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Approximate formulae for $L(1, \chi)$, II

by

OLIVIER RAMARÉ (Lille)

1. Introduction and results. Upper bounds of $|L(1, \chi)|$ are mainly useful in number theory to study class numbers of algebraic extensions. In [1]–[3] Louboutin establishes bounds for $|L(1, \chi)|$ that take into account the behavior of χ at small primes. His method uses special representations of $L(1, \chi)$ and does not extend to odd characters. For instance in [2] he uses $L(1, \chi) = 2 \sum_n \sum_{l \leq n} \chi(l)/(n(n+1)(n+2))$ which comes from an integration by parts; such a formula fails in the odd case. But the effect of this integration by parts is in fact similar to the introduction of a smoothing, something we did in [5], the only difficulty being to handle properly the Fourier transform of functions behaving like $1/t$ near ∞ . This method gives good numerical results in a uniform way.

In this note we improve on the results given in [2] and [3] and extend them to the odd character case. Let us mention that we take this opportunity to correct several typos occurring in [5].

We first state a general formula.

THEOREM. *Let χ be a primitive Dirichlet character modulo q and let h be an integer prime to q . Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(t) = F(t)/t$ is in $C^2(\mathbb{R})$ (also at 0), vanishes at $\pm\infty$ and f' and f'' are in $\mathcal{L}^1(\mathbb{R})$. Assume also that F is even if χ is odd, and odd if χ is even. Then, for every $\delta > 0$, we have*

$$\prod_{p|h} \left(1 - \frac{\chi(p)}{p}\right) L(1, \chi) = \sum_{\substack{n \geq 1 \\ (n, h) = 1}} \chi(n) \frac{1 - F(\delta n)}{n} \\ + \frac{\chi(-h)\tau(\chi)}{qh} \sum_{m \geq 1} c_h(m) \bar{\chi}(m) \int_{-\infty}^{\infty} \frac{F(t)}{t} e(mt/(\delta qh)) dt.$$

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Here the Gauss sum $\tau(\chi)$ is defined by

$$(1) \quad \tau(\chi) = \sum_{a \bmod q} \chi(a)e(a/q)$$

and the Ramanujan sums $c_h(m)$ by

$$(2) \quad c_h(m) = \sum_{a \bmod^* h} e(ma/q).$$

Of course $e(\cdot) = e^{2i\pi\cdot}$, and $a \bmod^* h$ denotes summation over all invertible residue classes modulo h . We further restrict our attention to square-free h .

Here are two interesting choices for F which we take directly from Proposition 2 of [5]. Set

$$(3) \quad F_3(t) = \left(\frac{\sin \pi t}{\pi}\right)^2 \left(\frac{2}{t} + \sum_{m \in \mathbb{Z}} \frac{\operatorname{sgn}(m)}{(t-m)^2}\right),$$

$$(4) \quad j(u) = \int_{-\infty}^{\infty} \frac{F_3(t)}{t} e(ut) dt = \mathbb{1}_{[-1,1]}(u) \int_{|u|}^1 (\pi(1-t) \cot \pi t + 1) dt,$$

$$(5) \quad F_4(t) = 1 - \left(\frac{\sin \pi t}{\pi t}\right)^2$$

which satisfies

$$(6) \quad \int_{-\infty}^{\infty} \frac{F_4(t)}{t} e(ut) dt = -i\pi(1-|u|)^2 \mathbb{1}_{[-1,1]}(u).$$

Notice furthermore that F_3 and F_4 take their values in $[0, 1]$.

In order to compute efficiently the resulting sums we select several levels of hypotheses, starting by the most general ones. We use the Euler ϕ -function and the number $\omega(t)$ of distinct prime factors of t .

COROLLARY 1. *Let χ be a primitive Dirichlet character modulo q and h an integer prime to q . Assume q is divisible by a square-free k and set $\kappa_\chi = 0$ if χ is even, and $\kappa_\chi = 5 - 2 \log 6 = 1.41648\dots$ if χ is odd. Then*

$$\left| \prod_{p|h} \left(1 - \frac{\chi(p)}{p}\right) L(1, \chi) \right| - \frac{\phi(hk)}{2hk} \left[\log q + 2 \sum_{p|hk} \frac{\log p}{p-1} + \omega(h) \log 4 + \kappa_\chi \right]$$

