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Abstract

Most of scientists describe gravitation with two theories depending on the desired accuracy: Newton's law of gravitation and general relativity. In previous work, we found two formulas that could give the Planck constant and the speed of light according to fundamental constants and vacuum properties. If they are not coincidences, they suggest that gravitation stems from a physical deformation process. We propose here the mathematical description of this theory. Based on four-dimensional continuum mechanics, this approach yields a similar law to Newton's law of gravitation. However, this theory would imply that the mathematical expression of gravitation might be different for high masses.

Keywords: Gravitation, Newton's law, deformation, four-dimensional

1 Introduction

Many theories of gravitation have been built throughout history, from the ancient Greek period to present. Among them, Newton's law of gravitation marked a turning point in this physics field. In *Philosophiae Naturalis Principia Mathematica*, he assumed that the gravitational forces experienced by planets were reciprocally proportional to the squares of their distances. The proportional constant was first measured by Henry Cavendish. In 1915, Newton's theory was then superseded by general relativity. Einstein's theory met with strong success: the perihelion precession of Mercury [1], [2], [3], the bending of light by the Sun [4], [5], [6], gravitational redshift [7], gravitational waves [8], [9], *etc.*

In 1913, V.M. Slipher observed the universe's expansion [10]. This will be confirmed by Hubble in 1929 [11]. In 1998, two independent teams discovered that this expansion is accelerating [12], [13]. Scientists explain this phenomenon with dark energy [14] but its physical nature remains unknown. Similarly, Newton's law of gravitation and general relativity requires the existence of dark matter [15]. This unknown matter would explain the galaxy rotation curves [16]. Different investigations are being carried out to observe this putative matter.

In a previous paper [17], we found two relations which seem to give the speed of light and the Planck constant. We then developed a novel theory under the assumption that both formulas were not coincidences. Gravitation would stem from the accelerating expansion of the universe and the deformation of four-dimensional matter. The theoretical framework is summarized in the first section of the present article. This section also discusses about the continuum mechanics theory to choose to describe the physical deformation of the hypothetical four-dimensional material. The second section addresses the mathematical part of the gravitation theory. In particular, the resolution of the equations agrees with Newton's law of gravitation. Finally, we touch upon the dark universe issue in the framework of the theory.

2 Theory

2.1 Framework

This study relies on the previous paper [17] which presents the following relation

$$\sqrt{10\frac{(\varepsilon_0 e^{-2})^3(k_B T)^4}{\rho_c}} \sim 3\times 10^8 \ m.s^{-1} \tag{2.1}$$

where ε_0 is the vacuum permittivity, e is the elementary charge, k_B is the Boltzmann constant, T is the temperature of the CMB (Cosmic Microwave Background) and ρ_c is the critical density of the universe. The consequences of this relation are discussed in [17] and outlined here. This formula, which is homogeneous to $\sqrt{\frac{P}{\rho}}$ (usual relation for the speed of material waves), seems to indicate that light and gravitational waves propagate in matter. The low density of vacuum drove us to reject a simple three-dimensional propagation. Instead, we pointed out in [17] that additional spatial dimensions did not affect relation (2.1) while being able to increase the value of the multidimensional density. Being consistent with Einstein's feeling, we assumed that our living world was a 3-sphere. In the framework of the theory, the universe would thus be a 4-ball containing matter and we would live at its surface. In adhering to the strong equivalence principle, gravitation would correspond to acceleration. Since the only acceleration that affects the whole universe is the acceleration of the expansion, we assumed that gravitation derives from this acceleration. In that respect, we proposed another definition of gravitation. Due to the accelerating expansion, heavy bodies in the 3-sphere would be maintained against four-dimensional matter and deform it. A light body in the vicinity of a heavy one would then experience the acceleration of the expansion on a deformed three-dimensional surface. It would therefore be "attracted" by the heavy body from the three-dimensional point of view. The deformation of spacetime in general relativity could then be linked to a tangible deformation of matter. Fig. 1 presents this hypothetical behaviour.

2.2 The deformation profile

This section handles the deformation profile associated with Newton's law of universal gravitation (Fig. 1b). We note f the surface equation. The z-coordinate of the light body is $z = f(r,\theta,\phi)$. Due to the spherical symmetry, the equation is independent on θ (polar angle) and ϕ (azimuthal angle) so $z = f(r)$. According to Newton, the acceleration experienced by a body in the gravitational field of a heavy body is $a = \frac{GM}{r^2}$ outside it. Within it, under the assumption that the high-mass body is homogenous, $a = \frac{GMr}{R^3}$ (R is the high-mass body radius). With the notations in Fig. 1b, $tan\ \Psi = \frac{\mathrm{d}z}{\mathrm{d}r}$ and $sin\ \Psi = \frac{a}{g}$.
Both relations yield

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \begin{cases} \tan\left(\arcsin\left(\frac{GMr}{gR^3}\right)\right) & \text{if } r \le R \\ \tan\left(\arcsin\left(\frac{GM}{gr^2}\right)\right) & \text{if } r \ge R \end{cases} \tag{2.2}$$

By assuming that $\frac{GM}{g} \ll 1$,

$$\frac{\mathrm{d}z}{\mathrm{d}r} \sim \begin{cases} \frac{GMr}{gR^3} & \text{if } r \le R \\ \frac{GM}{gr^2} & \text{if } r \ge R \end{cases} \tag{2.3}$$

The deformation profile would then be given by $z \sim \begin{cases} \frac{GMr^2}{2gR^3} - \frac{3}{2}\frac{GM}{gR} & \text{if } r \le R \\ -\frac{GM}{gr} & \text{if } r \ge R \end{cases}$.

