



HAL
open science

Modular affine Hecke category and regular unipotent centralizer, I

Roman Bezrukavnikov, Simon Riche, Laura Rider

► **To cite this version:**

Roman Bezrukavnikov, Simon Riche, Laura Rider. Modular affine Hecke category and regular unipotent centralizer, I. 2020. hal-02571688

HAL Id: hal-02571688

<https://hal.science/hal-02571688>

Preprint submitted on 13 May 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

MODULAR AFFINE HECKE CATEGORY AND REGULAR UNIPOTENT CENTRALIZER, I

ROMAN BEZRUKAVNIKOV, SIMON RICHE, AND LAURA RIDER

ABSTRACT. In this paper we provide, under some mild explicit assumptions, a geometric description of the category of representations of the centralizer of a regular unipotent element in a reductive algebraic group in terms of perverse sheaves on the Langlands dual affine flag variety. This equivalence is suggested and motivated by the “geometric Langlands” philosophy.

1. INTRODUCTION

1.1. Towards an equivalence between the two modular categorifications of the affine Hecke algebra. The present paper is the first one in a series dedicated to proving a modular variant (i.e. a variant for coefficients of positive characteristic) of the equivalence between the two categorifications of the affine Hecke algebra constructed by the first author for $\overline{\mathbb{Q}}_\ell$ -coefficients [Be2], and to derive a proof of the Finkelberg–Mirković equivalence describing the principal block of representations of a connected reductive algebraic group in terms of perverse sheaves on the Langlands dual affine Grassmannian [FM].

The origin of this work is the fact that the Hecke algebra of the affine Weyl group attached to a semisimple algebraic group admits two natural (and useful) geometric realizations: one (due to Iwahori–Matsumoto [IM]) in terms of Iwahori–bi-invariant functions on the loop group of the given group, and one (due to Kazhdan–Lusztig [KL] and Ginzburg [CG]) in terms of the equivariant K-theory of the Steinberg variety of the Langlands dual group. The equivalence of [Be2] “lifts” the isomorphism between these algebras to an equivalence of monoidal categories relating the Iwahori-equivariant derived category of the affine flag variety of the given group to a certain derived category of equivariant coherent sheaves on the Langlands dual Steinberg variety, both of the categories being defined over $\overline{\mathbb{Q}}_\ell$. (The precise statement of the equivalence requires some care, and will not be reviewed here.)

The construction of this equivalence in [Be2] was based on a previous work of S. Arkhipov and the first author [AB] which provided an equivalence of categories “lifting” an isomorphism between two geometric realizations of the *antispherical* module for the affine Hecke algebra; see the introduction of [AB] for more details. It turns out that adapting the constructions of [AB] to the modular setting presents several technical issues that we cannot overcome at the moment in general.¹ Although it will be based on similar ingredients, our strategy will therefore be different: in addition to the constructions of local geometric Langlands theory,

¹In [AR3], P. Achar and the second author explain how to generalize this construction in case $G = \mathrm{GL}(n)$ and $\ell > \frac{1}{2} \binom{n}{\lfloor n/2 \rfloor}$.

we will employ Soergel's method of describing the categories we want to compare in algebro-combinatorial terms.

1.2. The regular quotient and representations of the regular unipotent centralizer. The goal of the present paper and its sequel is to lay the foundation for this algebro-combinatorial description. The role of Soergel bimodules in our approach is played by the category of representations of the centralizer of a regular unipotent element; this is motivated by [AB] where a similar construction is carried out for $\overline{\mathbb{Q}}_\ell$ -coefficients, see also [Dod] where affine Soergel bimodules are related to centralizers of regular elements. Thus, following the pattern of [So], we establish an equivalence between a Serre quotient of the category of \mathbb{k} -perverse sheaves on the affine flag variety with one (if G is simply connected) simple object and representations of the regular unipotent centralizer in the dual group defined over \mathbb{k} ; this can be thought of as an analogue of Soergel's Endomorphismensatz, in its interpretation from [BBM]. (Here \mathbb{k} is an algebraically closed field of positive characteristic.) In a future work we plan to bootstrap this to an equivalence between the derived categories of constructible \mathbb{k} -sheaves on affine flags and coherent sheaves on the Steinberg variety, establishing an analogue of Soergel's Struktursatz.

To describe more explicitly the equivalence that we prove here, let us introduce more notation. We will denote by G a connected reductive algebraic group over an algebraically closed field of characteristic $p > 0$. Let us choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. The choice of B determines an Iwahori subgroup Iw in the loop group L^+G of G , and we denote by Fl_G the associated affine flag variety. This ind-scheme admits a natural action of Iw , and we can therefore consider, for \mathbb{k} an algebraically closed field of characteristic $\ell \neq p$ (with $\ell > 0$) the Iw -equivariant derived category $D_{Iw}^b(Fl_G, \mathbb{k})$. This category admits a natural convolution product, denoted \star_{Iw} . Let P_{Iw} denote the heart of the perverse t-structure on $D_{Iw}^b(Fl_G, \mathbb{k})$. Note that \star_{Iw} does *not* restrict to a monoidal product on P_{Iw} ; however we can construct an abelian monoidal category out of these data as follows. If we denote by P_{Iw}^0 the quotient of the category P_{Iw} by the Serre subcategory generated by the simple objects whose support has positive dimension, then it is not difficult to check that the assignment $(\mathcal{F}, \mathcal{G}) \mapsto {}^p\mathcal{H}^0(\mathcal{F} \star_{Iw} \mathcal{G})$ descends to an exact monoidal product on P_{Iw}^0 , which we denote by \star_{Iw}^0 .

On the dual side we consider the Langlands dual reductive group $G_{\mathbb{k}}^\vee$ over \mathbb{k} , a regular unipotent element u , and the corresponding centralizer subgroup $Z_{G_{\mathbb{k}}^\vee}(u)$. We denote by $\text{Rep}(Z_{G_{\mathbb{k}}^\vee}(u))$ the abelian category of finite-dimensional representations of $Z_{G_{\mathbb{k}}^\vee}(u)$; this category has a natural monoidal structure given by the tensor product of representations.

Let us denote by \mathfrak{R} the root system of (G, T) , and by \mathfrak{R}^\vee the corresponding coroot system. We will assume that

- either ℓ is very good, or $X^*(T)/\mathbb{Z}\mathfrak{R}$ is free (i.e. $G_{\mathbb{k}}^\vee$ has simply connected derived subgroup) and $X_*(T)/\mathbb{Z}\mathfrak{R}^\vee$ has no ℓ -torsion;
- ℓ is bigger than some explicit bound depending on \mathfrak{R} and given in Figure 5.1 below.

Under these assumptions, the main result of this paper (Theorem 5.4) states that there exists an equivalence of abelian monoidal categories

$$(1.1) \quad (P_{Iw}^0, \star_{Iw}^0) \cong (\text{Rep}(Z_{G_{\mathbb{k}}^\vee}(u)), \otimes_{\mathbb{k}}).$$

Remark 1.1. In case $\mathbb{k} = \overline{\mathbb{Q}}_\ell$, the equivalence (1.1) (or in fact a slightly less explicit variant) is a special case of the main result of [Be1], and it plays a technical role in a proof in [AB]. The equivalence itself is stated in a remark in [AB], and is proved explicitly in [AR3, §7.2].

The category $\text{Rep}(Z_{G_{\mathbb{k}}}^\vee(\mathbf{u}))$ is related to coherent sheaves on the Steinberg variety because it identifies with the category of $G_{\mathbb{k}}^\vee$ -equivariant coherent sheaves on the regular unipotent orbit of $G_{\mathbb{k}}^\vee$. In a sequel to this paper we will “extend” our equivalence to describe the category of equivariant coherent sheaves on the regular part of the “Grothendieck resolution” version of the Steinberg variety supported set-theoretically on the regular unipotent orbit. On the constructible side this requires replacing Iw-equivariant perverse sheaves on Fl_G by certain monodromic perverse sheaves on the natural T -torsor $\widetilde{\text{Fl}}_G \rightarrow \text{Fl}_G$.

1.3. Strategy of proof. Our construction of the equivalence (1.1) will rely in a crucial way on the geometric Satake equivalence and Gaitsgory’s theory of *central sheaves*. Let Gr_G denote the affine Grassmannian of G . This ind-scheme admits a natural action of L^+G , and we can consider the category $\text{Perv}_{\text{sph}}(\text{Gr}_G, \mathbb{k})$ of L^+G -equivariant étale \mathbb{k} -perverse sheaves on Gr_G . This category admits a natural convolution product \star , and the geometric Satake equivalence [MV] provides an equivalence of abelian monoidal categories

$$(1.2) \quad (\text{Perv}_{\text{sph}}(\text{Gr}_G, \mathbb{k}), \star) \xrightarrow{\sim} (\text{Rep}(G_{\mathbb{k}}^\vee), \otimes_{\mathbb{k}}).$$

The relation between perverse sheaves on Gr_G and on Fl_G is provided by Gaitsgory’s functor $Z : \text{Perv}_{\text{sph}}(\text{Gr}_G, \mathbb{k}) \rightarrow \text{P}_{\text{Iw}}$. This functor, defined in terms of nearby cycles, provides a “categorical lift” of Bernstein’s description of the center of the affine Hecke algebra, and the perverse sheaves in its image have a number of very favorable properties, studied in particular in [G1, G2, AB] and reviewed in [AR3]. Using a general lemma from [Be1], the properties of this functor provide a unipotent element $\mathbf{u} \in G_{\mathbb{k}}^\vee$, a subgroup scheme $H \subset Z_{G_{\mathbb{k}}}^\vee(\mathbf{u})$, and an equivalence of monoidal categories between a certain subcategory $\widetilde{\text{P}}_{\text{Iw}}^0$ of P_{Iw}^0 and the category of finite-dimensional representations of H , under which the composition of Z with the quotient functor $\text{P}_{\text{Iw}} \rightarrow \text{P}_{\text{Iw}}^0$ corresponds to the composition of the Satake equivalence (1.2) with the restriction functor $\text{Rep}(G_{\mathbb{k}}^\vee) \rightarrow \text{Rep}(H)$. Most of the new material in this paper will then be used to prove the following claims:

- (1) \mathbf{u} is regular;
- (2) the subcategory $\widetilde{\text{P}}_{\text{Iw}}^0$ is the whole of P_{Iw}^0 ;
- (3) the embedding $H \subset Z_{G_{\mathbb{k}}}^\vee(\mathbf{u})$ is an equality.

These proofs will use modular variants of some technical constructions from [AB]. In particular we will prove that, under appropriate assumptions, the images in the category of Iwahori-Whittaker perverse sheaves of the objects $Z(\mathcal{F})$ with \mathcal{F} tilting in $\text{Perv}_{\text{sph}}(\text{Gr}_G, \mathbb{k})$ are tilting.

1.4. Contents. In Section 2 we prove some preliminary results on centralizers of regular unipotent elements in reductive groups; in particular we study when this group scheme is smooth, and provide a general criterion on subgroup schemes of such centralizers that will be used to prove statement (3) from §1.3. In Section 3 we prove a slight extension of the general lemma from [Be1] alluded to in §1.3. Section 4 provides a reminder on the geometric Satake equivalence and Gaitsgory’s

central functor. In Section 5 we state our main result more precisely, and explain the construction of \mathbf{u} and H . In Section 6 we prove statement (1) from §1.3. In Section 7 we introduce the category of Iwahori-Whittaker perverse sheaves on Fl_G , and explain the relation with \mathbf{P}_{Iw} and $\mathbf{P}_{\mathrm{Iw}}^0$. In Section 8 we prove that the images of certain central sheaves in the category of Iwahori-Whittaker perverse sheaves are tilting. Finally, in Section 9 we use this to prove statements (2) and (3) from §1.3, and thus complete the proof of our main result.

1.5. Acknowledgements. R.B. was supported by NSF Grant No. DMS-1601953. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (S.R., grant agreement No. 677147). L.R. was supported by NSF Grant No. DMS-1802378.

We thank M. Korhonen for explaining the proof of Lemma 2.16 to one of us and for a helpful discussion on centralizers of regular unipotent elements, A. Bouthier for another useful discussion on centralizers of regular unipotent elements, and M. Finkelberg for spotting some typos.

2. PRELIMINARIES ON REDUCTIVE GROUPS

In this section we collect a number of general results on the geometry or representations of reductive algebraic groups which will be needed later in our arguments.

2.1. Quotients of group schemes. Let \mathbb{K} be an algebraically closed field. In our considerations below we will have to consider some quotients of affine \mathbb{K} -group schemes of finite type by closed subgroup schemes, without any reducedness assumptions. For the reader’s convenience, in this subsection we briefly recall the properties of this construction that we will use.

Let \mathbf{H} be an affine \mathbb{K} -group scheme of finite type.

- For any subgroup scheme $\mathbf{H}' \subset \mathbf{H}$, the fppf-sheafification of the functor $R \mapsto \mathbf{H}(R)/\mathbf{H}'(R)$ is represented by a \mathbb{K} -scheme of finite type, denoted \mathbf{H}/\mathbf{H}' ; see [DG, III, §3, Théorème 5.4].
- We have $\dim(\mathbf{H}/\mathbf{H}') = \dim(\mathbf{H}) - \dim(\mathbf{H}')$; see [DG, III, §3, Remarque 5.5(a)].
- If \mathbf{H}' is smooth then the natural map $\mathbf{H} \rightarrow \mathbf{H}/\mathbf{H}'$ is smooth; see [DG, III, §3, Corollaire 2.6].
- If \mathbf{H} is smooth, then the scheme \mathbf{H}/\mathbf{H}' is smooth; see [DG, III, §3, Proposition 2.7].
- If \mathbf{H}' is a normal subgroup, then the quotient \mathbf{H}/\mathbf{H}' is an affine group scheme (of finite type); see [DG, III, §3, Théorème 5.6].

If \mathbf{H} acts on a \mathbb{K} -scheme \mathbf{X} of finite type, and if $x \in \mathbf{X}(\mathbb{K})$, then the centralizer $Z_{\mathbf{H}}(x)$ of x in \mathbf{H} is the scheme-theoretic fiber over x of the morphism $\mathbf{H} \rightarrow \mathbf{X}$ defined by $g \mapsto g \cdot x$. By [DG, III, §3, Proposition 5.2], this morphism factors through a locally closed embedding $\mathbf{H}/Z_{\mathbf{H}}(x) \rightarrow \mathbf{X}$.

2.2. Smoothness of centralizers. From now on we assume that \mathbf{G} is a smooth affine group scheme of finite type over \mathbb{K} . For any $g \in \mathbf{G}$ we can consider the centralizer $Z_{\mathbf{G}}(g)$ relative to the adjoint action of \mathbf{G} on itself. We will denote by \mathfrak{g} the Lie algebra of \mathbf{G} . This vector space is endowed with the adjoint action of \mathbf{G} , and we set $\mathfrak{g}^g := \{v \in \mathfrak{g} \mid g \cdot v = v\}$. We start by recalling the following standard characterization of smoothness of centralizers.

Lemma 2.1. *The group scheme $Z_{\mathbf{G}}(g)$ is smooth iff we have*

$$\dim(Z_{\mathbf{G}}(g)) = \dim(\mathfrak{g}^g).$$

Proof. First, assume that $\dim(Z_{\mathbf{G}}(g)) = \dim(\mathfrak{g}^g)$. Let $Z_{\mathbf{G}}(g)_{\text{red}}$ be the reduced (hence smooth) group scheme associated with $Z_{\mathbf{G}}(g)$. Then we have embeddings

$$\text{Lie}(Z_{\mathbf{G}}(g)_{\text{red}}) \subset \text{Lie}(Z_{\mathbf{G}}(g)) \subset \mathfrak{g}^g.$$

Here we have $\dim(\text{Lie}(Z_{\mathbf{G}}(g)_{\text{red}})) = \dim(Z_{\mathbf{G}}(g)_{\text{red}}) = \dim(Z_{\mathbf{G}}(g))$, hence our assumption implies that these inclusions are equality. This implies in particular that $\dim(\text{Lie}(Z_{\mathbf{G}}(g))) = \dim(Z_{\mathbf{G}}(g))$, hence that $Z_{\mathbf{G}}(g)$ is smooth.

Assume now that $Z_{\mathbf{G}}(g)$ is smooth. The morphism $\mathbf{G} \rightarrow \mathbf{G}$ defined by $h \mapsto hgh^{-1}g^{-1}$ factors as a composition

$$\mathbf{G} \rightarrow \mathbf{G}/Z_{\mathbf{G}}(g) \rightarrow \mathbf{G},$$

where first morphism is smooth by our assumption and the second morphism is a locally closed embedding (see §2.1). Moreover, the quotient $\mathbf{G}/Z_{\mathbf{G}}(g)$ is smooth of dimension $\dim(\mathbf{G}) - \dim(Z_{\mathbf{G}}(g))$. We deduce a factorisation of the induced map on tangent spaces (namely, the difference of id and the adjoint action of g) as

$$\mathfrak{g} \rightarrow T_{[e]}(\mathbf{G}/Z_{\mathbf{G}}(g)) \rightarrow \mathfrak{g},$$

where $[e]$ is the base point of $\mathbf{G}/Z_{\mathbf{G}}(g)$. Here the first map is surjective and the second one is injective. We deduce that

$$\dim(\mathfrak{g}^g) = \dim(\mathfrak{g}) - \dim(\mathbf{G}/Z_{\mathbf{G}}(g)) = \dim(Z_{\mathbf{G}}(g)),$$

proving the desired equality. \square

As explained in §2.1, the morphism $\mathbf{G} \rightarrow \mathbf{G}$ defined by $h \mapsto hgh^{-1}$ factors through a locally closed embedding $\mathbf{G}/Z_{\mathbf{G}}(g) \rightarrow \mathbf{G}$. The image of this embedding is denoted $\mathcal{O}(g)$, and called the adjoint orbit of g . It is a smooth locally closed subscheme of \mathbf{G} , and by [DG, III, §1, Remarque 1.15], its set of \mathbb{K} -points is $\{hgh^{-1} : h \in \mathbf{G}\}$. In particular, this definition coincides with that given e.g. in [Hu, §1.5].

2.3. The case of regular unipotent elements in reductive groups. From now on we assume that \mathbf{G} is connected reductive, and denote by u a regular unipotent element. We denote by $\mathbf{B} \subset \mathbf{G}$ the unique Borel subgroup containing u , by \mathbf{U} its unipotent radical (so that $u \in \mathbf{U}$) and choose a maximal torus $\mathbf{T} \subset \mathbf{B}$. We will denote by $\mathfrak{b}, \mathfrak{t}, \mathfrak{u}$ the Lie algebras of \mathbf{B}, \mathbf{T} and \mathbf{U} respectively. We will also denote by $\mathfrak{R} \subset X^*(\mathbf{T})$ the root system of (\mathbf{G}, \mathbf{T}) , by $\mathfrak{R}^\vee \subset X_*(\mathbf{T})$ the associated coroots, and by $\mathbb{Z}\mathfrak{R} \subset X^*(\mathbf{T})$ and $\mathbb{Z}\mathfrak{R}^\vee \subset X_*(\mathbf{T})$ the root and coroot lattices. We will denote by r the rank of \mathbf{G} (equal to $\dim(\mathbf{T})$), by r_{ss} its semisimple rank (i.e. the rank of $\mathbb{Z}\mathfrak{R}$), and by ℓ the characteristic of \mathbb{K} .

Let $Z(\mathbf{G})$ be the (scheme-theoretic) center of \mathbf{G} . We start with the following observation.

Lemma 2.2. *The group scheme $Z(\mathbf{G})$ is smooth iff the quotient $X^*(\mathbf{T})/\mathbb{Z}\mathfrak{R}$ has no ℓ -torsion.*

Proof. It is well known that $Z(\mathbf{G})$ is the scheme-theoretic intersection of the kernels of the roots of (\mathbf{G}, \mathbf{T}) ; in other words it is the diagonalizable \mathbb{K} -group scheme whose group of characters is $X^*(\mathbf{T})/\mathbb{Z}\mathfrak{R}$. This implies the desired claim. \square

Our main concern in this subsection is to determine when $Z_{\mathbf{G}}(u)$ is smooth. Our starting point is the following result due to Steinberg.

Proposition 2.3 (Steinberg). *Assume that $X_*(\mathbf{T})/\mathbb{Z}\mathfrak{R}^\vee$ has no ℓ -torsion. Then $Z_{\mathbf{U}}(u)$ is smooth of dimension r_{ss} , and we have*

$$\mathfrak{g}^u = \mathfrak{b}^u = \mathfrak{z} \oplus \mathfrak{u}^u,$$

where $\mathfrak{z} \subset \mathfrak{t}$ is the center of \mathfrak{g} , i.e. the intersection of the kernels of the differentials of the roots of (\mathbf{G}, \mathbf{T}) . In particular, $Z_{\mathbf{G}}(u)$ is smooth iff $Z(\mathbf{G})$ is smooth, and in this case multiplication induces an isomorphism of group schemes

$$Z(\mathbf{G}) \times Z_{\mathbf{U}}(u) \xrightarrow{\sim} Z_{\mathbf{G}}(u).$$

Proof. It is proved in [St] (see also [Hu, §§4.4–4.5]) that under our assumptions we have $\mathfrak{g}^u = \mathfrak{b}^u = \mathfrak{z} \oplus \mathfrak{u}^u$, and that $\dim(\mathfrak{u}^u) = \dim(Z_{\mathbf{U}}(u)_{\text{red}}) = r_{\text{ss}}$. (In these references \mathbf{G} is assumed to be semisimple and simply connected, but the proof applies in the present generality.) This implies that $\dim(Z_{\mathbf{U}}(u)) = \dim(\mathfrak{u}^u)$, hence that $Z_{\mathbf{U}}(u)$ is smooth by Lemma 2.1. These results also imply that $Z_{\mathbf{G}}(u)$ is smooth iff $\dim(\mathfrak{z}) = r - r_{\text{ss}}$, i.e. $\dim(\mathfrak{z}) = \dim(Z(\mathbf{G}))$. Now it is easily seen that $\text{Lie}(Z(\mathbf{G})) \subset \mathfrak{z}$, so that this condition is equivalent to $Z(\mathbf{G})$ being smooth. If this condition is satisfied, since all the group schemes involved are smooth, the isomorphism $Z(\mathbf{G}) \times Z_{\mathbf{U}}(u) \xrightarrow{\sim} Z_{\mathbf{G}}(u)$ can be checked at the level of varieties, where it is well known (see e.g. [Hu, End of §4.1]). \square

The other result we will need (for which we do not need any assumption) is the following.

Lemma 2.4. *The group scheme $Z_{\mathbf{U}}(u)$ is smooth, of dimension r_{ss} , and abelian.*

Proof. First we assume that \mathbf{G} is semisimple and simply connected. In this case, the fact that $Z_{\mathbf{U}}(u)$ is smooth of dimension r_{ss} has been established in Proposition 2.3. Once this fact is established, to prove that this group scheme is abelian one can work at the level of varieties. In this setting, this result is discussed in [Hu, §4.7]. (In case ℓ is good this result is due to Springer [Sp], and follows from the fact that $Z_{\mathbf{U}}(u)$ is connected; the case of bad characteristics, which will not be used below, was worked out by Lou.)

We deduce the general case as follows. Let $\mathscr{D}(\mathbf{G})$ be the derived subgroup of \mathbf{G} , and let $\mathbf{G}' \rightarrow \mathscr{D}(\mathbf{G})$ be its simply connected cover. Let \mathbf{B}' be the preimage of \mathbf{B} under the composite morphism $\nu : \mathbf{G}' \rightarrow \mathbf{G}$, and \mathbf{U}' be its unipotent radical. Then ν restricts to a group scheme isomorphism $\mathbf{U}' \xrightarrow{\sim} \mathbf{U}$. In particular, there exists a unique element $u' \in \mathbf{U}'$ mapping to u , and we obtain a group scheme isomorphism

$$Z_{\mathbf{U}'}(u') \xrightarrow{\sim} Z_{\mathbf{U}}(u).$$

By the case treated above we know that $Z_{\mathbf{U}'}(u')$ is smooth, of dimension r_{ss} , and abelian. We deduce that $Z_{\mathbf{U}}(u)$ has the same properties. \square

In the rest of this subsection, for completeness we explain how Proposition 2.3 can be slightly generalized to cover more general reductive groups. First we consider the case of simple groups. (Here by a simple group we mean a semisimple group of adjoint type whose root system is irreducible.)

Lemma 2.5. *Assume that \mathbf{G} is simple, and that $\ell \neq 2$ if \mathbf{G} is of type \mathbf{B} , \mathbf{C} , \mathbf{D} or \mathbf{E}_7 , and $\ell \neq 3$ if \mathbf{G} is of type \mathbf{E}_6 . Then we have $\mathfrak{g}^u = \mathfrak{b}^u$.*

Proof. If \mathbf{G} is of type **A** then the claim can be checked by a matrix calculation. Otherwise, our assumption ensures that $X_*(\mathbf{T})/\mathbb{Z}\mathfrak{R}^\vee$ has no p -torsion, so that the claim follows from Proposition 2.3. \square

We deduce the following more general criterion.

Proposition 2.6. (1) *We have*

$$\mathfrak{b}^u = \mathfrak{z} \oplus \mathfrak{u}^u,$$

where $\mathfrak{z} \subset \mathfrak{t}$ is the center of \mathfrak{g} , i.e. the intersection of the kernels of the differentials of the roots of (\mathbf{G}, \mathbf{T}) . As a consequence, if $X^*(\mathbf{T})/\mathbb{Z}\mathfrak{R}$ has ℓ -torsion then $Z_{\mathbf{G}}(u)$ is not smooth.

(2) *Assume either that $X_*(\mathbf{T})/\mathbb{Z}\mathfrak{R}^\vee$ has no ℓ -torsion, or that ℓ satisfies the following condition: $\ell \neq 2$ if \mathfrak{R} has a component of type **B**, **C**, **D** or **E**₇, and $\ell \neq 3$ if \mathfrak{R} has a component of type **E**₆. Then if $X^*(\mathbf{T})/\mathbb{Z}\mathfrak{R}$ has no ℓ -torsion, the group scheme $Z_{\mathbf{G}}(u)$ is smooth, and multiplication induces an isomorphism of group schemes*

$$Z(\mathbf{G}) \times Z_{\mathbf{U}}(u) \xrightarrow{\sim} Z_{\mathbf{G}}(u).$$

Proof. (1) The equality $\mathfrak{b}^u = \mathfrak{z} \oplus \mathfrak{u}^u$ is easily checked using the fact that u has nontrivial components on any root subgroup corresponding to a simple root. Now the dimension of \mathfrak{z} is the codimension of the image of $\mathbb{k} \otimes_{\mathbb{Z}} \mathbb{Z}\mathfrak{R}$ in \mathfrak{t}^* . If $X^*(\mathbf{T})/\mathbb{Z}\mathfrak{R}$ has ℓ -torsion then it follows that $\dim(\mathfrak{b}^u) > r - r_{\text{ss}} + \dim(\mathfrak{u}^u)$. In view of Lemma 2.4 this implies that $\dim(\mathfrak{g}^u) > r$, hence that $Z_{\mathbf{G}}(u)$ is not smooth by Lemma 2.1.

(2) We assume that $X^*(\mathbf{T})/\mathbb{Z}\mathfrak{R}$ has no ℓ -torsion. In this case $Z(\mathbf{G})$ is smooth by Lemma 2.2. Therefore, as in the proof of Proposition 2.3, to prove the claims it suffices to prove that if one of the two conditions of the statement hold the group scheme $Z_{\mathbf{G}}(u)$ is smooth. In case $X_*(\mathbf{T})/\mathbb{Z}\mathfrak{R}^\vee$ has no ℓ -torsion, this has been proved in Proposition 2.3.

Now we assume that $\ell \neq 2$ if \mathfrak{R} has a component of type **B**, **C**, **D** or **E**₇, and $\ell \neq 3$ if \mathfrak{R} has a component of type **E**₆. By (1), since $X^*(\mathbf{T})/\mathbb{Z}\mathfrak{R}$ has no ℓ -torsion we have $\dim(\mathfrak{b}^u) = r$. Hence, in view of Lemma 2.1, $Z_{\mathbf{G}}(u)$ is smooth iff $\mathfrak{g}^u = \mathfrak{b}^u$. Consider the adjoint semisimple group $\mathbf{G}/Z(\mathbf{G})$; then we have $\text{Lie}(\mathbf{G}/Z(\mathbf{G})) = \mathfrak{g}/\mathfrak{z}$. The kernel of the quotient map $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{z}$ is contained in \mathfrak{b} ; therefore, to prove that $\mathfrak{g}^u = \mathfrak{b}^u$ it suffices to prove that $(\mathfrak{g}/\mathfrak{z})^{\bar{u}} = (\mathfrak{b}/\mathfrak{z})^{\bar{u}}$, where \bar{u} is the image of u in $\mathbf{G}/Z(\mathbf{G})$ (a regular unipotent element). Here $\mathbf{G}/Z(\mathbf{G})$ is a product of simple groups; therefore the desired equality follows from Lemma 2.5. \square

Remark 2.7. In case \mathbf{G} is an adjoint group of type **E**₆ and $\ell = 3$, or \mathbf{G} is an adjoint group of type **E**₇ and $\ell = 2$, the group scheme $Z_{\mathbf{G}}(u)$ is *not* smooth, as can be checked from the tables of [La] and Lemma 2.1. (In particular, Lemma 2.5 does not hold in these cases either.) We do not know what happens for types **B**, **C**, **D** in characteristic 2.

2.4. The unipotent cone and the multiplicative Springer resolution. We continue to assume that \mathbf{G} is a connected reductive group. We fix a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ and a maximal torus $\mathbf{T} \subset \mathbf{B}$, and denote by \mathbf{U} the unipotent radical of \mathbf{B} . The *multiplicative Springer resolution* is the induced variety

$$\tilde{\mathcal{U}} := \mathbf{G} \times^{\mathbf{B}} \mathbf{U},$$

where \mathbf{B} acts on \mathbf{U} by conjugation. Let us denote by $\mathcal{U} \subset \mathbf{G}$ the unipotent cone, i.e. the closed subvariety whose \mathbb{K} -points are the unipotent elements in \mathbf{G} . The adjoint action of \mathbf{G} on itself induces a morphism $\tilde{\mathcal{U}} \rightarrow \mathbf{G}$ which factors through a morphism

$$\pi : \tilde{\mathcal{U}} \rightarrow \mathcal{U}.$$

This morphism is surjective (see [Bo, Theorem 11.10]), hence induces an embedding of \mathbf{G} -modules

$$(2.1) \quad \mathcal{O}(\mathcal{U}) \hookrightarrow \mathcal{O}(\tilde{\mathcal{U}}).$$

We will also denote by $\mathcal{U}_{\text{reg}} \subset \mathcal{U}$ the unique open \mathbf{G} -orbit, which consists of the regular unipotent elements. If $u \in \mathcal{U}_{\text{reg}}$, then using the notation of §2.2 we have $\mathcal{U}_{\text{reg}} = \mathcal{O}(u)$; therefore, the adjoint action induces an isomorphism

$$(2.2) \quad \mathbf{G}/Z_{\mathbf{G}}(u) \xrightarrow{\sim} \mathcal{U}_{\text{reg}}.$$

The choice of \mathbf{B} determines a system of positive roots for (\mathbf{G}, \mathbf{T}) , chosen as the opposites of the roots appearing in the Lie algebra of \mathbf{B} , and therefore a notion of dominant weights. For $\lambda \in X^*(\mathbf{T})$ we denote by $\mathcal{O}_{\tilde{\mathcal{U}}}(\lambda)$ the pullback under the projection $\tilde{\mathcal{U}} \rightarrow \mathbf{G}/\mathbf{B}$ of the line bundle on \mathbf{G}/\mathbf{B} naturally associated with λ .

