Log } \zeta(\alpha)} = 1 - u$$

for a constant  $c$  and where  $s = \alpha + it$  with  $K(\alpha-1)T_0 \leq |t| \leq T_1 = \beta^3$ . We feed this information in Lemma 9.4. The resulting error term is  $\mathcal{O}$  of:

$$\frac{(\alpha-1)^{\lambda(1-\sqrt{1-u})/2}}{\sqrt{1-\lambda}} \text{Log } \beta + \frac{1}{\beta} + \frac{1}{\text{Log } x} + \frac{\beta \text{Log}(x/w_0)}{(x/w_0) \text{Log } x} + K \frac{\text{Log } w_0}{\sqrt{1-\lambda} \text{Log } x}$$

To compare with [5], our way of handling the smoothing yields a  $\text{Log } \beta$  instead of a  $\beta$ . To compare with [15], we would get a  $\text{Log } \beta$  but only for  $\beta \gg \text{Log } x \text{Log } \text{Log } x$ .

We select

$$(35) \quad \beta = \kappa = \frac{2 \text{Log } x}{\text{Log } 2w_0}.$$

We can majorize  $K$  by  $\sqrt{4K^2+1}$  and the latter by  $c_2(\text{Log } x)^{2u}$  for some constant  $c_2$  since

$$2u \text{Log } x + \mathcal{O}(1) = \frac{1}{2} \text{Log}(4K^2+1) - \text{Log}(1 + c(\alpha-1)\sqrt{K^2+1}\text{Log } \beta)$$

and we assume  $K = \mathcal{O}(\text{Log } x / \text{Log } \beta)$ . Indeed  $K$  will be chosen of size about  $(\text{Log } x)^{2u} = \kappa^{1 - \frac{2}{9 + \sqrt{81 - 32\xi}}}$  which is at most  $\kappa^{8/9}$ . Furthermore  $K \gg (\text{Log } x)^{2u} / \text{Log } \kappa$ . We majorize  $w$  by  $w_0$  and solve

$$(\alpha-1)^{(1-\sqrt{1-u})/2} = (\text{Log } x)^{2u} / \kappa.$$

Recall that  $\kappa = (\text{Log } x)^\xi$ . We thus need  $1 - \sqrt{1-u} = 2\xi - 4u$  i.e.  $1 + 4u - \sqrt{1-u} = 2\xi$ . If  $1 + 4u - 2\xi \geq 0$ , it is equivalent to

$$(1 + 4u)^2 + (-4\xi + \frac{1}{4})(1 + 4u) + 4\xi^2 - \frac{5}{4} = 0.$$

This gives

$$(36) \quad u = \frac{16\xi - 9 + \sqrt{81 - 32\xi}}{32}.$$

Next,  $1 + 4u - 2\xi \geq 0$  iff  $\sqrt{81 - 32\xi} \geq 1$  i.e.  $\xi \leq 5/4$  which is always true. We get the upper bound

$$(37) \quad \mathcal{O}((\log x)^{-1} + \beta^{-1} + \kappa^{-(1-2u\xi)\lambda}/\sqrt{1-\lambda})$$

where the exponent reads

$$1 - 2u/\xi = \frac{9 - \sqrt{81 - 32\xi}}{16\xi} = \frac{2}{9 + \sqrt{81 - 32\xi}}.$$

Finally, the choice  $\lambda = 1 - (\log \kappa)^{-1}$  is nearly optimal.  $\diamond \diamond \diamond$

From (33), we deduce

$$\frac{x^{i\tau}}{1+i\tau} \sum_{n \leq x} g(n)n^{-i\tau} = \frac{(x/w)^{i\tau} w^{1+i\tau}}{1+i\tau} \sum_{n \leq x/w} g(n)n^{-i\tau} + \mathcal{O}(x \cdot \varepsilon/(1+|\tau|)).$$

On using Lemma 2.2 for  $g_\tau$  and  $b = \tau$ , we get

$$\sum_{n \leq y} g(n) = \frac{y^{i\tau}}{1+i\tau} \sum_{n \leq y} g(n)n^{-i\tau} + \mathcal{O}(y \cdot (e^{L_y}/\log y)^{1/2} \log^2(2\beta))$$

for all  $y \leq x$  and where  $L_y = \sum_{p \leq y} |1 - g(p)p^{-i\tau}|/p$ . [5] proves that

$$L_y^2 \leq 2 \log \log(3y) ((1 - \sqrt{B}) \log \log x + \mathcal{O}(1))$$

where  $B$  is defined in Lemma 6.5 with  $T_0 = K(\alpha - 1)$  and  $T_1 = \beta^3$ . Recalling (31), we discover that  $B = 1 - u$ . Note that  $1 - \sqrt{1 - u} = 2\xi - 4u \leq 2\xi/8$ , and thus (see (1))

$$L_y^2 \leq \frac{1}{2} \log \log(3y) (\log \kappa + \mathcal{O}(1))$$

and thus

$$\sum_{n \leq y} g(n) = \frac{y^{i\tau}}{1+i\tau} \sum_{n \leq y} g(n)n^{-i\tau} + \mathcal{O}\left(y \cdot \kappa^{\frac{1}{2\sqrt{2}}\sqrt{\eta}(1+\mathcal{O}(1/\log \kappa)) - \frac{1}{2}\eta} \log^2(2\beta)\right)$$

with  $\eta \log \kappa = \log \log 3y$ . We need the exponent of  $\kappa$  to be strictly less than  $-1/8$ . To prove Theorem 1, it is enough to consider the case when  $\kappa$  is large enough. This implies in turn that  $\eta$  can be assumed large enough to ensure that the exponent is indeed  $\leq 1/7$ .

This argument shows also that  $S(g, x)$  and  $S(g, x/w)$  are smaller than the error term in Theorem 1.1 when  $\tau$  is larger than the saving there, namely  $\kappa^{\frac{2}{9+\sqrt{81-32\xi}}} (\log \kappa)^{-3/2}$ .

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