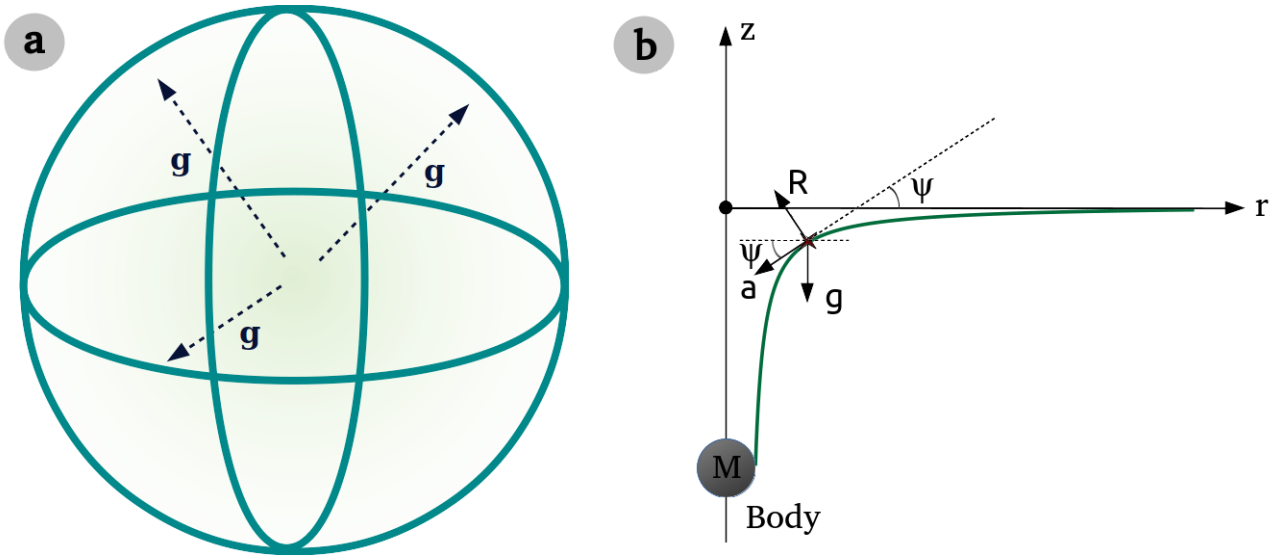

Figure 1: **a.** The universe would correspond to a 4-ball in expansion (its acceleration is noted g). **b.** Cutaway view of the three-dimensional profile (in green) which is deformed because of a high-mass body (grey disk). According to Newton's law of gravitation, the acceleration a experienced by a light body (red star) is equal to $\frac{GM}{r^2}$. g is the radial acceleration of the universe and $\mathbf{R}$ is the support reaction.

2.3 Which material properties?

This section focuses on the choice of the most suitable theory to describe the hypothetical deformation process. First, Newton's law of universal gravitation states that gravitation forces are proportional to the mass of bodies. Then, the force experienced by a light body in the gravitational field of two heavy bodies corresponds to the sum of both. The theory of deformation must thus be linear. Second, no gravitational field has been observed without matter. So the deformation due to bodies disappears when they move away. The deformation must then be elastic. Third, the gravitational attraction due to a body does not depend on its radius. Therefore, the hypothetical deformation of four-dimensional matter cannot derive from the surface tension phenomenon of a fluid. The deformation of a fluid by this process indeed depends on the load surface. For a solid, the deformation does not always depend on the load surface. For instance, the deformation of thin plates or beams depends only on the applied force magnitude (with the material and geometrical properties). Thus, for the deformation to be independent on the body radius, it seems that the hypothetical four-dimensional material must be thin in the fourth dimension. Given the previous considerations, continuum mechanics with Hooke's law will be the adopted theory. We thus consider a linear elastic thin four-dimensional solid material in the paper.

3 Results

3.1 The set of equations

The static equations of continuum mechanics correspond to $\boldsymbol{\nabla}.\boldsymbol{\sigma} = \mathbf{0}$ with $\boldsymbol{\sigma}$ the Cauchy stress tensor. Due to the spherical symmetry of gravitation, the spherindrical coordinates (equivalent to the cylindrical coordinates in four dimensions) lend themselves to this problem (Fig. A.1).

The tensor analysis in spherindrical coordinates is presented in the appendix.

The Cauchy stress tensor is calculated from Hooke's law

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda tr(\boldsymbol{\varepsilon})\mathbf{I} \tag{3.1}$$

where λ and μ are the Lamé coefficients and $\boldsymbol{\varepsilon} = \frac{1}{2}\left({}^t\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}\right)$ with $\mathbf{u}$ the displacement.

Due to the symmetries, we assume that $\mathbf{u} = u_r(r,z)\mathbf{e_r} + u_z(r,z)\mathbf{e_z}$. Relation (A.5) then provides the gradient of the displacement and the strain tensor

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \partial_r u_r & 0 & 0 & \frac{1}{2}(\partial_r u_z + \partial_z u_r) \\ 0 & \frac{u_r}{r} & 0 & 0 \\ 0 & 0 & \frac{u_r}{r} & 0 \\ \frac{1}{2}(\partial_r u_z + \partial_z u_r) & 0 & 0 & \partial_z u_z \end{pmatrix} \tag{3.2}$$

Considering the symmetries, the strain tensor expression and Hooke's law, relation (A.15) leads to

$$\boldsymbol{\nabla}.\boldsymbol{\sigma} = \begin{pmatrix} \partial_r\sigma_{rr} + \partial_z\sigma_{rz} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + \frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r} \\ 0 \\ 0 \\ \partial_r\sigma_{rz} + \partial_z\sigma_{zz} + \frac{2}{r}\sigma_{rz} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.3}$$