is bounded from above if χ is even and $q \geq k^2 4^{\omega(h)}$ by

$$\frac{\phi(h) 2^{\omega(k)-1}}{h\sqrt{q}} \times \begin{cases} \log(q 4^{-\omega(h)+1}) & \text{if } q \geq k^2 4^{\omega(h)}, \\ 1.81 + \omega(h) \log 4 - \log q & \text{if } k = 1, \end{cases}$$

and if χ is odd by

$$\frac{3\pi\phi(hk)}{2hkg} \prod_{p|hk} \frac{p^2-1}{4p^2} + \begin{cases} \frac{\pi\phi(h)2^{\omega(k)}}{2h\sqrt{q}} & \text{if } k^2 \max\left(\frac{11}{10} \cdot 4^{\omega(h)}, h^2 4^{-\omega(h)+1}\right), \\ 0 & \text{if } k = 1. \end{cases}$$

This improves on Theorems 1, 4 and 5 of [3] in the quality of the bounds and in their range, and also by the fact that it covers the case of odd characters. For instance in Theorem 5 of [3], where Louboutin studies separately the cases $h = 3$ and $k = 2$, he gets the upper bound $\frac{1}{6}(\log q + 4.83\dots + o(1))$ for even characters, while we get $\frac{1}{6}(\log q + 3.87\dots + 3(\log q)/\sqrt{q})$. Recently in [4], by generalizing his method introduced in [2], Louboutin has reached a similar result for the case of even characters, albeit with a slightly larger constant $\kappa_\chi = 2 + \gamma - \log(4\pi) = 0.046\dots$ instead of $\kappa_\chi = 0$. This enabled him to replace $\frac{1}{6}(\log q + 4.83\dots + o(1))$ by $\frac{1}{6}(\log q + 3.91\dots)$.

Notice that the upper bound in the case of even characters is non-positive when $k = 1$ as soon as $q \geq 6.2 \cdot 4^{\omega(h)}$.

When $h = 2$ we can get slightly more precise results:

COROLLARY 2. *Let χ be a primitive Dirichlet character modulo odd q . Then*

$$|(1 - \chi(2)/2)L(1, \chi)| \leq \frac{1}{4}(\log q + \kappa(\chi))$$

where $\kappa(\chi) = 4 \log 2$ if χ is even, and $\kappa(\chi) = 5 - 2 \log(3/2)$ otherwise.

In [2], the value $\kappa(\chi) \simeq 2.818\dots$ is proved to hold true for even characters while $4 \log 2 = 2.772\dots$

We introduce the character ψ induced by χ modulo qh . Furthermore (m, t) denotes the gcd of m and t .

As for the typos in [5], first, Proposition 2 gives a wrong formula for $L(1, \chi)$ if χ is even: the sign preceding $\tau(\chi)$ should be $+$ and not $-$. Then Lemma 8 gives a fancy value for ϱ_4 . In fact $\varrho_4(t) = -i\pi(1 - |t|)^2 \mathbf{1}_{[-1,1]}(t)$, which is what is proved and used throughout the paper! Finally, in the 6th line of page 264, it is written, “and this last summand is non-negative”, while this summand is without any doubt non-positive.

We thank the referee for his careful reading and for improving Lemma 11.

2. Lemmas. We essentially combine Louboutin’s proof [2] and ours [5], while generalizing both situations.

First here is a generalization of the new part in Louboutin’s paper [2]:

LEMMA 1. *For every m in \mathbb{Z} , we have*

$$\sum_{a \bmod qh} \psi(a)e(am/(qh)) = c_h(m)\chi(h)\bar{\chi}(m)\tau(\chi).$$

Proof. By the Chinese remainder theorem,

$$\begin{aligned} \sum_{a \bmod hq} \psi(a)e(am/(hq)) &= \sum_{x \bmod h} \sum_{y \bmod q} \psi(xq + yh)e((xq + yh)m/(hq)) \\ &= \sum_{x \bmod^* h} e(xm/h) \sum_{y \bmod q} \chi(yh)e(ym/q) \\ &= c_h(m)\chi(h)\bar{\chi}(m)\tau(\chi), \end{aligned}$$

where $c_h(m)$ is the Ramanujan sum defined by (2).

Now, Lemma 3 of [5] can be extended to

LEMMA 2. *The sum $\sum_n^w f(\delta n)\chi(n)$ exists in the restricted sense given in [5] and*

$$\sum_{n \in \mathbb{Z}}^w f(\delta n)\psi(n) = \frac{\chi(-h)\tau(\chi)}{qh} \sum_{m \in \mathbb{Z} \setminus \{0\}} c_h(m)\bar{\chi}(m) \int_{-\infty}^{\infty} f(\delta t)e(mt/(qh)) dt.$$

Note: $\int_{-\infty}^{\infty} g(t)e(ut) dt = \lim_{T \rightarrow \infty} \int_{-T}^T g(t)e(ut) dt$ for $u \neq 0$.