Lemma 2.8. *If $\lambda \in X^*(\mathbf{T})$ is dominant, then there exists an embedding of \mathbf{G} -modules $\Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)) \hookrightarrow \Gamma(\tilde{\mathcal{U}}, \mathcal{O}_{\tilde{\mathcal{U}}}(\lambda))$. In particular, $\Gamma(\tilde{\mathcal{U}}, \mathcal{O}_{\tilde{\mathcal{U}}}(\lambda)) \neq 0$.*

Proof. The pushforward of the structure sheaf under the projection $\tilde{\mathcal{U}} \rightarrow \mathbf{G}/\mathbf{B}$ is the quasi-coherent sheaf on \mathbf{G}/\mathbf{B} associated with the \mathbf{B} -module $\mathcal{O}(\mathbf{U})$. In particular, the natural morphism from $\mathcal{O}_{\mathbf{G}/\mathbf{B}}$ is induced by the embedding $\mathbb{k} \hookrightarrow \mathcal{O}(\mathbf{U})$, hence is injective. We deduce an embedding $\Gamma(\mathbf{G}/\mathbf{B}, \mathcal{O}_{\mathbf{G}/\mathbf{B}}(\lambda)) \hookrightarrow \Gamma(\tilde{\mathcal{U}}, \mathcal{O}_{\tilde{\mathcal{U}}}(\lambda))$. \square

We denote by $\tilde{\mathcal{U}}_{\text{reg}}$ the inverse image of \mathcal{U}_{reg} in $\tilde{\mathcal{U}}$. For each simple root α we denote by $\mathbf{U}_{(\alpha)} \subset \mathbf{U}$ the subgroup generated by the root subgroups associated with roots different from $-\alpha$ (in other words, the unipotent radical of the parabolic subgroup of \mathbf{G} containing \mathbf{B} associated with the subset $\{\alpha\}$ of the set of simple roots). Then $\mathbf{U}_{(\alpha)}$ is stable under the \mathbf{B} -action, so that we can define

$$\mathcal{D}_{\alpha} := \mathbf{G} \times^{\mathbf{B}} \mathbf{U}_{(\alpha)}.$$

The subvariety \mathcal{D}_{α} is a divisor in $\tilde{\mathcal{U}}$, and we have

$$\mathcal{O}_{\tilde{\mathcal{U}}}(\mathcal{D}_{\alpha}) = \mathcal{O}_{\tilde{\mathcal{U}}}(-\alpha).$$

For any sum of positive roots λ , written as $\lambda = \sum_{\alpha} n_{\alpha} \cdot \alpha$ (where α runs over the simple roots), we set

$$\mathcal{D}_{\lambda} := \sum_{\alpha} n_{\alpha} \cdot \mathcal{D}_{\alpha}.$$

Then \mathcal{D}_{λ} is an effective divisor in $\tilde{\mathcal{U}}$ with

$$\mathcal{O}_{\tilde{\mathcal{U}}}(\mathcal{D}_{\lambda}) = \mathcal{O}_{\tilde{\mathcal{U}}}(-\lambda),$$

and if each coefficient n_{α} is positive the associated closed subscheme $\mathcal{Y}_{\lambda} \subset \tilde{\mathcal{U}}$ satisfies

$$\tilde{\mathcal{U}} \setminus \mathcal{Y}_{\lambda} = \tilde{\mathcal{U}}_{\text{reg}},$$

see e.g. [Hu, §4.1]. It follows that if $j_{\text{reg}} : \tilde{\mathcal{U}}_{\text{reg}} \hookrightarrow \tilde{\mathcal{U}}$ is the embedding, then we have

$$(2.3) \quad (j_{\text{reg}})_* \mathcal{O}_{\tilde{\mathcal{U}}_{\text{reg}}} = \varinjlim_{m \in \mathbb{Z}_{\geq 0}} \mathcal{O}_{\tilde{\mathcal{U}}}(m \cdot \mathcal{D}_\lambda)$$

(where the maps $\mathcal{O}_{\tilde{\mathcal{U}}}(m \cdot \mathcal{D}_\lambda) \rightarrow \mathcal{O}_{\tilde{\mathcal{U}}}((m+1) \cdot \mathcal{D}_\lambda)$ are induced by the natural embedding $\mathcal{O}_{\tilde{\mathcal{U}}} \hookrightarrow \mathcal{O}_{\tilde{\mathcal{U}}}(\mathcal{D}_\lambda)$).

Lemma 2.9. *Let $\lambda = \sum_\alpha n_\alpha \cdot \alpha$, where $n_\alpha > 0$ for all α . Then restriction to $\tilde{\mathcal{U}}_{\text{reg}}$ induces an isomorphism*

$$\varinjlim_{m \in \mathbb{Z}_{\geq 0}} \text{Ext}_{\text{Coh } \mathfrak{G}(\tilde{\mathcal{U}})}^1(\mathcal{O}_{\tilde{\mathcal{U}}}(m \cdot \lambda), \mathcal{O}_{\tilde{\mathcal{U}}}) \xrightarrow{\sim} \text{Ext}_{\text{Coh } \mathfrak{G}(\tilde{\mathcal{U}}_{\text{reg}})}^1(\mathcal{O}_{\tilde{\mathcal{U}}_{\text{reg}}}, \mathcal{O}_{\tilde{\mathcal{U}}_{\text{reg}}}).$$

Proof. The morphism j_{reg} is affine. Therefore, using adjunction we obtain an isomorphism

$$\begin{aligned} \text{Ext}_{\text{Coh } \mathfrak{G}(\tilde{\mathcal{U}}_{\text{reg}})}^1(\mathcal{O}_{\tilde{\mathcal{U}}_{\text{reg}}}, \mathcal{O}_{\tilde{\mathcal{U}}_{\text{reg}}}) &= \text{Ext}_{\text{Coh } \mathfrak{G}(\tilde{\mathcal{U}}_{\text{reg}})}^1(j_{\text{reg}}^* \mathcal{O}_{\tilde{\mathcal{U}}}, \mathcal{O}_{\tilde{\mathcal{U}}_{\text{reg}}}) \\ &\cong \text{Ext}_{\text{QCoh } \mathfrak{G}(\tilde{\mathcal{U}})}^1(\mathcal{O}_{\tilde{\mathcal{U}}}, (j_{\text{reg}})_* \mathcal{O}_{\tilde{\mathcal{U}}_{\text{reg}}}). \end{aligned}$$

The derived functor $R\Gamma(\tilde{\mathcal{U}}, -)$ commutes with filtered direct limits (see e.g. [Li, Lemma 3.9.3.1]), and so does the functor of derived \mathfrak{G} -invariants, see [J1, Lemma I.4.17]. Since the functor $R\text{Hom}_{\text{QCoh } \mathfrak{G}(\tilde{\mathcal{U}})}(\mathcal{O}_{\tilde{\mathcal{U}}}, -)$ is the composition of these two functors (see [MR1, Proposition A.6]), it also satisfies this property, which implies the desired claim in view of (2.3). \square

2.5. Tilting modules and subgroups of the centralizer of a regular unipotent element. From now on we assume that:

- (1) \mathfrak{G} has simply connected derived subgroup (i.e. $X_*(\mathbf{T})/\mathbb{Z}\mathfrak{R}^\vee$ is free);
- (2) ℓ is good for \mathfrak{G} ;
- (3) $X^*(\mathbf{T})/\mathbb{Z}\mathfrak{R}$ has no ℓ -torsion.

The assumption of the derived subgroup of \mathfrak{G} has the following consequence.

Lemma 2.10. (1) *The morphism*

$$\mathcal{O}(\mathcal{U}) \rightarrow \mathcal{O}(\mathcal{U}_{\text{reg}})$$

induced by restriction is an isomorphism.

- (2) *The restriction of π to $\tilde{\mathcal{U}}_{\text{reg}}$ is an isomorphism of varieties $\tilde{\mathcal{U}}_{\text{reg}} \xrightarrow{\sim} \mathcal{U}_{\text{reg}}$.*

Proof. Since \mathcal{U} and $\tilde{\mathcal{U}}$ are unchanged if \mathfrak{G} is replaced by its derived subgroup, we can assume \mathfrak{G} is semisimple (and simply connected). Then (1) follows from the normality of \mathcal{U} (see [Hu, Theorem 4.24(iii)]) and the fact that the complement of \mathcal{U}_{reg} in \mathcal{U} has codimension at least 2 (which itself follows from the facts that \mathcal{U} has finitely many orbits and that non-regular orbits have dimension at most $\dim(\mathcal{U}) - 2$, see [Hu, Remark in §4.1].) For (2), see [Hu, §6.3–6.4]. \square

Lemma 2.11. *We have*

$$H^1(\tilde{\mathcal{U}}, \mathcal{O}_{\tilde{\mathcal{U}}}) = 0,$$

and moreover the \mathfrak{G} -module $\mathcal{O}(\tilde{\mathcal{U}})$ admits a good filtration.

Proof. Assume first that \mathfrak{G} is semisimple. Under our assumption on ℓ , $\tilde{\mathcal{U}}$ identifies with the usual (additive) Springer resolution (see [KLT, Proposition 3]), so that our claims are special cases of [KLT, Theorem 2] and [KLT, Theorem 7]. The general

case follows, using the fact that $\tilde{\mathcal{U}}$ is unchanged if \mathbf{G} is replaced by its derived subgroup $\mathcal{D}(\mathbf{G})$, and that a \mathbf{G} -module V admits a good filtration iff $V|_{\mathcal{D}(\mathbf{G})}$ admits a good filtration. \square

Now we fix $u \in \mathcal{U}_{\text{reg}}$. Under our assumptions, Proposition 2.3 ensures that the centralizer $Z_{\mathbf{G}}(u)$ is smooth. By Lemma 2.2, the center $Z(\mathbf{G})$ is also smooth.

Lemma 2.12. *Let $\mathbf{K} \subset Z_{\mathbf{G}}(u)$ be a subgroup scheme containing $Z(\mathbf{G})$. If for any finite-dimensional tilting \mathbf{G} -module V the embedding*

$$V^{Z_{\mathbf{G}}(u)} \hookrightarrow V^{\mathbf{K}}$$

is an equality, then $\mathbf{K} = Z_{\mathbf{G}}(u)$.

Proof. We will show that if $\mathbf{K} \subset Z_{\mathbf{G}}(u)$ is a proper subgroup scheme containing $Z(\mathbf{G})$, then there exists a finite-dimensional tilting \mathbf{G} -module V such that the embedding $V^{Z_{\mathbf{G}}(u)} \hookrightarrow V^{\mathbf{K}}$ is strict.

We assume (as we may, by surjectivity of π) that $u \in \mathbf{U}$. By Proposition 2.3, multiplication induces a group scheme isomorphism

$$Z_{\mathbf{U}}(u) \times Z(\mathbf{G}) \xrightarrow{\sim} Z_{\mathbf{G}}(u).$$

Since \mathbf{K} is contained in $Z_{\mathbf{G}}(u)$ and contains $Z(\mathbf{G})$, we deduce that we also have

$$(Z_{\mathbf{U}}(u) \cap \mathbf{K}) \times Z(\mathbf{G}) \xrightarrow{\sim} \mathbf{K}.$$

By Lemma 2.4, $Z_{\mathbf{U}}(u)$ is abelian; therefore the same is true for $Z_{\mathbf{G}}(u)$, hence \mathbf{K} is automatically a normal subgroup. We deduce that the quotient $Z_{\mathbf{G}}(u)/\mathbf{K}$ is a smooth affine group scheme of finite type (see §2.1). Since this group scheme is nontrivial by assumption, it follows that $\mathcal{O}(Z_{\mathbf{G}}(u)/\mathbf{K}) \neq \mathbb{k}$.

Let $L \subset \mathcal{O}(Z_{\mathbf{G}}(u)/\mathbf{K})/\mathbb{k}$ be a simple $Z_{\mathbf{G}}(u)/\mathbf{K}$ -submodule (where the action on $\mathcal{O}(Z_{\mathbf{G}}(u)/\mathbf{K})$ is induced by right multiplication), and denote by E its inverse image in $\mathcal{O}(Z_{\mathbf{G}}(u)/\mathbf{K})$. Since $Z_{\mathbf{U}}(u)$ is unipotent, the group $Z_{\mathbf{G}}(u)/\mathbf{K} \cong Z_{\mathbf{U}}(u)/Z_{\mathbf{U}}(u) \cap \mathbf{K}$ is also unipotent; therefore $L \cong \mathbb{k}$, and E fits in an exact sequence of $Z_{\mathbf{G}}(u)/\mathbf{K}$ -modules

$$(2.4) \quad \mathbb{k} \hookrightarrow E \rightarrow \mathbb{k}.$$

Recall that any $Z_{\mathbf{G}}(u)$ -module M determines a \mathbf{G} -equivariant quasi-coherent sheaf $\mathcal{L}_{\mathbf{G}/Z_{\mathbf{G}}(u)}(M)$ on $\mathbf{G}/Z_{\mathbf{G}}(u)$, see [J1, §I.5.8]. In particular, if $f : \mathbf{G}/\mathbf{K} \rightarrow \mathbf{G}/Z_{\mathbf{G}}(u)$ is the projection, then we have

$$f_* \mathcal{O}_{\mathbf{G}/\mathbf{K}} \cong \mathcal{L}_{\mathbf{G}/Z_{\mathbf{G}}(u)}(\mathcal{O}(Z_{\mathbf{G}}(u)/\mathbf{K})),$$

see [J1, §I.5.12, Remark 3]. The functor $\mathcal{L}_{\mathbf{G}/Z_{\mathbf{G}}(u)}(-)$ is exact; in particular, the short exact sequence (2.4) provides a short exact sequence of \mathbf{G} -equivariant coherent sheaves

$$(2.5) \quad \mathcal{O}_{\mathbf{G}/Z_{\mathbf{G}}(u)} \hookrightarrow \mathcal{L}_{\mathbf{G}/Z_{\mathbf{G}}(u)}(E) \rightarrow \mathcal{O}_{\mathbf{G}/Z_{\mathbf{G}}(u)},$$

and the embedding $E \hookrightarrow \mathcal{O}(Z_{\mathbf{G}}(u)/\mathbf{K})$ provides an embedding

$$(2.6) \quad \mathcal{L}_{\mathbf{G}/Z_{\mathbf{G}}(u)}(E) \hookrightarrow f_* \mathcal{O}_{\mathbf{G}/\mathbf{K}}.$$

In view of (2.2), the exact sequence (2.5) can be considered as an exact sequence in $\text{Coh}^{\mathbf{G}}(\mathcal{U}_{\text{reg}})$, or equivalently (see Lemma 2.10(2)) in $\text{Coh}^{\mathbf{G}}(\tilde{\mathcal{U}}_{\text{reg}})$. After choosing a weight λ as in Lemma 2.9, which we furthermore assume to be dominant, this lemma ensures that there exists $m \in \mathbb{Z}_{\geq 0}$ and a short exact sequence

$$(2.7) \quad \mathcal{O}_{\tilde{\mathcal{U}}} \hookrightarrow \mathcal{E} \rightarrow \mathcal{O}_{\tilde{\mathcal{U}}}(m \cdot \lambda)$$

for some $m \in \mathbb{Z}_{\geq 0}$ whose image under j_{reg}^* is (2.5). In particular, \mathcal{E} is a vector bundle on $\tilde{\mathcal{U}}$ such that $j_{\text{reg}}^* \mathcal{E} = \mathcal{L}_{\mathbf{G}/Z_{\mathbf{G}}(u)}(E)$. Since $\tilde{\mathcal{U}}_{\text{reg}}$ is open (hence dense) in $\tilde{\mathcal{U}}$, the morphism

$$\mathcal{E} \rightarrow (j_{\text{reg}})_* \mathcal{L}_{\mathbf{G}/Z_{\mathbf{G}}(u)}(E)$$

induced by adjunction is injective. Taking global sections we deduce an embedding of \mathbf{G} -modules

$$\Gamma(\tilde{\mathcal{U}}, \mathcal{E}) \hookrightarrow \Gamma(\mathbf{G}/Z_{\mathbf{G}}(u), \mathcal{L}_{\mathbf{G}/Z_{\mathbf{G}}(u)}(E)).$$

Composing this map with the morphism obtained from (2.6) by taking global sections, we deduce an embedding of \mathbf{G} -modules

$$(2.8) \quad \Gamma(\tilde{\mathcal{U}}, \mathcal{E}) \hookrightarrow \mathcal{O}(\mathbf{G}/\mathbf{K}).$$

Let us now come back to (2.7), and denote by $\text{Rep}^\infty(\mathbf{G})$ the category of all (not necessarily finite-dimensional) algebraic \mathbf{G} -modules. By Lemma 2.11 we have $H^1(\tilde{\mathcal{U}}, \mathcal{O}_{\tilde{\mathcal{U}}}) = 0$. Therefore, by taking global sections we obtain an exact sequence of \mathbf{G} -modules

$$(2.9) \quad \mathcal{O}(\tilde{\mathcal{U}}) \hookrightarrow \Gamma(\tilde{\mathcal{U}}, \mathcal{E}) \rightarrow \Gamma(\tilde{\mathcal{U}}, \mathcal{O}_{\tilde{\mathcal{U}}}(m \cdot \lambda)).$$

Now the second statement in Lemma 2.11 implies that for any finite-dimensional tilting \mathbf{G} -module V we have

$$\text{Ext}_{\text{Rep}^\infty(\mathbf{G})}^1(V, \mathcal{O}(\tilde{\mathcal{U}})) = 0.$$

On the other hand, using Lemma 2.8 we see that there exists a (finite-dimensional) tilting \mathbf{G} -module V (e.g. the indecomposable tilting \mathbf{G} -module with highest weight $m\lambda$) such that

$$\text{Hom}_{\text{Rep}^\infty(\mathbf{G})}(V, \Gamma(\tilde{\mathcal{U}}, \mathcal{O}_{\tilde{\mathcal{U}}}(m \cdot \lambda))) \neq 0.$$

Applying the functor $\text{Hom}_{\text{Rep}^\infty(\mathbf{G})}(V, -)$ to (2.9) we deduce a strict embedding

$$\text{Hom}_{\text{Rep}^\infty(\mathbf{G})}(V, \mathcal{O}(\tilde{\mathcal{U}})) \hookrightarrow \text{Hom}_{\text{Rep}^\infty(\mathbf{G})}(V, \Gamma(\tilde{\mathcal{U}}, \mathcal{E}))$$

and then, using (2.1) and (2.8), a strict embedding

$$\text{Hom}_{\text{Rep}^\infty(\mathbf{G})}(V, \mathcal{O}(\mathcal{U})) \hookrightarrow \text{Hom}_{\text{Rep}^\infty(\mathbf{G})}(V, \mathcal{O}(\mathbf{G}/\mathbf{K})).$$

By Lemma 2.10(1) and (2.2), this map can be considered as a strict embedding

$$\text{Hom}_{\text{Rep}^\infty(\mathbf{G})}(V, \mathcal{O}(\mathbf{G}/Z_{\mathbf{G}}(u))) \hookrightarrow \text{Hom}_{\text{Rep}^\infty(\mathbf{G})}(V, \mathcal{O}(\mathbf{G}/\mathbf{K})).$$

Now, by Frobenius reciprocity we have isomorphisms

$$\begin{aligned} \text{Hom}_{\text{Rep}^\infty(\mathbf{G})}(V, \mathcal{O}(\mathbf{G}/Z_{\mathbf{G}}(u))) &\cong (V^*)^{Z_{\mathbf{G}}(u)}, \\ \text{Hom}_{\text{Rep}^\infty(\mathbf{G})}(V, \mathcal{O}(\mathbf{G}/\mathbf{K})) &\cong (V^*)^{\mathbf{K}}. \end{aligned}$$

Since V^* is a tilting module, this finishes the proof. \square

2.6. The Springer resolution. We continue with the assumptions of §2.5.

We denote by \mathfrak{g} , \mathfrak{b} , \mathfrak{u} , \mathfrak{t} the Lie algebras of \mathbf{G} , \mathbf{B} , \mathbf{U} , \mathbf{T} . We now consider the more traditional “additive” Springer resolution

$$\tilde{\mathcal{N}} := \mathbf{G} \times^{\mathbf{B}} \mathfrak{u},$$

and the Grothendieck resolution

$$\tilde{\mathfrak{g}} := \mathbf{G} \times^{\mathbf{B}} \mathfrak{b}.$$

Both $\tilde{\mathcal{N}}$ and $\tilde{\mathfrak{g}}$ are vector bundles over the flag variety \mathbf{G}/\mathbf{B} . Moreover there exists a natural morphism $\tilde{\mathfrak{g}} \rightarrow \mathfrak{t}$ induced by the quotient morphism $\mathfrak{b} \rightarrow \mathfrak{b}/\mathfrak{u} \cong \mathfrak{t}$, and $\tilde{\mathcal{N}}$ identifies with the fiber of this map over 0. We will also denote by $\mathfrak{t}_{\text{reg}} \subset \mathfrak{t}$ the open subset of regular elements, i.e. the complement of the union of the kernels of the differentials of the roots of (\mathbf{G}, \mathbf{T}) . (Note that our assumption (3) implies that the differentials of roots are nonzero, so that this open subset is nonempty.)

Lemma 2.13. *There exists a canonical isomorphism of \mathbf{G} -varieties*

$$\mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\text{reg}} \xrightarrow{\sim} \tilde{\mathfrak{g}} \times_{\mathfrak{t}} \mathfrak{t}_{\text{reg}}.$$

Proof. As explained in [J2, p. 188], the adjoint action induces an isomorphism

$$\mathbf{U} \times \mathfrak{t}_{\text{reg}} \xrightarrow{\sim} \mathfrak{b} \times_{\mathfrak{t}} \mathfrak{t}_{\text{reg}},$$

for the morphism $\mathfrak{b} \rightarrow \mathfrak{t}$ considered above. Inducing from \mathbf{B} to \mathbf{G} we deduce the desired isomorphism. \square

Below we will use the following well-known properties of the Springer resolution.

Lemma 2.14. (1) *For any $i > 0$, we have $H^i(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}) = 0$.*

(2) *We have*

$$\mathcal{O}(\tilde{\mathcal{N}}) \cong \text{Ind}_{Z_{\mathbf{G}}(\mathfrak{u})}^{\mathbf{G}}(\mathbb{k}),$$

and moreover this \mathbf{G} -module admits a good filtration.

(3) *We have a canonical isomorphism of $\mathcal{O}(\mathfrak{t}_{\text{reg}})$ -modules*

$$\mathcal{O}(\mathfrak{t}_{\text{reg}}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}(\tilde{\mathfrak{g}}) \cong \mathcal{O}(\mathfrak{t}_{\text{reg}}) \otimes_{\mathbb{k}} \text{Ind}_{\mathbf{T}}^{\mathbf{G}}(\mathbb{k}).$$

Proof. In case \mathbf{G} is semisimple, claim (1) is proved in [KLT, Theorem 2], and the second assertion in (2) is proved in [KLT, Theorem 7]. As in the proof of Lemma 2.11, these statements follow in the general case.

Let now $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone, i.e. the closed subvariety whose \mathbb{K} -points are the nilpotent elements in \mathfrak{g} . Our assumptions imply that the prime number ℓ is *pretty good* in the sense of [Her], see [Her, Lemma 2.12]. In view of [Her, Corollary 5.5], we deduce that there exists a \mathbf{G} -equivariant “Springer isomorphism”

$$\mathcal{U} \xrightarrow{\sim} \mathcal{N}.$$

By [McN, Remark 10], such an isomorphism necessarily restricts to a \mathbf{B} -equivariant isomorphism $\mathbf{U} \xrightarrow{\sim} \mathfrak{u}$, which itself induces a \mathbf{G} -equivariant isomorphism $\tilde{\mathcal{U}} \xrightarrow{\sim} \tilde{\mathcal{N}}$. Hence to finish the proof of (2) it suffices to construct an isomorphism of \mathbf{G} -modules $\mathcal{O}(\tilde{\mathcal{U}}) \cong \text{Ind}_{Z_{\mathbf{G}}(\mathfrak{u})}^{\mathbf{G}}(\mathbb{k})$. However, \mathcal{U} is a normal variety (see the proof of Lemma 2.10); therefore by Zariski’s Main Theorem the projective and birational morphism π induces an isomorphism $\mathcal{O}(\mathcal{U}) \xrightarrow{\sim} \mathcal{O}(\tilde{\mathcal{U}})$, so that the claim follows from Lemma 2.10(1) and (2.2).

Claim (3) follows Lemma 2.13 by taking global sections. \square

\mathbf{B}_n ($n \geq 2$)	\mathbf{C}_n ($n \geq 3$)	\mathbf{D}_n ($n \geq 4$)	\mathbf{E}_6	\mathbf{E}_7	\mathbf{E}_8	\mathbf{F}_4	\mathbf{G}_2
$\ell \neq 2$	$\ell \nmid n$	$\ell \neq 2$	$\ell \neq 3$	$\ell \neq 2$	\emptyset	$\ell \neq 3$	$\ell \neq 2$

 FIGURE 2.1. Conditions on ℓ

2.7. Fixed points of the centralizer of a regular unipotent element. We continue with the assumptions of §§2.5–2.6.

Lemma 2.15. *For any finite-dimensional \mathbf{G} -module V which admits a good filtration and any regular unipotent element $u \in \mathbf{G}$, we have*

$$\dim(V^{Z_{\mathbf{G}}(u)}) = \dim(V_0).$$

Proof. Lemma 2.14(1) and standard arguments imply that the \mathbf{G} -module $\mathcal{O}(\tilde{\mathfrak{g}})$ admits a filtration with associated graded isomorphic to $\mathcal{O}(\mathfrak{t}) \otimes \mathcal{O}(\tilde{\mathcal{N}})$; see [BMR, Proposition 3.4.1] or [MR1, Lemma 4.12] for similar considerations. In particular, this space is free as an $\mathcal{O}(\mathfrak{t})$ -module, it admits a good filtration (by Lemma 2.14(2)), and we have a canonical isomorphism

$$(2.10) \quad \mathbb{k} \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}(\tilde{\mathfrak{g}}) \cong \mathcal{O}(\tilde{\mathcal{N}}).$$

Now, we consider the complex $R\mathbb{I}^{\mathbf{G}}(V \otimes \mathcal{O}(\tilde{\mathfrak{g}}))$, where $\mathbb{I}^{\mathbf{G}}$ is the functor of \mathbf{G} -invariants. Since V and $\mathcal{O}(\tilde{\mathfrak{g}})$ admit good filtrations, the \mathbf{G} -module $V \otimes \mathcal{O}(\tilde{\mathfrak{g}})$ also admits a good filtration (see [J1, Proposition II.4.21]), which implies that our complex is concentrated in degree 0 (see [J1, Proposition II.4.13]). Using [MR1, Proposition A.8], Lemma 2.14(2) and (2.10) we see that

$$\mathbb{k} \otimes_{\mathcal{O}(\mathfrak{t})}^L R\mathbb{I}^{\mathbf{G}}(V \otimes \mathcal{O}(\tilde{\mathfrak{g}})) \cong R\mathbb{I}^{\mathbf{G}}(V \otimes \text{Ind}_{Z_{\mathbf{G}}(u)}^{\mathbf{G}}(\mathbb{k})).$$

As above the right-hand side is concentrated in degree 0, and isomorphic to $V^{Z_{\mathbf{G}}(u)}$. In particular, from this computation one deduces that $(V \otimes \mathcal{O}(\tilde{\mathfrak{g}}))^{\mathbf{G}}$ is free over $\mathcal{O}(\mathfrak{t})$, and that its rank is $\dim(V^{Z_{\mathbf{G}}(u)})$.

On the other hand, using Lemma 2.14(3) we see that

$$\mathcal{O}(\mathfrak{t}_{\text{reg}}) \otimes_{\mathcal{O}(\mathfrak{t})} (V \otimes \mathcal{O}(\tilde{\mathfrak{g}}))^{\mathbf{G}} \cong (V \otimes \mathcal{O}(\mathfrak{t}_{\text{reg}}) \otimes \text{Ind}_{\mathbf{T}}^{\mathbf{G}}(\mathbb{k}))^{\mathbf{G}} \cong V_0 \otimes \mathcal{O}(\mathfrak{t}_{\text{reg}}),$$

so that this $\mathcal{O}(\mathfrak{t}_{\text{reg}})$ -module is free of rank $\dim(V_0)$. The desired equality follows. \square

2.8. Fixed points of regular unipotent elements in tilting modules with quasi-minuscule highest weight. In this subsection we drop our previous assumptions, but assume instead that \mathbf{G} is quasi-simple and simply connected, not of type **A**. We will denote by α_0 its highest short root (i.e. its unique quasi-minuscule dominant weight).

Lemma 2.16. *Assume that $\ell := \text{char}(\mathbb{K})$ satisfies the conditions in Figure 2.1. Then if V is the indecomposable tilting module of highest weight α_0 and $u \in \mathbf{G}$ is regular unipotent, we have*

$$\dim(V^u) = \dim(V_0).$$

Proof. ² The conditions in Figure 2.1 are exactly those which guarantee that the Weyl module of highest weight α_0 is simple, hence tilting (see [Lüb], see also [Ko, Table 1]).

²This proof was explained to one of us by M. Korhonen. Our earlier proof required stronger assumptions on ℓ .

Assume first that \mathbf{G} is of exceptional type. Then the Jordan blocks of u acting on V (in particular their number, which coincides with $\dim(V^u)$) are described in [La]. From these tables we obtain that the number of Jordan blocks is in each case equal to the number of short simple roots in \mathfrak{R} , i.e. to $\dim(V_0)$.

We assume now that \mathbf{G} is of classical type. If \mathbf{G} is of type \mathbf{B}_n then $\mathbf{G} \cong \mathrm{Spin}(2n+1, \mathbb{k})$ and V is the natural $(2n+1)$ -dimensional representation of $\mathrm{SO}(2n+1, \mathbb{k})$, seen as a \mathbf{G} -module through the canonical surjection $\mathrm{Spin}(2n+1, \mathbb{k}) \twoheadrightarrow \mathrm{SO}(2n+1, \mathbb{k})$; the desired claim is then clear. If \mathbf{G} is of type \mathbf{D}_n then V is the adjoint representation. By Proposition 2.6 the dimension of \mathfrak{g}^u is n , which of course coincides with $\dim(\mathfrak{g}_0) = \dim(\mathfrak{t})$.