3.2 Resolution

3.2.1 General solution

The low thickness hypothesis yields $\partial_z u_z = 0$. The second equation of (3.3) becomes

$$\mu\left[\partial_r\left(\partial_r u_z + \partial_z u_r\right) + \frac{2}{r}\left(\partial_r u_z + \partial_z u_r\right)\right] + \lambda\partial_z\left(\partial_r u_r + 2\frac{u_r}{r}\right) = 0 \tag{3.4}$$

$$\Rightarrow(\mu+\lambda)\left(\partial^2_{rz}u_r + \frac{2}{r}\partial_z u_r\right) + \mu\left(\partial^2_r u_z + \frac{2}{r}\partial_r u_z\right) = 0 \tag{3.5}$$

$$\overset{\times r^2}{\Rightarrow}(\mu+\lambda)\left(r^2\partial^2_{rz}u_r + 2r\partial_z u_r\right) + \mu\left(r^2\partial^2_r u_z + 2r\partial_r u_z\right) = 0 \tag{3.6}$$

$$\Rightarrow(\mu+\lambda)\partial_r\left(r^2\partial_z u_r\right) + \mu\partial_r\left(r^2\partial_r u_z\right) = 0 \tag{3.7}$$

$$\Rightarrow(\mu+\lambda)\partial_z u_r + \mu\partial_r u_z = \frac{\phi(z)}{r^2} \tag{3.8}$$

with ϕ an unknown function which only depends on z. We deduce

$$\sigma_{rz} = \mu\left(\partial_r u_z + \partial_z u_r\right) = \mu\left(\partial_r u_z + \frac{\phi(z)}{(\mu+\lambda)r^2} - \frac{\mu}{\mu+\lambda}\partial_r u_z\right) \tag{3.9}$$

$$\Rightarrow\partial_z\sigma_{rz} = \frac{\mu}{\mu+\lambda}\frac{\phi'(z)}{r^2} \tag{3.10}$$

since $\partial_z u_z = 0$.

The first equation thus yields

$$2\mu\partial_r^2 u_r + \lambda\partial_r\left(\partial_r u_r + 2\frac{u_r}{r}\right) + \frac{\mu}{\mu+\lambda}\frac{\phi'(z)}{r^2} + \frac{4\mu}{r}\left(\partial_r u_r - \frac{u_r}{r}\right) = 0 \tag{3.11}$$

$$\Rightarrow(2\mu+\lambda)\partial_r\left(\partial_r u_r + 2\frac{u_r}{r}\right) + \frac{\mu}{\mu+\lambda}\frac{\phi'(z)}{r^2} = 0 \tag{3.12}$$

$$\Rightarrow\partial_r u_r + 2\frac{u_r}{r} = \frac{\mu}{(2\mu+\lambda)(\mu+\lambda)}\frac{\phi'(z)}{r} + 3\alpha(z) \tag{3.13}$$

The solution of this equation is $u_r = \alpha(z)r + \frac{\beta(z)}{r^2} + \frac{\mu}{2(2\mu+\lambda)(\mu+\lambda)}\phi'(z)$ with α and β two unknown functions. The equation (3.8) gives the vertical displacement

$$\partial_r u_z = \frac{\phi(z)}{\mu r^2} - \frac{\mu+\lambda}{\mu}\partial_z u_r \tag{3.14}$$

$$\Rightarrow\partial_r u_z = \frac{\phi(z) - (\mu+\lambda)\beta'(z)}{\mu r^2} - \frac{\mu+\lambda}{\mu}\alpha'(z)r - \frac{\phi''(z)}{2(2\mu+\lambda)} \tag{3.15}$$

$$u_z = \frac{(\mu+\lambda)\beta'(z) - \phi(z)}{\mu r} - \frac{\mu+\lambda}{2\mu}\alpha'(z)r^2 - \frac{\phi''(z)}{2(2\mu+\lambda)}r + \gamma \tag{3.16}$$

with γ a constant since $\partial_z u_z = 0$.

3.2.2 Determination of the unknown functions

We segregate two cases, inside and oustide the high-mass body.

1. **Outside**

   - $\lim_{r\to+\infty} u_r(r) = 0 \Rightarrow \begin{cases} \alpha = 0 \\ \phi' = 0 \end{cases}$
   - $\lim_{r\to+\infty} u_z(r) = 0 \Rightarrow \begin{cases} \alpha' = 0 \\ \phi'' = 0 \\ \gamma = 0 \end{cases}$

   The boundary conditions on the displacement when $r \to \infty$ lead to

$$\begin{cases} u_r = \frac{\beta(z)}{r^2} \\ u_z = \frac{(\mu+\lambda)\beta'(z) - \phi(z)}{\mu r} \end{cases} \tag{3.17}$$

   - $\boldsymbol{\sigma}.\mathbf{e_z} = \mathbf{0}$ for $z = 0$ and $z = -l$ (l is the material thickness). This leads to $\sigma_{rz} = 0$ and $\sigma_{zz} = 0$ for $z = 0$ and $z = -l$.
     - $\sigma_{rz} = 0 \Leftrightarrow \partial_r u_z = -\partial_z u_r \Leftrightarrow \phi - (\mu+\lambda)\beta'(z) = -\mu\beta'(z)$. This yields

$$\beta'(0) = \beta'(-l) = \frac{\phi}{\lambda} \tag{3.18}$$

     - $\sigma_{zz} = 0 \Leftrightarrow \partial_r u_r + 2\frac{u_r}{r} = 0$: always true

In addition, the profile of the horizontal displacement along the thickness of a thin material is well known. The profile is linear and the displacement is negative at the top and positive at the bottom. Then, we assume $u_r(r,z) = -\left(\frac{2z}{l}+1\right)|u_{r,max}(r)|$ with $|u_{r,max}|$ the absolute value of the maximum displacement when $z=0,-l$. This means that $\beta(z) = -a\left(\frac{2z}{l}+1\right)$ with a a constant. Then, $\phi = -2\frac{a\lambda}{l}$. The displacements are thus given by

$$\begin{cases} u_r = -a\left(\frac{2z}{l}+1\right)\frac{1}{r^2} \\ u_z = \frac{-2a}{lr} \end{cases} \tag{3.19}$$

