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# Approximate formulae for $L(1, \chi)$, II 

by<br>Olivier Ramaré (Lille)

1. Introduction and results. Upper bounds of $|L(1, \chi)|$ are mainly useful in number theory to study class numbers of algebraic extensions. In [1]-[3] Louboutin establishes bounds for $|L(1, \chi)|$ that take into account the behavior of $\chi$ at small primes. His method uses special representations of $L(1, \chi)$ and does not extend to odd characters. For instance in [2] he uses $L(1, \chi)=2 \sum_{n} \sum_{l<n} \chi(l) /(n(n+1)(n+2))$ which comes from an integration by parts; such a formula fails in the odd case. But the effect of this integration by parts is in fact similar to the introduction of a smoothing, something we did in [5], the only difficulty being to handle properly the Fourier transform of functions behaving like $1 / t$ near $\infty$. This method gives good numerical results in a uniform way.

In this note we improve on the results given in [2] and [3] and extend them to the odd character case. Let us mention that we take this opportunity to correct several typos occurring in [5].

We first state a general formula.

Theorem. Let $\chi$ be a primitive Dirichlet character modulo $q$ and let $h$ be an integer prime to $q$. Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(t)=F(t) / t$ is in $C^{2}(\mathbb{R})$ (also at 0 ), vanishes at $\pm \infty$ and $f^{\prime}$ and $f^{\prime \prime}$ are in $\mathcal{L}^{1}(\mathbb{R})$. Assume also that $F$ is even if $\chi$ is odd, and odd if $\chi$ is even. Then, for every $\delta>0$, we have

$$
\begin{aligned}
\prod_{p \mid h}\left(1-\frac{\chi(p)}{p}\right) L(1, \chi) & =\sum_{\substack{n \geq 1 \\
(n, h)=1}} \chi(n) \frac{1-F(\delta n)}{n} \\
& +\frac{\chi(-h) \tau(\chi)}{q h} \sum_{m \geq 1} c_{h}(m) \bar{\chi}(m) \int_{-\infty}^{\infty} \frac{F(t)}{t} e(m t /(\delta q h)) d t
\end{aligned}
$$

[^0]Here the Gauss sum $\tau(\chi)$ is defined by

$$
\begin{equation*}
\tau(\chi)=\sum_{a \bmod q} \chi(a) e(a / q) \tag{1}
\end{equation*}
$$

and the Ramanujan sums $c_{h}(m)$ by

$$
\begin{equation*}
c_{h}(m)=\sum_{a \bmod ^{*} h} e(m a / q) \tag{2}
\end{equation*}
$$

Of course $e(\cdot)=e^{2 i \pi \cdot}$, and $a \bmod ^{*} h$ denotes summation over all invertible residue classes modulo $h$. We further restrict our attention to square-free $h$.

Here are two interesting choices for $F$ which we take directly from Proposition 2 of [5]. Set

$$
\begin{equation*}
F_{3}(t)=\left(\frac{\sin \pi t}{\pi}\right)^{2}\left(\frac{2}{t}+\sum_{m \in \mathbb{Z}} \frac{\operatorname{sgn}(m)}{(t-m)^{2}}\right) \tag{3}
\end{equation*}
$$

$$
\begin{align*}
j(u)=\int_{-\infty}^{\infty} \frac{F_{3}(t)}{t} e(u t) d t & =\mathbb{1}_{[-1,1]}(u) \int_{|u|}^{1}(\pi(1-t) \cot \pi t+1) d t  \tag{4}\\
F_{4}(t) & =1-\left(\frac{\sin \pi t}{\pi t}\right)^{2}
\end{align*}
$$

which satisfies

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{F_{4}(t)}{t} e(u t) d t=-i \pi(1-|u|)^{2} \mathbb{1}_{[-1,1]}(u) \tag{6}
\end{equation*}
$$

Notice furthermore that $F_{3}$ and $F_{4}$ take their values in $[0,1]$.
In order to compute efficiently the resulting sums we select several levels of hypotheses, starting by the most general ones. We use the Euler $\phi$-function and the number $\omega(t)$ of distinct prime factors of $t$.