Now we state and prove lemmas that give approximations of the relevant quantities.

LEMMA 3. *For $\delta > 0$ and $hk \geq 2$ we have*

$$\frac{hk}{\phi(hk)} \sum_{\substack{n \geq 1 \\ (n, hk) = 1}} \frac{1 - F_3(\delta n)}{n} = -\log \delta - 1 + \sum_{p|hk} \frac{\log p}{p-1}.$$

Proof. We have

$$\begin{aligned} \sum_{\substack{n \geq 1 \\ (n, hk) = 1}} \frac{1 - F_3(\delta n)}{n} &= \sum_{d|hk} \mu(d) \sum_{\substack{n \geq 1 \\ d|n}} \frac{1 - F_3(\delta n)}{n} \\ &= \sum_{d|hk} \frac{\mu(d)}{d} \sum_{n \geq 1} \frac{1 - F_3(d\delta n)}{n}. \end{aligned}$$

Lemma 16 of [5] gives the value of the above if $hk = 1$, which is $-\log \delta - 1 + \delta$. This equality is stated only for $\delta \leq 1$ but since only analytic functions are involved, it naturally extends to $\delta > 0$. We infer that

$$\begin{aligned} \sum_{\substack{n \geq 1 \\ (n, hk) = 1}} \frac{1 - F_3(\delta n)}{n} &= \sum_{d|hk} \frac{\mu(d)}{d} (-\log(d\delta) - 1 + d\delta) \\ &= -\frac{\phi(hk)}{hk} \log \delta - \frac{\phi(hk)}{hk} + \frac{\phi(hk)}{hk} \sum_{p|hk} \frac{\log p}{p-1} \end{aligned}$$

provided $hk \geq 2$.

LEMMA 4. For $\delta uq \geq 1$ we have

$$\delta uq - 2 \log(e\delta uq) \leq \sum_{1 \leq m \leq \delta uq} j(m/(\delta uq)) \leq \delta uq - \log(2\pi\delta uq/e).$$

The upper bound is proved between (6.3) and (6.4) in [5]. There also the restriction $\delta \leq 1$ can be dispensed with. The lower bound comes simply from a comparison to an integral since j is non-increasing and since $j(t) \leq -2 \log |t|$ for $t \leq 1$ (shown to be true in Lemma 7 of [5]),

$$(7) \quad \int_0^r j(t) dt \leq -2(r \log r - r) \quad (r \in [0, 1]).$$

LEMMA 5. For $\delta > 0$ and $h' = h/(2, h)$ we have

$$\sum_{1 \leq m \leq \delta q} \frac{\phi((m, h))}{\phi(h)} j(m/(\delta hq)) \leq 2^{\omega(h)} \delta q + 1 - \log(2\pi\delta q) + \frac{H(h')}{\phi(h)} \sum_{p|h'} \frac{\log p}{p-2}.$$

Proof. Let us introduce the non-negative multiplicative function $H = \mu \star \phi$. We have $H(p) = p - 2$. We get

$$\begin{aligned} \sum_{1 \leq m \leq \delta q} \phi((m, h)) j(m/(\delta q)) &= \sum_{d|h} H(d) \sum_{1 \leq m \leq \delta q/d} j(dm/(\delta q)) \\ &\leq \sum_{d|h} \frac{hH(d)}{d} \delta q + \phi(h)(1 - \log(2\pi\delta hq)) + \sum_{d|h} H(d) \log d. \end{aligned}$$

Now and since h is square-free we see that $\sum_{d|h} hH(d)/d = 2^{\omega(h)} \phi(h)$.

LEMMA 6. For $\delta \geq k/q$ we have

$$\sum_{\substack{1 \leq m \leq \delta q \\ (m, k)=1}} \frac{\phi((m, h))}{\phi(h)} j(m/(\delta hq)) \leq 2^{\omega(h)} \frac{\phi(k)}{k} \delta q + 2^{\omega(k)} \log(e\delta q/2).$$

Proof. Following the proof of Lemma 5, our sum equals

$$\begin{aligned} \sum_{d|h} H(d) \sum_{l|k} \mu(l) \sum_{1 \leq m \leq \delta q/(dl)} j(dlm/(\delta hq)) \\ \leq \delta q 2^{\omega(h)} \phi(h) \frac{\phi(k)}{k} + \sum_{d|h} H(d) \sum_{\substack{l|k \\ \mu(l)=-1}} 2 \log(e\delta q/(dl)) \\ \leq \delta q 2^{\omega(h)} \phi(h) \frac{\phi(k)}{k} + \phi(h) 2^{\omega(k)} \log(e\delta q/2) \end{aligned}$$

provided that $\delta q/k \geq 1$.