Finally, we consider the case when \mathbf{G} is of type \mathbf{C}_n , so that $\mathbf{G} \cong \mathrm{Sp}(2n, \mathbb{k})$. If E is the natural $2n$ -dimensional representation of this group, then V is the direct summand of $\wedge^2 E$ (as a \mathbf{G} -module) given by the kernel of the natural morphism $\wedge^2 E \rightarrow \mathbb{k}$ sending $v \otimes v'$ to $\chi(v, v')$, where χ is the alternating form used to define the symplectic group (see e.g. [PS]); from this we see that to conclude it suffices to prove that $\dim((\wedge^2 E)^u) = n$. Now u acting on E has a single Jordan block, so that the desired claim follows from [Lin] or [Ba, Theorem 2.1(2)]. \square

Remark 2.17. M. Korhonen informed us that Lemma 2.16 holds for all characteristics, except maybe in type \mathbf{D} for $\ell = 2$. For instance in type \mathbf{B} for $\ell = 2$, the tilting module is the restriction of the natural module for the group of type \mathbf{D}_{n+1} , and a regular unipotent element for the group of type \mathbf{B}_n is regular in the group of type \mathbf{D}_{n+1} , hence acts on this module with two Jordan blocks.

3. RECONSTRUCTING A SUBGROUP FROM THE RESTRICTION FUNCTOR

In this section we prove a slight extension of [Be1, Proposition 1] which will be required for our arguments.

3.1. Statement. Let (\mathbf{A}, \star) be a monoidal category. Recall that the *Drinfeld center* $Z(\mathbf{A})$ of (\mathbf{A}, \star) is the category whose objects are pairs (X, ι) with X an object in \mathbf{A} and

$$\iota : X \star (-) \xrightarrow{\sim} (-) \star X$$

an isomorphism of functors such that

$$\iota_{Y \star Z} = (\mathrm{id} \star \iota_Z) \circ (\iota_Y \star \mathrm{id})$$

for any Y, Z in \mathbf{A} (where we omit the associativity isomorphism for simplicity), and whose morphisms from (X, ι) to (X', ι') are morphisms $X \rightarrow X'$ in \mathbf{A} which are compatible with the morphisms ι and ι' in the obvious way. The Drinfeld center of a monoidal category admits a natural structure of braided monoidal category, with monoidal product induced by that of \mathbf{A} , and the braiding $(X, \iota) \star (X', \iota') \xrightarrow{\sim} (X', \iota') \star (X, \iota)$ given by ι .

If (\mathbf{B}, \otimes) is a *symmetric* monoidal category, then as explained in [Be1, §2.1] the datum of a braided monoidal functor from \mathbf{B} to $Z(\mathbf{A})$ is equivalent to that of a *central functor* from \mathbf{B} to \mathbf{A} , i.e. a pair consisting of a monoidal functor $F : \mathbf{B} \rightarrow \mathbf{A}$ (with ‘‘monoidality’’ isomorphisms $\phi_{-, -}$) together with bifunctorial isomorphisms

$$\sigma_{X, Y} : F(X) \star Y \xrightarrow{\sim} Y \star F(X)$$

for X in \mathbf{B} and Y in \mathbf{A} , which satisfy the following properties.

- (1) For X, X' in \mathbf{B} , the isomorphism $\sigma_{X, F(X')}$ coincides with the composition

$$F(X) \star F(X') \xrightarrow[\sim]{\phi_{X, X'}^{-1}} F(X \otimes X') \xrightarrow{\sim} F(X' \otimes X) \xrightarrow[\sim]{\phi_{X', X}} F(X') \star F(X)$$

where the middle isomorphism is the image under F of the commutativity constraint of \mathbf{B} (applied to (X, X')).

- (2) For Y_1, Y_2 in \mathbf{A} and X in \mathbf{B} , the composition

$$F(X) \star Y_1 \star Y_2 \xrightarrow[\sim]{\sigma_{X, Y_1} \star \text{id}_{Y_2}} Y_1 \star F(X) \star Y_2 \xrightarrow[\sim]{\text{id}_{Y_1} \star \sigma_{X, Y_2}} Y_1 \star Y_2 \star F(X)$$

coincides with $\sigma_{X, Y_1 \star Y_2}$ (where we omit the associativity constraint of \mathbf{A}).

- (3) For Y in \mathbf{A} and X_1, X_2 in \mathbf{B} , the composition

$$\begin{aligned} F(X_1 \otimes X_2) \star Y &\xrightarrow[\sim]{\phi_{X_1, X_2} \star Y} F(X_1) \star F(X_2) \star Y \xrightarrow[\sim]{\text{id}_{F(X_1)} \star \sigma_{X_2, Y}} F(X_1) \star Y \star F(X_2) \\ &\xrightarrow[\sim]{\sigma_{X_1, Y} \star \text{id}_{F(X_2)}} Y \star F(X_1) \star F(X_2) \xrightarrow[\sim]{\text{id}_Y \star \phi_{X_1, X_2}^{-1}} Y \star F(X_1 \otimes X_2) \end{aligned}$$

coincides with $\sigma_{X_1 \otimes X_2, Y}$.

Our goal in this section is to prove the following statement. Here, we let \mathbb{K} be an algebraically closed field, and for \mathbf{H} an affine \mathbb{K} -group scheme of finite type we denote by $\text{Rep}(\mathbf{H})$ the category of finite-dimensional \mathbf{H} -modules (in other words, $\mathcal{O}(\mathbf{H})$ -comodules). This category admits a natural symmetric monoidal structure, with product given by tensor product over \mathbb{K} .

Proposition 3.1. *Let \mathbf{G} be an affine \mathbb{K} -group scheme of finite type. Let (\mathbf{A}, \otimes) be a \mathbb{K} -linear abelian monoidal category with unit object $\mathbb{1}$ which satisfies $\text{End}(\mathbb{1}) = \mathbb{K}$, and such that \otimes is exact in each variable. We assume furthermore that $\mathbb{1}$ is a simple object of \mathbf{A} , and that $\text{Hom}_{\mathbf{A}}(\mathbb{1}, X)$ is finite-dimensional for all X in \mathbf{A} . Assume we are given an exact, central functor*

$$F : \text{Rep}(\mathbf{G}) \rightarrow \mathbf{A}$$

such that each object in \mathbf{A} is isomorphic to a subquotient of an object $F(V)$ with V in $\text{Rep}(\mathbf{G})$. Then there exists a subgroup scheme $\mathbf{H} \subset \mathbf{G}$ and an equivalence of monoidal categories

$$\Phi : \text{Rep}(\mathbf{H}) \xrightarrow{\sim} \mathbf{A}$$

which satisfies $F \cong \Phi \circ \text{Res}_{\mathbf{H}}^{\mathbf{G}}$, where $\text{Res}_{\mathbf{H}}^{\mathbf{G}} : \text{Rep}(\mathbf{G}) \rightarrow \text{Rep}(\mathbf{H})$ is the restriction functor.

Remark 3.2. (1) The assumption that $\mathbb{1}$ is simple follows from the condition that $\text{End}(\mathbb{1}) = \mathbb{K}$ in case \mathbf{A} is rigid, see [DM, Proposition 1.17].

- (2) The subgroup $\mathbf{H} \subset \mathbf{G}$ is unique only up to conjugation; using the notation introduced in the proof of the proposition, it depends on the subobject \mathcal{J} , which cannot be chosen in any canonical way.

This proposition appears as [Be1, Proposition 1] with slightly different assumptions. (Namely, it is not assumed in [Be1] that $\mathbb{1}$ is simple, but one imposes the extra assumption that \mathbb{K} is uncountable.) Below we show how to adapt the proof in [Be1] in order to fit with the present assumptions. (In later sections we will need to apply this proposition in case \mathbb{K} is an algebraic closure of a finite field, hence in particular *is* countable.) No detail of this proof will be used in later sections.

3.2. Preliminaries. The main ingredient needed to extend the proof of [Bel] will be the following easy commutative algebra lemma.

Lemma 3.3. *Let \mathbb{K} be an algebraically closed field, and let \mathbb{L}/\mathbb{K} be a field extension. Assume there exists a field extension \mathbb{K}'/\mathbb{K} , a finitely generated commutative \mathbb{K}' -algebra A , and an embedding of \mathbb{K}' -algebras $\mathbb{K}' \otimes_{\mathbb{K}} \mathbb{L} \subset A$. Then $\mathbb{L} = \mathbb{K}$.*

Proof. One can of course assume that \mathbb{K}' is algebraically closed. Assume for a contradiction that $\mathbb{L} \neq \mathbb{K}$, and let $a \in \mathbb{L} \setminus \mathbb{K}$. Let $\tilde{a} = 1 \otimes a \in \mathbb{K}' \otimes_{\mathbb{K}} \mathbb{L} \subset A$. One can consider \tilde{a} as a scheme morphism $\text{Spec}(A) \rightarrow \mathbb{A}_{\mathbb{K}'}^1$. The image of this morphism is constructible by Chevalley's theorem, hence is either a finite subset of \mathbb{K}' or the complement of such a subset. Now $\tilde{a} - t$ is invertible in A for any $t \in \mathbb{K}$, so that this image does not intersect \mathbb{K} . We deduce that it is finite, so that there exist $a_1, \dots, a_r \in \mathbb{K}'$ such that the morphism $\text{Spec}(A) \rightarrow \mathbb{A}_{\mathbb{K}'}^1$ associated with the element $\prod_{i=1}^r (\tilde{a} - a_i)$ has image $\{0\}$, i.e. is nilpotent in A . Hence for some $N \geq 1$ the elements $1, \tilde{a}, \dots, \tilde{a}^N$ are linearly dependent (over \mathbb{K}') in $\mathbb{K}' \otimes_{\mathbb{K}} \mathbb{L}$. Then the elements $1, a, \dots, a^N$ must be linearly dependent (over \mathbb{K}) in \mathbb{L} , so that a is algebraic over \mathbb{K} . This is impossible since \mathbb{K} is algebraically closed and $a \notin \mathbb{K}$. \square

We will also require the following lemma.

Lemma 3.4. *Let \mathbb{K} be a field, and let (\mathbf{A}, \otimes) be a \mathbb{K} -linear abelian monoidal category which satisfies the following assumptions:*

- (1) *the unit object $\mathbb{1}$ is simple and satisfies $\text{End}(\mathbb{1}) = \mathbb{K}$;*
- (2) *for any X in \mathbf{A} , the \mathbb{K} -vector space $\text{Hom}_{\mathbf{A}}(\mathbb{1}, X)$ is finite-dimensional;*
- (3) *\otimes is exact in each variable.*

Then for all objects M and N in \mathbf{A} , the morphism

$$\text{Hom}_{\mathbf{A}}(\mathbb{1}, M) \otimes_{\mathbb{K}} \text{Hom}_{\mathbf{A}}(\mathbb{1}, N) \rightarrow \widetilde{\text{Hom}}_{\mathbf{A}}(\mathbb{1}, M \otimes N)$$

which, for $f \in \text{Hom}_{\mathbf{A}}(\mathbb{1}, M)$ and $g \in \text{Hom}_{\mathbf{A}}(\mathbb{1}, N)$, sends $f \otimes g$ to the morphism

$$\mathbb{1} = \mathbb{1} \otimes \mathbb{1} \xrightarrow{f \otimes g} M \otimes N,$$

is injective.

Proof. For any M in \mathbf{A} , since $\dim \text{Hom}_{\mathbf{A}}(\mathbb{1}, M) < \infty$ we can consider the object $\text{Hom}_{\mathbf{A}}(\mathbb{1}, M) \otimes_{\mathbb{K}} \mathbb{1}$, and we have an obvious morphism

$$(3.1) \quad \text{Hom}_{\mathbf{A}}(\mathbb{1}, M) \otimes_{\mathbb{K}} \mathbb{1} \rightarrow M.$$

We claim that this morphism is injective. In fact, its kernel is a subobject of the semisimple object $\text{Hom}_{\mathbf{A}}(\mathbb{1}, M) \otimes_{\mathbb{K}} \mathbb{1}$, hence is itself a direct sum of copies of $\mathbb{1}$. If it were nonzero, then there would exist a morphism $\mathbb{1} \rightarrow \text{Hom}_{\mathbf{A}}(\mathbb{1}, M) \otimes_{\mathbb{K}} \mathbb{1}$ whose composition with (3.1) vanishes. Now we have

$$\text{Hom}_{\mathbf{A}}(\mathbb{1}, \text{Hom}_{\mathbf{A}}(\mathbb{1}, M) \otimes_{\mathbb{K}} \mathbb{1}) \cong \text{Hom}_{\mathbf{A}}(\mathbb{1}, M),$$

and for any nonzero f in the right-hand side, the composition of its image in the left-hand side with (3.1) is f , hence is nonzero. The kernel under consideration must therefore vanish.

Now, consider a second object N in \mathbf{A} . Tensoring (3.1) on the right with N (which is an exact functor by assumption) we deduce an embedding

$$\text{Hom}_{\mathbf{A}}(\mathbb{1}, M) \otimes_{\mathbb{K}} N \hookrightarrow M \otimes N.$$

Finally, applying the left-exact functor $\mathrm{Hom}_{\mathbf{A}}(\mathbb{1}, -)$ we deduce an embedding

$$\mathrm{Hom}_{\mathbf{A}}(\mathbb{1}, M) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathbf{A}}(\mathbb{1}, N) \hookrightarrow \mathrm{Hom}_{\mathbf{A}}(\mathbb{1}, M \otimes N),$$

which is easily seen to be given by the formula in the statement. \square

3.3. Proof of Proposition 3.1. In this subsection we assume that the conditions in Proposition 3.1 are satisfied, and we explain how to prove [Be1, Lemma 1] without relying on \mathbb{K} being uncountable. Once that is in place, the argument for Proposition 3.1 follows in the same manner as for [Be1, Proposition 1].

We recall the necessary notation from *loc. cit.* Let $\underline{\mathcal{O}}_{\mathbf{G}}$ denote the ring of regular functions on \mathbf{G} regarded as a commutative ring object in the symmetric monoidal category $\mathrm{Ind}(\mathrm{Rep}(\mathbf{G}))$ of ind-objects in $\mathrm{Rep}(\mathbf{G})$ with \mathbf{G} -action coming from left multiplication of \mathbf{G} on itself. Then, still denoting by F the natural extension of this functor to ind-objects, $F(\underline{\mathcal{O}}_{\mathbf{G}})$ inherits a structure of ring object in the category $\mathrm{Ind}(\mathbf{A})$ of ind-objects in \mathbf{A} . Let $\mathcal{J} \subset F(\underline{\mathcal{O}}_{\mathbf{G}})$ be a maximal left ideal subobject; then as explained in [Be1, p. 73], \mathcal{J} is automatically a right ideal subobject also, so that $\underline{\mathcal{O}}_{\mathbf{H}} := F(\underline{\mathcal{O}}_{\mathbf{G}})/\mathcal{J}$ acquires a natural ring object (in $\mathrm{Ind}(\mathbf{A})$) structure. (For details justifying the existence of a maximal left ideal subobject, see [AR3].) The multiplication map for this algebra will be denoted $m_{\underline{\mathcal{O}}_{\mathbf{H}}} : \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}} \rightarrow \underline{\mathcal{O}}_{\mathbf{H}}$, and the unit (inherited from the unital structure of $\underline{\mathcal{O}}_{\mathbf{G}}$) will be denoted $\iota : \mathbb{1} \rightarrow \underline{\mathcal{O}}_{\mathbf{H}}$.

We may now consider the category of (left) $\underline{\mathcal{O}}_{\mathbf{H}}$ -module ind-objects in \mathbf{A} ; we denote morphisms in this category by $\mathrm{Hom}_{\underline{\mathcal{O}}_{\mathbf{H}}}(-, -)$. Since $\underline{\mathcal{O}}_{\mathbf{H}}$ is simple when regarded as an $\underline{\mathcal{O}}_{\mathbf{H}}$ -module, the \mathbb{K} -algebra $\mathbb{L} := \mathrm{End}_{\underline{\mathcal{O}}_{\mathbf{H}}}(\underline{\mathcal{O}}_{\mathbf{H}})$ is a division algebra. The content of [Be1, Lemma 1] is then the following.

Lemma 3.5. *We have $\mathbb{L} = \mathbb{K}$.*

The proof of this lemma will be given at the end of the subsection, after some preliminary results that we now consider.

Lemma 3.6. *There exists a canonical isomorphism*

$$\varsigma : \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}} \xrightarrow{\sim} \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}}$$

which satisfies the following equalities:

- (1) for any morphism $f : \delta^0 \rightarrow \underline{\mathcal{O}}_{\mathbf{H}}$ we have $\varsigma \circ (\mathrm{id}_{\underline{\mathcal{O}}_{\mathbf{H}}} \otimes f) = f \otimes \mathrm{id}_{\underline{\mathcal{O}}_{\mathbf{H}}}$ (where we identify $\mathbb{1} \otimes \underline{\mathcal{O}}_{\mathbf{H}}$ and $\underline{\mathcal{O}}_{\mathbf{H}} \otimes \mathbb{1}$ with $\underline{\mathcal{O}}_{\mathbf{H}}$ using the unit constraint);
- (2) $m_{\underline{\mathcal{O}}_{\mathbf{H}}} \circ \varsigma = m_{\underline{\mathcal{O}}_{\mathbf{H}}}$;
- (3) $(\mathrm{id} \otimes m_{\underline{\mathcal{O}}_{\mathbf{H}}}) \circ (\varsigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varsigma) = \varsigma \circ (m_{\underline{\mathcal{O}}_{\mathbf{H}}} \otimes \mathrm{id})$;
- (4) $(m_{\underline{\mathcal{O}}_{\mathbf{H}}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varsigma) \circ (\varsigma \otimes \mathrm{id}) = \varsigma \circ (\mathrm{id} \otimes m_{\underline{\mathcal{O}}_{\mathbf{H}}})$;

Proof. As part of the central structure on F , we have a canonical isomorphism

$$\sigma_{\underline{\mathcal{O}}_{\mathbf{G}}, F(\underline{\mathcal{O}}_{\mathbf{G}})} : F(\underline{\mathcal{O}}_{\mathbf{G}}) \otimes F(\underline{\mathcal{O}}_{\mathbf{G}}) \xrightarrow{\sim} F(\underline{\mathcal{O}}_{\mathbf{G}}) \otimes F(\underline{\mathcal{O}}_{\mathbf{G}}).$$

By functoriality this isomorphism sends $F(\underline{\mathcal{O}}_{\mathbf{G}}) \otimes \mathcal{J}$ to $\mathcal{J} \otimes F(\underline{\mathcal{O}}_{\mathbf{G}})$. Moreover, this isomorphism is also (via the appropriate identifications) the image under F of the commutativity constraint in $\mathrm{Rep}(\mathbf{G})$, which is symmetric; therefore it satisfies $\sigma_{\underline{\mathcal{O}}_{\mathbf{G}}, F(\underline{\mathcal{O}}_{\mathbf{G}})} \circ \sigma_{F(\underline{\mathcal{O}}_{\mathbf{G}}), \underline{\mathcal{O}}_{\mathbf{G}}} = \mathrm{id}$, hence also sends $\mathcal{J} \otimes F(\underline{\mathcal{O}}_{\mathbf{G}})$ to $F(\underline{\mathcal{O}}_{\mathbf{G}}) \otimes \mathcal{J}$. It follows that this isomorphism sends $\mathcal{J} \otimes F(\underline{\mathcal{O}}_{\mathbf{G}}) + F(\underline{\mathcal{O}}_{\mathbf{G}}) \otimes \mathcal{J}$ to itself, so that it induces the wished-for isomorphism

$$\varsigma : \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}} \xrightarrow{\sim} \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}}.$$

All the desired equalities hold for the corresponding structures on $\underline{\mathcal{O}}_{\mathbf{G}}$ in the category $\text{Ind}(\text{Rep}(\mathbf{G}))$; hence they hold for $F(\underline{\mathcal{O}}_{\mathbf{G}})$ in $\text{Ind}(\mathbf{A})$, and finally for its quotient $\underline{\mathcal{O}}_{\mathbf{H}}$. \square

Corollary 3.7. *The division algebra \mathbb{L} is commutative, hence a field.*

Proof. Composition with ι defines an isomorphism

$$\mathbb{L} \xrightarrow{\sim} \text{Hom}_{\text{Ind}(\mathbf{A})}(\mathbb{1}, \underline{\mathcal{O}}_{\mathbf{H}}),$$

with inverse given by $f \mapsto m_{\underline{\mathcal{O}}_{\mathbf{H}}} \circ (\text{id}_{\underline{\mathcal{O}}_{\mathbf{H}}} \otimes f)$. In these terms multiplication can be described as follows: given $f, g : \mathbb{1} \rightarrow \underline{\mathcal{O}}_{\mathbf{H}}$, the product $f \cdot g$ is the composition

$$\mathbb{1} = \mathbb{1} \otimes \mathbb{1} \xrightarrow{f \otimes g} \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}} \xrightarrow{m_{\underline{\mathcal{O}}_{\mathbf{H}}}} \underline{\mathcal{O}}_{\mathbf{H}}.$$

Using the isomorphism $\varsigma_{\underline{\mathcal{O}}_{\mathbf{H}}}$ and properties (1)–(2) in Lemma 3.6, from this description it is not difficult to check that \mathbb{L} is commutative, hence a field. \square

Lemma 3.8. (1) *For any V in $\text{Rep}(\mathbf{G})$ we have a canonical isomorphism of $\underline{\mathcal{O}}_{\mathbf{H}}$ -modules*

$$\underline{\mathcal{O}}_{\mathbf{H}} \otimes F(V) \cong \underline{\mathcal{O}}_{\mathbf{H}} \otimes_{\mathbb{K}} V.$$

(2) *For any M in \mathbf{A} , the $\underline{\mathcal{O}}_{\mathbf{H}}$ -module $\underline{\mathcal{O}}_{\mathbf{H}} \otimes M$ is isomorphic to a finite direct sum of copies of $\underline{\mathcal{O}}_{\mathbf{H}}$.*

(3) *The functor $M \mapsto \text{Hom}_{\text{Ind}(\mathbf{A})}(\mathbb{1}, \underline{\mathcal{O}}_{\mathbf{H}} \otimes M)$ is exact.*

Proof. (1) We have a canonical isomorphism of $\underline{\mathcal{O}}_{\mathbf{G}}$ -modules

$$\underline{\mathcal{O}}_{\mathbf{G}} \otimes V \xrightarrow{\sim} \underline{\mathcal{O}}_{\mathbf{G}} \otimes_{\mathbb{K}} \text{For}^{\mathbf{G}}(V),$$

where in the left-hand side we consider the diagonal \mathbf{G} -action and on the right-hand side we consider the action on $\underline{\mathcal{O}}_{\mathbf{G}}$ only. Applying F we deduce an isomorphism of $F(\underline{\mathcal{O}}_{\mathbf{G}})$ -modules

$$F(\underline{\mathcal{O}}_{\mathbf{G}}) \otimes F(V) \xrightarrow{\sim} F(\underline{\mathcal{O}}_{\mathbf{G}}) \otimes_{\mathbb{K}} V.$$

Being an isomorphism of $F(\underline{\mathcal{O}}_{\mathbf{G}})$ -modules, this morphism must send

$$\mathcal{J} \cdot (F(\underline{\mathcal{O}}_{\mathbf{G}}) \otimes F(V)) = \mathcal{J} \otimes F(V)$$

to

$$\mathcal{J} \cdot (F(\underline{\mathcal{O}}_{\mathbf{G}}) \otimes_{\mathbb{K}} V) = \mathcal{J} \otimes_{\mathbb{K}} V;$$

we deduce the desired isomorphism by passing to the quotients by these submodules.

(2) By assumption, M is a subquotient of $F(V)$ for some V in $\text{Rep}(\mathbf{G})$. Then, in view of (1) the $\underline{\mathcal{O}}_{\mathbf{H}}$ -module $\underline{\mathcal{O}}_{\mathbf{H}} \otimes M$ is a subquotient of $\underline{\mathcal{O}}_{\mathbf{H}} \otimes_{\mathbb{K}} V$. The latter object being a direct sum of copies of the simple module $\underline{\mathcal{O}}_{\mathbf{H}}$, $\underline{\mathcal{O}}_{\mathbf{H}} \otimes M$ must also be a direct sum of copies of this module.

(3) Consider an exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ in \mathbf{A} . By exactness of \otimes , tensoring with $\underline{\mathcal{O}}_{\mathbf{H}}$ we deduce an exact sequence

$$(3.2) \quad 0 \rightarrow \underline{\mathcal{O}}_{\mathbf{H}} \otimes M_1 \rightarrow \underline{\mathcal{O}}_{\mathbf{H}} \otimes M_2 \rightarrow \underline{\mathcal{O}}_{\mathbf{H}} \otimes M_3 \rightarrow 0$$

in $\text{Ind}(\mathbf{A})$. Then, applying the left-exact functor $\text{Hom}_{\text{Ind}(\mathbf{A})}(\mathbb{1}, -)$ we deduce a left exact sequence

$$0 \rightarrow \text{Hom}_{\text{Ind}(\mathbf{A})}(\mathbb{1}, \underline{\mathcal{O}}_{\mathbf{H}} \otimes M_1) \rightarrow \text{Hom}_{\text{Ind}(\mathbf{A})}(\mathbb{1}, \underline{\mathcal{O}}_{\mathbf{H}} \otimes M_2) \rightarrow \text{Hom}_{\text{Ind}(\mathbf{A})}(\mathbb{1}, \underline{\mathcal{O}}_{\mathbf{H}} \otimes M_3).$$

Now, each $\text{Hom}_{\text{Ind}(\mathbf{A})}(\mathbb{1}, \underline{\mathcal{O}}_{\mathbf{H}} \otimes M_i)$ has a natural structure of \mathbb{L} -vector space, and each map in this sequence is \mathbb{L} -linear. The dimension of $\text{Hom}_{\text{Ind}(\mathbf{A})}(\mathbb{1}, \underline{\mathcal{O}}_{\mathbf{H}} \otimes M_i)$ is the number of copies of $\underline{\mathcal{O}}_{\mathbf{H}}$ appearing in $\underline{\mathcal{O}}_{\mathbf{H}} \otimes M_i$. The exact sequence (3.2)

shows that the number of copies in $\underline{\mathcal{O}}_{\mathbf{H}} \otimes M_2$ is the sum of the number of copies in $\underline{\mathcal{O}}_{\mathbf{H}} \otimes M_1$ and $\underline{\mathcal{O}}_{\mathbf{H}} \otimes M_3$. Therefore the morphism $\mathrm{Hom}_{\mathrm{Ind}(\mathbf{A})}(\mathbb{1}, \underline{\mathcal{O}}_{\mathbf{H}} \otimes M_2) \rightarrow \mathrm{Hom}_{\mathrm{Ind}(\mathbf{A})}(\mathbb{1}, \underline{\mathcal{O}}_{\mathbf{H}} \otimes M_3)$ must be surjective, showing that the above sequence is exact. \square

Proof of Lemma 3.5. We set

$$R := \mathrm{Hom}_{\mathrm{Ind}(\mathbf{A})}(\mathbb{1}, \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}}).$$

Properties (3)–(4) in Lemma 3.6 ensure that the composition

$$(\underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}}) \otimes (\underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}}) \xrightarrow{\mathrm{id} \otimes \iota \otimes \mathrm{id}} \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}} \xrightarrow{m_{\underline{\mathcal{O}}_{\mathbf{H}}} \otimes m_{\underline{\mathcal{O}}_{\mathbf{H}}}} \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}}$$

defines a structure of ring-object on $\underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}}$; hence, copying the description of the product in $\mathrm{Hom}_{\mathrm{Ind}(\mathbf{A})}(\mathbb{1}, \underline{\mathcal{O}}_{\mathbf{H}})$, we see that R admits a natural ring structure. As for \mathbb{L} one sees that this ring is commutative.

The embeddings

$$\underline{\mathcal{O}}_{\mathbf{H}} = \underline{\mathcal{O}}_{\mathbf{H}} \otimes \mathbb{1} \xrightarrow{\mathrm{id} \otimes \iota} \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}} \quad \text{and} \quad \underline{\mathcal{O}}_{\mathbf{H}} = \mathbb{1} \otimes \underline{\mathcal{O}}_{\mathbf{H}} \xrightarrow{\iota \otimes \mathrm{id}} \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}}$$

induce two \mathbb{K} -algebra morphisms $\mathbb{L} \rightarrow R$. We use the first one to turn R into an \mathbb{L} -algebra. Considering also the second one we obtain an \mathbb{L} -linear morphism

$$\mathbb{L} \otimes_{\mathbb{K}} \mathbb{L} \rightarrow \mathrm{Hom}_{\mathbf{A}}(\mathbb{1}, \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}})$$

which, for $f, g : \mathbb{1} \rightarrow \underline{\mathcal{O}}_{\mathbf{H}}$, sends $f \otimes g$ to the morphism

$$\mathbb{1} = \mathbb{1} \otimes \mathbb{1} \xrightarrow{f \otimes g} \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}}.$$

By Lemma 3.4 this morphism is injective. It is also easily seen to be an \mathbb{L} -algebra morphism.

Now by exactness of \otimes we have a surjective morphism

$$\underline{\mathcal{O}}_{\mathbf{H}} \otimes F(\underline{\mathcal{O}}_{\mathbf{G}}) \twoheadrightarrow \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}},$$

which by Lemma 3.8(1) can be interpreted as a surjective morphism

$$\underline{\mathcal{O}}_{\mathbf{H}} \otimes_{\mathbb{K}} \mathcal{O}(\mathbf{G}) \twoheadrightarrow \underline{\mathcal{O}}_{\mathbf{H}} \otimes \underline{\mathcal{O}}_{\mathbf{H}}.$$

This morphism is an algebra-object morphism, where the structure on the left-hand side is the tensor product of the ring structures on $\underline{\mathcal{O}}_{\mathbf{H}}$ and $\mathcal{O}(\mathbf{G})$, and the structure on the right-hand side is as above. Applying the functor $\mathrm{Hom}_{\mathrm{Ind}(\mathbf{A})}(\mathbb{1}, -)$, in view of Lemma 3.8(3) we deduce a surjective \mathbb{L} -algebra morphism

$$\mathbb{L} \otimes_{\mathbb{K}} \mathcal{O}(\mathbf{G}) \twoheadrightarrow R,$$

which shows that R is finitely generated over \mathbb{L} .