Corollary 1. Let $\chi$ be a primitive Dirichlet character modulo $q$ and $h$ an integer prime to $q$. Assume $q$ is divisible by a square-free $k$ and set $\kappa_{\chi}=0$ if $\chi$ is even, and $\kappa_{\chi}=5-2 \log 6=1.41648 \ldots$ if $\chi$ is odd. Then

$$
\left|\prod_{p \mid h}\left(1-\frac{\chi(p)}{p}\right) L(1, \chi)\right|-\frac{\phi(h k)}{2 h k}\left[\log q+2 \sum_{p \mid h k} \frac{\log p}{p-1}+\omega(h) \log 4+\kappa_{\chi}\right]
$$

is bounded from above if $\chi$ is even and $q \geq k^{2} 4^{\omega(h)}$ by

$$
\frac{\phi(h) 2^{\omega(k)-1}}{h \sqrt{q}} \times \begin{cases}\log \left(q 4^{-\omega(h)+1}\right) & \text { if } q \geq k^{2} 4^{\omega(h)} \\ 1.81+\omega(h) \log 4-\log q & \text { if } k=1\end{cases}
$$

and if $\chi$ is odd by

$$
\frac{3 \pi \phi(h k)}{2 h k q} \prod_{p \mid h k} \frac{p^{2}-1}{4 p^{2}}+ \begin{cases}\frac{\pi \phi(h) 2^{\omega(k)}}{2 h \sqrt{q}} & \text { if } k^{2} \max \left(\frac{11}{10} \cdot 4^{\omega(h)}, h^{2} 4^{-\omega(h)+1}\right), \\ 0 & \text { if } k=1 .\end{cases}
$$

This improves on Theorems 1, 4 and 5 of [3] in the quality of the bounds and in their range, and also by the fact that it covers the case of odd characters. For instance in Theorem 5 of [3], where Louboutin studies separately the cases $h=3$ and $k=2$, he gets the upper bound $\frac{1}{6}(\log q+4.83 \ldots+o(1))$ for even characters, while we get $\frac{1}{6}(\log q+3.87 \ldots+3(\log q) / \sqrt{q})$. Recently in [4], by generalizing his method introduced in [2], Louboutin has reached a similar result for the case of even characters, albeit with a slightly larger constant $\kappa_{\chi}=2+\gamma-\log (4 \pi)=0.046 \ldots$ instead of $\kappa_{\chi}=0$. This enabled him to replace $\frac{1}{6}(\log q+4.83 \ldots+o(1))$ by $\frac{1}{6}(\log q+3.91 \ldots)$.

Notice that the upper bound in the case of even characters is non-positive when $k=1$ as soon as $q \geq 6.2 \cdot 4^{\omega(h)}$.

When $h=2$ we can get slightly more precise results:
Corollary 2. Let $\chi$ be a primitive Dirichlet character modulo odd $q$. Then

$$
|(1-\chi(2) / 2) L(1, \chi)| \leq \frac{1}{4}(\log q+\kappa(\chi))
$$

where $\kappa(\chi)=4 \log 2$ if $\chi$ is even, and $\kappa(\chi)=5-2 \log (3 / 2)$ otherwise.
In [2], the value $\kappa(\chi) \simeq 2.818 \ldots$ is proved to hold true for even characters while $4 \log 2=2.772 \ldots$

We introduce the character $\psi$ induced by $\chi$ modulo $q h$. Furthermore ( $m, t$ ) denotes the gcd of $m$ and $t$.

As for the typos in [5], first, Proposition 2 gives a wrong formula for $L(1, \chi)$ if $\chi$ is even: the sign preceding $\tau(\chi)$ should be + and not - . Then Lemma 8 gives a fancy value for $\varrho_{4}$. In fact $\varrho_{4}(t)=-i \pi(1-|t|)^{2} \mathbb{1}_{[-1,1]}(t)$, which is what is proved and used throughout the paper! Finally, in the 6th line of page 264, it is written, "and this last summand is non-negative", while this summand is without any doubt non-positive.