2. **Inside**

- $u_r(r=0,z)=0 \Rightarrow \begin{cases} \beta = 0 \\ \phi' = 0 \end{cases}$
- $\partial_r u_z = 0$ for $r=0$. This involves $\begin{cases} \phi = (\mu+\lambda)\beta' \\ \phi'' = 0 \end{cases}$

The boundary conditions on the displacement when $r \to \infty$ lead to

$$\begin{cases} u_r = \alpha(z) r \\ u_z = -\frac{\mu+\lambda}{2\mu}\alpha'(z) r^2 + \gamma \end{cases} \tag{3.20}$$

For the same reason as outside the high-mass body, $\alpha(z) = -b\left(\frac{2z}{l}+1\right)$ with b a constant. In addition, $\boldsymbol{\sigma}.\mathbf{e_z} = -\frac{Mg}{\frac{4}{3}\pi R^3}\mathbf{e_z}$ for $z=0$ leads to $\sigma_{zz} = -\frac{Mg}{\frac{4}{3}\pi R^3} \Rightarrow \alpha(0) = -\frac{Mg}{4\pi R^3\lambda}$. Then, $b = \frac{Mg}{4\pi R^3\lambda}$ and

$$\begin{cases} u_r = -\frac{Mg}{4\pi R^3\lambda}\left(\frac{2z}{l}+1\right) r \\ u_z = \frac{\mu+\lambda}{l\lambda\mu}\frac{Mg}{4\pi R^3} r^2 + \gamma \end{cases} \tag{3.21}$$

3. $\mathbf{r} = \mathbf{R}$

- $u_r(R,z) = -\frac{Mg}{4\pi R^3\lambda}\left(\frac{2z}{l}+1\right) R = -a\left(\frac{2z}{l}+1\right)\frac{1}{R^2} \Rightarrow a = \frac{Mg}{4\pi\lambda}$
- $u_z(R) = \frac{\mu+\lambda}{l\lambda\mu}\frac{Mg}{4\pi R^3}R^2 + \gamma = \frac{-2a}{lR} = -\frac{Mg}{2\pi Rl\lambda} \Rightarrow \gamma = -\frac{Mg}{4\pi Rl\lambda}\left(2+\frac{\mu+\lambda}{\mu}\right)$

Eventually, the solution is

$$u_r(r,z) = \begin{cases} -\frac{Mg}{4\pi R^3\lambda}\left(\frac{2z}{l}+1\right) r & \text{if } r \le R \\ -\frac{Mg}{4\pi\lambda}\left(\frac{2z}{l}+1\right)\frac{1}{r^2} & \text{if } r \ge R \end{cases} \tag{3.22}$$

$$u_z(r) = \begin{cases} \frac{\mu+\lambda}{\mu\lambda}\frac{Mg}{4\pi l}\frac{r^2}{R^3} - \frac{3\mu+\lambda}{\mu\lambda}\frac{Mg}{4\pi lR} & \text{if } r \le R \\ \frac{-Mg}{2\pi\lambda lr} & \text{if } r \ge R \end{cases} \tag{3.23}$$

The solution can be compared with Newton's law of gravitation

$$u_z(r) \sim \begin{cases} \frac{GMr^2}{2gR^3} - \frac{3}{2}\frac{GM}{gR} & \text{if } r \le R \\ -\frac{GM}{gr} & \text{if } r \ge R \end{cases} \tag{3.24}$$

The law outside the high-mass body requires $G = \frac{g^2}{2\pi\lambda l}$. This definition avoids having the same law inside the body, except if $\mu \gg \lambda$. The assumption $\mu \gg \lambda$ is equivalent to a very weak Poisson ratio. In this case, $\frac{\mu+\lambda}{\mu\lambda} \sim \frac{1}{\lambda}$ and $\frac{3\mu+\lambda}{\mu\lambda} \sim \frac{3}{\lambda}$ which leads exactly to Newton's law

of gravitation.

Note that there is no proof that Newton's law of gravitation is accurate inside bodies. The assumption about Poisson's ratio to fit with Newton's law could thus be unreliable.

4 Dark matter

In that respect, Newton's law of gravitation would rely on the linear elastic assumption. For low masses, this law agrees with observations. However, for black holes or galaxies, the deformation of the space could be different from the profile provided in the paper. In this theory, gravitation could be given by other equations since the linear elastic assumption could become unreliable. Then, the equations from general relativity that are based on Newton's law could also fail to describe correctly gravitation for high mass bodies. Therefore, the assessment of dark matter quantities based on the current theories of gravitation might be impossible according to the present theory. The very existence of dark matter might also be questioned.

5 Conclusion

In prior work [17], based on two formulas giving the speed of light and the Planck constant, we developed a qualitative theory of gravitation. We assumed that gravitational acceleration derived from the acceleration of the universe's expansion and the deformation of four-dimensional matter. In that respect, Newton's law of gravitation would result from these processes. We have developed here the mathematical part of this theory in order to find back Newton's theory. To fit with physical evidence, we considered the deformation of a linear elastic thin four-dimensional solid material. Continuum mechanics with Hooke's law was then selected to describe the deformation process.

Under these hypotheses, the deformation profile outside a high-mass body is proportional to the reciprocal function and is parabolic inside it. Then, in a four-dimensional space experiencing an accelerating expansion, a light body in the vicinity of a high-mass body at the three-dimensional surface could be described by Newton's law of gravitation. Therefore, this theory agrees with Newton's one with the appropriate definition of the gravitational constant.

According to this theory, Newton's law describes the linear elastic deformation of four-dimensional matter. When the mass of a body is high enough, the process could become non-linear. Then, Newton's law of gravitation could become unreliable for heavy bodies or galaxies. In that sense, dark matter models might also be inaccurate.

A Appendix

Let us note $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4})$ the natural basis of $\mathbb{R}^4$ and $(\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\varphi, \mathbf{e_z})$ the spherindrical basis (Fig. A.1).