We have now checked the assumptions of Lemma 3.3 for $A = R$ and $\mathbb{K}' = \mathbb{L}$; this lemma implies that $\mathbb{L} = \mathbb{K}$, which finishes the proof. \square

4. GEOMETRIC SATAKE, CENTRAL SHEAVES AND WAKIMOTO SHEAVES

In this section we provide a reminder on the geometric Satake equivalence and the construction of “central sheaves” on the affine flag variety.

4.1. The affine Grassmannian and the geometric Satake equivalence. We let G be a connected reductive algebraic group over an algebraically closed field \mathbb{F} of characteristic $p > 0$, and \mathbb{k} be an algebraic closure of a finite field of characteristic $\ell \neq p$.

We will denote by LG , resp. L^+G , the functor from \mathbb{F} -algebras to groups defined by

$$R \mapsto G(R((z))), \quad \text{resp.} \quad G(R[[z]]),$$

where z is an indeterminate. It is well known that the functor LG is represented by an ind-affine group ind-scheme (which will be denoted similarly), and that L^+G is represented by an affine group scheme (which will also be denoted similarly). One can then define the affine Grassmannian Gr_G as the fppf quotient

$$\text{Gr}_G = (LG/L^+G)_{\text{fppf}}.$$

It is well known also that Gr_G is represented by an ind-projective ind-scheme of ind-finite type, which will once again be denoted similarly.

If \mathbb{k}' is a finite subfield of \mathbb{k} , then we can consider the L^+G -equivariant derived category of \mathbb{k}' -sheaves on Gr_G , which will be denoted $D_{\text{sph}}^{\text{b}}(\text{Gr}_G, \mathbb{k}')$. (The definition of this category requires a little bit of care; see [G1, §6] or [BR, §1.16.4] for discussions of these subtleties.) Taking the direct limit of these categories over all finite subfields of \mathbb{k} one obtains a category which will be denoted $D_{\text{sph}}^{\text{b}}(\text{Gr}_G, \mathbb{k})$, and which we will consider informally as the L^+G -equivariant derived category of \mathbb{k} -sheaves on Gr_G . This category admits a natural convolution product \star , and an associativity constraint φ for this product. The skyscraper sheaf δ_{Gr} at the base point of Gr_G is a unit for \star , so that we obtain a monoidal category $(D_{\text{sph}}^{\text{b}}(\text{Gr}_G, \mathbb{k}), \star, \varphi, \delta_{\text{Gr}})$.

On $D_{\text{sph}}^{\text{b}}(\text{Gr}_G, \mathbb{k})$ we also have a natural perverse t-structure, whose heart will be denoted $\text{Perv}_{\text{sph}}(\text{Gr}_G, \mathbb{k})$. The bifunctor \star restricts to an exact bifunctor

$$\text{Perv}_{\text{sph}}(\text{Gr}_G, \mathbb{k}) \times \text{Perv}_{\text{sph}}(\text{Gr}_G, \mathbb{k}) \rightarrow \text{Perv}_{\text{sph}}(\text{Gr}_G, \mathbb{k}),$$

see [BR, §1.6.3 and §1.10.3] for details and references; moreover the restriction of \star to the category $\text{Perv}_{\text{sph}}(\text{Gr}_G, \mathbb{k})$ admits a commutativity constraint which can be constructed (following an idea of Drinfeld) using a description of this product as a “fusion product;” see [BR, §1.7] for details. In this way, $\text{Perv}_{\text{sph}}(\text{Gr}_G, \mathbb{k})$ comes equipped with the structure of an abelian symmetric monoidal category.

Using a detailed study of the so-called “weight functors,” Mirković–Vilonen construct in [MV] an affine \mathbb{k} -group scheme $\tilde{G}_{\mathbb{k}}$ and an equivalence of abelian symmetric monoidal categories

$$(4.1) \quad \mathcal{S} : \text{Perv}_{\text{sph}}(\text{Gr}_G, \mathbb{k}) \xrightarrow{\sim} \text{Rep}(\tilde{G}_{\mathbb{k}}),$$

where the right-hand side denotes the category of finite-dimensional $\tilde{G}_{\mathbb{k}}$ -modules. Following earlier work of Lusztig [Lu], Ginzburg [Gi] and Beilinson–Drinfeld [BD], they then prove the following.

Theorem 4.1 (Geometric Satake equivalence, [MV]). *The group scheme $\tilde{G}_{\mathbb{k}}$ is a connected reductive group, and its root datum is dual to that of G .*

See [BR] for a detailed study of the proof of Theorem 4.1. In view of this theorem, in the rest of the paper we will denote the reductive group $\tilde{G}_{\mathbb{k}}$ by $G_{\mathbb{k}}^{\vee}$. In the course of the proof of Theorem 4.1, Mirković and Vilonen construct a canonical maximal torus of $G_{\mathbb{k}}^{\vee}$; we will denote this maximal torus by $T_{\mathbb{k}}^{\vee}$. (More precisely, given a choice of

Borel subgroup $B \subset G$ and of maximal torus $T \subset B$, one obtains “weight functors” which allow to define a maximal torus $T_{\mathbb{k}}^{\vee} \subset G_{\mathbb{k}}^{\vee}$ with a canonical identification $X^*(T_{\mathbb{k}}^{\vee}) = X_*(T)$. The authors then prove that the resulting subgroup $T_{\mathbb{k}}^{\vee}$ does not depend on the choice of B and T , see [BR, §1.5.5] for details.)

4.2. The affine flag variety and Gaitsgory’s central functor. We now choose a Borel subgroup $B \subset G$. We have a natural group scheme morphism

$$\mathrm{ev} : L^+G \rightarrow G,$$

induced by the evaluation of z at 0. The preimage of B under this morphism will be denoted Iw . One can then define the *affine flag variety* as the fppf quotient

$$\mathrm{Fl}_G = (LG/\mathrm{Iw})_{\mathrm{fppf}}.$$

As for Gr_G , this functor is represented by an ind-projective ind-scheme of ind-finite type. Moreover the natural projection $\pi : \mathrm{Fl}_G \rightarrow \mathrm{Gr}_G$ is a smooth projective morphism, all of whose fibers are isomorphic to the flag variety G/B .

As for the category $D_{\mathrm{sph}}^{\mathrm{b}}(\mathrm{Gr}_G, \mathbb{k})$, there exists a natural structure of monoidal category on the Iw-equivariant derived category $D_{\mathrm{Iw}}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k})$, whose product will be denoted \star_{Iw} , and whose unit object is the skyscraper sheaf δ_{Fl} at the base point of Fl_G . This category also admits a perverse t-structure, whose heart will be denoted P_{Iw} ; however in this setting it is *not* true that a convolution of perverse sheaves is perverse.

Following an earlier construction of Gaitsgory [G1] together with a remark of Heinloth [Hei], Zhu constructs in [Zh] an ind-scheme

$$(4.2) \quad \mathrm{Gr}_G^{\mathrm{Cen}} \rightarrow \mathbb{A}^1$$

whose fiber over 0 identifies canonically with Fl_G , and whose restriction to $\mathbb{A}^1 \setminus \{0\}$ identifies with $\mathrm{Gr}_G \times (\mathbb{A}^1 \setminus \{0\})$. Using this ind-scheme one can then define the “central functor”

$$\mathrm{Z} : \mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k}) \rightarrow \mathrm{Perv}(\mathrm{Fl}_G, \mathbb{k})$$

by setting

$$\mathrm{Z}(\mathcal{A}) := \Psi_{\mathrm{Gr}_G^{\mathrm{Cen}}}(\mathcal{A} \boxtimes_{\mathbb{k}} \mathbb{k}_{\mathbb{A}^1 \setminus \{0\}}[1]),$$

where $\Psi_{\mathrm{Gr}_G^{\mathrm{Cen}}}$ is the nearby cycles functor associated with the morphism (4.2). (Here $\mathrm{Perv}(\mathrm{Fl}_G, \mathbb{k})$ denotes the category of all \mathbb{k} -perverse sheaves on Fl_G .)

This functor will be fundamental for our constructions. Its main properties are summarized in the following statement.

Theorem 4.2 (Gaitsgory). *(1) For any $\mathcal{A} \in \mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$ the perverse sheaf $\mathrm{Z}(\mathcal{A})$ is Iw-equivariant; in other words, the functor Z factors through a functor*

$$\mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k}) \rightarrow \mathrm{P}_{\mathrm{Iw}},$$

which will be denoted similarly.

(2) For any $\mathcal{A} \in \mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$ the perverse sheaf $\mathrm{Z}(\mathcal{A})$ is convolution-exact, in the sense that for any $\mathcal{F} \in \mathrm{Perv}(\mathrm{Fl}_G, \mathbb{k})$ the convolution product $\mathcal{F} \star_{\mathrm{Iw}} \mathrm{Z}(\mathcal{A})$ is a perverse sheaf.

(3) For any $\mathcal{A} \in \mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$ and $\mathcal{F} \in D_{\mathrm{Iw}}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k})$, there exists a canonical (in particular, bifunctorial) isomorphism

$$\sigma_{\mathcal{A}, \mathcal{F}} : \mathrm{Z}(\mathcal{A}) \star_{\mathrm{Iw}} \mathcal{F} \xrightarrow{\sim} \mathcal{F} \star_{\mathrm{Iw}} \mathrm{Z}(\mathcal{A}).$$

(4) *The composition*

$$\mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k}) \xrightarrow{\mathbb{Z}} \mathrm{P}_{\mathrm{Iw}} \rightarrow D_{\mathrm{Iw}}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k})$$

has a canonical monoidal structure; in other words we have an identification $\mathbb{Z}(\delta_{\mathrm{Gr}}) = \delta_{\mathrm{Fl}}$, for $\mathcal{A}, \mathcal{B} \in \mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$ there is a canonical isomorphism

$$\phi_{\mathcal{A}, \mathcal{B}} : \mathbb{Z}(\mathcal{A} \star \mathcal{B}) \xrightarrow{\sim} \mathbb{Z}(\mathcal{A}) \star_{\mathrm{Iw}} \mathbb{Z}(\mathcal{B}),$$

and these isomorphisms intertwine the associativity constraints of the monoidal categories $\mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$ and $D_{\mathrm{Iw}}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k})$ in the obvious way.

(5) For $\mathcal{A} \in \mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$, we have a canonical isomorphism

$$\pi_*(\mathbb{Z}(\mathcal{A})) \xrightarrow{\sim} \mathcal{A}.$$

For the original proof of these properties (in the setting where \mathbb{k} is replaced by $\overline{\mathbb{Q}}_\ell$, and using a different version of the ind-scheme $\mathrm{Gr}_G^{\mathrm{Cen}}$), see [G1].³ For a modification of these proofs using the present definition of $\mathrm{Gr}_G^{\mathrm{Cen}}$ (but still in the setting where \mathbb{k} is replaced by $\overline{\mathbb{Q}}_\ell$), see [Zh, §7]. For a detailed review of this proof, which allows more general coefficient rings, see [AR3].

Recall the notion of a central functor (see §3.1). By Theorem 4.2 the functor \mathbb{Z} admits a natural monoidal structure, and we have the natural isomorphisms $\sigma_{\mathcal{A}, \mathcal{F}}$ for \mathcal{A} in $\mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$ and \mathcal{F} in $D_{\mathrm{Iw}}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k})$. These data are exactly the ingredients that enter the definition of central functors. The following statement says that this triple indeed is a central functor.

Theorem 4.3 (Gaitsgory). *The triple $(\mathbb{Z}, \phi_{-, -}, \sigma_{-, -})$ is a central functor from $\mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$ to $D_{\mathrm{Iw}}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k})$.*

See [G2] for the original proof of Theorem 4.3, written in the setting where \mathbb{k} is replaced by $\overline{\mathbb{Q}}_\ell$. See [AR3] for a detailed review of this proof, adapted to the present setting.

4.3. Monodromy. For any \mathcal{A} in $\mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$, since the perverse sheaf $\mathbb{Z}(\mathcal{A})$ is defined in terms of nearby cycles, it comes equipped with a canonical *monodromy automorphism*⁴

$$\mathfrak{m}_{\mathcal{A}} : \mathbb{Z}(\mathcal{A}) \xrightarrow{\sim} \mathbb{Z}(\mathcal{A}).$$

Below we will need the following properties of these automorphisms.

Proposition 4.4. (1) For any \mathcal{A}, \mathcal{B} in $\mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$, under the isomorphism $\phi_{\mathcal{A}, \mathcal{B}}$ we have

$$\mathfrak{m}_{\mathcal{A} \star \mathcal{B}} = \mathfrak{m}_{\mathcal{A}} \star_{\mathrm{Iw}} \mathfrak{m}_{\mathcal{B}}.$$

(2) For any \mathcal{A} in $\mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$, the automorphism $\mathfrak{m}_{\mathcal{A}}$ is unipotent.

(3) For any $\mathcal{A}, \mathcal{B} \in \mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$ and $f \in \mathrm{Hom}_{\mathrm{P}_{\mathrm{Iw}}}(\mathbb{Z}(\mathcal{A}), \mathbb{Z}(\mathcal{B}))$ we have

$$\mathfrak{m}_{\mathcal{B}} \circ f = f \circ \mathfrak{m}_{\mathcal{A}}.$$

³In [G1] the isomorphism $\sigma_{\mathcal{A}, \mathcal{F}}$ is constructed only in the special case when \mathcal{F} is perverse, which causes complications for our later considerations.

⁴More precisely, $\mathbb{Z}(\mathcal{A})$ comes equipped with a canonical action of $\mathbb{Z}_\ell(1)$, the inverse limit of the groups of ℓ^n -th roots of unity in \mathbb{F} for all n . Below we fix once and for all a topological generator of this group, and consider the action of this element.

Here the proof of (1) is easy, see [AR3, Proposition 3.4.2]. For the original proof of (2), in the setting where \mathbb{k} is replaced by $\overline{\mathbb{Q}}_\ell$, see [G1, Proposition 7]. This proof does not easily adapt to our present setting; however an alternative proof, which also implies property (3), is provided in [BM]. This proof does work in our present setting, see [AR3, Proposition 2.4.6] for details.

The proof of Proposition 4.4(2)–(3) requires a different (but equivalent) description of the automorphisms $\mathfrak{m}_{\mathcal{A}}$, which we now explain for later use. Consider the “loop rotation” action of \mathbb{G}_m on Fl_G , induced by the action on $\mathbb{F}((z))$ which “rescales” z . Since the Iw-orbits on Fl_G are stable under this action, every object \mathcal{F} of P_{Iw} is monodromic for this action in the sense of [Ve]; therefore it admits a canonical automorphism $\mathfrak{M}_{\mathcal{F}}$ ⁵ (see [BM] or [AR3, §9.5.3] for details and more precise references), and $\mathfrak{M}_{(-)}$ defines an endomorphism of the identity functor $\mathrm{id}_{\mathrm{P}_{\mathrm{Iw}}}$. As explained in [BM, §5.2], for any \mathcal{A} in $\mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$ we have

$$(4.3) \quad \mathfrak{m}_{\mathcal{A}} = \mathfrak{M}_{Z(\mathcal{A})}^{-1}.$$

4.4. Iwahori orbits. From now on we fix a choice of maximal torus $T \subset B$, and set $\mathbf{X}^\vee := X_*(T)$. We then obtain a canonical identification $\mathbf{X}^\vee = X^*(T_{\mathbb{k}}^\vee)$, see §4.1. We will denote by \mathfrak{R} the root system of (G, T) . The choice of B determines a choice of positive roots $\mathfrak{R}_+ \subset \mathfrak{R}$ (such that the T -weights in the Lie algebra of B are the *negative* roots), hence a subset $\mathbf{X}_+^\vee \subset \mathbf{X}^\vee$ of dominant coweights. We will also denote by $B_{\mathbb{k}}^\vee \subset G_{\mathbb{k}}^\vee$ the Borel subgroup containing $T_{\mathbb{k}}^\vee$ such that the $T_{\mathbb{k}}^\vee$ -weights on the Lie algebra of $B_{\mathbb{k}}^\vee$ are the negative coroots of G .

We denote by $W_{\mathfrak{f}}$ the Weyl group of (G, T) . Then $W_{\mathfrak{f}}$ acts canonically on \mathbf{X}^\vee , and the semi-direct product

$$W := W_{\mathfrak{f}} \rtimes \mathbf{X}^\vee$$

is called the (extended) affine Weyl group. This group parametrizes the Iw-orbits on Fl_G in the following way. For $\lambda \in \mathbf{X}^\vee$, we will denote by $\mathfrak{t}(\lambda)$ the image of λ in W . Then if $w = \mathfrak{t}(\lambda)x$ for some $\lambda \in \mathbf{X}^\vee$ and $x \in W_{\mathfrak{f}}$, we denote by $\mathrm{Fl}_{G,w} \subset \mathrm{Fl}_G$ the Iw-orbit of the coset $z^\lambda \dot{x} \mathrm{Iw}$, where:

- z^λ is the image of z under the morphism $\mathbb{F}((z))^\times \rightarrow LG$ induced by λ ;
- \dot{x} is any lift of x in $N_G(T)$.

Then it is well known that we have

$$(\mathrm{Fl}_G)_{\mathrm{red}} = \bigsqcup_{w \in W} \mathrm{Fl}_{G,w}.$$

For $w \in W$ we will denote by $j_w : \mathrm{Fl}_{G,w} \rightarrow \mathrm{Fl}_G$ the embedding, and set

$$\ell(w) := \dim(\mathrm{Fl}_{G,w}).$$

A formula due to Iwahori–Matsumoto [IM] states that if $w = x\mathfrak{t}(\lambda)$ with $x \in W_{\mathfrak{f}}$ and $\lambda \in \mathbf{X}^\vee$, we have

$$(4.4) \quad \ell(w) = \sum_{\substack{\alpha \in \mathfrak{R}_+ \\ x(\alpha) \in \mathfrak{R}_+}} |\langle \lambda, \alpha \rangle| + \sum_{\substack{\alpha \in \mathfrak{R}_+ \\ x(\alpha) \in -\mathfrak{R}_+}} |\langle \lambda, \alpha \rangle + 1|.$$

Set

$$W^{\mathrm{Cox}} := W_{\mathfrak{f}} \rtimes \mathbb{Z}\mathfrak{R}^\vee,$$

⁵More precisely, \mathcal{F} admits a canonical action of $\mathbb{Z}_\ell(1)$; we deduce a canonical automorphism by taking the image of our fixed topological generator of this group.

where $\mathfrak{R}^\vee \subset \mathbf{X}^\vee$ is the coroot system of (G, T) and $\mathbb{Z}\mathfrak{R}^\vee \subset \mathbf{X}^\vee$ is the coroot lattice. If we set $S = \{w \in W^{\text{Cox}} \mid \ell(w) = 1\}$, then it is well known that (W^{Cox}, S) is a Coxeter system, whose length function is the restriction of ℓ . Moreover, if we set $\Omega := \{w \in W \mid \ell(w) = 0\}$, then for any $\omega \in \Omega$ conjugation by ω stabilizes S , hence acts on W^{Cox} as a Coxeter group automorphism. Finally, the product induces a group isomorphism

$$\Omega \ltimes W^{\text{Cox}} \xrightarrow{\sim} W,$$

such that $\ell(\omega w) = \ell(w)$ for $\omega \in \Omega$ and $w \in W^{\text{Cox}}$.

For any $w \in W$, we define the *standard* and *costandard* perverse sheaves associated with w as

$$\Delta_w^{\text{Iw}} := j_{w!} \mathbb{k}_{\text{Fl}_{G,w}}[\ell(w)], \quad \nabla_w^{\text{Iw}} := j_{w*} \mathbb{k}_{\text{Fl}_{G,w}}[\ell(w)].$$

These complexes are indeed perverse sheaves because j_w is an affine embedding, and are Iw-equivariant. It is well known that if $w, y \in W$ are such that $\ell(wy) = \ell(w) + \ell(y)$ there exist canonical isomorphisms

$$(4.5) \quad \Delta_w^{\text{Iw}} \star_{\text{Iw}} \Delta_y^{\text{Iw}} \cong \Delta_{wy}^{\text{Iw}}, \quad \nabla_w^{\text{Iw}} \star_{\text{Iw}} \nabla_y^{\text{Iw}} \cong \nabla_{wy}^{\text{Iw}},$$

and that for any $w \in W$ there exist isomorphisms

$$(4.6) \quad \Delta_w^{\text{Iw}} \star_{\text{Iw}} \nabla_{w^{-1}}^{\text{Iw}} \cong \delta_{\text{Fl}} \cong \nabla_{w^{-1}}^{\text{Iw}} \star_{\text{Iw}} \Delta_w^{\text{Iw}}.$$

In particular, the objects Δ_w^{Iw} and ∇_w^{Iw} are invertible (in the monoidal category $(D_{\text{Iw}}^{\text{b}}(\text{Fl}_G, \mathbb{k}), \star_{\text{Iw}})$) for any $w \in W$.

For any $w \in W$, the intersection cohomology complex of $\text{Fl}_{G,w}$ (i.e. the image of the unique—up to scalar—nonzero morphism $\Delta_w^{\text{Iw}} \rightarrow \nabla_w^{\text{Iw}}$) will be denoted \mathcal{IC}_w ; then the assignment $w \mapsto \mathcal{IC}_w$ induces a bijection between W and the set of isomorphism classes of simple objects in P_{Iw} .

The following lemma is an analogue of [BBM, Lemma 2.1] for Fl_G . The same arguments as in [BBM] apply.

Lemma 4.5. *Let $w \in W$, and assume that $w = \omega x$ for some $\omega \in \Omega$ and $x \in W^{\text{Cox}}$. Then the socle of Δ_w^{Iw} is \mathcal{IC}_ω , and the cokernel of the embedding $\mathcal{IC}_\omega \hookrightarrow \Delta_w^{\text{Iw}}$ does not contain any object of the form $\mathcal{IC}_{\omega'}$ with $\omega' \in \Omega$ as a composition factor. Dually, the top of ∇_w^{Iw} is \mathcal{IC}_ω , and the kernel of the surjection $\nabla_w^{\text{Iw}} \twoheadrightarrow \mathcal{IC}_\omega$ does not contain any object of the form $\mathcal{IC}_{\omega'}$ with $\omega' \in \Omega$ as a composition factor.*

4.5. Wakimoto sheaves. Fix $\lambda \in \mathbf{X}^\vee$. Given $\mu, \nu \in \mathbf{X}_+^\vee$ such that $\lambda = \mu - \nu$ and \mathcal{F} in $D_{\text{Iw}}^{\text{b}}(\text{Fl}_G, \mathbb{k})$, we can consider the \mathbb{k} -vector space

$$\text{Hom}_{D_{\text{Iw}}^{\text{b}}(\text{Fl}_G, \mathbb{k})}(\nabla_{\mathfrak{t}(\mu)}^{\text{Iw}}, \nabla_{\mathfrak{t}(\nu)}^{\text{Iw}} \star_{\text{Iw}} \mathcal{F}).$$

If $\mu', \nu' \in \mathbf{X}_+^\vee$ are such that $\lambda = \mu' - \nu'$ and $\mu' - \mu \in \mathbf{X}_+^\vee$, then convolution with $\nabla_{\mathfrak{t}(\mu' - \mu)}^{\text{Iw}}$ induces a canonical isomorphism

$$\text{Hom}_{D_{\text{Iw}}^{\text{b}}(\text{Fl}_G, \mathbb{k})}(\nabla_{\mathfrak{t}(\mu)}^{\text{Iw}}, \nabla_{\mathfrak{t}(\nu)}^{\text{Iw}} \star_{\text{Iw}} \mathcal{F}) \xrightarrow{\sim} \text{Hom}_{D_{\text{Iw}}^{\text{b}}(\text{Fl}_G, \mathbb{k})}(\nabla_{\mathfrak{t}(\mu')}^{\text{Iw}}, \nabla_{\mathfrak{t}(\nu')}^{\text{Iw}} \star_{\text{Iw}} \mathcal{F}).$$

One can check that if (μ'', ν'') is another pair such that $\lambda = \mu'' - \nu''$ and $\mu'' - \mu' \in \mathbf{X}_+^\vee$, the two isomorphisms

$$\text{Hom}_{D_{\text{Iw}}^{\text{b}}(\text{Fl}_G, \mathbb{k})}(\nabla_{\mathfrak{t}(\mu)}^{\text{Iw}}, \nabla_{\mathfrak{t}(\nu)}^{\text{Iw}} \star_{\text{Iw}} \mathcal{F}) \xrightarrow{\sim} \text{Hom}_{D_{\text{Iw}}^{\text{b}}(\text{Fl}_G, \mathbb{k})}(\nabla_{\mathfrak{t}(\mu'')}^{\text{Iw}}, \nabla_{\mathfrak{t}(\nu'')}^{\text{Iw}} \star_{\text{Iw}} \mathcal{F})$$

obtained by first convolving with $\nabla_{\mathfrak{t}(\mu'-\mu)}^{\text{Iw}}$ and then with $\nabla_{\mathfrak{t}(\mu''-\mu')}^{\text{Iw}}$, or directly with $\nabla_{\mathfrak{t}(\mu''-\mu)}^{\text{Iw}}$, coincide; we can therefore consider the (filtrant) direct limit

$$\varinjlim_{\substack{(\mu, \nu) \in (\mathbf{X}_+^\vee)^2 \\ \lambda = \mu - \nu}} \text{Hom}_{D_{\text{Iw}}^b(\text{Fl}_G, \mathbb{k})}(\nabla_{\mathfrak{t}(\mu)}^{\text{Iw}}, \nabla_{\mathfrak{t}(\nu)}^{\text{Iw}} \star_{\text{Iw}} \mathcal{F})$$

with respect to the order such that $(\mu, \nu) \preceq (\mu', \nu')$ if $\mu' - \mu \in \mathbf{X}_+^\vee$. We denote by $\mathcal{W}_\lambda \in D_{\text{Iw}}^b(\text{Fl}_G, \mathbb{k})$ the object which represents the functor

$$\mathcal{F} \mapsto \varinjlim_{\substack{(\mu, \nu) \in (\mathbf{X}_+^\vee)^2 \\ \lambda = \mu - \nu}} \text{Hom}_{D_{\text{Iw}}^b(\text{Fl}_G, \mathbb{k})}(\nabla_{\mathfrak{t}(\mu)}^{\text{Iw}}, \nabla_{\mathfrak{t}(\nu)}^{\text{Iw}} \star_{\text{Iw}} \mathcal{F}).$$

(This functor is indeed representable since $\nabla_{\mathfrak{t}(\nu)}^{\text{Iw}}$ is invertible, see (4.6).)

Following Mirković (to whom this definition is due), the objects \mathcal{W}_λ are called *Wakimoto sheaves*. Their properties are investigated in [BM] (in the setting where \mathbb{k} is replaced by $\overline{\mathbb{Q}}_\ell$) and in [AR3] (in the general setting). Those which will be useful for us are the following. (Here, we denote by \preceq the order on \mathbf{X}^\vee such that $\lambda \preceq \mu$ iff $\mu - \lambda$ is a sum of positive roots.)

- (1) If $\lambda = \mu - \nu$ with $\mu, \nu \in \mathbf{X}_+^\vee$ we have *noncanonical* isomorphisms

$$\mathcal{W}_\lambda \cong \Delta_{\mathfrak{t}(-\nu)}^{\text{Iw}} \star_{\text{Iw}} \nabla_{\mathfrak{t}(\mu)}^{\text{Iw}} \cong \nabla_{\mathfrak{t}(\mu)}^{\text{Iw}} \star_{\text{Iw}} \Delta_{\mathfrak{t}(-\nu)}^{\text{Iw}}.$$

- (2) For any $\lambda, \lambda' \in \mathbf{X}^\vee$ there exists a canonical isomorphism

$$\mathcal{W}_\lambda \star_{\text{Iw}} \mathcal{W}_{\lambda'} \xrightarrow{\sim} \mathcal{W}_{\lambda + \lambda'}.$$

- (3) For any $\lambda \in \mathbf{X}^\vee$, the object \mathcal{W}_λ is a perverse sheaf supported on $\overline{\text{Fl}_{G, \mathfrak{t}(\lambda)}}$, and its restriction to $\text{Fl}_{G, \mathfrak{t}(\lambda)}$ is isomorphic to $\mathbb{k}_{\text{Fl}_{G, \mathfrak{t}(\lambda)}}[\ell(\mathfrak{t}(\lambda))]$.

- (4) For $\lambda, \lambda' \in \mathbf{X}^\vee$ and $n \in \mathbb{Z}$ we have

$$\text{Hom}_{D_{\text{Iw}}^b(\text{Fl}_G, \mathbb{k})}(\mathcal{W}_\lambda, \mathcal{W}_{\lambda'}[n]) = 0 \quad \text{unless } \lambda' \preceq \lambda.$$

- (5) For $\lambda \in \mathbf{X}^\vee$ we have

$$\text{Hom}_{\mathbf{P}_{\text{Iw}}}(\mathcal{W}_\lambda, \mathcal{W}_\lambda) = \mathbb{k} \quad \text{and} \quad \text{Ext}_{\mathbf{P}_{\text{Iw}}}^1(\mathcal{W}_\lambda, \mathcal{W}_\lambda) = 0.$$

Remark 4.6. Following [Zh], in [AR3] Wakimoto sheaves are defined in a more general setting, where \mathbf{X}_+^\vee is replaced by the intersection of \mathbf{X}^\vee with any fixed choice of Weyl chamber in $\mathbb{R} \otimes_{\mathbb{Z}} \mathbf{X}^\vee$. All the properties listed above remain true in this generality, with the appropriate replacement for (1) and for the order in (4). Below we will also use the Wakimoto sheaves associated with the *anti-dominant* chamber, which will be denoted $\mathcal{W}_\lambda^{\text{op}}$ ($\lambda \in \mathbf{X}^\vee$). So, if $\lambda = \mu - \nu$ with $\mu, \nu \in \mathbf{X}_+^\vee$ then we have

$$\mathcal{W}_\lambda^{\text{op}} \cong \nabla_{\mathfrak{t}(-\nu)}^{\text{Iw}} \star_{\text{Iw}} \Delta_{\mathfrak{t}(\mu)}^{\text{Iw}} \cong \Delta_{\mathfrak{t}(\mu)}^{\text{Iw}} \star_{\text{Iw}} \nabla_{\mathfrak{t}(-\nu)}^{\text{Iw}}.$$

We also have $\text{Hom}_{D_{\text{Iw}}^b(\text{Fl}_G, \mathbb{k})}(\mathcal{W}_\lambda^{\text{op}}, \mathcal{W}_{\lambda'}^{\text{op}}[n]) = 0$ unless $\lambda' \succeq \lambda$.