We thank the referee for his careful reading and for improving Lemma 11.
2. Lemmas. We essentially combine Louboutin's proof [2] and ours [5], while generalizing both situations.

First here is a generalization of the new part in Louboutin's paper [2]:
Lemma 1. For every $m$ in $\mathbb{Z}$, we have

$$
\sum_{a \bmod q h} \psi(a) e(a m /(q h))=c_{h}(m) \chi(h) \bar{\chi}(m) \tau(\chi) .
$$

Proof. By the Chinese remainder theorem,

$$
\begin{aligned}
\sum_{a \bmod h q} \psi(a) e(a m /(h q)) & =\sum_{x \bmod h y \bmod q} \sum_{x} \psi(x q+y h) e((x q+y h) m /(h q)) \\
& =\sum_{x \bmod ^{*} h} e(x m / h) \sum_{y \bmod q} \chi(y h) e(y m / q) \\
& =c_{h}(m) \chi(h) \bar{\chi}(m) \tau(\chi),
\end{aligned}
$$

where $c_{h}(m)$ is the Ramanujan sum defined by (2).
Now, Lemma 3 of [5] can be extended to
Lemma 2. The sum $\sum_{n}^{w} f(\delta n) \chi(n)$ exists in the restricted sense given in [5] and

$$
\sum_{n \in \mathbb{Z}}^{w} f(\delta n) \psi(n)=\frac{\chi(-h) \tau(\chi)}{q h} \sum_{m \in \mathbb{Z} \backslash\{0\}} c_{h}(m) \bar{\chi}(m) \int_{-\infty}^{\infty} f(\delta t) e(m t /(q h)) d t
$$

Note: $\int_{-\infty}^{\infty} g(t) e(u t) d t=\lim _{T \rightarrow \infty} \int_{-T}^{T} g(t) e(u t) d t$ for $u \neq 0$.
Now we state and prove lemmas that give approximations of the relevant quantities.

Lemma 3. For $\delta>0$ and $h k \geq 2$ we have

$$
\frac{h k}{\phi(h k)} \sum_{\substack{n \geq 1 \\(n, h \bar{k})=1}} \frac{1-F_{3}(\delta n)}{n}=-\log \delta-1+\sum_{p \mid h k} \frac{\log p}{p-1}
$$

Proof. We have

$$
\begin{aligned}
\sum_{\substack{n \geq 1 \\
(n, h k)=1}} \frac{1-F_{3}(\delta n)}{n} & =\sum_{d \mid h k} \mu(d) \sum_{\substack{n \geq 1 \\
d \mid n}} \frac{1-F_{3}(\delta n)}{n} \\
& =\sum_{d \mid h k} \frac{\mu(d)}{d} \sum_{n \geq 1} \frac{1-F_{3}(d \delta n)}{n} .
\end{aligned}
$$

Lemma 16 of [5] gives the value of the above if $h k=1$, which is $-\log \delta-1+\delta$. This equality is stated only for $\delta \leq 1$ but since only analytic functions are involved, it naturally extends to $\delta>0$. We infer that

$$
\begin{aligned}
\sum_{\substack{n \geq 1 \\
(n, h k)=1}} \frac{1-F_{3}(\delta n)}{n} & =\sum_{d \mid h k} \frac{\mu(d)}{d}(-\log (d \delta)-1+d \delta) \\
& =-\frac{\phi(h k)}{h k} \log \delta-\frac{\phi(h k)}{h k}+\frac{\phi(h k)}{h k} \sum_{p \mid h k} \frac{\log p}{p-1}
\end{aligned}
$$

provided $h k \geq 2$.