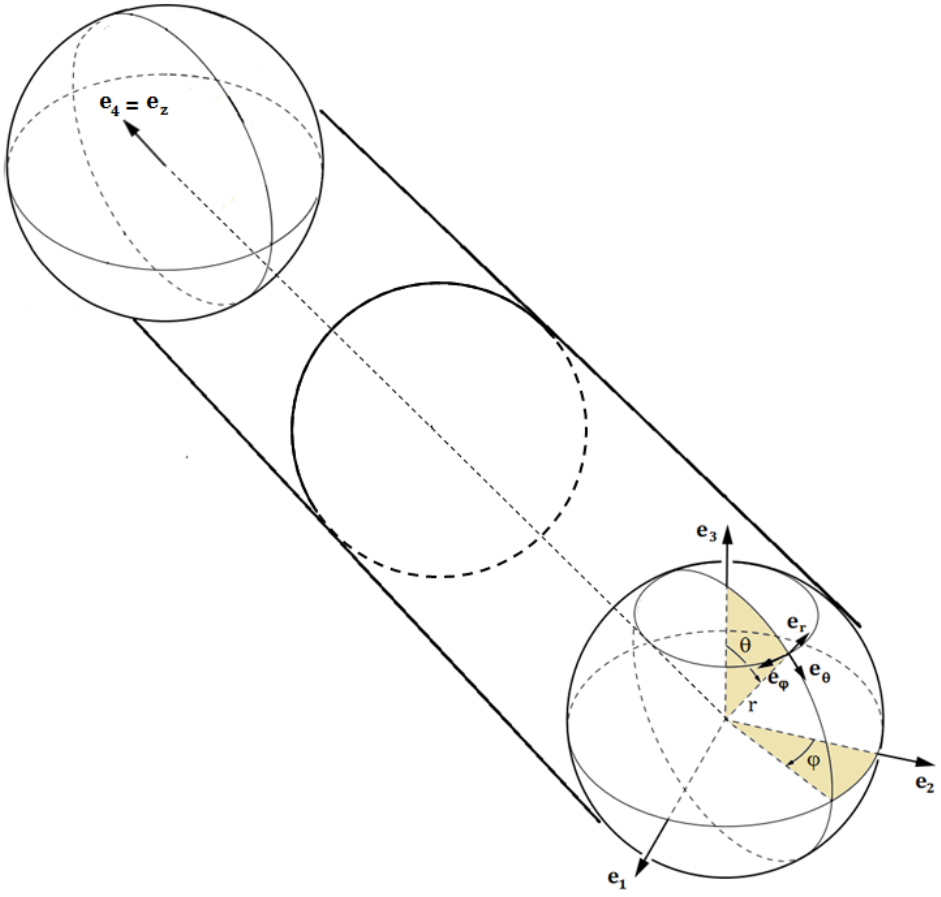

Figure A.1: Spherindrical coordinates.

We consider the spherindrical coordinates (r, θ, φ, z) such as $x_1 = r\sin\theta\sin\varphi$, $x_2 = r\sin\theta\cos\varphi$, $x_3 = r\cos\theta$ and $x_4 = z$. The spherindrical basis vectors can be expressed in the natural basis such as:

$$\begin{aligned}
\mathbf{e_r} &= \sin\theta\sin\varphi\ \mathbf{e_1} + \sin\theta\cos\varphi\ \mathbf{e_2} + \cos\theta\ \mathbf{e_3} \\
\mathbf{e}_\theta &= \cos\theta\sin\varphi\ \mathbf{e_1} + \cos\theta\cos\varphi\ \mathbf{e_2} - \sin\theta\ \mathbf{e_3} \\
\mathbf{e}_\varphi &= \cos\varphi\ \mathbf{e_1} - \sin\varphi\ \mathbf{e_2} \\
\mathbf{e_z} &= \mathbf{e_4}
\end{aligned} \tag{A.1}$$

The nabla operator is given by $\boldsymbol{\nabla} = \mathbf{e_r}\partial_r + \mathbf{e}_\theta\frac{1}{r}\partial_\theta + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\partial_\varphi + \mathbf{e_z}\partial_z$. The symbol ∂_α denotes the partial derivative with respect to the variable α.
Let us calculate the partial derivatives of the spherindrical basis vectors.

$$\begin{aligned}
&\partial_r\mathbf{e}_\alpha = \mathbf{0},\ \forall\ \alpha \in [r, \theta, \varphi, z] \\
&\partial_\theta\mathbf{e_r} = \mathbf{e}_\theta,\ \partial_\theta\ \mathbf{e}_\theta = -\mathbf{e_r},\ \partial_\theta\mathbf{e}_\varphi = \partial_\theta\mathbf{e_z} = \mathbf{0} \\
&\partial_\varphi\mathbf{e_r} = \sin\theta\ \mathbf{e}_\varphi,\ \partial_\varphi\mathbf{e}_\theta = \cos\theta\ \mathbf{e}_\varphi,\ \partial_\varphi\mathbf{e}_\varphi = -\sin\theta\ \mathbf{e_r} - \cos\theta\ \mathbf{e}_\theta,\ \partial_\varphi\mathbf{e_z} = \mathbf{0} \\
&\partial_z\mathbf{e}_\alpha = \mathbf{0},\ \forall\ \alpha \in [r, \theta, \varphi, z]
\end{aligned} \tag{A.2}$$