4.6. Wakimoto filtrations. We will denote by $\mathbf{P}_{\text{Iw}}^{\mathcal{W}}$ the full subcategory of \mathbf{P}_{Iw} whose objects are the perverse sheaves which admit a filtration by Wakimoto sheaves, i.e. a filtration whose subquotients are of the form \mathcal{W}_λ with $\lambda \in \mathbf{X}^\vee$. This category admits a natural structure of exact category, inherited from the ambient abelian category \mathbf{P}_{Iw} . Since a convolution of Wakimoto sheaves is a Wakimoto sheaf (in particular, is perverse), $\mathbf{P}_{\text{Iw}}^{\mathcal{W}}$ is a monoidal subcategory of $(D_{\text{Iw}}^b(\text{Fl}_G, \mathbb{k}), \star_{\text{Iw}})$.

It follows from properties (4)–(5) in §4.5 that for any \mathcal{F} in $\mathbf{P}_{\mathrm{Iw}}^{\mathcal{W}}$ and any ideal $\mathbf{Y} \subset \mathbf{X}^\vee$ (i.e. any subset such that if $\lambda \in \mathbf{Y}$ and $\mu \preceq \lambda$ then $\mu \in \mathbf{Y}$) there exists a unique subobject $\mathcal{F}_{\mathbf{Y}} \subset \mathcal{F}$ which is an extension of objects \mathcal{W}_λ with $\lambda \in \mathbf{Y}$ and such that the quotient $\mathcal{F}/\mathcal{F}_{\mathbf{Y}}$ is an extension of objects \mathcal{W}_μ with $\mu \in \mathbf{X}^\vee \setminus \mathbf{Y}$; moreover the assignment $\mathcal{F} \mapsto \mathcal{F}_{\mathbf{Y}}$ is functorial.

In particular, given $\lambda \in \mathbf{X}^\vee$ one can choose an ideal $\mathbf{Y} \subset \mathbf{X}^\vee$ such that $\lambda \in \mathbf{Y}$ is maximal, and consider the quotient

$$\mathrm{gr}_\lambda(\mathcal{F}) := \mathcal{F}_{\mathbf{Y}} / \mathcal{F}_{\mathbf{Y} \setminus \{\lambda\}}.$$

By construction $\mathrm{gr}_\lambda(\mathcal{F})$ is a direct sum of copies of \mathcal{W}_λ , and we set

$$\mathrm{Grad}_\lambda(\mathcal{F}) := \overline{\mathrm{Hom}}_{\mathbf{P}_{\mathrm{Iw}}}(\mathcal{W}_\lambda, \mathrm{gr}_\lambda(\mathcal{F})).$$

Lemma 4.7. *The object $\mathrm{gr}_\lambda(\mathcal{F})$ does not depend on the choice of \mathbf{Y} . Moreover, the functor $\mathcal{F} \mapsto \mathrm{Grad}_\lambda(\mathcal{F})$ is exact (for the exact structure on $\mathbf{P}_{\mathrm{Iw}}^{\mathcal{W}}$ considered above).*

Proof. For the first claim, see [AR3, Lemma 4.3.4]. For the second claim, see [AR3, Proposition 4.6.1]. \square

The following result is due to Arkhipov and the first author [AB, Theorem 4(a)] in the setting where \mathbb{k} is replaced by \mathbb{Q}_ℓ . The proof is also reproduced in [Zh, §7.3] and (in the present setting) in [AR3].

Theorem 4.8 (Arkhipov–Bezrukavnikov). *For any \mathcal{F} in $\mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$, the perverse sheaf $\mathbf{Z}(\mathcal{F})$ belongs to $\mathbf{P}_{\mathrm{Iw}}^{\mathcal{W}}$.*

Remark 4.9. In [Zh] and [AR3], Theorem 4.8 is proved with the more general definition of Wakimoto sheaves alluded to in Remark 4.6. In particular, for any \mathcal{F} in $\mathrm{Perv}_{\mathrm{sph}}(\mathrm{Gr}_G, \mathbb{k})$ the perverse sheaf $\mathbf{Z}(\mathcal{F})$ also admits a filtration with subquotients of the form $\mathcal{W}_\lambda^{\mathrm{op}}$ with $\lambda \in \mathbf{X}^\vee$.

To simplify notation, from now on we will use the notation

$$\mathcal{Z} := \mathbf{Z} \circ \mathcal{S}^{-1} : \mathrm{Rep}(G_{\mathbb{k}}^\vee) \rightarrow \mathbf{P}_{\mathrm{Iw}},$$

where \mathcal{S} is as in (4.1), and write \mathfrak{m}_V for $\mathfrak{m}_{\mathcal{S}^{-1}(V)}$.

Thanks to Theorem 4.8, we can consider for any V in $\mathrm{Rep}(G_{\mathbb{k}}^\vee)$ and any $\lambda \in \mathbf{X}^\vee$ the vector space

$$\mathrm{Grad}_\lambda(\mathcal{Z}(V)).$$

By [AR3, Lemma 4.8.1], we have a canonical identification

$$(4.7) \quad \mathrm{Grad}_\lambda(\mathcal{Z}(V)) \cong V_{w_0(\lambda)},$$

where w_0 is the longest element in $W_{\mathfrak{f}}$. (Here, in the right-hand side we consider the weight space for the action of the canonical maximal torus $T_{\mathbb{k}}^\vee \subset G_{\mathbb{k}}^\vee$.)

5. THE REGULAR QUOTIENT

In this section we introduce the main player of the paper, namely the “regular quotient” of the category \mathbf{P}_{Iw} .

5.1. The regular quotient. We consider the Serre subcategory $\langle \mathcal{IC}_w : \ell(w) > 0 \rangle_{\text{Serre}} \subset \mathbf{P}_{\text{Iw}}$ generated by the objects \mathcal{IC}_w with $\ell(w) > 0$, and the Serre quotient

$$\mathbf{P}_{\text{Iw}}^0 := \mathbf{P}_{\text{Iw}} / \langle \mathcal{IC}_w : \ell(w) > 0 \rangle_{\text{Serre}}.$$

The quotient functor $\mathbf{P}_{\text{Iw}} \rightarrow \mathbf{P}_{\text{Iw}}^0$ will be denoted Π^0 . Every object in the category \mathbf{P}_{Iw}^0 has finite length, and the assignment $w \mapsto \Pi^0(\mathcal{IC}_w)$ induces a bijection between $\Omega = \{w \in W \mid \ell(w) = 0\}$ and the set of isomorphism classes of simple objects in \mathbf{P}_{Iw}^0 .

The proof of the following lemma can be copied verbatim from [AR3, §6.5.6].

Lemma 5.1. (1) *If \mathcal{F} belongs to $\langle \mathcal{IC}_w : \ell(w) > 0 \rangle_{\text{Serre}}$ and \mathcal{G} is any object of \mathbf{P}_{Iw} , then for any $n \in \mathbb{Z}$ the perverse sheaves ${}^{\mathbf{p}}\mathcal{H}^n(\mathcal{F} \star_{\text{Iw}} \mathcal{G})$ and ${}^{\mathbf{p}}\mathcal{H}^n(\mathcal{G} \star_{\text{Iw}} \mathcal{F})$ belong to $\langle \mathcal{IC}_w : \ell(w) > 0 \rangle_{\text{Serre}}$.*
 (2) *For \mathcal{F}, \mathcal{G} in \mathbf{P}_{Iw} and any $n \in \mathbb{Z} \setminus \{0\}$ we have $\Pi^0({}^{\mathbf{p}}\mathcal{H}^n(\mathcal{F} \star_{\text{Iw}} \mathcal{G})) = 0$.*

Lemma 5.1 has the following consequence.

Proposition 5.2. (1) *The bifunctor*

$$\mathbf{P}_{\text{Iw}} \times \mathbf{P}_{\text{Iw}} \rightarrow \mathbf{P}_{\text{Iw}}^0$$

sending a pair $(\mathcal{F}, \mathcal{G})$ to $\Pi^0({}^{\mathbf{p}}\mathcal{H}^0(\mathcal{F} \star_{\text{Iw}} \mathcal{G}))$ factors through a bifunctor

$$(-) \star_{\text{Iw}}^0 (-) : \mathbf{P}_{\text{Iw}}^0 \times \mathbf{P}_{\text{Iw}}^0 \rightarrow \mathbf{P}_{\text{Iw}}^0.$$

(2) *The bifunctor \star_{Iw}^0 is exact on both sides, and admits natural associativity and unit constraints.*

Proof. Lemma 5.1 implies that the bifunctor $(\mathcal{F}, \mathcal{G}) \mapsto \Pi^0({}^{\mathbf{p}}\mathcal{H}^0(\mathcal{F} \star_{\text{Iw}} \mathcal{G}))$ is exact on both sides, and vanishes on $\langle \mathcal{IC}_w : \ell(w) > 0 \rangle_{\text{Serre}} \times \mathbf{P}_{\text{Iw}}$ and on $\mathbf{P}_{\text{Iw}} \times \langle \mathcal{IC}_w : \ell(w) > 0 \rangle_{\text{Serre}}$. Therefore it factors through a bifunctor \star_{Iw}^0 , which is exact on both sides (see [Gab, Corollaires 2–3, p. 368–369]). This observation also shows that for $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ in \mathbf{P}_{Iw}^0 we have canonical isomorphisms

$$(\mathcal{F}_1 \star_{\text{Iw}}^0 \mathcal{F}_2) \star_{\text{Iw}}^0 \mathcal{F}_3 \cong \Pi^0({}^{\mathbf{p}}\mathcal{H}^0(\mathcal{F}_1 \star_{\text{Iw}} \mathcal{F}_2 \star_{\text{Iw}} \mathcal{F}_3)) \cong \mathcal{F}_1 \star_{\text{Iw}}^0 (\mathcal{F}_2 \star_{\text{Iw}}^0 \mathcal{F}_3),$$

which provides the desired associativity constraint. The unit object for this product is $\delta^0 := \Pi^0(\delta_{\text{Fl}})$. \square

Thanks to Proposition 5.2, we can now consider the abelian monoidal category $(\mathbf{P}_{\text{Iw}}^0, \star_{\text{Iw}}^0)$. We set

$$\mathcal{Z}^0 := \Pi^0 \circ \mathcal{Z} : \text{Rep}(G_{\mathbb{k}}^{\vee}) \rightarrow \mathbf{P}_{\text{Iw}}^0,$$

and for V in $\text{Rep}(G_{\mathbb{k}}^{\vee})$ we set $\mathfrak{m}_V^0 := \Pi^0(\mathfrak{m}_V) \in \text{End}_{\mathbf{P}_{\text{Iw}}^0}(\mathcal{Z}^0(V))$.

The following claim is an immediate consequence of Proposition 4.4(3).

Lemma 5.3. *For any V, V' in $\text{Rep}(G_{\mathbb{k}}^{\vee})$ and any $f \in \text{Hom}_{\mathbf{P}_{\text{Iw}}^0}(\mathcal{Z}^0(V), \mathcal{Z}^0(V'))$ we have $f \circ \mathfrak{m}_V^0 = \mathfrak{m}_{V'}^0 \circ f$.*

5.2. Main result: statement and strategy of proof. The main result of the present paper can be stated as follows.

Theorem 5.4. *Assume that the following conditions hold:*

- (1) *either $X^*(T)/\mathbb{Z}\mathfrak{R}$ is free and $X_*(T)/\mathbb{Z}\mathfrak{R}^{\vee}$ has no ℓ -torsion, or ℓ is very good;*
- (2) *for any indecomposable factor of the root system \mathfrak{R} , the prime ℓ is strictly bigger than the corresponding value in the table of Figure 5.1.*

\mathbf{A}_n ($n \geq 1$)	\mathbf{B}_n ($n \geq 2$)	\mathbf{C}_n ($n \geq 3$)	\mathbf{D}_n ($n \geq 4$)	\mathbf{E}_6	\mathbf{E}_7	\mathbf{E}_8	\mathbf{F}_4	\mathbf{G}_2
1	n	2	2	3	19	31	3	3

FIGURE 5.1. Bounds on ℓ

Then there exists a regular unipotent element $\mathbf{u} \in G_{\mathbb{k}}^{\vee}$ and an equivalence of monoidal categories

$$\Phi^0 : (\mathbf{P}_{\mathrm{Iw}}^0, \star_{\mathrm{Iw}}^0) \xrightarrow{\sim} (\mathrm{Rep}(Z_{G_{\mathbb{k}}^{\vee}}(\mathbf{u})), \otimes)$$

such that $\Phi^0 \circ \mathcal{Z}^0 \cong \mathrm{For}_{Z_{G_{\mathbb{k}}^{\vee}}(\mathbf{u})}^{G_{\mathbb{k}}^{\vee}}$.

Remark 5.5. Let us briefly discuss the assumptions in Theorem 5.4. The condition that $X^*(T)/\mathbb{Z}\mathfrak{A}$ is free is equivalent to requiring that the derived subgroup of $G_{\mathbb{k}}^{\vee}$ is simply connected. (Alternatively, since the quotient $X^*(T)/\mathbb{Z}\mathfrak{A}$ identifies with the group of characters of the center $Z(G)$ of G , this assumption is also equivalent to the condition that $Z(G)$ is a torus.) The assumption that $X_*(T)/\mathbb{Z}\mathfrak{A}^{\vee}$ has no ℓ -torsion guarantees that $Z(G_{\mathbb{k}}^{\vee})$ and $Z_{G_{\mathbb{k}}^{\vee}}(\mathbf{u})$ are smooth group schemes, see §2.3. (This assumption in fact follows from (2) if \mathfrak{A} has no component of type \mathbf{A} .) Finally, assumption (2) is an artefact of our proof of the statement considered in (ii) below; it is likely that it can be refined (at least) to the condition that ℓ is good for G .

In the rest of the paper we will prove Theorem 5.4 assuming that (2) and the first condition in (1) hold. The latter condition can be replaced a posteriori by the condition that ℓ is very good for G , as follows. The diagonalizable group $Z(G)$ can be written as the product of a torus and a finite group. If G' is the quotient of G by this finite group, then if ℓ is very good for G Theorem 5.4 applies to G' . Now Proposition 2.6 guarantees that the centralizer of a regular unipotent element in $G_{\mathbb{k}}^{\vee}$ is the quotient of the corresponding subgroup of $(G')_{\mathbb{k}}^{\vee}$ by a subgroup of $Z((G')_{\mathbb{k}}^{\vee})$. One can check that the equivalence of Theorem 5.4 for G' then restricts to a similar equivalence for G .

Our proof of Theorem 5.4 will use Proposition 3.1, applied to the abelian monoidal category $(\mathbf{P}_{\mathrm{Iw}}^0, \star_{\mathrm{Iw}}^0)$ and the functor $\mathcal{Z}^0 : \mathrm{Rep}(G_{\mathbb{k}}^{\vee}) \rightarrow \mathbf{P}_{\mathrm{Iw}}^0$. For this we will need to construct a central structure on this functor; this will be done in §5.3, and does not require any assumption. The first real difficulty appears when trying to show that any object in $\mathbf{P}_{\mathrm{Iw}}^0$ is a subquotient of an object $\mathcal{Z}^0(V)$. In fact we will not show this directly; instead we will provisionally replace the category $\mathbf{P}_{\mathrm{Iw}}^0$ by the full subcategory $\tilde{\mathbf{P}}_{\mathrm{Iw}}^0$ whose objects are precisely the subquotients of objects $\mathcal{Z}^0(V)$. With this replacement Proposition 3.1 applies, and provides a closed subgroup $H \subset G_{\mathbb{k}}^{\vee}$, see §5.4. The rest of the paper will then be devoted to the proof of the following two claims (under the assumptions of Theorem 5.4), which will imply the theorem:

- (a) the embedding $\tilde{\mathbf{P}}_{\mathrm{Iw}}^0 \subset \mathbf{P}_{\mathrm{Iw}}^0$ is an equality;
- (b) the subgroup H is the centralizer of a regular unipotent element.

These properties will be proved in Section 9. The main two ingredients of these proofs are the following:

- (i) the construction of the unipotent element appearing in (b) (based on the use of the monodromy automorphisms \mathfrak{m}_V^0) and the proof that this element is regular, which will be explained in Section 6;

- (ii) the proof that the images of the central sheaves in the Iwahori–Whittaker category $\mathbf{P}_{\mathcal{I}\mathcal{W}}$ are tilting perverse sheaves, which will be obtained in Section 8 after some preliminaries (including the definition of the category $\mathbf{P}_{\mathcal{I}\mathcal{W}}$) in Section 7.

Here (i) only requires the assumption that $X^*(T)/\mathbb{Z}\mathfrak{R}$ is free; the proof of (ii) will however require that ℓ is bigger than the corresponding values in Figure 5.1. The other assumption in Theorem 5.4 will be used to deduce (a)–(b) from (i)–(ii).

5.3. Central structure on \mathcal{Z}^0 . The following claim is an easy consequence of Theorem 4.3.

Lemma 5.6. *The functor \mathcal{Z}^0 admits a canonical structure of central functor from $\text{Rep}(G_{\mathbb{k}}^{\vee})$ to \mathbf{P}_{Iw}^0 .*

Proof. Let us first explain how to define the “monoidality” and “centrality” isomorphisms for the functor \mathcal{Z}^0 . Given V, V' in $\text{Rep}(G_{\mathbb{k}}^{\vee})$, we have the canonical isomorphism

$$\phi_{\mathcal{S}^{-1}(V), \mathcal{S}^{-1}(V')} : \mathbf{Z}(\mathcal{S}^{-1}(V) \star \mathcal{S}^{-1}(V')) \xrightarrow{\sim} \mathbf{Z}(\mathcal{S}^{-1}(V)) \star_{\text{Iw}} \mathbf{Z}(\mathcal{S}^{-1}(V')).$$

see Theorem 4.2(4). Using the monoidal structure on the functor \mathcal{S}^{-1} , one can interpret that isomorphism as an identification

$$\mathcal{Z}(V \otimes V') \xrightarrow{\sim} \mathcal{Z}(V) \star_{\text{Iw}} \mathcal{Z}(V').$$

These complexes are perverse sheaves; applying the functor Π^0 we deduce the wished-for isomorphism

$$\phi_{V, V'}^0 : \mathcal{Z}^0(V \otimes V') \xrightarrow{\sim} \mathcal{Z}^0(V) \star_{\text{Iw}}^0 \mathcal{Z}^0(V').$$

It is easily seen that these isomorphisms define a monoidal structure on \mathcal{Z}^0 .

Next, for V in $\text{Rep}(G_{\mathbb{k}}^{\vee})$ and \mathcal{F} in \mathbf{P}_{Iw}^0 , we must define a canonical isomorphism

$$\sigma_{V, \mathcal{F}}^0 : \mathcal{Z}^0(V) \star_{\text{Iw}}^0 \mathcal{F} \xrightarrow{\sim} \mathcal{F} \star_{\text{Iw}}^0 \mathcal{Z}^0(V).$$

However, the objects of \mathbf{P}_{Iw}^0 and \mathbf{P}_{Iw} are the same; \mathcal{F} can therefore also be considered as an object in \mathbf{P}_{Iw} , for which we have the canonical isomorphism

$$\sigma_{\mathcal{S}^{-1}(V), \mathcal{F}} : \mathbf{Z}(\mathcal{S}^{-1}(V)) \star_{\text{Iw}} \mathcal{F} \xrightarrow{\sim} \mathcal{F} \star_{\text{Iw}} \mathbf{Z}(\mathcal{S}^{-1}(V)),$$

see Theorem 4.2(3). Here again both sides are perverse sheaves, see Theorem 4.2(2), and applying the functor Π^0 we obtain the wished-for isomorphism $\sigma_{V, \mathcal{F}}^0$.

We must now check that these data satisfy properties (1)–(3) from §3.1. Properties (1) and (3) immediately follow from the corresponding properties for the functor \mathbf{Z} by applying the functor Π^0 , since all the involved objects are perverse. In order to check property (2), we consider some V in $\text{Rep}(G_{\mathbb{k}}^{\vee})$ and \mathcal{F}, \mathcal{G} in \mathbf{P}_{Iw}^0 . As above \mathcal{F} and \mathcal{G} can also be considered as objects in \mathbf{P}_{Iw} , and we know that the composition

$$\begin{aligned} \mathcal{Z}(V) \star_{\text{Iw}} \mathcal{F} \star_{\text{Iw}} \mathcal{G} &\xrightarrow{\sigma_{\mathcal{S}^{-1}(V), \mathcal{F}} \star_{\text{Iw}} \text{id}} \mathcal{F} \star_{\text{Iw}} \mathcal{Z}(V) \star_{\text{Iw}} \mathcal{G} \\ &\xrightarrow{\text{id} \star_{\text{Iw}} \sigma_{\mathcal{S}^{-1}(V), \mathcal{G}}} \mathcal{F} \star_{\text{Iw}} \mathcal{G} \star_{\text{Iw}} \mathcal{Z}(V) \end{aligned}$$

coincides with $\sigma_{S^{-1}(V), \mathcal{F} \star_{Iw} \mathcal{G}}$. Let us apply the functor $\Pi^0 \circ {}^p\mathcal{H}^0$. By exactness of the functors $\mathcal{L}(V) \star_{Iw} (-)$ and $(-) \star_{Iw} \mathcal{L}(V)$ (see Theorem 4.2(2)), the domain, resp. codomain of the morphism we obtain identify with

$$\Pi^0(\mathcal{L}(V) \star_{Iw} {}^p\mathcal{H}^0(\mathcal{F} \star_{Iw} \mathcal{G})) = \mathcal{L}^0(V) \star_{Iw}^0 \mathcal{F} \star_{Iw}^0 \mathcal{G},$$

resp. with

$$\Pi^0({}^p\mathcal{H}^0(\mathcal{F} \star_{Iw} \mathcal{G}) \star_{Iw} \mathcal{L}(V)) = \mathcal{F} \star_{Iw}^0 \mathcal{G} \star_{Iw}^0 \mathcal{L}^0(V).$$

It is clear also that under these identifications we have $\Pi^0 \circ {}^p\mathcal{H}^0(\sigma_{S^{-1}(V), \mathcal{F} \star_{Iw} \mathcal{G}}) = \sigma_{V, \mathcal{F} \star_{Iw}^0 \mathcal{G}}^0$. On the other hand, since $\mathcal{L}(V) \star_{Iw} \mathcal{F}$ and $\mathcal{F} \star_{Iw} \mathcal{L}(V)$ are perverse we have

$$\Pi^0 \circ {}^p\mathcal{H}^0(\mathcal{L}(V) \star_{Iw} \mathcal{F} \star_{Iw} \mathcal{G}) = \Pi^0(\mathcal{L}(V) \star_{Iw} \mathcal{F}) \star_{Iw}^0 \mathcal{G}$$

and

$$\Pi^0 \circ {}^p\mathcal{H}^0(\mathcal{F} \star_{Iw} \mathcal{L}(V) \star_{Iw} \mathcal{G}) = \Pi^0(\mathcal{F} \star_{Iw} \mathcal{L}(V)) \star_{Iw}^0 \mathcal{G},$$

and under these identifications we have $\Pi^0 \circ {}^p\mathcal{H}^0(\sigma_{S^{-1}(V), \mathcal{F} \star_{Iw} \text{id}_{\mathcal{G}}}) = \sigma_{V, \mathcal{F} \star_{Iw}^0 \text{id}_{\mathcal{G}}}^0$. Similarly we have

$$\Pi^0 \circ {}^p\mathcal{H}^0(\mathcal{F} \star_{Iw} \mathcal{L}(V) \star_{Iw} \mathcal{G}) = \mathcal{F} \star_{Iw}^0 \Pi^0(\mathcal{L}(V) \star_{Iw} \mathcal{G})$$

and

$$\Pi^0 \circ {}^p\mathcal{H}^0(\mathcal{F} \star_{Iw} \mathcal{G} \star_{Iw} \mathcal{L}(V)) = \mathcal{F} \star_{Iw}^0 \Pi^0(\mathcal{G} \star_{Iw} \mathcal{L}(V)),$$

and under these identifications we have $\Pi^0 \circ {}^p\mathcal{H}^0(\text{id}_{\mathcal{F} \star_{Iw} \sigma_{S^{-1}(V), \mathcal{G}}}) = \text{id}_{\mathcal{F} \star_{Iw}^0 \sigma_{V, \mathcal{G}}^0}$. Combining these descriptions we obtain that our data satisfy property (2), which finishes the proof. \square

5.4. Description of a subcategory of the regular quotient as a category of representations. We will denote by $\tilde{\mathcal{P}}_{Iw}^0$ the full abelian subcategory of \mathcal{P}_{Iw}^0 whose objects are the subquotients of objects of the form $\mathcal{L}^0(V)$ with V in $\text{Rep}(G_{\mathbb{k}}^{\vee})$. It is not difficult (using in particular Lemma 4.5) to check that this subcategory contains all the simple objects of \mathcal{P}_{Iw}^0 . Obviously, the functor \mathcal{L}^0 factors through a functor $\text{Rep}(G_{\mathbb{k}}^{\vee}) \rightarrow \tilde{\mathcal{P}}_{Iw}^0$, which we will denote similarly.

Lemma 5.7. *If \mathcal{F} and \mathcal{G} belong to $\tilde{\mathcal{P}}_{Iw}^0$, then $\mathcal{F} \star_{Iw}^0 \mathcal{G}$ also belongs to $\tilde{\mathcal{P}}_{Iw}^0$.*

Proof. If \mathcal{F} is a subquotient of $\mathcal{L}^0(V)$ and \mathcal{G} a subquotient of $\mathcal{L}^0(V')$, then by exactness of the product \star_{Iw}^0 (see Proposition 5.2) the object $\mathcal{F} \star_{Iw}^0 \mathcal{G}$ is a subquotient of $\mathcal{L}^0(V) \star_{Iw}^0 \mathcal{L}^0(V') \cong \mathcal{L}^0(V \otimes V')$, hence belongs to $\tilde{\mathcal{P}}_{Iw}^0$. \square

As a consequence of Lemma 5.7, $\tilde{\mathcal{P}}_{Iw}^0$ admits a natural structure of abelian monoidal category. Moreover, Lemma 5.6 implies that $\mathcal{L}^0 : \text{Rep}(G_{\mathbb{k}}^{\vee}) \rightarrow \tilde{\mathcal{P}}_{Iw}^0$ has a canonical structure of central functor.

Proposition 5.8. *There exist*

- (1) a subgroup scheme $H \subset G_{\mathbb{k}}^{\vee}$;
- (2) an element $\mathbf{u} \in G_{\mathbb{k}}^{\vee}$ such that $H \subset Z_{G_{\mathbb{k}}^{\vee}}(\mathbf{u})$;
- (3) an equivalence of monoidal categories

$$\Phi^0 : (\tilde{\mathcal{P}}_{Iw}^0, \star_{Iw}^0) \xrightarrow{\sim} (\text{Rep}(H), \otimes);$$

- (4) and an isomorphism of functors $\eta : \Phi^0 \circ \mathcal{L}^0 \xrightarrow{\sim} \text{For}_H^{G_{\mathbb{k}}^{\vee}}$

such that for any V in $\text{Rep}(G_{\mathbb{k}}^{\vee})$, the endomorphism $\eta(\Phi^0(\mathbf{m}_V^0))$ coincides with the action of \mathbf{u} on V .

Proof. Using Proposition 3.1 we obtain the subgroup scheme $H \subset G_{\mathbb{k}}^{\vee}$ and the equivalence of monoidal categories Φ^0 such that $\Phi^0 \circ \mathcal{Z}^0 \cong \text{For}_H^{G_{\mathbb{k}}^{\vee}}$. Lemma 5.3 implies in particular that the morphisms $(\mathfrak{m}_V^0 : V \in \text{Rep}(G_{\mathbb{k}}^{\vee}))$ define an automorphism of \mathcal{Z}^0 as a monoidal functor; composing with $\text{For}^H \circ \Phi^0$ (where $\text{For}^H : \text{Rep}(H) \rightarrow \text{Vect}_{\mathbb{k}}$ is the forgetful functor) we obtain a tensor automorphism of $\text{For}^H \circ \Phi^0 \circ \mathcal{Z}^0 \cong \text{For}^{G_{\mathbb{k}}^{\vee}}$, hence an element $\mathfrak{u} \in G_{\mathbb{k}}^{\vee}$ (see [DM, Proposition 2.8]). The fact that this automorphism is induced by an automorphism of $\text{For}_H^{G_{\mathbb{k}}^{\vee}}$ means that \mathfrak{u} commutes with H , i.e. that $H \subset Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})$. \square

Remark 5.9. The monodromy automorphism with respect to loop rotation action defines, for any object \mathcal{F} of \mathbf{P}_{Iw}^0 , an automorphism $\mathfrak{M}_{\mathcal{F}}^0$. It follows from (4.3) that for V in $\text{Rep}(G_{\mathbb{k}}^{\vee})$ we have $\mathfrak{m}_V^0 = (\mathfrak{M}_{\mathcal{Z}^0(V)}^0)^{-1}$. This shows in particular that the automorphism of the functor $\text{For}^{G_{\mathbb{k}}^{\vee}}$ defining \mathfrak{u} is induced by an automorphism of $\text{For}^H : \text{Rep}(H) \rightarrow \text{Vect}_{\mathbb{k}}$. Hence \mathfrak{u} in fact belongs to the subgroup H . (This observation will not play any role below.)

6. REGULARITY OF \mathfrak{u}

In the rest of the paper we assume that $X^*(T)/\mathbb{Z}\mathfrak{R}$ is free.

6.1. Statement. The following statement will be crucial for us.

Theorem 6.1. *The element $\mathfrak{u} \in G_{\mathbb{k}}^{\vee}$ is regular unipotent.*

In order to prove this statement we will construct a Borel subgroup $\tilde{B}^{\vee} \subset G_{\mathbb{k}}^{\vee}$ containing \mathfrak{u} in its unipotent radical \tilde{U}^{\vee} , and then show that the image of \mathfrak{u} in $\tilde{U}^{\vee}/(\tilde{U}^{\vee}, \tilde{U}^{\vee})$ has nontrivial components in each (simple) root subgroup, see Proposition 6.2 below.