Lemma 4. For $\delta u q \geq 1$ we have

$$
\delta u q-2 \log (e \delta u q) \leq \sum_{1 \leq m \leq \delta u q} j(m /(\delta u q)) \leq \delta u q-\log (2 \pi \delta u q / e)
$$

The upper bound is proved between (6.3) and (6.4) in [5]. There also the restriction $\delta \leq 1$ can be dispensed with. The lower bound comes simply from a comparison to an integral since $j$ is non-increasing and since $j(t) \leq$ $-2 \log |t|$ for $t \leq 1$ (shown to be true in Lemma 7 of [5]),

$$
\begin{equation*}
\int_{0}^{r} j(t) d t \leq-2(r \log r-r) \quad(r \in[0,1]) \tag{7}
\end{equation*}
$$

Lemma 5. For $\delta>0$ and $h^{\prime}=h /(2, h)$ we have

$$
\sum_{1 \leq m \leq \delta q} \frac{\phi((m, h))}{\phi(h)} j(m /(\delta h q)) \leq 2^{\omega(h)} \delta q+1-\log (2 \pi \delta q)+\frac{H\left(h^{\prime}\right)}{\phi(h)} \sum_{p \mid h^{\prime}} \frac{\log p}{p-2} .
$$

Proof. Let us introduce the non-negative multiplicative function $H=$ $\mu \star \phi$. We have $H(p)=p-2$. We get

$$
\begin{aligned}
& \sum_{1 \leq m \leq \delta q} \phi((m, h)) j(m /(\delta q))=\sum_{d \mid h} H(d) \sum_{1 \leq m \leq \delta q / d} j(d m /(\delta q)) \\
& \leq \sum_{d \mid h} \frac{h H(d)}{d} \delta q+\phi(h)(1-\log (2 \pi \delta h q))+\sum_{d \mid h} H(d) \log d
\end{aligned}
$$

Now and since $h$ is square-free we see that $\sum_{d \mid h} h H(d) / d=2^{\omega(h)} \phi(h)$.
Lemma 6. For $\delta \geq k / q$ we have

$$
\sum_{\substack{1 \leq m \leq \delta q \\(m, k)=1}} \frac{\phi((m, h))}{\phi(h)} j(m /(\delta h q)) \leq 2^{\omega(h)} \frac{\phi(k)}{k} \delta q+2^{\omega(k)} \log (e \delta q / 2)
$$

Proof. Following the proof of Lemma 5, our sum equals

$$
\begin{aligned}
\sum_{d \mid h} H(d) \sum_{l \mid k} \mu(l) & \sum_{1 \leq m \leq \delta q /(d l)} j(d l m /(\delta h q)) \\
& \leq \delta q 2^{\omega(h)} \phi(h) \frac{\phi(k)}{k}+\sum_{d \mid h} H(d) \sum_{\substack{l \mid k \\
\mu(l)=-1}} 2 \log (e \delta q /(d l)) \\
& \leq \delta q 2^{\omega(h)} \phi(h) \frac{\phi(k)}{k}+\phi(h) 2^{\omega(k)} \log (e \delta q / 2)
\end{aligned}
$$

provided that $\delta q / k \geq 1$.

Lemma 7. For $\delta>0$ and $h k \geq 2$ we have

$$
\begin{aligned}
\frac{h k}{\phi(h k)} \sum_{\substack{n \geq 1 \\
(n, h k)=1}} \frac{1-F_{4}(\delta n)}{n}= & \log \delta+\frac{3}{2}-\log (2 \pi)+\sum_{p \mid h k} \frac{\log p}{p-1} \\
& +\frac{2 \phi(h k)}{h k} \sum_{d \mid h k} \mu(d) \int_{0}^{1}(1-t) \log \left|\frac{\pi d \delta t}{\sin (\pi d \delta t)}\right| \frac{d t}{d}
\end{aligned}
$$

When $h k=2$ the last summand is non-positive, and in general if $\delta \leq$ $1 /(2 h k)$, it is not more than $\frac{\pi^{3}}{6} \delta^{2} \prod_{p \mid h k}\left(p^{2}-1\right) / p^{2}$.