Let us consider a vector $\mathbf{v} = v_r\ \mathbf{e_r} + v_\theta\ \mathbf{e}_\theta + v_\varphi\ \mathbf{e}_\varphi + v_z\ \mathbf{e_z}$ and calculate the gradient of this vector $\mathbf{G} = \boldsymbol{\nabla}\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \alpha} \otimes \mathbf{e}_\alpha$. Then,

$$\mathbf{G}.\mathbf{e_r} = \frac{\partial \mathbf{v}}{\partial r} = \frac{\partial}{\partial r}(v_r\ \mathbf{e_r} + v_\theta\ \mathbf{e}_\theta + v_\varphi\ \mathbf{e}_\varphi + v_z\ \mathbf{e_z}) = \partial_r v_r\ \mathbf{e_r} + \partial_r v_\theta\ \mathbf{e}_\theta + \partial_r v_\varphi\ \mathbf{e}_\varphi + \partial_r v_z\ \mathbf{e_z} \quad \text{(A.3)}$$

because $\partial_r \mathbf{e}_\alpha = \mathbf{0},\ \forall\ \alpha \in [r, \theta, \varphi, z]$ according to relation (A.2). Then,

$$\begin{array}{l} \mathbf{G}.\mathbf{e}_\theta = \frac{1}{r}\frac{\partial \mathbf{v}}{\partial \theta} = \frac{1}{r}\left[(\partial_\theta v_r - v_\theta)\,\mathbf{e_r} + (v_r + \partial_\theta v_\theta)\,\mathbf{e}_\theta + \partial_\theta v_\varphi \mathbf{e}_\varphi + \partial_\theta v_z \mathbf{e_z}\right] \\ \mathbf{G}.\mathbf{e}_\varphi = \frac{1}{r\sin\theta}\frac{\partial \mathbf{v}}{\partial \varphi} = \frac{1}{r\sin\theta}\left[(\partial_\varphi v_r - \sin\theta v_\varphi)\,\mathbf{e_r} + (\partial_\varphi v_\theta - \cos\theta v_\varphi)\,\mathbf{e}_\theta + \right. \\ \left. (\partial_\varphi v_\varphi + \sin\theta v_r + \cos\theta v_\theta)\,\mathbf{e}_\varphi + \partial_\varphi v_z \mathbf{e_z}\right] \\ \mathbf{G}.\mathbf{e_z} = \frac{\partial \mathbf{v}}{\partial z} = \partial_z v_r\ \mathbf{e_r} + \partial_z v_\theta\ \mathbf{e}_\theta + \partial_z v_\varphi\ \mathbf{e}_\varphi + \partial_z v_z\ \mathbf{e_z} \end{array} \quad \text{(A.4)}$$

Then, the gradient of $\mathbf{v}$ can be written

$$\boldsymbol{\nabla}\mathbf{v} = \begin{pmatrix} \partial_r v_r & \frac{1}{r}\partial_\theta v_r - \frac{v_\theta}{r} & \frac{1}{r\sin\theta}\partial_\varphi v_r - \frac{v_\varphi}{r} & \partial_z v_r \\ \partial_r v_\theta & \frac{1}{r}\partial_\theta v_\theta + \frac{v_r}{r} & \frac{1}{r\sin\theta}\partial_\varphi v_\theta - \cot\theta\frac{v_\varphi}{r} & \partial_z v_\theta \\ \partial_r v_\varphi & \frac{1}{r}\partial_\theta v_\varphi & \frac{1}{r\sin\theta}\partial_\varphi v_\varphi + \frac{v_r}{r} + \cot\theta\frac{v_\theta}{r} & \partial_z v_\varphi \\ \partial_r v_z & \frac{1}{r}\partial_\theta v_z & \frac{1}{r\sin\theta}\partial_\varphi v_z & \partial_z v_z \end{pmatrix} \quad \text{(A.5)}$$

The divergence of the vector $\mathbf{v}$ is given by

$$\boldsymbol{\nabla}.\mathbf{v} = tr(\boldsymbol{\nabla}\mathbf{v}) = \frac{1}{r^2}\partial_r(r^2 v_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\partial_\varphi v_\varphi + \partial_z v_z \quad \text{(A.6)}$$

Now, let us consider a symmetric second-order tensor field $\boldsymbol{\sigma}$ and calculate its divergence. With ":" denoting the double dot product and $\mathbf{I} = \mathbf{e_r} \otimes \mathbf{e_r} + \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \mathbf{e_z} \otimes \mathbf{e_z}$ the second order identity tensor, $\boldsymbol{\nabla}.\boldsymbol{\sigma} = \boldsymbol{\nabla}\boldsymbol{\sigma} : \mathbf{I}$. With $\mathbf{A}$ a third-order tensor field and $\mathbf{b}$ and $\mathbf{c}$ two vectors, we have $\mathbf{A} : (\mathbf{b} \otimes \mathbf{c}) = (\mathbf{A}.\mathbf{b}).\mathbf{c}$. The divergence of $\boldsymbol{\sigma}$ can then be expressed as

$$\begin{aligned} \boldsymbol{\nabla}.\boldsymbol{\sigma} &= (\boldsymbol{\nabla}\boldsymbol{\sigma}.\mathbf{e_r}).\mathbf{e_r} + (\boldsymbol{\nabla}\boldsymbol{\sigma}.\mathbf{e}_\theta).\mathbf{e}_\theta + (\boldsymbol{\nabla}\boldsymbol{\sigma}.\mathbf{e}_\varphi).\mathbf{e}_\varphi + (\boldsymbol{\nabla}\boldsymbol{\sigma}.\mathbf{e_z}).\mathbf{e_z} \\ &= (\partial_r\boldsymbol{\sigma}).\mathbf{e_r} + \frac{1}{r}(\partial_\theta\boldsymbol{\sigma}).\mathbf{e}_\theta + \frac{1}{r\sin\theta}(\partial_\varphi\boldsymbol{\sigma}).\mathbf{e}_\varphi + (\partial_z\boldsymbol{\sigma}).\mathbf{e_z} \end{aligned} \quad \text{(A.7)}$$