Recall (see §4.6) that for any V in $\text{Rep}(G_{\mathbb{k}}^{\vee})$ and any ideal $\mathbf{Y} \subset \mathbf{X}$ we have a canonical subobject $\mathcal{Z}(V)_{\mathbf{Y}} \subset \mathcal{Z}(V)$. We define an increasing \mathbb{Z} -filtration on $\mathcal{Z}(V)$ by setting, for $n \in \mathbb{Z}$,

$$F^n \mathcal{Z}(V) := \mathcal{Z}(V)_{\{\lambda \in \mathbf{X}^{\vee} \mid \langle \lambda, 2\rho \rangle \leq n\}}.$$

(Here, 2ρ denotes the sum of the positive roots of G .) Composing with the exact functor Π^0 we deduce a filtration on $\mathcal{Z}^0(V)$, and then applying the fiber functor $\text{For}^H \circ \Phi^0$ we obtain an exact \otimes -filtration on the fiber functor $\text{For}^H \circ \Phi^0 \circ \mathcal{Z}^0 \cong \text{For}^{G_{\mathbb{k}}^{\vee}}$ in the sense of [SR, Chap. IV, §2.1]. (In fact the associated graded for this filtration is the functor sending V to

$$(6.1) \quad \bigoplus_{\lambda \in \mathbf{X}^{\vee}} \text{Grad}_{\lambda}(\mathcal{Z}(V)) \otimes_{\mathbb{k}} (\Phi^0 \circ \Pi^0(\mathcal{W}_{\lambda})),$$

which is exact by Lemma 4.7.) By [SR, Théorème in Chap. IV, §2.4] (a general result due to Deligne), such a filtration is automatically “splittable.” Hence [SR, Chap. IV, Prop. 2.2.5] ensures that the subgroup scheme \tilde{B}^{\vee} of $G_{\mathbb{k}}^{\vee}$ consisting of tensor automorphisms of $\text{For}^{G_{\mathbb{k}}^{\vee}}$ which are compatible with this filtration (see [SR, Chap. IV, §2.1.3]) is a parabolic subgroup, with unipotent radical the subgroup \tilde{U}^{\vee} of automorphisms inducing the identity on the associated graded. The Lie algebra of \tilde{B}^{\vee} is also described in [SR, Chap. IV, Prop. 2.2.5]; from the dimension of this Lie algebra we see that in our case \tilde{B}^{\vee} is in fact a Borel subgroup in $G_{\mathbb{k}}^{\vee}$. (In this

computation we use (4.7) and the fact that in (6.1) the vector space $\Phi^0 \circ \Pi^0(\mathscr{W}_\lambda)$ is 1-dimensional, since it is invertible in the monoidal category of vector spaces.)

For V in $\text{Rep}(G_{\mathbb{k}}^\vee)$, the automorphism \mathfrak{m}_V^0 is induced by the automorphism \mathfrak{m}_V of $\mathscr{Z}(V)$, which must respect the filtration $F^\bullet \mathscr{Z}(V)$ by functoriality of this filtration. Therefore \mathfrak{m}_V^0 respects the filtration of $\mathscr{Z}^0(V)$, showing that \mathfrak{u} belongs to \tilde{B}^\vee . Moreover, since \mathscr{W}_λ is equivariant under the loop-rotation action, (4.3) implies that the action of \mathfrak{m}_V^0 on the associated graded for our filtration is trivial. Therefore, \mathfrak{u} even belongs to \tilde{U}^\vee .

We now consider the quotient

$$\bar{U}^\vee := \tilde{U}^\vee / (\tilde{U}^\vee, \tilde{U}^\vee),$$

a commutative algebraic group. This group admits a natural action (by group automorphisms) of the ‘‘universal maximal torus’’ A^\vee . (This group is defined as the quotient of any Borel subgroup by its unipotent radical; so in particular we have $\tilde{B}^\vee / \tilde{U}^\vee \cong A^\vee$ canonically.) This action ‘‘breaks’’ \bar{U}^\vee as a product of subgroups \bar{U}_α^\vee isomorphic to \mathbb{G}_a and naturally parametrized by the opposites of the simple roots of $(G_{\mathbb{k}}^\vee, T_{\mathbb{k}}^\vee)$, seen as characters of $T_{\mathbb{k}}^\vee \cong B_{\mathbb{k}}^\vee / (B_{\mathbb{k}}^\vee, B_{\mathbb{k}}^\vee) \cong A^\vee$.

By a well-known characterization of regular unipotent elements (see e.g. [Hu, §4.1]), Theorem 6.1 will follow from the following claim.

Proposition 6.2. *For any simple root α , the component $\mathfrak{u}_{-\alpha}$ of \mathfrak{u} in $\bar{U}_{-\alpha}^\vee$ is non-trivial.*

The proof of this proposition will require some preparation; it will be given in §6.3 below.

6.2. Computation in rank 1. In this subsection we fix a simple root α of (G, T) , and set $s := s_\alpha$. We also choose an element $\varpi \in \mathbf{X}_+^\vee$ such that $\langle \varpi, \alpha \rangle = 1$ (or equivalently $s(\varpi) = \varpi - \alpha^\vee$). (Our running assumption guarantees that such a coweight exists.)

Lemma 6.3. *We have*

$$\mathscr{W}_\varpi \cong \nabla_{\mathfrak{t}(\varpi)}^{\text{Iw}} \cong \nabla_{\mathfrak{t}(\varpi)s}^{\text{Iw}} \star_{\text{Iw}} \nabla_s^{\text{Iw}}, \quad \mathscr{W}_{\varpi - \alpha^\vee} \cong \Delta_s^{\text{Iw}} \star_{\text{Iw}} \nabla_{\mathfrak{t}(\varpi)s}^{\text{Iw}}.$$

Proof. The equality $\mathscr{W}_\varpi = \nabla_{\mathfrak{t}(\varpi)}^{\text{Iw}}$ follows from the definitions (and the fact that ϖ is dominant), and the isomorphism $\nabla_{\mathfrak{t}(\varpi)s}^{\text{Iw}} \star_{\text{Iw}} \nabla_s^{\text{Iw}} \cong \nabla_{\mathfrak{t}(\varpi)}^{\text{Iw}}$ from the fact that $\ell(\mathfrak{t}(\varpi)s) = \ell(\mathfrak{t}(\varpi)) - 1$ (which can be checked using (4.4) and (4.5)). Next, we observe that $\varpi - \alpha^\vee = (2\varpi - \alpha^\vee) - \varpi$, with $2\varpi - \alpha^\vee \in \mathbf{X}_+^\vee$. Hence

$$\mathscr{W}_{\varpi - \alpha^\vee} \cong \Delta_{\mathfrak{t}(-\varpi)}^{\text{Iw}} \star_{\text{Iw}} \nabla_{\mathfrak{t}(2\varpi - \alpha^\vee)}^{\text{Iw}}.$$

Now as above we have $\ell(\mathfrak{t}(\varpi)s) = \ell(\mathfrak{t}(\varpi)) - 1$, hence $\Delta_{\mathfrak{t}(-\varpi)}^{\text{Iw}} \cong \Delta_s^{\text{Iw}} \star_{\text{Iw}} \Delta_{\mathfrak{st}(-\varpi)}^{\text{Iw}}$. On the other hand we have $\ell(\mathfrak{t}(2\varpi - \alpha^\vee)) = \ell(\mathfrak{t}(2\varpi)) - 2 = 2\ell(\mathfrak{t}(\varpi)) - 2$, and $\mathfrak{t}(2\varpi - \alpha^\vee) = (\mathfrak{t}(\varpi)s)^2$. It follows that

$$\nabla_{\mathfrak{t}(2\varpi - \alpha^\vee)}^{\text{Iw}} \cong \nabla_{\mathfrak{t}(\varpi)s}^{\text{Iw}} \star_{\text{Iw}} \nabla_{\mathfrak{t}(\varpi)s}^{\text{Iw}}.$$

We deduce as desired that

$$\mathscr{W}_{\varpi - \alpha^\vee} \cong \Delta_s^{\text{Iw}} \star_{\text{Iw}} \nabla_{\mathfrak{t}(\varpi)s}^{\text{Iw}},$$

since $\Delta_{\mathfrak{st}(-\varpi)}^{\text{Iw}} \star_{\text{Iw}} \nabla_{\mathfrak{t}(\varpi)s}^{\text{Iw}} \cong \delta_{\text{F1}}$ by (4.6). \square

We now set

$$X_1 := \mathrm{Fl}_{G, \mathfrak{t}(\varpi)}, \quad X_2 := \mathrm{Fl}_{G, \mathfrak{t}(\varpi - \alpha^\vee)}, \quad Y := \mathrm{Fl}_{G, \mathfrak{t}(\varpi)_s}$$

and $X := X_1 \sqcup X_2 \sqcup Y$. Then X_1 , X_2 and Y are isomorphic to affine spaces, X is a locally closed subvariety in Fl_G , X_1 and X_2 are open in X , and Y is closed in X . The corresponding embeddings will be denoted

$$j : X \hookrightarrow \mathrm{Fl}, \quad j_1 : X_1 \hookrightarrow X, \quad j_2 : X_2 \hookrightarrow X, \quad i : Y \rightarrow X.$$

Corollary 6.4. *We have*

$$\mathcal{W}_\varpi = j_*(j_{1*}\mathbb{k}_{X_1}[\ell(\mathfrak{t}(\varpi))]), \quad \mathcal{W}_{\varpi - \alpha^\vee} = j_*(j_{2*}\mathbb{k}_{X_2}[\ell(\mathfrak{t}(\varpi))]).$$

Proof. The first isomorphism is clear from the first isomorphism in Lemma 6.3, since $j \circ j_1 = j_{\mathfrak{t}(\varpi)}$. For the description of $\mathcal{W}_{\varpi - \alpha^\vee}$, we start with the isomorphism $\mathcal{W}_{\varpi - \alpha^\vee} \cong \Delta_s^{\mathrm{Iw}} \star_{\mathrm{Iw}} \nabla_{\mathfrak{t}(\varpi)_s}^{\mathrm{Iw}}$ from Lemma 6.3. By definition of convolution, we have

$$\Delta_s^{\mathrm{Iw}} \star_{\mathrm{Iw}} \nabla_{\mathfrak{t}(\varpi)_s}^{\mathrm{Iw}} = f_*(\Delta_s^{\mathrm{Iw}} \widetilde{\boxtimes}_{\mathbb{k}_Y}[\dim(Y)]),$$

where $f : \overline{\mathrm{Fl}_{G,s}} \times^{\mathrm{Iw}} Y \rightarrow \mathrm{Fl}_G$ is the morphism induced by multiplication in LG . Now f induces an isomorphism $\overline{\mathrm{Fl}_{G,s}} \times^{\mathrm{Iw}} Y \xrightarrow{\sim} Y \sqcup X_2$, under which the open subvariety $\mathrm{Fl}_{G,s} \times^{\mathrm{Iw}} Y$ identifies with X_2 . Hence $\Delta_s^{\mathrm{Iw}} \star_{\mathrm{Iw}} \nabla_{\mathfrak{t}(\varpi)_s}^{\mathrm{Iw}}$ is obtained from $\mathbb{k}_{X_2}[\ell(\mathfrak{t}(\varpi))]$ by taking $!$ -pushforward under the embedding $X_2 \hookrightarrow Y \sqcup X_2$, and then $*$ -pushforward under the embedding $Y \sqcup X_2 \hookrightarrow \mathrm{Fl}$. The latter map can be written as the composition of the closed embedding $Y \sqcup X_2 \hookrightarrow X$ followed by j , and we deduce the desired identification. \square

In the following lemma we denote by $\mathrm{Perv}_{\mathbb{G}_m^{\mathrm{rot}} \times \mathrm{Iw}}(\mathrm{Fl}_G, \mathbb{k})$ the category of $\mathbb{G}_m \times \mathrm{Iw}$ -equivariant perverse sheaves on Fl_G , where \mathbb{G}_m acts on Iw and Fl_G by loop rotation.

Proposition 6.5. *We have*

$$\mathrm{Ext}_{\mathrm{Perv}_{\mathbb{G}_m^{\mathrm{rot}} \times \mathrm{Iw}}(\mathrm{Fl}_G, \mathbb{k})}^1(\mathcal{W}_\varpi, \mathcal{W}_{\varpi - \alpha^\vee}) = 0.$$

Proof. Using Corollary 6.4 and adjunction, we see that

$$\mathrm{Ext}_{\mathrm{Perv}_{\mathbb{G}_m^{\mathrm{rot}} \times \mathrm{Iw}}(\mathrm{Fl}_G, \mathbb{k})}^1(\mathcal{W}_\varpi, \mathcal{W}_{\varpi - \alpha^\vee}) \cong \mathrm{Ext}_{\mathrm{Perv}_{\mathbb{G}_m^{\mathrm{rot}} \times \mathrm{Iw}}(X, \mathbb{k})}^1(j_{1*}\mathbb{k}_{X_1}[d], j_{2*}\mathbb{k}_{X_2}[d]),$$

where $d := \ell(\mathfrak{t}(\varpi))$. We denote by $\bar{j}_2 : \overline{X_2} = X_2 \sqcup Y \hookrightarrow X$ the embedding of the closure $\overline{X_2}$ (a smooth closed subvariety) in X , and consider the natural distinguished triangle

$$j_{2*}\mathbb{k}_{X_2}[d] \rightarrow \bar{j}_{2*}\mathbb{k}_{\overline{X_2}}[d] \rightarrow i_*\mathbb{k}_Y[d] \xrightarrow{[1]}.$$

Here the second term is a simple perverse sheaf, which we will denote by $\mathcal{I}\mathcal{C}_2$, and the third term is the shift by 1 of a simple perverse sheaf which will be denoted $\mathcal{I}\mathcal{C}_Y$. Hence this triangle induces a short exact sequence of $\mathbb{G}_m^{\mathrm{rot}} \times \mathrm{Iw}$ -equivariant perverse sheaves

$$(6.2) \quad \mathcal{I}\mathcal{C}_Y \hookrightarrow j_{2*}\mathbb{k}_{X_2}[d] \rightarrow \mathcal{I}\mathcal{C}_2$$

on X . Similarly, we have a short exact sequence of $\mathbb{G}_m^{\mathrm{rot}} \times \mathrm{Iw}$ -equivariant perverse sheaves

$$(6.3) \quad \mathcal{I}\mathcal{C}_Y \hookrightarrow j_{1*}\mathbb{k}_{X_1}[d] \rightarrow \mathcal{I}\mathcal{C}_1$$

where $\mathcal{I}\mathcal{C}_1$ is the pushforward of the constant sheaf on $X_1 \sqcup Y$ shifted by d . Applying Verdier duality to (6.2)–(6.3) we deduce short exact sequences of $\mathbb{G}_m^{\text{rot}} \times \text{Iw}$ -equivariant perverse sheaves

$$(6.4) \quad \mathcal{I}\mathcal{C}_2 \hookrightarrow j_{2*}\mathbb{k}_{X_2}[d] \twoheadrightarrow \mathcal{I}\mathcal{C}_Y, \quad \mathcal{I}\mathcal{C}_1 \hookrightarrow j_{1*}\mathbb{k}_{X_1}[d] \twoheadrightarrow \mathcal{I}\mathcal{C}_Y.$$

Applying the functor $\text{Hom}_{\text{Perv}_{\mathbb{G}_m^{\text{rot}} \times \text{Iw}}(\text{Fl}_{G, \mathbb{k}})}(-, j_{2!}\mathbb{k}_{X_2}[d])$ to the second exact sequence in (6.4) we obtain an exact sequence

$$(6.5) \quad \text{Ext}^1(\mathcal{I}\mathcal{C}_Y, j_{2!}\mathbb{k}_{X_2}[d]) \rightarrow \text{Ext}^1(j_{1*}\mathbb{k}_{X_1}[d], j_{2!}\mathbb{k}_{X_2}[d]) \\ \rightarrow \text{Ext}^1(\mathcal{I}\mathcal{C}_1, j_{2!}\mathbb{k}_{X_2}[d]) \rightarrow \text{Hom}_{D_{\mathbb{G}_m^{\text{rot}} \times \text{Iw}}^b(X, \mathbb{k})}(\mathcal{I}\mathcal{C}_Y, j_{2!}\mathbb{k}_{X_2}[d+2]).$$

Hence to conclude, it suffices to prove that

$$(6.6) \quad \text{Ext}^1(\mathcal{I}\mathcal{C}_Y, j_{2!}\mathbb{k}_{X_2}[d]) = 0;$$

$$(6.7) \quad \text{the third map in (6.5) is injective.}$$

For this, consider (a portion of) the long exact sequence obtained by applying the functor $\text{Hom}(\mathcal{I}\mathcal{C}_Y, -)$ to (6.2):

$$(6.8) \quad \text{Ext}^1(\mathcal{I}\mathcal{C}_Y, \mathcal{I}\mathcal{C}_Y) \rightarrow \text{Ext}^1(\mathcal{I}\mathcal{C}_Y, j_{2!}\mathbb{k}_{X_1}[d]) \rightarrow \text{Ext}^1(\mathcal{I}\mathcal{C}_Y, \mathcal{I}\mathcal{C}_2) \\ \rightarrow \text{Hom}_{D_{\mathbb{G}_m^{\text{rot}} \times \text{Iw}}^b(X, \mathbb{k})}(\mathcal{I}\mathcal{C}_Y, \mathcal{I}\mathcal{C}_Y[2]) \rightarrow \text{Hom}_{D_{\mathbb{G}_m^{\text{rot}} \times \text{Iw}}^b(X, \mathbb{k})}(\mathcal{I}\mathcal{C}_Y, j_{2!}\mathbb{k}_{X_2}[d+2]) \\ \rightarrow \text{Hom}_{D_{\mathbb{G}_m^{\text{rot}} \times \text{Iw}}^b(X, \mathbb{k})}(\mathcal{I}\mathcal{C}_Y, \mathcal{I}\mathcal{C}_2[2]).$$

The first, resp. fourth, term in this sequence identifies with $H_{\mathbb{G}_m^{\text{rot}} \times \text{Iw}}^1(Y, \mathbb{k}) = 0$, resp. with $H_{\mathbb{G}_m^{\text{rot}} \times \text{Iw}}^2(Y, \mathbb{k}) = \mathbb{k} \otimes_{\mathbb{Z}} X^*(\mathbb{G}_m \times T)$. It is also easy to check using adjunction that we have

$$\text{Ext}^1(\mathcal{I}\mathcal{C}_Y, \mathcal{I}\mathcal{C}_2) \cong H_{\mathbb{G}_m^{\text{rot}} \times \text{Iw}}^0(Y, \mathbb{k}) = \mathbb{k}$$

and

$$\text{Hom}_{D_{\mathbb{G}_m^{\text{rot}} \times I}^b(X, \mathbb{k})}(\mathcal{I}\mathcal{C}_Y, \mathcal{I}\mathcal{C}_2[2]) \cong H_{\mathbb{G}_m^{\text{rot}} \times \text{Iw}}^1(Y, \mathbb{k}) = 0.$$

For any $\beta \in \mathfrak{R}^+$ we have a root subgroup $U_\beta \subset G$, and for $m \in \mathbb{Z}$ we denote by $U_{\beta, m} \subset LG$ the subgroup which identifies, for any choice of isomorphism $u_\beta : \mathbb{G}_a \rightarrow U_\beta$, with the image of $t \mapsto u_\beta(tz^m)$. We set

$$J := \prod_{\substack{\beta \in \mathfrak{R}^+ \setminus \{\alpha\} \\ 0 < n \leq \langle \varpi, \beta \rangle}} U_{\beta, n},$$

where the product is taken in an arbitrary fixed order. Then the map $j \mapsto jz^\varpi \dot{s}I/I$ induces an isomorphism $J \xrightarrow{\sim} Y$, and the map $(u, j) \mapsto u j z^\varpi \dot{s}I/I$ induces an isomorphism between $U_\alpha \times J$ and an open neighborhood of Y in $Y \sqcup X_2$. Hence, arguing as in [BeR, Proof of Lemma 6.5], we see that the image of the map $\text{Ext}^1(\mathcal{I}\mathcal{C}_Y, \mathcal{I}\mathcal{C}_2) \rightarrow \text{Hom}_{D_{\mathbb{G}_m^{\text{rot}} \times I}^b(X, \mathbb{k})}(\mathcal{I}\mathcal{C}_Y, \mathcal{I}\mathcal{C}_Y[2])$ in (6.8) is $\mathbb{k} \otimes \alpha$. It follows that the third map in (6.8) is injective, proving (6.6), and providing moreover an isomorphism

$$(6.9) \quad \text{Hom}_{D_{\mathbb{G}_m^{\text{rot}} \times I}^b(X, \mathbb{k})}(\mathcal{I}\mathcal{C}_Y, j_{2!}\mathbb{k}_{X_1}[d][2]) \cong (\mathbb{k} \otimes_{\mathbb{Z}} X^*(\mathbb{G}_m \times T))/\mathbb{k} \otimes \alpha.$$

Now, let us consider (a portion of) the long exact sequence obtained by applying $\mathrm{Hom}(\mathcal{I}\mathcal{C}_1, -)$ to (6.2):

$$\begin{aligned} \mathrm{Hom}(\mathcal{I}\mathcal{C}_1, \mathcal{I}\mathcal{C}_2) &\rightarrow \mathrm{Ext}^1(\mathcal{I}\mathcal{C}_1, \mathcal{I}\mathcal{C}_Y) \rightarrow \\ &\mathrm{Ext}^1(\mathcal{I}\mathcal{C}_1, j_{2!}\underline{\mathbb{k}}_{X_2}[d]) \rightarrow \mathrm{Ext}^1(\mathcal{I}\mathcal{C}_1, \mathcal{I}\mathcal{C}_2). \end{aligned}$$

Here the first term vanishes. The fourth term also vanishes: in fact this follows from the observation that any extension of $\mathcal{I}\mathcal{C}_1$ by $\mathcal{I}\mathcal{C}_2$ must be the intermediate extension of its restriction to the open subset $X_1 \sqcup X_2$ (because so are $\mathcal{I}\mathcal{C}_1$ and $\mathcal{I}\mathcal{C}_2$); hence this extension must split. These considerations show that the natural morphism $\mathrm{Ext}^1(\mathcal{I}\mathcal{C}_1, \mathcal{I}\mathcal{C}_Y) \rightarrow \mathrm{Ext}^1(\mathcal{I}\mathcal{C}_1, j_{2!}\underline{\mathbb{k}}_{X_2}[d])$ is an isomorphism, which allows to identify the third map in (6.5) with the composition

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{I}\mathcal{C}_1, \mathcal{I}\mathcal{C}_Y) &\rightarrow \mathrm{Hom}_{D_{\mathbb{G}_m^{\mathrm{rot}} \times \mathrm{Iw}}^{\mathrm{b}}(X, \mathbb{k})}(\mathcal{I}\mathcal{C}_Y, \mathcal{I}\mathcal{C}_Y[2]) \\ &\rightarrow \mathrm{Hom}_{D_{\mathbb{G}_m^{\mathrm{rot}} \times \mathrm{Iw}}^{\mathrm{b}}(X, \mathbb{k})}(\mathcal{I}\mathcal{C}_Y, j_{2!}\underline{\mathbb{k}}_{X_2}[d][2]). \end{aligned}$$

As above the first term is 1-dimensional, the middle term identifies with $\mathbb{k} \otimes_{\mathbb{Z}} X^*(\mathbb{G}_m \times T)$, and the first map is injective with image $\mathbb{k} \otimes (\alpha - \hbar)$, where \hbar is the tautological character of \mathbb{G}_m . (To see this, one observes that the map $(j, u) \mapsto jz^\varpi u s I / I$ induces an isomorphism between $J \times U_{-\alpha}$ and an open neighborhood of Y in $Y \sqcup X_1$.) In view of (6.9) we deduce that our composition is indeed injective, finishing the proof of (6.7), hence of the proposition. \square

6.3. Proof of Proposition 6.2. We can now give the proof of Proposition 6.2. We fix a simple root α , with corresponding simple reflection s . We also fix an element $\varpi \in X_*(T) = X^*(T_{\mathbb{k}}^{\vee})$ such that $\langle \varpi, \alpha \rangle = 1$ and $\langle \varpi, \beta \rangle = 0$ for any simple root β different from α , and use it as our choice of cocharacter denoted similarly in §6.2. To simplify notation, we set $r := \langle \varpi, 2\rho \rangle = \ell(\varpi)$.

Let $V := \mathrm{Ind}_{\tilde{B}_{\mathbb{k}}^{\vee}}^{G_{\mathbb{k}}^{\vee}}(\varpi)$ (where ϖ is identified with a character of $A^{\vee} = \tilde{B}^{\vee} / \tilde{U}^{\vee}$ via the canonical isomorphism $T_{\mathbb{k}}^{\vee} \cong A^{\vee}$, see §6.1). For the filtration on V constructed in §6.1 we have $V = F^r(V)$, $V/F^{r-2}(V)$ is one-dimensional (with weight ϖ for the action of A^{\vee}), and so is $F^{r-2}(V)/F^{r-4}(V)$ (with weight $\varpi - \alpha^{\vee}$ for the action of A^{\vee}). This shows that the perverse sheaf

$$\mathcal{F} := \mathcal{Z}(V) / F^{r-4} \mathcal{Z}(V)$$

fits in an exact sequence of Iw-equivariant perverse sheaves

$$(6.10) \quad \mathcal{W}_{\varpi - \alpha^{\vee}} \hookrightarrow \mathcal{F} \rightarrow \mathcal{W}_{\varpi}.$$

Lemma 6.6. *The extension (6.10) is nonsplit.*

Proof. Let $W(\varpi)$ be the set of $T_{\mathbb{k}}^{\vee}$ -weights of V . Then since $\mathcal{Z}(V)$ admits a filtration whose subquotients are of the form \mathcal{W}_{λ} with $\lambda \in W(\varpi)$ (see §4.6), in view of Property (3) in §4.5 its support must be

$$\bigcup_{\lambda \in W(\varpi)} \overline{\mathrm{Fl}_{G, t(\lambda)}}.$$

It is clear that the subvariety X of §6.2 is contained (and open) in the closed subvariety $\overline{\mathrm{Fl}_{G, t(\varpi)}} \cup \overline{\mathrm{Fl}_{G, t(\varpi - \alpha^{\vee})}}$. Moreover, X does not intersect any subvariety $\overline{\mathrm{Fl}_{G, t(\lambda)}}$ with $\lambda \in W(\varpi) \setminus \{\varpi, \varpi - \alpha^{\vee}\}$; in fact this is clear for dimension reasons

if $\lambda \notin W_f \cdot \varpi$, and if $\lambda \in W_f \cdot \varpi \setminus \{\varpi, \varpi - \alpha^\vee\}$ then $\mathbf{t}(\varpi)s$ is not smaller than $\mathbf{t}(\lambda)$ in the Bruhat order.

From this observation we deduce that the restriction of $\mathcal{Z}(V)$ to X coincides with that of \mathcal{F} , and is a perverse sheaf. Restricting (6.10) to X and using Corollary 6.4 we see moreover that we have an exact sequence

$$(6.11) \quad (j_2)_! \mathbb{k}_{X_2}[r] \hookrightarrow \mathcal{Z}(V)|_X \twoheadrightarrow (j_1)_* \mathbb{k}_{X_1}[r].$$

Now, similar considerations using the ‘‘opposite’’ Wakimoto sheaves (see Remark 4.6 and Remark 4.9) show that $\mathcal{Z}(V)$ has a canonical subobject \mathcal{F}' such that $\mathcal{Z}(V)|_X = \mathcal{F}'|_X$ and which fits in a short exact sequence $\mathcal{W}_\varpi^{\text{op}} \hookrightarrow \mathcal{F}' \twoheadrightarrow \mathcal{W}_{\varpi - \alpha^\vee}^{\text{op}}$. Restricting to X and using the ‘‘opposite’’ version of Corollary 6.4 we obtain a short exact sequence

$$(j_1)_! \mathbb{k}_{X_1}[r] \hookrightarrow \mathcal{Z}(V)|_X \twoheadrightarrow (j_2)_* \mathbb{k}_{X_2}[r].$$

The existence of this exact sequence forces (6.11) (and hence (6.10)) to be nonsplit. (For instance, if (6.11) were split there would exist no injective map $(j_1)_! \mathbb{k}_{X_1}[r] \hookrightarrow \mathcal{Z}(V)|_X$.) \square

Now we consider the action of \mathfrak{u} on the 2-dimensional \tilde{B}^\vee -module $V/F^{r-4}(V)$. (Since \mathfrak{u} belongs to \tilde{B}^\vee , it stabilizes $F^{r-4}(V)$, hence indeed induces an endomorphism of this quotient.) It is clear from the description of this module that this action coincides with the action of the component $\mathfrak{u}_{-\alpha}$. Hence to conclude the proof it suffices to prove that this action is nontrivial, or in other words that the endomorphism of $\Pi^0(\mathcal{F})$ induced by \mathfrak{m}_V^0 is nontrivial. In view of (4.3), for this it suffices to prove that $\Pi^0(\mathfrak{M}_\mathcal{F})$ is nontrivial.

First, we claim that $\mathfrak{M}_\mathcal{F}$ is nontrivial. In fact, in view of [BeR, Lemma 2.6] (see also [BeR, §10.1] for a discussion of the étale version) this is equivalent to saying that \mathcal{F} is not equivariant under the action of $\mathbb{G}_m^{\text{rot}}$. However, by construction \mathcal{F} is Iw-equivariant. If it were $\mathbb{G}_m^{\text{rot}}$ -equivariant, then it would be $\mathbb{G}_m^{\text{rot}} \times \text{Iw}$ -equivariant, which is impossible by Proposition 6.5 and Lemma 6.6.

Now that this claim is established, we know that the nilpotent endomorphism $\mathfrak{M}_\mathcal{F} - \text{id}_\mathcal{F}$ is nonzero. This endomorphism stabilizes the filtration (6.10), and induces the zero endomorphism on the associated graded. It therefore factors as a composition

$$\mathcal{F} \twoheadrightarrow \mathcal{W}_\varpi \xrightarrow{f} \mathcal{W}_{\varpi - \alpha^\vee} \hookrightarrow \mathcal{F},$$

where $f \neq 0$. Now we have $\mathcal{W}_\varpi = \nabla_\varpi^{\text{Iw}}$, whose top is isomorphic to $\mathcal{I}\mathcal{C}_\omega$ for some $\omega \in \Omega$ by Lemma 4.5. Hence the image of f admits $\mathcal{I}\mathcal{C}_\omega$ as a composition factor. Since $\mathcal{I}\mathcal{C}_\omega$ is not killed by Π^0 it follows that $\Pi^0(f) \neq 0$, so that $\Pi^0(\mathfrak{M}_\mathcal{F}) \neq \text{id}_{\Pi^0(\mathcal{F})}$, which finishes the proof.

7. ANTISPHERICAL AND IWAHORI–WHITTAKER CATEGORIES

We continue with the assumption of Section 6, namely that $X^*(T)/\mathbb{Z}\mathfrak{X}$ is free.

7.1. Iwahori–Whittaker perverse sheaves. We denote by $B^+ \subset G$ the Borel subgroup containing T and opposite to B , and by $\text{Iw}^+ \subset L^+G$ the corresponding Iwahori subgroup. Let also Iw_u^+ be the pro-unipotent radical of Iw^+ , i.e. the inverse image of the unipotent radical U^+ of B^+ under the natural map $\text{Iw}^+ \rightarrow B^+$. We choose a nondegenerate character $\chi' : U^+ \rightarrow \mathbb{G}_a$ (i.e. a group morphism nontrivial on each simple root subgroup), and denote by χ its composition with the projection

$\mathrm{Iw}_u^+ \rightarrow U^+$. Finally we choose an Artin–Schreier \mathbb{k} -local system $\mathcal{L}_{\mathrm{AS}}$ on \mathbb{G}_a , and consider the derived category $D_{\mathcal{I}\mathcal{W}}^b(\mathrm{Fl}_G, \mathbb{k})$ of Iwahori–Whittaker sheaves on Fl_G , i.e. the $(\mathrm{Iw}_u^+, \chi^*(\mathcal{L}_{\mathrm{AS}}))$ -equivariant derived category of \mathbb{k} -sheaves on Fl_G . (As for the L^+G -equivariant derived category of Gr_G or the Iw-equivariant derived category of Fl_G , the “true” definition of this category requires restricting to finite unions of orbits, and considering a finite-type quotient of Iw_u^+ through which the action factors; this procedure is standard, and will be omitted.) This category is endowed with the perverse t-structure, whose heart will be denoted $\mathcal{P}_{\mathcal{I}\mathcal{W}}$.