Proof. Lemma 17 of [5] gives us

$$
\sum_{n \geq 1} \frac{1-F_{4}(\delta n)}{n}=-\log \delta+\frac{3}{2}-\log (2 \pi)+2 \int_{0}^{1}(1-t) \log \left|\frac{\pi \delta t}{\sin (\pi \delta t)}\right| d t
$$

and we use the same technique as in the previous lemma. The error term is non-positive if $h k=2$ as shown in [5] between (7.2) and (7.3). Furthermore the integral is shown there (in Lemma 18) to be not more than $\pi^{3} \delta^{2} / 12$ as soon as $\delta \leq 1 / 2$.

A simple comparison to an integral yields:
Lemma 8. For $\delta u q \geq 1$ we have

$$
\frac{\delta u q}{3}-1 \leq \sum_{1 \leq m \leq \delta u q}\left(1-\frac{m}{\delta u q}\right)^{2} \leq \frac{\delta u q}{3}
$$

Lemma 9. For $\delta \geq k / q$ we have

$$
\sum_{\substack{1 \leq m \leq \delta h q \\(m, k)=1}} \frac{\phi((m, h))}{\phi(h)}\left(1-\frac{m}{\delta h q}\right)^{2} \leq \frac{\phi(k)}{k} \frac{\delta q}{3} 2^{\omega(h)}+2^{\omega(k)-1}
$$

where the last summand can be omitted if $k=1$.
Proof. We proceed as in Lemma 6 to deduce that our sum is

$$
\sum_{d \mid h} H(d) \sum_{l \mid k} \mu(l) \sum_{1 \leq m \leq \delta q /(d l)}\left(1-\frac{d l m}{\delta q}\right)^{2}
$$

and the conclusion follows readily.
From $[6,(3.22),(2.11)$ and (3.26)], we get

Lemma 10. We have

$$
\begin{aligned}
& \sum_{1<p \leq X} \frac{\log p}{p} \leq \log X-1.332+\frac{1}{2 \log X} \quad(X \geq 319), \\
& \prod_{2<p \leq X} \frac{p-1}{p} \leq \frac{2 e^{-\gamma}}{\log X}\left(1+\frac{1}{2 \log ^{2} X}\right) \quad(X>1)
\end{aligned}
$$

where $\gamma$ is Euler's constant.
Lemma 11. For $h>1$, we have

$$
\prod_{2<p \mid h} \frac{p-2}{p-1} \sum_{2<p \mid h} \frac{\log p}{p-2} \leq 0.7414
$$

Proof. First writing $h=h_{1} p_{1}$ where $p_{1}$ is a prime factor, the reader readily checks that our quantity is a non-increasing function of $p_{1}$. We thus find that its maximum is obtained when $h=\prod_{2<p \leq X} p$. As a function of $X$, it numerically seems increasing and GP/PARI needs at most 10 seconds to prove it is $\leq 0.72$ if the product is taken over primes $\leq 10^{6}$. Using Lemma 10, we get

$$
\begin{aligned}
S(X) & =\sum_{2<p \leq X} \frac{\log p}{p-2}=\sum_{2<p \leq X} \frac{2 \log p}{p(p-2)}+\sum_{1<p \leq X} \frac{\log p}{p}-\frac{\log 2}{2} \\
& \leq 1.27+\log X-1.332+\frac{1}{2 \log X}-0.346 \\
& \leq \log X-0.4
\end{aligned}
$$

for $X \geq 10^{6}$. Furthermore, still invoking Lemma 10 , we have

$$
\begin{aligned}
\Pi(X) & =\prod_{2<p \leq X} \frac{p-2}{p-1} \\
& \leq \prod_{2<p \leq X}\left(1-\frac{1}{(p-1)^{2}}\right) \prod_{2<p \leq X} \frac{p-1}{p} \\
& \leq \prod_{2<p \leq 10^{6}}\left(1-\frac{1}{(p-1)^{2}}\right) \frac{2 e^{-\gamma}}{\log X}\left(1+\frac{1}{2 \log ^{2} X}\right)
\end{aligned}
$$

also for $X \geq 10^{6}$. Since $(1-0.4 y)\left(1+0.5 y^{2}\right) \leq 1$ if $0 \leq y \leq 0.4$, our function is not more than

$$
\begin{equation*}
2 e^{-\gamma} \prod_{2<p \leq 10^{6}}\left(1-\frac{1}{(p-1)^{2}}\right) \leq 0.7414 . \tag{8}
\end{equation*}
$$