The calculation of $\partial_\gamma\boldsymbol{\sigma}$ can be divided into two parts:

$$\begin{aligned} \partial_\gamma\boldsymbol{\sigma} &= \partial_\gamma \sum_{\alpha=r,\theta,\varphi,z}\ \sum_{\beta=r,\theta,\varphi,z} \sigma_{\alpha\beta}\ \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \\ &= \sum_{\alpha=r,\theta,\varphi,z}\ \sum_{\beta=r,\theta,\varphi,z} (\partial_\gamma\sigma_{\alpha\beta})\mathbf{e}_\alpha \otimes \mathbf{e}_\beta + \sum_{\alpha=r,\theta,\varphi,z}\ \sum_{\beta=r,\theta,\varphi,z} \sigma_{\alpha\beta}[(\partial_\gamma\mathbf{e}_\alpha) \otimes \mathbf{e}_\beta + \mathbf{e}_\alpha \otimes (\partial_\gamma\mathbf{e}_\beta)] \end{aligned} \quad \text{(A.8)}$$

The calculation of the derivatives of the basis vectors are provided by relations (A.2). Since $\partial_r\mathbf{e}_\alpha = \mathbf{0},\ \forall\ \alpha \in [r, \theta, \varphi, z]$, $\gamma = r$ yields $\partial_r\boldsymbol{\sigma} = \sum_{\alpha=r,\theta,\varphi,z}\ \sum_{\beta=r,\theta,\varphi,z} (\partial_r\sigma_{\alpha\beta})\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$.

So,

$$(\partial_r\boldsymbol{\sigma}).\mathbf{e_r} = \sum_{\beta=r,\theta,\varphi,z} (\partial_r\sigma_{r\beta})\mathbf{e}_\beta = (\partial_r\sigma_{rr})\mathbf{e_r} + (\partial_r\sigma_{r\theta})\mathbf{e}_\theta + (\partial_r\sigma_{r\varphi})\mathbf{e}_\varphi + (\partial_r\sigma_{rz})\mathbf{e_z} \quad \text{(A.9)}$$

When $\gamma = \theta$,

$$\begin{aligned} &\sum_{\alpha=r,\theta,\varphi,z}\ \sum_{\beta=r,\theta,\varphi,z} \sigma_{\alpha\beta}[(\partial_\theta\mathbf{e}_\alpha) \otimes \mathbf{e}_\beta + \mathbf{e}_\alpha \otimes (\partial_\theta\mathbf{e}_\beta)] = \\ &\sigma_{rr}(\mathbf{e}_\theta \otimes \mathbf{e_r} + \mathbf{e_r} \otimes \mathbf{e}_\theta) + \sigma_{r\theta}(\mathbf{e}_\theta \otimes \mathbf{e}_\theta - \mathbf{e_r} \otimes \mathbf{e_r} - \mathbf{e_r} \otimes \mathbf{e_r} + \mathbf{e}_\theta \otimes \mathbf{e}_\theta) \\ &+ \sigma_{r\varphi}(\mathbf{e}_\theta \otimes \mathbf{e}_\varphi + \mathbf{e}_\varphi \otimes \mathbf{e}_\theta) + \sigma_{rz}(\mathbf{e}_\theta \otimes \mathbf{e_z} + \mathbf{e_z} \otimes \mathbf{e}_\theta) - \sigma_{\theta\theta}(\mathbf{e_r} \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e_r}) \\ &- \sigma_{\theta\varphi}(\mathbf{e_r} \otimes \mathbf{e}_\varphi + \mathbf{e}_\varphi \otimes \mathbf{e_r}) - \sigma_{\theta z}(\mathbf{e_r} \otimes \mathbf{e_z} + \mathbf{e_z} \otimes \mathbf{e_r}) \end{aligned} \quad \text{(A.10)}$$

We directly replaced the derivatives and used the symmetry of $\boldsymbol{\sigma}$. Then,

$$\begin{aligned}(\partial_\theta \boldsymbol{\sigma}).\mathbf{e}_\theta &= \sum_{\beta=r,\theta,\varphi,z} (\partial_\theta \sigma_{\theta\beta})\mathbf{e}_\beta + (\sigma_{rr}-\sigma_{\theta\theta})\mathbf{e_r} + 2\sigma_{r\theta}\mathbf{e}_\theta + \sigma_{r\varphi}\mathbf{e}_\varphi + \sigma_{rz}\mathbf{e_z} \\ &= (\sigma_{rr}-\sigma_{\theta\theta}+\partial_\theta\sigma_{r\theta})\mathbf{e_r} + (2\sigma_{r\theta}+\partial_\theta\sigma_{\theta\theta})\mathbf{e}_\theta + (\sigma_{r\varphi}+\partial_\theta\sigma_{\theta\varphi})\mathbf{e}_\varphi + (\sigma_{rz}+\partial_z\sigma_{\theta z})\mathbf{e_z}\end{aligned} \tag{A.11}$$