LEMMA 7. For $\delta > 0$ and $hk \geq 2$ we have

$$\begin{aligned} \frac{hk}{\phi(hk)} \sum_{\substack{n \geq 1 \\ (n, hk) = 1}} \frac{1 - F_4(\delta n)}{n} &= \log \delta + \frac{3}{2} - \log(2\pi) + \sum_{p|hk} \frac{\log p}{p-1} \\ &+ \frac{2\phi(hk)}{hk} \sum_{d|hk} \mu(d) \int_0^1 (1-t) \log \left| \frac{\pi d \delta t}{\sin(\pi d \delta t)} \right| \frac{dt}{d}. \end{aligned}$$

When $hk = 2$ the last summand is non-positive, and in general if $\delta \leq 1/(2hk)$, it is not more than $\frac{\pi^3}{6} \delta^2 \prod_{p|hk} (p^2 - 1)/p^2$.

Proof. Lemma 17 of [5] gives us

$$\sum_{n \geq 1} \frac{1 - F_4(\delta n)}{n} = -\log \delta + \frac{3}{2} - \log(2\pi) + 2 \int_0^1 (1-t) \log \left| \frac{\pi \delta t}{\sin(\pi \delta t)} \right| dt$$

and we use the same technique as in the previous lemma. The error term is non-positive if $hk = 2$ as shown in [5] between (7.2) and (7.3). Furthermore the integral is shown there (in Lemma 18) to be not more than $\pi^3 \delta^2 / 12$ as soon as $\delta \leq 1/2$.

A simple comparison to an integral yields:

LEMMA 8. For $\delta uq \geq 1$ we have

$$\frac{\delta uq}{3} - 1 \leq \sum_{1 \leq m \leq \delta uq} \left(1 - \frac{m}{\delta uq}\right)^2 \leq \frac{\delta uq}{3}.$$

LEMMA 9. For $\delta \geq k/q$ we have

$$\sum_{\substack{1 \leq m \leq \delta hq \\ (m, k) = 1}} \frac{\phi((m, h))}{\phi(h)} \left(1 - \frac{m}{\delta hq}\right)^2 \leq \frac{\phi(k)}{k} \frac{\delta q}{3} 2^{\omega(h)} + 2^{\omega(k)-1}$$

where the last summand can be omitted if $k = 1$.

Proof. We proceed as in Lemma 6 to deduce that our sum is

$$\sum_{d|h} H(d) \sum_{l|k} \mu(l) \sum_{1 \leq m \leq \delta q/(dl)} \left(1 - \frac{dlm}{\delta q}\right)^2$$

and the conclusion follows readily.

From [6, (3.22), (2.11) and (3.26)], we get

LEMMA 10. *We have*

$$\sum_{1 < p \leq X} \frac{\log p}{p} \leq \log X - 1.332 + \frac{1}{2 \log X} \quad (X \geq 319),$$

$$\prod_{2 < p \leq X} \frac{p-1}{p} \leq \frac{2e^{-\gamma}}{\log X} \left(1 + \frac{1}{2 \log^2 X}\right) \quad (X > 1),$$

where γ is Euler's constant.

LEMMA 11. *For $h > 1$, we have*

$$\prod_{2 < p|h} \frac{p-2}{p-1} \sum_{2 < p|h} \frac{\log p}{p-2} \leq 0.7414.$$

Proof. First writing $h = h_1 p_1$ where p_1 is a prime factor, the reader readily checks that our quantity is a non-increasing function of p_1 . We thus find that its maximum is obtained when $h = \prod_{2 < p \leq X} p$. As a function of X , it numerically seems increasing and GP/PARI needs at most 10 seconds to prove it is ≤ 0.72 if the product is taken over primes $\leq 10^6$. Using Lemma 10, we get

$$\begin{aligned} S(X) &= \sum_{2 < p \leq X} \frac{\log p}{p-2} = \sum_{2 < p \leq X} \frac{2 \log p}{p(p-2)} + \sum_{1 < p \leq X} \frac{\log p}{p} - \frac{\log 2}{2} \\ &\leq 1.27 + \log X - 1.332 + \frac{1}{2 \log X} - 0.346 \\ &\leq \log X - 0.4 \end{aligned}$$