The Iw_u^+ -orbits on Fl_G are parametrized in the standard way by W . Those which support a nonzero $(\mathrm{Iw}_u^+, \chi^*(\mathcal{L}_{\mathrm{AS}}))$ -equivariant local system are those corresponding to elements in the subset ${}^fW \subset W$ of elements w which are minimal in $W_f w$. For any $\lambda \in \mathbf{X}^\vee$ we will denote by w_λ the minimal length element in $W_f \cdot \mathfrak{t}(\lambda)$, by $\mathrm{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}}$ the corresponding orbit, and by $\mathcal{L}_{\chi,\lambda}$ the unique rank-1 $(\mathrm{Iw}_u^+, \chi^*(\mathcal{L}_{\mathrm{AS}}))$ -equivariant local system on $\mathrm{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}}$. We also denote by $j_\lambda^{\mathcal{I}\mathcal{W}} : \mathrm{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}} \rightarrow \mathrm{Fl}_G$ the embedding, and set

$$\Delta_\lambda^{\mathcal{I}\mathcal{W}} := (j_\lambda^{\mathcal{I}\mathcal{W}})_! \mathcal{L}_{\chi,\lambda}[\dim(\mathrm{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}})], \quad \nabla_\lambda^{\mathcal{I}\mathcal{W}} := (j_\lambda^{\mathcal{I}\mathcal{W}})_* \mathcal{L}_{\chi,\lambda}[\dim(\mathrm{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}})].$$

(These objects are perverse sheaves because $j_\lambda^{\mathcal{I}\mathcal{W}}$ is an affine embedding.) As usual we have

$$(7.1) \quad \mathrm{Hom}_{D_{\mathcal{I}\mathcal{W}}^b(\mathrm{Fl}_G, \mathbb{k})}(\Delta_\lambda^{\mathcal{I}\mathcal{W}}, \nabla_\mu^{\mathcal{I}\mathcal{W}}[n]) \cong \begin{cases} \mathbb{k} & \text{if } \lambda = \mu \text{ and } n = 0; \\ 0 & \text{otherwise,} \end{cases}$$

and the image of any nonzero morphism $\Delta_\lambda^{\mathcal{I}\mathcal{W}} \rightarrow \nabla_\lambda^{\mathcal{I}\mathcal{W}}$ is simple; this simple object will be denoted $\mathcal{S}_{\mathcal{C}_\lambda^{\mathcal{I}\mathcal{W}}}$. It is well known also that the category $\mathcal{P}_{\mathcal{I}\mathcal{W}}$ has a natural structure of highest weight category, with weight poset \mathbf{X}^\vee (for the order $\leq^{\mathcal{I}\mathcal{W}}$ defined by $\lambda \leq^{\mathcal{I}\mathcal{W}} \mu$ iff $\mathrm{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}} \subset \overline{\mathrm{Fl}_{G,\mu}^{\mathcal{I}\mathcal{W}}}$) and standard, resp. costandard, objects $(\Delta_\lambda^{\mathcal{I}\mathcal{W}} : \lambda \in \mathbf{X}^\vee)$, resp. $(\nabla_\lambda^{\mathcal{I}\mathcal{W}} : \lambda \in \mathbf{X}^\vee)$.

In Section 8 we will need some properties of the associated order on \mathbf{X}^\vee . Here, for $\lambda \in \mathbf{X}^\vee$, with dominant W_f -conjugate λ^+ , we set

$$W(\lambda) = \{\mu \in \mathbf{X}^\vee \mid \mu^+ \preceq \lambda^+\},$$

where \preceq is as in §4.5. In other words, $W(\lambda)$ is the set of $T_{\mathbb{k}}^\vee$ -weights of the Weyl module for $G_{\mathbb{k}}^\vee$ of highest weight λ^+ (so that this notation extends the notation used already in the proof of Lemma 6.6). The other standard description of these weights shows that $W(\lambda)$ coincides with the set denoted $\mathrm{conv}(\lambda)$ in [AR2], i.e. the intersection of the convex hull of $W_f \lambda$ with $\lambda + \mathbb{Z}\mathfrak{R}^\vee$.

Lemma 7.1. (1) *If $\lambda, \mu \in \mathbf{X}^\vee$ and $\mathrm{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}} \subset \overline{\mathrm{Fl}_{G,\mu}^{\mathcal{I}\mathcal{W}}}$, then $\lambda \in W(\mu)$.*

(2) *If $\lambda \in \mathbf{X}_+^\vee$ and $\mu \in W_f(\lambda)$, then $\mathrm{Fl}_{G,\mu}^{\mathcal{I}\mathcal{W}} \subset \overline{\mathrm{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}}}$.*

Proof. It is not difficult to see that for any $\nu \in \mathbf{X}^\vee$ we have

$$\overline{\mathrm{Fl}_{G,\nu}^{\mathcal{I}\mathcal{W}}} = \overline{\mathrm{Fl}_{G,w_0 w_\nu}},$$

where (as in §4.6) w_0 is the longest element in W_f . Hence $\mathrm{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}} \subset \overline{\mathrm{Fl}_{G,\mu}^{\mathcal{I}\mathcal{W}}}$ iff $w_0 w_\lambda \leq w_0 w_\mu$ in the Bruhat order. In view of [Dou, Lemma 2.2], this is equivalent to the property that $w_\lambda \leq w_\mu$. Hence (1) follows from [AR2, Lemma 9.12(3)], and (2) from the fact that $w_\mu \leq w_\lambda = \mathfrak{t}(\lambda)$ in this case (see e.g. [MR2, Lemma 2.4]). \square

7.2. Statement. Note that since the closure $\overline{\mathrm{Fl}_{G,0}^{\mathcal{I}\mathcal{W}}}$ does not contain any orbit $\mathrm{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}}$ with $\lambda \neq 0$, the natural morphism $\Delta_0^{\mathcal{I}\mathcal{W}} \rightarrow \nabla_0^{\mathcal{I}\mathcal{W}}$ is an isomorphism (and this complex coincides with $\mathcal{S}\mathcal{C}_0^{\mathcal{I}\mathcal{W}}$). We consider the functor

$$(7.2) \quad \mathrm{Av}_{\mathcal{I}\mathcal{W}} : D_{\mathrm{Iw}}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k}) \rightarrow D_{\mathcal{I}\mathcal{W}}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k})$$

defined by

$$\mathrm{Av}_{\mathcal{I}\mathcal{W}}(\mathcal{F}) = \Delta_0^{\mathcal{I}\mathcal{W}} \star_{\mathrm{Iw}} \mathcal{F}.$$

We will denote by $\mathrm{P}_{\mathrm{Iw}}^{\mathrm{asp}}$ the Serre quotient of the category P_{Iw} by the Serre subcategory generated by the simple objects $\mathcal{S}\mathcal{C}_w$ with $w \notin {}^{\mathrm{f}}W$. The associated quotient functor will be denoted $\Pi^{\mathrm{asp}} : \mathrm{P}_{\mathrm{Iw}} \rightarrow \mathrm{P}_{\mathrm{Iw}}^{\mathrm{asp}}$. (Here, “asp” standard for “anti-spherical.”)

The main result of the present section is the following statement.

Theorem 7.2. (1) *The functor $\mathrm{Av}_{\mathcal{I}\mathcal{W}}$ is t-exact for the perverse t-structures on $D_{\mathrm{Iw}}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k})$ and $D_{\mathcal{I}\mathcal{W}}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k})$.*
(2) *The functor*

$$\mathrm{Av}_{\mathcal{I}\mathcal{W}} : \mathrm{P}_{\mathrm{Iw}} \rightarrow \mathrm{P}_{\mathcal{I}\mathcal{W}}$$

factors through a fully-faithful functor

$$\mathrm{Av}_{\mathcal{I}\mathcal{W}}^{\mathrm{asp}} : \mathrm{P}_{\mathrm{Iw}}^{\mathrm{asp}} \rightarrow \mathrm{P}_{\mathcal{I}\mathcal{W}}.$$

Moreover, the essential image of this functor is closed under subquotients.

Remark 7.3. We will see later (see §9.1) that, under appropriate assumptions, the functor in Theorem 7.2(2) is an equivalence of categories.

The proof of Theorem 7.2 is very close to the corresponding proof in [AB, §2], which is repeated with more details in [AR3, Chap. 6]. We will not repeat the proofs that can simply be copied from these references.

7.3. Exactness. We start by proving Theorem 7.2(1).

The following lemma can be proved as in [AR3, Lemma 6.4.4 & Lemma 6.4.5].

Lemma 7.4. (1) *For $w \in W$ we have $\mathrm{Av}_{\mathcal{I}\mathcal{W}}(\mathcal{S}\mathcal{C}_w) = 0$ unless $w \in {}^{\mathrm{f}}W$.*
(2) *For any $w \in W$ we have*

$$\mathrm{Av}_{\mathcal{I}\mathcal{W}}(\Delta_w^{\mathrm{Iw}}) \cong \Delta_\lambda^{\mathcal{I}\mathcal{W}} \quad \text{and} \quad \mathrm{Av}_{\mathcal{I}\mathcal{W}}(\nabla_w^{\mathrm{Iw}}) \cong \nabla_\lambda^{\mathcal{I}\mathcal{W}},$$

where $\lambda \in \mathbf{X}^\vee$ is the unique element such that $W_{\mathrm{f}} \cdot w = W_{\mathrm{f}} \cdot w_\lambda$.

As in [AR3, Corollary 6.4.7] one deduces the following corollary, which in particular establishes Theorem 7.2(1).

Corollary 7.5. *The functor $\mathrm{Av}_{\mathcal{I}\mathcal{W}}$ is t-exact. Moreover, for any $\lambda \in \mathbf{X}$ we have*

$$\mathrm{Av}_{\mathcal{I}\mathcal{W}}(\mathcal{S}\mathcal{C}_{w_\lambda}) \cong \mathcal{S}\mathcal{C}_\lambda^{\mathcal{I}\mathcal{W}}.$$

7.4. An adjoint functor. The functor $\mathrm{Av}_{\mathcal{I}\mathcal{W}}$ admits another description. Namely, we will denote by Iw_0 the intersection of Iw and Iw_+^\dagger , or in other words the kernel of the morphism $\mathrm{ev} : L^+G \rightarrow G$ from §4.2. We can then consider the associated equivariant derived category $D_{\mathrm{Iw}_0}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k})$, and the functor

$${}^{\dagger}\mathrm{Ind}_{\mathrm{Iw}_0}^{\mathcal{I}\mathcal{W}} : D_{\mathrm{Iw}_0}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k}) \rightarrow D_{\mathcal{I}\mathcal{W}}^{\mathrm{b}}(\mathrm{Fl}_G, \mathbb{k})$$

sending a complex \mathcal{F} to $a_!(\chi^*(\mathcal{L}_{AS}) \boxtimes \mathcal{F})[\dim(U^+)]$, where $a : \mathrm{Iw}_u^+ \times^{\mathrm{Iw}_0} \mathrm{Fl}_G \rightarrow \mathrm{Fl}_G$ is the action morphism. It is not difficult to see that we have

$$\mathrm{Av}_{\mathcal{I}\mathcal{W}} = {}^!\mathrm{Ind}_{\mathrm{Iw}_0}^{\mathcal{I}\mathcal{W}} \circ \mathrm{For}_{\mathrm{Iw}_0}^{\mathrm{Iw}},$$

where $\mathrm{For}_{\mathrm{Iw}_0}^{\mathrm{Iw}} : D_{\mathrm{Iw}_0}^b(\mathrm{Fl}_G, \mathbb{k}) \rightarrow D_{\mathrm{Iw}_0}^b(\mathrm{Fl}_G, \mathbb{k})$ is the forgetful functor, and moreover that the functor ${}^!\mathrm{Ind}_{\mathrm{Iw}_0}^{\mathcal{I}\mathcal{W}}$ is left adjoint to $\mathrm{For}_{\mathrm{Iw}_0}^{\mathcal{I}\mathcal{W}}[\dim(U^+)]$, where $\mathrm{For}_{\mathrm{Iw}_0}^{\mathcal{I}\mathcal{W}} : D_{\mathcal{I}\mathcal{W}}^b(\mathrm{Fl}_G, \mathbb{k}) \rightarrow D_{\mathrm{Iw}_0}^b(\mathrm{Fl}_G, \mathbb{k})$ is the forgetful functor (see e.g. [AR1, Lemma A.3] for similar considerations).

Now we also consider the induction functor

$${}^*\mathrm{Ind}_{\mathrm{Iw}_0}^{\mathrm{Iw}} : D_{\mathrm{Iw}_0}^b(\mathrm{Fl}_G, \mathbb{k}) \rightarrow D_{\mathrm{Iw}}^b(\mathrm{Fl}, \mathbb{k})$$

sending a complex \mathcal{F} to $b_*(\mathbb{k} \boxtimes \mathcal{F})[\dim(B)]$, where $b : \mathrm{Iw} \times^{\mathrm{Iw}_0} \mathrm{Fl}_G \rightarrow \mathrm{Fl}_G$ is the action morphism. This functor is right adjoint to the functor $\mathrm{For}_{\mathrm{Iw}_0}^{\mathrm{Iw}}[-\dim(B)]$. Therefore, the functor

$${}^*\mathrm{Ind}_{\mathrm{Iw}_0}^{\mathrm{Iw}} \circ \mathrm{For}_{\mathrm{Iw}_0}^{\mathcal{I}\mathcal{W}}[-\dim(T)]$$

is right adjoint to the functor $\mathrm{Av}_{\mathcal{I}\mathcal{W}}$.

As seen in Corollary 7.5 the functor $\mathrm{Av}_{\mathcal{I}\mathcal{W}}$ is t-exact; therefore its right adjoint ${}^*\mathrm{Ind}_{\mathrm{Iw}_0}^{\mathrm{Iw}} \circ \mathrm{For}_{\mathrm{Iw}_0}^{\mathcal{I}\mathcal{W}}[-\dim(T)]$ is left t-exact, and the functor

$$(7.3) \quad \mathrm{F} : \mathrm{P}_{\mathcal{I}\mathcal{W}} \rightarrow \mathrm{P}_{\mathrm{Iw}}$$

defined by

$$\mathrm{F}(\mathcal{F}) = {}^{\mathrm{P}}\mathcal{H}^0({}^*\mathrm{Ind}_{\mathrm{Iw}_0}^{\mathrm{Iw}} \circ \mathrm{For}_{\mathrm{Iw}_0}^{\mathcal{I}\mathcal{W}}(\mathcal{F})[-\dim(T)])$$

is right adjoint to the functor $\mathrm{Av}_{\mathcal{I}\mathcal{W}} : \mathrm{P}_{\mathrm{Iw}} \rightarrow \mathrm{P}_{\mathcal{I}\mathcal{W}}$.

7.5. Some properties of tilting objects. As explained in §7.1 the category $\mathrm{P}_{\mathcal{I}\mathcal{W}}$ has a natural highest weight structure, with weight poset \mathbf{X}^\vee (which we have identified with ${}^{\mathrm{f}}W$). In such a category we can consider the tilting objects, and the general theory of highest weight categories (see e.g. [Ri, §7]) shows that the set of isomorphism classes of indecomposable tilting objects in $\mathrm{P}_{\mathcal{I}\mathcal{W}}$ is in a natural bijection with \mathbf{X}^\vee ; the object associated with λ will be denoted $\mathcal{T}_\lambda^{\mathcal{I}\mathcal{W}}$. In this subsection we prove some properties of these objects which will not play any role in the proof of Theorem 7.2, but which will be needed in Section 9 below.

We begin with the following claim. Here we denote by $\mathrm{P}_{\mathcal{I}\mathcal{W}}^0$ the quotient of the abelian category $\mathrm{P}_{\mathcal{I}\mathcal{W}}$ by the Serre subcategory generated by the objects of the form $\mathcal{S}\mathcal{C}_\lambda^{\mathcal{I}\mathcal{W}}$ with $\ell(w_\lambda) > 0$; the associated quotient functor will be denoted $\Pi_{\mathcal{I}\mathcal{W}}^0 : \mathrm{P}_{\mathcal{I}\mathcal{W}} \rightarrow \mathrm{P}_{\mathcal{I}\mathcal{W}}^0$.

Lemma 7.6. *For any tilting objects \mathcal{T} and \mathcal{T}' in $\mathrm{P}_{\mathcal{I}\mathcal{W}}$, the quotient functor $\Pi_{\mathcal{I}\mathcal{W}}^0$ induces an isomorphism*

$$\mathrm{Hom}_{\mathrm{P}_{\mathcal{I}\mathcal{W}}}(\mathcal{T}, \mathcal{T}') \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{P}_{\mathcal{I}\mathcal{W}}^0}(\Pi_{\mathcal{I}\mathcal{W}}^0(\mathcal{T}), \Pi_{\mathcal{I}\mathcal{W}}^0(\mathcal{T}')).$$

Proof. The idea of this proof is copied from [BBM, §2.1]. From the definition of morphism spaces in a quotient category, the lemma will follow if we prove that the socle, resp. top, of each object $\Delta_\lambda^{\mathcal{I}\mathcal{W}}$, resp. $\nabla_\lambda^{\mathcal{I}\mathcal{W}}$, is a sum of objects $\mathcal{S}\mathcal{C}_\mu^{\mathcal{I}\mathcal{W}}$ with $\ell(w_\mu) = 0$. Here it suffices to prove the claim for $\nabla_\lambda^{\mathcal{I}\mathcal{W}}$; the other one will follow

by Verdier duality. If we choose $w \in W$ such that $W_f \cdot w = W_f \cdot w_\lambda$, then for any $\mu \in \mathbf{X}^\vee$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{P}_{\mathcal{I}\mathcal{W}}}(\nabla_\lambda^{\mathcal{I}\mathcal{W}}, \mathcal{S}\mathcal{C}_\mu^{\mathcal{I}\mathcal{W}}) &\cong \mathrm{Hom}_{\mathbf{P}_{\mathcal{I}\mathcal{W}}}(\mathrm{Av}_{\mathcal{I}\mathcal{W}}(\nabla_w^{\mathcal{I}\mathcal{W}}), \mathcal{S}\mathcal{C}_\mu^{\mathcal{I}\mathcal{W}}) \\ &\cong \mathrm{Hom}_{\mathbf{P}_{\mathrm{I}w}}(\nabla_w^{\mathrm{I}w}, \mathbf{F}(\mathcal{S}\mathcal{C}_\mu^{\mathcal{I}\mathcal{W}})) \end{aligned}$$

by Lemma 7.4(2) and adjunction. By Lemma 4.5, the top of $\nabla_w^{\mathrm{I}w}$ is $\mathcal{S}\mathcal{C}_\omega^{\mathrm{I}w}$, where $\omega \in \Omega$ is the unique element such that $\omega W^{\mathrm{Cox}} = w W^{\mathrm{Cox}}$. If $\ell(w_\mu) > 0$, and if $s \in S$ is such that $\ell(w_\mu s) < \ell(w_\mu)$, then $w_\mu s \in {}^f W$, and $\mathcal{S}\mathcal{C}_\mu^{\mathcal{I}\mathcal{W}}$ is the shifted pullback of a simple perverse sheaf on the partial affine flag variety Fl_G^s associated with s . Then $\mathbf{F}(\mathcal{S}\mathcal{C}_\mu^{\mathcal{I}\mathcal{W}})$ also the shifted pullback of a perverse sheaf on Fl_G^s , which shows that

$$\mathrm{Hom}_{\mathbf{P}_{\mathrm{I}w}}(\nabla_w^{\mathrm{I}w}, \mathbf{F}(\mathcal{S}\mathcal{C}_\mu^{\mathcal{I}\mathcal{W}})) = 0$$

and finishes the proof. \square

The other result we will need is the following.

Lemma 7.7. *For any $\lambda \in \mathbf{X}^\vee$, the object $\mathcal{T}_\lambda^{\mathcal{I}\mathcal{W}}$ is isomorphic to a subobject of a direct sum of objects of the form $\mathcal{S}_\mu^{\mathcal{I}\mathcal{W}}$ with $\mu \in \mathbf{X}_+^\vee$.*

Proof. We denote by $\mathrm{I}w_u$ the pro-unipotent radical of $\mathrm{I}w$, by $\mathrm{Perv}_{\mathrm{I}w_u}(\mathrm{Fl}_G, \mathbb{k})$ the category of $\mathrm{I}w_u$ -equivariant perverse sheaves on Fl_G (or equivalently the category of perverse sheaves which are constructible with respect to the stratification by $\mathrm{I}w$ -orbits), and set for $w \in W$

$$\Delta_w^{\mathrm{I}w_u} := \mathrm{For}_{\mathrm{I}w_u}^{\mathrm{I}w}(\Delta_w^{\mathrm{I}w}), \quad \nabla_w^{\mathrm{I}w_u} := \mathrm{For}_{\mathrm{I}w_u}^{\mathrm{I}w}(\nabla_w^{\mathrm{I}w}),$$

where $\mathrm{For}_{\mathrm{I}w_u}^{\mathrm{I}w} : D_{\mathrm{I}w_u}^b(\mathrm{Fl}_G, \mathbb{k}) \rightarrow D_{\mathrm{I}w_u}^b(\mathrm{Fl}_G, \mathbb{k})$ is the forgetful functor. Then the category $\mathrm{Perv}_{\mathrm{I}w_u}(\mathrm{Fl}_G, \mathbb{k})$ admits a structure of highest weight category with weight poset W (for the Bruhat order) and standard, resp. costandard, objects the objects $\Delta_w^{\mathrm{I}w_u}$ with $w \in W$, resp. $\nabla_w^{\mathrm{I}w_u}$ with $w \in W$.

The setup of §7.4 allows us to construct a functor $\mathrm{Av}'_{\mathcal{I}\mathcal{W}} : D_{\mathrm{I}w_u}^b(\mathrm{Fl}_G, \mathbb{k}) \rightarrow D_{\mathcal{I}\mathcal{W}}^b(\mathrm{Fl}_G, \mathbb{k})$ such that $\mathrm{Av}_{\mathcal{I}\mathcal{W}} = \mathrm{Av}'_{\mathcal{I}\mathcal{W}} \circ \mathrm{For}_{\mathrm{I}w_u}^{\mathrm{I}w}$. Then by Lemma 7.4(2), for any $w \in W$ we have

$$\mathrm{Av}'_{\mathcal{I}\mathcal{W}}(\Delta_w^{\mathrm{I}w_u}) \cong \Delta_\lambda^{\mathcal{I}\mathcal{W}} \quad \text{and} \quad \mathrm{Av}'_{\mathcal{I}\mathcal{W}}(\nabla_w^{\mathrm{I}w_u}) \cong \nabla_\lambda^{\mathcal{I}\mathcal{W}},$$

where $\lambda \in \mathbf{X}^\vee$ is the unique element such that $W_f \cdot w = W_f \cdot w_\lambda$. These formulas show that $\mathrm{Av}'_{\mathcal{I}\mathcal{W}}$ is t-exact, and sends tilting objects in $\mathrm{Perv}_{\mathrm{I}w_u}(\mathrm{Fl}_G, \mathbb{k})$ to tilting objects in $\mathbf{P}_{\mathcal{I}\mathcal{W}}$. Moreover, if we denote by $\mathcal{T}_w^{\mathrm{I}w_u}$ the indecomposable tilting object in $\mathrm{Perv}_{\mathrm{I}w_u}(\mathrm{Fl}_G, \mathbb{k})$ associated with $w \in W$, then one can check that the indecomposable tilting object in $\mathbf{P}_{\mathcal{I}\mathcal{W}}$ associated with $\lambda \in \mathbf{X}^\vee$ is a direct summand in $\mathrm{Av}'_{\mathcal{I}\mathcal{W}}(\mathcal{T}_{w_\lambda}^{\mathrm{I}w_u})$. Therefore, to conclude we only have to prove that any indecomposable tilting object in $\mathrm{Perv}_{\mathrm{I}w_u}(\mathrm{Fl}_G, \mathbb{k})$ is isomorphic to a subobject of a tilting object whose image under $\mathrm{Av}'_{\mathcal{I}\mathcal{W}}$ is a direct sum of tilting objects associated with dominant weights.

Let us fix an indecomposable tilting object \mathcal{T} in $\mathrm{Perv}_{\mathrm{I}w_u}(\mathrm{Fl}_G, \mathbb{k})$. Denote by $\widetilde{\mathrm{Fl}}_G$ the “extended” affine flag variety, i.e. the ind-scheme representing the fppf quotient $(LG/\mathrm{I}w_u)_{\mathrm{fppf}}$; then we have a T -bundle $\pi : \widetilde{\mathrm{Fl}}_G \rightarrow \mathrm{Fl}_G$. Recall the “free pro-unipotently monodromic” tilting perverse sheaves on $\widetilde{\mathrm{Fl}}_G$, see [BY] or [BR, §5.4]. By [BR, Proposition 5.12] (see also [BY, Lemma A.7.3] for a closely related statement) there exists an indecomposable free pro-unipotently monodromic tilting

perverse sheaf $\widehat{\mathcal{F}}$ such that $\mathcal{T} = \pi_! \widehat{\mathcal{F}}[\dim(T)]$. Now we have a convolution action $\widehat{\star}$ of free pro-unipotently monodromic tilting perverse sheaves on the category $D_{\text{Iw}_u}^b(\text{Fl}_G, \mathbb{k})$ (see [BR, §7.3] for a discussion of a similar construction in the context of usual flag varieties), and the considerations in the proof of [BR, Lemma 7.8] show that for any free pro-unipotently monodromic tilting perverse sheaf $\widehat{\mathcal{F}}'$ the functor $\widehat{\mathcal{F}}' \widehat{\star} (-)$ is t-exact and sends tilting perverse sheaves to tilting perverse sheaves. In particular, since $\Delta_e^{\text{Iw}_u}$ is a subobject of $\mathcal{T}_{w_0}^{\text{Iw}_u}$ by Lemma 4.5, the tilting object $\mathcal{T} = \pi_! \widehat{\mathcal{F}}[\dim(T)] = \widehat{\mathcal{F}} \widehat{\star} \Delta_e^{\text{Iw}_u}$ is a subobject of the tilting object $\mathcal{T}' = \widehat{\mathcal{F}} \widehat{\star} \mathcal{T}_{w_0}^{\text{Iw}_u}$.

What remains to be proved is that $\text{Av}'_{\mathcal{I}\mathcal{W}}(\mathcal{T}')$ is a direct sum of tilting objects associated with dominant weights. However if $\lambda \in \mathbf{X}^\vee$ is not dominant then there exists $s \in S \cap W_f$ such that $w_\lambda s > w_\lambda$ and $w_\lambda s \in {}^f W$ (see e.g. [MR1, Lemma 2.4 and its proof]). Then if q_s is the projection from Fl_G to the partial affine flag variety Fl_G^s associated with s , the indecomposable tilting object in $\mathcal{P}_{\mathcal{I}\mathcal{W}}$ associated with λ is not killed by the functor $(q_s)_!$ (because the restriction of its image to the Iw_u^+ -orbit associated with w_λ is nonzero). Hence to prove the claim we only have to observe that $(q_s)_! \mathcal{T}' = 0$ for any $s \in S \cap W_f$. This fact follows from the properties that $(q_s)_! \mathcal{T}_{w_0}^{\text{Iw}_u} = 0$ (which is standard, and can e.g. be deduced from the fact that $\mathcal{T}_{w_0}^{\text{Iw}_u}$ is the projective cover of $\Delta_e^{\text{Iw}_u}$ in $\text{Perv}_{\text{Iw}_u}(L^+G/I)$, see [AR1, Proposition 5.26]), and that the functor $\widehat{\mathcal{F}} \widehat{\star} (-)$ commutes with $(q_s)_!$ in the appropriate sense. \square

Recall that in a highest weight category, any object is isomorphic to a subquotient of a tilting object (see e.g. [Ri, Proposition 7.17]). Therefore, Lemma 7.7 has the following consequence.

Corollary 7.8. *Any object in $\mathcal{P}_{\mathcal{I}\mathcal{W}}$ is isomorphic to a subquotient of a direct sum of objects of the form $\mathcal{T}_\mu^{\mathcal{I}\mathcal{W}}$ with $\mu \in \mathbf{X}_+^\vee$.*

7.6. Proof of Theorem 7.2(2). We have now introduced all the tools (needed for the proof of Theorem 7.2(2)). We begin with a general lemma.

Lemma 7.9. *Let \mathbf{A} and \mathbf{B} be abelian categories, let \mathbf{M} be a Serre subcategory of \mathbf{A} , let \mathbf{N} be a Serre subcategory of \mathbf{B} , and consider the associated quotient functors $\Pi_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A}/\mathbf{M}$ and $\Pi_{\mathbf{B}} : \mathbf{B} \rightarrow \mathbf{B}/\mathbf{N}$. Assume we are given functors $F : \mathbf{A} \rightarrow \mathbf{B}$, $G : \mathbf{B} \rightarrow \mathbf{A}$ and $F^0 : \mathbf{A}/\mathbf{M} \rightarrow \mathbf{B}/\mathbf{N}$, $G^0 : \mathbf{B}/\mathbf{N} \rightarrow \mathbf{A}/\mathbf{M}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{A} & \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} & \mathbf{B} \\ \Pi_{\mathbf{A}} \downarrow & & \downarrow \Pi_{\mathbf{B}} \\ \mathbf{A}/\mathbf{M} & \begin{array}{c} \xrightarrow{F^0} \\ \xleftarrow{G^0} \end{array} & \mathbf{B}/\mathbf{N}. \end{array}$$

If F is left adjoint to G , then F^0 is left adjoint to G^0 .