When $\gamma = \varphi$,

$$\begin{aligned}&\sum_{\alpha=r,\theta,\varphi,z}\sum_{\beta=r,\theta,\varphi,z} \sigma_{\alpha\beta}[(\partial_\varphi \mathbf{e}_\alpha)\otimes\mathbf{e}_\beta + \mathbf{e}_\alpha\otimes(\partial_\varphi\mathbf{e}_\beta)] = \\ &\sigma_{rr}\sin\theta(\mathbf{e_r}\otimes\mathbf{e}_\varphi+\mathbf{e}_\varphi\otimes\mathbf{e_r}) + \sigma_{r\theta}(\cos\theta\mathbf{e_r}\otimes\mathbf{e}_\varphi+\sin\theta\mathbf{e}_\varphi\otimes\mathbf{e}_\theta+\cos\theta\mathbf{e}_\varphi\otimes\mathbf{e_r}+\sin\theta\mathbf{e}_\theta\otimes\mathbf{e}_\varphi) \\ &+\sigma_{r\varphi}(2\sin\theta\mathbf{e}_\varphi\otimes\mathbf{e}_\varphi-2\sin\theta\mathbf{e_r}\otimes\mathbf{e_r}-\cos\theta(\mathbf{e_r}\otimes\mathbf{e}_\theta+\mathbf{e}_\theta\otimes\mathbf{e_r})) + \sigma_{rz}\sin\theta(\mathbf{e}_\varphi\otimes\mathbf{e_z}+\mathbf{e_z}\otimes\mathbf{e}_\varphi) \\ &+\sigma_{\theta\theta}\cos\theta(\mathbf{e}_\theta\otimes\mathbf{e}_\varphi+\mathbf{e}_\varphi\otimes\mathbf{e}_\theta) + \sigma_{\theta\varphi}(2\cos\theta\mathbf{e}_\varphi\otimes\mathbf{e}_\varphi-2\cos\theta\mathbf{e}_\theta\otimes\mathbf{e}_\theta-\sin\theta(\mathbf{e}_\theta\otimes\mathbf{e_r}+\mathbf{e_r}\otimes\mathbf{e}_\theta)) \\ &+\sigma_{\theta z}\cos\theta(\mathbf{e}_\varphi\otimes\mathbf{e_z}+\mathbf{e_z}\otimes\mathbf{e}_\varphi) - \sigma_{\varphi\varphi}(\sin\theta(\mathbf{e}_\varphi\otimes\mathbf{e_r}+\mathbf{e_r}\otimes\mathbf{e}_\varphi)+\cos\theta(\mathbf{e}_\varphi\otimes\mathbf{e}_\theta+\mathbf{e}_\theta\otimes\mathbf{e}_\varphi)) \\ &-\sigma_{\varphi z}(\sin\theta(\mathbf{e_r}\otimes\mathbf{e_z}+\mathbf{e_z}\otimes\mathbf{e_r})+\cos\theta(\mathbf{e}_\theta\otimes\mathbf{e_z}+\mathbf{e_z}\otimes\mathbf{e}_\theta))\end{aligned} \tag{A.12}$$

Then,

$$\begin{aligned}(\partial_\varphi\boldsymbol{\sigma}).\mathbf{e}_\varphi &= \sum_{\beta=r,\theta,\varphi,z}(\partial_\varphi\sigma_{\varphi\beta})\mathbf{e}_\beta + (\sin\theta\sigma_{rr}+\cos\theta\sigma_{r\theta}-\sin\theta\sigma_{\varphi\varphi})\mathbf{e_r} \\ &+(\sin\theta\sigma_{r\theta}+\cos\theta\sigma_{\theta\theta}-\cos\theta\sigma_{\varphi\varphi})\mathbf{e}_\theta + (2\sin\theta\sigma_{r\varphi}+2\cos\theta\sigma_{\theta\varphi})\mathbf{e}_\varphi + (\sin\theta\sigma_{rz}+\cos\theta\sigma_{\theta z})\mathbf{e_z} \\ &= (\sin\theta\sigma_{rr}+\cos\theta\sigma_{r\theta}-\sin\theta\sigma_{\varphi\varphi}+\partial_\varphi\sigma_{r\varphi})\mathbf{e_r} + (\sin\theta\sigma_{r\theta}+\cos\theta\sigma_{\theta\theta}-\cos\theta\sigma_{\varphi\varphi}+\partial_\varphi\sigma_{\theta\varphi})\mathbf{e}_\theta \\ &+(2\sin\theta\sigma_{r\varphi}+2\cos\theta\sigma_{\theta\varphi}+\partial_\varphi\sigma_{\varphi\varphi})\mathbf{e}_\varphi + (\sin\theta\sigma_{rz}+\cos\theta\sigma_{\theta z}+\partial_\varphi\sigma_{\varphi z})\mathbf{e_z}\end{aligned} \tag{A.13}$$

When $\gamma = z$,

$$(\partial_z\boldsymbol{\sigma}).\mathbf{e_z} = (\partial_z\sigma_{rz})\mathbf{e_r} + (\partial_z\sigma_{\theta z})\mathbf{e}_\theta + (\partial_z\sigma_{\varphi z})\mathbf{e}_\varphi + (\partial_z\sigma_{zz})\mathbf{e_z} \tag{A.14}$$

Finally, the divergence of $\boldsymbol{\sigma}$ given in the spherindrical basis is

$$\boldsymbol{\nabla}.\boldsymbol{\sigma} = \begin{pmatrix} \partial_r\sigma_{rr} + \frac{1}{r}\partial_\theta\sigma_{r\theta} + \frac{1}{r\sin\theta}\partial_\varphi\sigma_{r\varphi} + \partial_z\sigma_{rz} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + \frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r} + \frac{\cot\theta}{r}\sigma_{r\theta} \\ \partial_r\sigma_{r\theta} + \frac{1}{r}\partial_\theta\sigma_{\theta\theta} + \frac{1}{r\sin\theta}\partial_\varphi\sigma_{\theta\varphi} + \partial_z\sigma_{\theta z} + 3\frac{\sigma_{r\theta}}{r} + \cot\theta\frac{\sigma_{\theta\theta}-\sigma_{\varphi\varphi}}{r} \\ \partial_r\sigma_{r\varphi} + \frac{1}{r}\partial_\theta\sigma_{\theta\varphi} + \frac{1}{r\sin\theta}\partial_\varphi\sigma_{\varphi\varphi} + \partial_z\sigma_{\varphi z} + 3\frac{\sigma_{r\varphi}}{r} + 2\frac{\cot\theta}{r}\sigma_{\theta\varphi} \\ \partial_r\sigma_{rz} + \frac{1}{r}\partial_\theta\sigma_{\theta z} + \frac{1}{r\sin\theta}\partial_\varphi\sigma_{\varphi z} + \partial_z\sigma_{zz} + 2\frac{\sigma_{rz}}{r} + \frac{\cot\theta}{r}\sigma_{\theta z} \end{pmatrix} \tag{A.15}$$

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