3. Proof of the Theorem. Let us start with

$$
\begin{equation*}
L(1, \psi)=\sum_{n \geq 1} \psi(n) \frac{1-F(\delta n)}{n}+\sum_{n \geq 1} \psi(n) \frac{F(\delta n)}{n} \tag{9}
\end{equation*}
$$

Thanks to the hypothesis concerning the respective parities of $F$ and $\chi$, we get

$$
\begin{equation*}
\sum_{n \geq 1} \psi(n) \frac{F(\delta n)}{n}=\frac{1}{2} \sum_{n \in \mathbb{Z}} \psi(n) \delta f(\delta n) \tag{10}
\end{equation*}
$$

to which we apply Lemma 2, and the Theorem follows readily.
4. Proofs of the corollaries. For even characters we take $F=F_{3}$. Combining the Theorem with Lemmas 3 and 6 , and noticing that $\left|c_{h}(m)\right| \leq$ $\phi((h, m))$, we get

$$
\begin{align*}
& \left|\prod_{p \mid h}\left(1-\frac{\chi(p)}{p}\right) L(1, \chi)\right| \frac{h k}{\phi(h k)}  \tag{11}\\
& \quad \leq-\log \delta-1+\sum_{p \mid h k} \frac{\log p}{p-1}+\frac{1}{\sqrt{q}}\left(2^{\omega(h)} \delta q+\frac{k 2^{\omega(k)}}{\phi(k)} \log (e \delta q / 2)\right)
\end{align*}
$$

provided $\delta \geq k / q$. We simply have to choose $\delta=1 /\left(2^{\omega(h)} \sqrt{q}\right)$ and the claimed formula follows readily.

For odd characters we use $F=F_{4}$ and Lemmas 7 and 9 to get

$$
\begin{align*}
& \left|\prod_{p \mid h}\left(1-\frac{\chi(p)}{p}\right) L(1, \chi)\right| \frac{h k}{\phi(h k)} \leq-\log \delta+\frac{3}{2}-\log (2 \pi)  \tag{12}\\
& \quad+\sum_{p \mid h k} \frac{\log p}{p-1}+\frac{\pi^{3}}{6} \delta^{2} \prod_{p \mid h k} \frac{p^{2}-1^{2}}{p}+\frac{\pi}{\sqrt{q}}\left(\frac{\delta 2^{\omega(h)} q}{3}+2^{\omega(k)-1} \frac{k}{\phi(k)}\right)
\end{align*}
$$

provided $\delta \in[k / q, 1 /(2 h k)]$. We take $\delta=3 /\left(2^{\omega(h)} \pi \sqrt{q}\right)$ and the claimed formula follows readily.

To prove the second corollary (i.e. with $k=1$ ), we simply adapt the above proof, but we can simplify the bound in the even case. We first obtain

$$
\begin{equation*}
\frac{1}{\sqrt{q}}\left(1-\log \left((2 \pi / e) \sqrt{q} 2^{-\omega(h)}\right)+\prod_{2<p \mid h} \frac{p-2}{p-1} \sum_{2<p \mid h} \frac{\log p}{p-2}\right) \tag{13}
\end{equation*}
$$

The last factor is bounded in Lemma 11 by 0.7414 , so the above term is not more than $(1.81+\omega(h) \log 4-\log q) /(2 \sqrt{q})$ as announced.

When $h=2$, the claimed upper bounds are proved if $q \geq 39$, in part because the term in $\delta^{2}$ appearing in (12) disappears by Lemma 7. We complete the verification by appealing to GP/PARI as indicated in [5]. The maximum of $\kappa(\chi)$ for even characters of module $\leq 1000$ is $\leq 1.705$, attained
for $q=109$, while the maximum of $\kappa(\chi)$ for odd characters of module $\leq 1000$ is $\leq 3.360$, attained for $q=131$.

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