for $X \geq 10^6$. Furthermore, still invoking Lemma 10, we have

$$\begin{aligned} \Pi(X) &= \prod_{2 < p \leq X} \frac{p-2}{p-1} \\ &\leq \prod_{2 < p \leq X} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{2 < p \leq X} \frac{p-1}{p} \\ &\leq \prod_{2 < p \leq 10^6} \left(1 - \frac{1}{(p-1)^2}\right) \frac{2e^{-\gamma}}{\log X} \left(1 + \frac{1}{2 \log^2 X}\right) \end{aligned}$$

also for $X \geq 10^6$. Since $(1 - 0.4y)(1 + 0.5y^2) \leq 1$ if $0 \leq y \leq 0.4$, our function is not more than

$$(8) \quad 2e^{-\gamma} \prod_{2 < p \leq 10^6} \left(1 - \frac{1}{(p-1)^2}\right) \leq 0.7414.$$

3. Proof of the Theorem. Let us start with

$$(9) \quad L(1, \psi) = \sum_{n \geq 1} \psi(n) \frac{1 - F(\delta n)}{n} + \sum_{n \geq 1} \psi(n) \frac{F(\delta n)}{n}.$$

Thanks to the hypothesis concerning the respective parities of F and χ , we get

$$(10) \quad \sum_{n \geq 1} \psi(n) \frac{F(\delta n)}{n} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \psi(n) \delta f(\delta n),$$

to which we apply Lemma 2, and the Theorem follows readily.

4. Proofs of the corollaries. For even characters we take $F = F_3$. Combining the Theorem with Lemmas 3 and 6, and noticing that $|c_h(m)| \leq \phi((h, m))$, we get

$$(11) \quad \left| \prod_{p|h} \left(1 - \frac{\chi(p)}{p}\right) L(1, \chi) \right| \frac{hk}{\phi(hk)} \\ \leq -\log \delta - 1 + \sum_{p|hk} \frac{\log p}{p-1} + \frac{1}{\sqrt{q}} \left(2^{\omega(h)} \delta q + \frac{k 2^{\omega(k)}}{\phi(k)} \log(e\delta q/2) \right)$$

provided $\delta \geq k/q$. We simply have to choose $\delta = 1/(2^{\omega(h)} \sqrt{q})$ and the claimed formula follows readily.

For odd characters we use $F = F_4$ and Lemmas 7 and 9 to get

$$(12) \quad \left| \prod_{p|h} \left(1 - \frac{\chi(p)}{p}\right) L(1, \chi) \right| \frac{hk}{\phi(hk)} \leq -\log \delta + \frac{3}{2} - \log(2\pi) \\ + \sum_{p|hk} \frac{\log p}{p-1} + \frac{\pi^3}{6} \delta^2 \prod_{p|hk} \frac{p^2 - 1}{p} + \frac{\pi}{\sqrt{q}} \left(\frac{\delta 2^{\omega(h)} q}{3} + 2^{\omega(k)-1} \frac{k}{\phi(k)} \right)$$

provided $\delta \in [k/q, 1/(2hk)]$. We take $\delta = 3/(2^{\omega(h)} \pi \sqrt{q})$ and the claimed formula follows readily.

To prove the second corollary (i.e. with $k = 1$), we simply adapt the above proof, but we can simplify the bound in the even case. We first obtain

$$(13) \quad \frac{1}{\sqrt{q}} \left(1 - \log((2\pi/e)\sqrt{q} 2^{-\omega(h)}) + \prod_{2 < p|h} \frac{p-2}{p-1} \sum_{2 < p|h} \frac{\log p}{p-2} \right).$$

The last factor is bounded in Lemma 11 by 0.7414, so the above term is not more than $(1.81 + \omega(h) \log 4 - \log q)/(2\sqrt{q})$ as announced.

When $h = 2$, the claimed upper bounds are proved if $q \geq 39$, in part because the term in δ^2 appearing in (12) disappears by Lemma 7. We complete the verification by appealing to GP/PARI as indicated in [5]. The maximum of $\kappa(\chi)$ for even characters of module ≤ 1000 is ≤ 1.705 , attained

for $q = 109$, while the maximum of $\kappa(\chi)$ for odd characters of module ≤ 1000 is ≤ 3.360 , attained for $q = 131$.

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