Proof. We have adjunction morphisms $F \circ G \rightarrow \text{id}$ and $\text{id} \rightarrow G \circ F$. Composing with $\Pi_{\mathbf{B}}$ and $\Pi_{\mathbf{A}}$ respectively on the left, we deduce morphisms of functors $F^0 \circ G^0 \circ \Pi_{\mathbf{B}} \rightarrow \Pi_{\mathbf{B}}$ and $\Pi_{\mathbf{A}} \rightarrow G^0 \circ F^0 \circ \Pi_{\mathbf{A}}$. Such morphisms must be induced by morphisms $F^0 \circ G^0 \rightarrow \text{id}$ and $\text{id} \rightarrow G^0 \circ F^0$. Since our original morphisms satisfy the zig-zag relations, so do these morphisms, which implies the desired claim. \square

By Lemma 7.4(1) and [Gab, Corollaires 2–3, p. 368–369], the restriction of $\text{Av}_{\mathcal{I}\mathcal{W}}$ to the hearts of the perverse t-structures factors through an exact functor

$$\text{Av}_{\mathcal{I}\mathcal{W}}^{\text{asp}} : \mathbb{P}_{\text{Iw}}^{\text{asp}} \rightarrow \mathbb{P}_{\mathcal{I}\mathcal{W}};$$

i.e. we have $\text{Av}_{\mathcal{I}\mathcal{W}} = \text{Av}_{\mathcal{I}\mathcal{W}}^{\text{asp}} \circ \Pi^{\text{asp}}$. Moreover, by Lemma 7.9, the functor $\mathbb{F}^{\text{asp}} := \Pi^{\text{asp}} \circ \mathbb{F}$ is right adjoint to $\text{Av}_{\mathcal{I}\mathcal{W}}^{\text{asp}}$.

The following lemma can be proved using the same arguments as in [AR3, Proof of Lemma 6.4.10].

Lemma 7.10. *Let \mathcal{F} in \mathbb{P}_{Iw} . If $\mathcal{G} \in D_{\text{Iw}}^{\text{b}}(\text{Fl}_G, \mathbb{k})$ belongs to the full subcategory generated under extensions by the objects of the form $\mathcal{S}\mathcal{C}_w[n]$ with $(w, n) \in W_{\text{f}} \times \mathbb{Z}_{\leq 0}$ and either $n < 0$ or $n = 0$ and $w \neq e$, then we have*

$$\Pi^{\text{asp}} \circ {}^{\text{p}}\mathcal{H}^{-1}(\mathcal{G} \star_{\text{Iw}} \mathcal{F}) = \Pi^{\text{asp}} \circ {}^{\text{p}}\mathcal{H}^0(\mathcal{G} \star_{\text{Iw}} \mathcal{F}) = 0.$$

The fact that the functor $\text{Av}_{\mathcal{I}\mathcal{W}}^{\text{asp}}$ is fully faithful is now equivalent to the following lemma.

Lemma 7.11. *The adjunction morphism $\text{id} \rightarrow \mathbb{F}^{\text{asp}} \circ \text{Av}_{\mathcal{I}\mathcal{W}}^{\text{asp}}$ is an isomorphism.*

Proof. We have to prove that for any \mathcal{F} in $\mathbb{P}_{\text{Iw}}^{\text{asp}}$ the morphism

$$\mathcal{F} \rightarrow \mathbb{F}^{\text{asp}} \circ \text{Av}_{\mathcal{I}\mathcal{W}}^{\text{asp}}(\mathcal{F})$$

is an isomorphism, or in other words that for any \mathcal{F} in \mathbb{P}_{Iw} the morphism

$$\Pi^{\text{asp}}(\mathcal{F}) \rightarrow \Pi^{\text{asp}} \circ \mathbb{F} \circ \text{Av}_{\mathcal{I}\mathcal{W}}(\mathcal{F})$$

induced by the adjunction morphism $\text{id} \rightarrow \mathbb{F} \circ \text{Av}_{\mathcal{I}\mathcal{W}}$ is an isomorphism. Now we have

$$\begin{aligned} \mathbb{F} \circ \text{Av}_{\mathcal{I}\mathcal{W}}(\mathcal{F}) &= {}^{\text{p}}\mathcal{H}^0(*\text{Ind}_{\text{Iw}_0}^{\text{Iw}}(\Delta_0^{\mathcal{I}\mathcal{W}} \star_{\text{Iw}} \mathcal{F})[-\dim(T)]) \\ &\cong {}^{\text{p}}\mathcal{H}^0(*\text{Ind}_{\text{Iw}_0}^{\text{Iw}}(\Delta_0^{\mathcal{I}\mathcal{W}}) \star_{\text{Iw}} \mathcal{F}[-\dim(T)]), \end{aligned}$$

and the adjunction morphism considered here is induced by the adjunction morphism

$$(7.4) \quad \Delta_e^{\text{Iw}} \rightarrow *\text{Ind}_{\text{Iw}_0}^{\text{Iw}}(\text{Av}_{\mathcal{I}\mathcal{W}}(\Delta_e^{\text{Iw}}))[-\dim(T)] = *\text{Ind}_{\text{Iw}_0}^{\text{Iw}}(\Delta_0^{\mathcal{I}\mathcal{W}})[- \dim(T)].$$

Hence to conclude we must show that for any \mathcal{F} in \mathbb{P}_{Iw} the latter morphism induces an isomorphism

$$\Pi^{\text{asp}}(\mathcal{F}) \rightarrow \Pi^{\text{asp}}\left({}^{\text{p}}\mathcal{H}^0(*\text{Ind}_{\text{Iw}_0}^{\text{Iw}}(\Delta_0^{\mathcal{I}\mathcal{W}}) \star_{\text{Iw}} \mathcal{F}[-\dim(T)])\right).$$

We start by analyzing the morphism (7.4). The G -action on the base point of Fl_G induces an isomorphism between $\overline{\text{Fl}}_{G, w_0} = L^+G/\text{Iw}$ and the flag variety G/B , from which we obtain equivalences of categories

$$\begin{aligned} D_{\text{Iw}}^{\text{b}}(\overline{\text{Fl}}_{w_0}, \mathbb{k}) &\cong D_B^{\text{b}}(G/B, \mathbb{k}), \\ D_{(\text{Iw}_0^+, \chi^*(\mathcal{L}_{\text{AS}}))}^{\text{b}}(\overline{\text{Fl}}_{G, w_0}, \mathbb{k}) &\cong D_{(U^+, (\chi')^*(\mathcal{L}_{\text{AS}}))}^{\text{b}}(G/B, \mathbb{k}). \end{aligned}$$

Using these equivalences, we can apply [BeR, Lemma 12.1], which guarantees that the complex on the right-hand side of (7.4) is concentrated in non-negative (perverse) degrees, and that moreover we have

$${}^{\text{p}}\mathcal{H}^0(*\text{Ind}_{\text{Iw}_0}^{\text{Iw}}(\Delta_0^{\mathcal{I}\mathcal{W}})[- \dim(T)]) = {}^{\text{p}}\mathcal{H}^{-\dim(T)}\left(*\text{Ind}_{\text{Iw}_0}^{\text{Iw}}(\Delta_0^{\mathcal{I}\mathcal{W}})\right) \cong \Delta_{w_0}^{\text{Iw}}.$$

It follows that (7.4) is induced by a nonzero morphism $\Delta_e^{\text{Iw}} \rightarrow \Delta_{w_0}^{\text{Iw}}$. Such a morphism is automatically injective, and the composition factors of its cokernel are of the form $\mathcal{S}\mathcal{C}_w$ with $w \in W_f \setminus \{e\}$, see Lemma 4.5. In particular, the cone \mathcal{C} of (7.4) belongs to the full subcategory of $D_{\text{Iw}}^b(\text{Fl}_G, \mathbb{k})$ generated under extensions by the objects of the form $\mathcal{S}\mathcal{C}_w[n]$ with $(w, n) \in W_f \times \mathbb{Z}_{\leq 0}$ and either $n < 0$ or $n = 0$ and $w \neq e$. By Lemma 7.10 it follows that we have

$$\Pi^{\text{asp}} \circ {}^p\mathcal{H}^{-1}(\mathcal{C} \star_{\text{Iw}} \mathcal{F}) = \Pi^{\text{asp}} \circ {}^p\mathcal{H}^0(\mathcal{C} \star_{\text{Iw}} \mathcal{F}) = 0.$$

The desired claim follows, using the long exact sequence in cohomology associated with the distinguished triangle

$$\mathcal{F} \rightarrow {}^*\text{Ind}_{\text{Iw}_0}^{\text{Iw}}(\Delta_0^{\text{IW}}) \star_{\text{Iw}} \mathcal{F}[-\dim(T)] \rightarrow \mathcal{C} \star_{\text{Iw}} \mathcal{F} \xrightarrow{[1]}$$

induced by (7.4) and the exactness of Π^{asp} . \square

To complete the proof of Theorem 7.2(2), it only remains to prove that the essential image of $\text{Av}_{\text{IW}}^{\text{asp}}$ is closed under subquotients. This property follows from Corollary 7.5 and the following general lemma.

Lemma 7.12. *Let \mathbf{A} and \mathbf{B} be abelian categories, and let $F : \mathbf{A} \rightarrow \mathbf{B}$ be an exact fully faithful functor. Assume that every object in \mathbf{A} has finite length, and that $F(M)$ is simple for any simple object M in \mathbf{A} . Then the essential image of F is closed under subquotients.*

Proof. Our assumption implies that every object in the essential image of F has finite length, and that each of its composition factors is of the form $F(M)$ with M simple in \mathbf{A} . Any subquotient of such an object can be obtained by repeatedly taking a cokernel of a morphism from an object of the form $F(M)$ with M simple in \mathbf{A} , or a kernel of a morphism to an object of this form. To prove the lemma it therefore suffices to show that if M, N are objects of \mathbf{A} with M simple and if $f : F(M) \rightarrow F(N)$, resp. $f : F(N) \rightarrow F(M)$ is an embedding, resp. a surjection, then $\text{cok}(f)$, resp. $\text{ker}(f)$, belongs to the essential image of F . However, since F is fully faithful we can write $f = F(g)$ for some morphism $g : M \rightarrow N$, resp. $g : N \rightarrow M$, and by exactness we then have $\text{cok}(f) = F(\text{cok}(g))$, resp. $\text{ker}(f) = F(\text{ker}(g))$. \square

Remark 7.13. Lemma 7.12 is stated in [BBD, Lemme 4.2.6.1] under the extra assumption that F admits left and right adjoints. Our arguments above show that this condition is in fact not necessary.

8. CENTRAL AND TILTING PERVERSE SHEAVES

We continue with the assumption of Sections 6–7, namely that $X^*(T)/\mathbb{Z}\mathfrak{R}$ has no torsion. (In fact, one can easily check that this assumption is not required for the proof of the main result of this section.)

8.1. Statement and strategy of proof. To simplify notation, we now set

$$\mathcal{Z}^{\text{IW}} := \text{Av}_{\text{IW}} \circ \mathcal{L} : \text{Rep}(G_{\mathbb{k}}^{\vee}) \rightarrow \text{P}_{\text{IW}}.$$

Our goal in this section is to prove the following result.

Theorem 8.1. *Assume that for any indecomposable factor of the root system \mathfrak{R} , the prime ℓ is strictly bigger than the corresponding value in the table of Figure 5.1. Then for any tilting $G_{\mathbb{k}}^{\vee}$ -module V , the perverse sheaf $\mathcal{Z}^{\text{IW}}(V)$ is tilting.*

Remark 8.2. It is likely that the assumption on ℓ in Theorem 8.1 can be weakened; however, in view of [JMW, §3.8], this extension will require a different strategy.

The first step towards Theorem 8.1 is the following claim, whose proof can be copied from [AB, Lemma 25] or [AR3, Proposition 6.5.7]. (This statement does not require any assumption on G or ℓ .)

Proposition 8.3. *If V, V' are tilting $G_{\mathbb{k}}^{\vee}$ -modules such that $\mathcal{Z}^{\text{TW}}(V)$ and $\mathcal{Z}^{\text{TW}}(V')$ are tilting, then $\mathcal{Z}^{\text{TW}}(V \otimes V')$ is tilting.*

Assume now for a moment that G is simple (i.e. quasi-simple of adjoint type), so that $G_{\mathbb{k}}^{\vee}$ is quasi-simple and simply-connected. Recall that in this context a dominant coweight $\lambda \in \mathbf{X}_{+}^{\vee}$ is called *minuscule* if for any root α we have $\langle \lambda, \alpha \rangle \in \{0, \pm 1\}$, and moreover that in this case we have $W(\lambda) = W_{\mathfrak{f}}\lambda$ (where the left-hand side is as in §7.1). We will also denote by α_0 the maximal root of (G, T) (for our given choice of positive roots), and by α_0^{\vee} the associated coroot.⁶ Then we have $W(\alpha_0^{\vee}) = W_{\mathfrak{f}}\alpha_0^{\vee} \cup \{0\}$.

The following result is proved in [JMW, §§3.6–3.7].

Proposition 8.4. *Assume that G is simple and that ℓ is strictly bigger than the value corresponding to its type in the table of Figure 5.1. Then any indecomposable tilting $G_{\mathbb{k}}^{\vee}$ -module appears as a direct summand of a tensor product of indecomposable tilting modules whose highest weights are either minuscule or equal to α_0^{\vee} . Moreover, if G is of type **A** then only minuscule coweights are needed.*

This proposition shows the importance of understanding the objects $\mathcal{Z}(V)$ (or $\mathcal{Z}^{\text{TW}}(V)$) when V is an indecomposable tilting module whose highest weight is either minuscule or equal to α_0^{\vee} . These objects will be studied in the rest of the section. In particular, in §8.3 we will prove the following claim.

Proposition 8.5. *Assume that G is simple, and let V be an indecomposable tilting $G_{\mathbb{k}}^{\vee}$ -module with highest weight λ . We assume that*

- (1) *either λ is minuscule;*
- (2) *or $\lambda = \alpha_0^{\vee}$, G is not of type **A**, and ℓ satisfies the conditions in Figure 2.1 with respect to the type of $G_{\mathbb{k}}^{\vee}$.*

Then $\mathcal{Z}^{\text{TW}}(V)$ is tilting.

For now, let us explain why these results imply Theorem 8.1.

Proof of Theorem 8.1. Consider the quotient group $G/Z(G)$ (which is semisimple of adjoint type), and the associated affine Grassmannian $\text{Gr}_{G/Z(G)}$. The quotient morphism $G \rightarrow G/Z(G)$ induces a morphism

$$(8.1) \quad \text{Gr}_G \rightarrow \text{Gr}_{G/Z(G)}.$$

We claim that this morphism restricts, on the reduced ind-subscheme associated with each connected component of Gr_G , to a universal homeomorphism onto the reduced ind-scheme associated with a connected component of $\text{Gr}_{G/Z(G)}$. In fact, since the connected components of Gr_G are permuted by the action of LG it suffices to check this claim for the reduced ind-scheme $(\text{Gr}_G)_{\text{red}}^0$ associated with the

⁶Note the slight conflict with the notation of §2.8. But here the group under consideration is $\mathbf{G} = G_{\mathbb{k}}^{\vee}$, whose highest *short* root is α_0^{\vee} .

connected component containing the base point. Now by [PR, Proposition 6.6], if $\mathcal{D}G$ is the derived subgroup of G , the embedding $\mathcal{D}G \hookrightarrow G$ induces an isomorphism

$$(\mathrm{Gr}_{\mathcal{D}G})_{\mathrm{red}}^0 \xrightarrow{\sim} (\mathrm{Gr}_G)_{\mathrm{red}}^0.$$

We have $Z(\mathcal{D}G) = Z(G) \cap \mathcal{D}G$, and the morphism $\mathcal{D}G/Z(\mathcal{D}G) \rightarrow G/Z(G)$ induced by the embedding $\mathcal{D}G \hookrightarrow G$ is an isomorphism, which reduces the proof of our claim to the case G is semisimple. In this case the claim follows from the results of [PR, §6]; see [BR, Footnote on p. 3]. (In case p does not divide the order of the fundamental group of $G/Z(G)$, then it is proved in [PR] that Gr_G and $\mathrm{Gr}_{G/Z(G)}$ are reduced, and that the above morphism restricts to an *isomorphism* on each connected component of Gr_G .)

Our claim implies that pushforward along the morphism (8.1) induces a fully faithful exact functor

$$\mathrm{Perv}_{L+G}(\mathrm{Gr}_G, \mathbb{k}) \rightarrow \mathrm{Perv}_{L+(G/Z(G))}(\mathrm{Gr}_{G/Z(G)}, \mathbb{k}).$$

(To justify the fact that this functor takes values in $L^+(G/Z(G))$ -equivariant perverse sheaves, we use the fact that equivariance follows from constructibility with respect to the stratification by $L^+(G/Z(G))$ -orbits; see [MV, Proposition 2.1] or [BR, Proposition 1.10.8].) It can be easily seen that this functor is monoidal; hence in view of Theorem 4.1 it provides a monoidal functor $\mathrm{Rep}(G_{\mathbb{k}}^{\vee}) \rightarrow \mathrm{Rep}((G/Z(G))_{\mathbb{k}}^{\vee})$. It is well known that this functor is induced by a natural morphism $(G/Z(G))_{\mathbb{k}}^{\vee} \rightarrow G_{\mathbb{k}}^{\vee}$ which identifies $(G/Z(G))_{\mathbb{k}}^{\vee}$ with the simply-connected cover of the derived subgroup of $G_{\mathbb{k}}^{\vee}$.

Since the morphism $\mathrm{Fl}_G \rightarrow \mathrm{Gr}_G$ is a (Zariski) locally trivial fibration with fibers G/B , and similarly for the morphism $\mathrm{Fl}_{G/Z(G)} \rightarrow \mathrm{Gr}_{G/Z(G)}$, we also have a similar claim as above for the natural morphism $\mathrm{Fl}_G \rightarrow \mathrm{Fl}_{G/Z(G)}$, which allows to identify, for any connected component of Fl_G , the subcategory of $\mathcal{P}_{\mathrm{Iw}}$ consisting of perverse sheaves supported on this orbit with a full category of the corresponding category for $G/Z(G)$. Since the construction of the functor \mathcal{Z} is compatible with these identifications in the obvious way, since $\mathcal{Z}(V)$ is supported on a single connected component if V is indecomposable, and since the pullback to $(G/Z(G))_{\mathbb{k}}^{\vee}$ of a tilting $G_{\mathbb{k}}^{\vee}$ -module is a tilting $(G/Z(G))_{\mathbb{k}}^{\vee}$ -module, this reduces the proof of the theorem to the case G is semisimple of adjoint type. Then, since such a group is a product of simple groups, we can further assume that G is simple.

Recall that (by the general theory of highest weight categories) a direct summand of a tilting object in $\mathcal{P}_{\mathcal{Iw}}$ is again tilting. If G is of type **A**, then by Proposition 8.4 any indecomposable tilting $G_{\mathbb{k}}^{\vee}$ -module is a direct summand of a tensor product of tilting modules with minuscule highest weights; hence in this case the theorem follows from Proposition 8.3 and Case (1) in Proposition 8.5. If G is not of type **A**, the claim follows similarly from Proposition 8.3, Proposition 8.4, and Proposition 8.5 (cases (1) and (2)). \square

8.2. Stalks and costalks of objects $\mathcal{Z}^{\mathcal{Iw}}(V)$. The proof of the following proposition can be copied from [AB, Lemma 27] or [AR3, Proposition 6.5.4].

Proposition 8.6. *For any $G_{\mathbb{k}}^{\vee}$ -module V and any $\mu \in \mathbf{X}^{\vee}$ we have*

$$\begin{aligned} \sum_{i \geq 0} (-1)^i \cdot \dim(\mathrm{Hom}_{D_{\mathbb{Z}\mathcal{W}}^b(\mathrm{Fl}_{G,\mathbb{k}})}(\Delta_{\mu}^{\mathcal{I}\mathcal{W}}, \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)[i])) &= \dim(V_{\mu}), \\ \sum_{i \geq 0} (-1)^i \cdot \dim(\mathrm{Hom}_{D_{\mathbb{Z}\mathcal{W}}^b(\mathrm{Fl}_{G,\mathbb{k}})}(\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V), \nabla_{\mu}^{\mathcal{I}\mathcal{W}}[i])) &= \dim(V_{\mu}). \end{aligned}$$

Remark 8.7. Once Theorem 8.1 will be proved, in view of (7.1) we will know that the morphism spaces

$$\mathrm{Hom}_{D_{\mathbb{Z}\mathcal{W}}^b(\mathrm{Fl}_{G,\mathbb{k}})}(\Delta_{\mu}^{\mathcal{I}\mathcal{W}}, \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)[i]) \text{ and } \mathrm{Hom}_{D_{\mathbb{Z}\mathcal{W}}^b(\mathrm{Fl}_{G,\mathbb{k}})}(\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V), \nabla_{\mu}^{\mathcal{I}\mathcal{W}}[i])$$

vanish for $i \neq 0$. Proposition 8.6 will then imply that for any V tilting and $\lambda \in \mathbf{X}^{\vee}$ we have

$$(\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V) : \Delta_{\lambda}^{\mathcal{I}\mathcal{W}}) = (\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V) : \nabla_{\lambda}^{\mathcal{I}\mathcal{W}}) = \dim(V_{\lambda}),$$

where the first, resp. second, expression means the multiplicity in a standard, resp. costandard, filtration. From this one can in particular deduce that if V has a highest weight $\lambda \in \mathbf{X}_{+}^{\vee}$, then the indecomposable tilting object $\mathcal{Z}_{\lambda}^{\mathcal{I}\mathcal{W}}$ is a direct summand of $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$.

Corollary 8.8. *Let V in $\mathrm{Rep}(G_{\mathbb{k}}^{\vee})$ and $\lambda \in \mathbf{X}_{+}^{\vee}$. If the $T_{\mathbb{k}}^{\vee}$ -weights of V all belong to $W(\lambda)$, then $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ is supported on the closed subvariety $\bigcup_{\mu \in W(\lambda) \cap \mathbf{X}_{+}^{\vee}} \overline{\mathrm{Fl}_{G,\mu}^{\mathcal{I}\mathcal{W}}}$.*

Proof. Let U be the open complement of $\bigcup_{\mu \in W(\lambda) \cap \mathbf{X}_{+}^{\vee}} \overline{\mathrm{Fl}_{G,\mu}^{\mathcal{I}\mathcal{W}}}$. Then Lemma 7.1 shows that $\mathrm{Fl}_{G,\nu}^{\mathcal{I}\mathcal{W}} \subset U$ iff $\nu \in \mathbf{X}^{\vee} \setminus W(\lambda)$. Our assumption and Proposition 8.6 show that the Euler characteristic of the restriction to any such orbit of the perverse sheaf $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)|_U$ vanishes. Hence this perverse sheaf is 0 (because the restriction of a perverse sheaf to a stratum open in its support is nonzero, and concentrated in one degree). \square

The proof of the following lemma is similar to that of [AB, Lemma 28] or [AR3, Lemma 6.5.11].

Lemma 8.9. *For any V in $\mathrm{Rep}(G_{\mathbb{k}}^{\vee})$, any $\lambda \in \mathbf{X}^{\vee}$, any $x \in W_{\mathfrak{f}}$ and any $n \in \mathbb{Z}$ we have*

$$\mathrm{Hom}_{D_{\mathbb{Z}\mathcal{W}}^b(\mathrm{Fl}_{G,\mathbb{k}})}(\Delta_{\lambda}^{\mathcal{I}\mathcal{W}}, \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)[n]) \cong \mathrm{Hom}_{D_{\mathbb{Z}\mathcal{W}}^b(\mathrm{Fl}_{G,\mathbb{k}})}(\Delta_{x(\lambda)}^{\mathcal{I}\mathcal{W}}, \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)[n])$$

and

$$\mathrm{Hom}_{D_{\mathbb{Z}\mathcal{W}}^b(\mathrm{Fl}_{G,\mathbb{k}})}(\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V), \nabla_{\lambda}^{\mathcal{I}\mathcal{W}}[n]) \cong \mathrm{Hom}_{D_{\mathbb{Z}\mathcal{W}}^b(\mathrm{Fl}_{G,\mathbb{k}})}(\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V), \nabla_{x(\lambda)}^{\mathcal{I}\mathcal{W}}[n]).$$

The next statement involves the element u from Proposition 5.8.

Lemma 8.10. *For any $G_{\mathbb{k}}^{\vee}$ -module V which admits a good filtration, we have*

$$\dim\left(\mathrm{Hom}_{\mathcal{P}_{\mathcal{I}\mathcal{W}}}(\Delta_0^{\mathcal{I}\mathcal{W}}, \mathrm{Av}_{\mathcal{I}\mathcal{W}}(\mathcal{Z}(V)))\right) \leq \dim(V^u).$$

For any $G_{\mathbb{k}}^{\vee}$ -module V which admits a Weyl filtration, we have

$$\dim\left(\mathrm{Hom}_{\mathcal{P}_{\mathcal{I}\mathcal{W}}}(\mathrm{Av}_{\mathcal{I}\mathcal{W}}(\mathcal{Z}(V)), \nabla_0^{\mathcal{I}\mathcal{W}})\right) \leq \dim(V^u).$$

Proof. We prove the first claim only; the second one can be treated similarly (using the fact that the invariants and coinvariants of u acting on V have the same dimension).

We have $\Delta_0^{\mathcal{I}\mathcal{W}} = \text{Av}_{\mathcal{I}\mathcal{W}}(\mathcal{I}\mathcal{C}_e)$; hence Theorem 7.2 provides an isomorphism

$$\text{Hom}_{\mathbf{P}_{\mathcal{I}\mathcal{W}}}(\Delta_0^{\mathcal{I}\mathcal{W}}, \text{Av}_{\mathcal{I}\mathcal{W}}(\mathcal{Z}(V))) \cong \text{Hom}_{\mathbf{P}_{\text{Iw}}^{\text{asp}}}(\Pi^{\text{asp}}(\mathcal{I}\mathcal{C}_e), \Pi^{\text{asp}}(\mathcal{Z}(V))).$$

Recall now the quotient functor $\Pi^0 : \mathbf{P}_{\text{Iw}} \rightarrow \mathbf{P}_{\text{Iw}}^0$. It is clear that this functor factors through an exact functor $\mathbf{P}_{\text{Iw}}^{\text{asp}} \rightarrow \mathbf{P}_{\text{Iw}}^0$; one can easily show that the latter functor identifies \mathbf{P}_{Iw}^0 with the Serre quotient of $\mathbf{P}_{\text{Iw}}^{\text{asp}}$ by the Serre subcategory generated by the simple objects $\Pi^{\text{asp}}(\mathcal{I}\mathcal{C}_w)$ with $w \in {}^f W \setminus \Omega$. Since $\Pi^{\text{asp}}(\mathcal{I}\mathcal{C}_e)$ is simple, any nonzero morphism $\Pi^{\text{asp}}(\mathcal{I}\mathcal{C}_e) \rightarrow \Pi^{\text{asp}}(\mathcal{Z}(V))$ is injective, and its image is not killed by the quotient functor $\mathbf{P}_{\text{Iw}}^{\text{asp}} \rightarrow \mathbf{P}_{\text{Iw}}^0$; we deduce that this functor induces an injective morphism

$$\text{Hom}_{\mathbf{P}_{\text{Iw}}^{\text{asp}}}(\Pi^{\text{asp}}(\mathcal{I}\mathcal{C}_e), \Pi^{\text{asp}}(\mathcal{Z}(V))) \hookrightarrow \text{Hom}_{\mathbf{P}_{\text{Iw}}^0}(\mathcal{Z}^0(\mathbb{k}), \mathcal{Z}^0(V)).$$

Next, since $\mathfrak{m}_{\mathbb{k}}^0 = \text{id}$, using Lemma 5.3 we see that

$$\text{Hom}_{\mathbf{P}_{\text{Iw}}^0}(\mathcal{Z}^0(\mathbb{k}), \mathcal{Z}^0(V)) = \text{Hom}_{\mathbf{P}_{\text{Iw}}^0}(\mathcal{Z}^0(\mathbb{k}), \ker(\mathfrak{m}_V^0 - \text{id})).$$

And then using Proposition 5.8 we obtain an injection

$$\text{Hom}_{\mathbf{P}_{\text{Iw}}^0}(\mathcal{Z}^0(\mathbb{k}), \ker(\mathfrak{m}_V^0 - \text{id})) \hookrightarrow \text{Hom}_{\mathbb{k}}(\mathbb{k}, V^u) = V^u.$$

The desired inequality follows. \square

8.3. Proof of Proposition 8.5. In this subsection we assume that G is simple, and give the proof of Proposition 8.5. We will use the easy observation that a perverse sheaf in $\text{Perv}_{\mathcal{I}\mathcal{W}}(\text{Fl}_G, \mathbb{k})$ is tilting iff its restriction and corestriction to each orbit $\text{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}}$ contained in its support are perverse.

Let us first consider case (1). Corollary 8.8 shows that $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ is supported on $\overline{\text{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}}}$, which contains the orbit $\text{Fl}_{G,\mu}^{\mathcal{I}\mathcal{W}}$ iff $\mu \in W_f \cdot \lambda$. The orbit $\text{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}}$ is open in this subvariety; hence the restriction and corestriction of the perverse sheaf $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ to this stratum are perverse. Using Lemma 8.9, we deduce the same property for each orbit $\text{Fl}_{G,\mu}^{\mathcal{I}\mathcal{W}}$ with $\mu \in W_f \cdot \lambda$, which completes the proof in this case.

Now we consider case (2). In particular we assume that $\lambda = \alpha_0^\vee$, that G is not of type **A**, and that ℓ satisfies the conditions in Figure 2.1 with respect to the type of $G_{\mathbb{k}}^\vee$. The perverse sheaf $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ is supported on $\overline{\text{Fl}_{G,\lambda}^{\mathcal{I}\mathcal{W}}} \cup \overline{\text{Fl}_{G,0}^{\mathcal{I}\mathcal{W}}}$, which contains the orbit $\text{Fl}_{G,\mu}^{\mathcal{I}\mathcal{W}}$ iff $\mu \in W_f \cdot \lambda \cup \{0\}$. As in case (1), the restriction and corestriction of $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ to any orbit $\text{Fl}_{G,\mu}^{\mathcal{I}\mathcal{W}}$ with $\mu \in W_f \cdot \lambda$ are perverse, so to conclude it suffices to prove the similar claim for $\text{Fl}_{G,0}^{\mathcal{I}\mathcal{W}}$. Let $i : \overline{\text{Fl}_{G,0}^{\mathcal{I}\mathcal{W}}} \rightarrow \text{Fl}_G$ be the embedding, and let j be the embedding of the open complement. Then we have distinguished triangles

$$\begin{aligned} j_! j^* \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V) &\rightarrow \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V) \rightarrow i_* i^* \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V) \xrightarrow{[1]}_{\rightarrow}, \\ i_* i^! \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V) &\rightarrow \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V) \rightarrow j_* j^* \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V) \xrightarrow{[1]}_{\rightarrow}. \end{aligned}$$

What we have proved so far implies that $j^* \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ admits a filtration with standard subquotients, and a filtration with costandard subquotients; hence both $j_! j^* \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ and $j_* j^* \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ are perverse sheaves; it follows that $i^* \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ is concentrated in perverse degrees 0 and -1 , while $i^! \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ is concentrated in perverse degrees 0 and 1.

By Lemma 8.10 we have

$$\dim \operatorname{Hom}_{\mathbf{P}_{\mathcal{I}\mathcal{W}}}(\Delta_0^{\mathcal{I}\mathcal{W}}, \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)) \leq \dim(V^u).$$

Now by Theorem 6.1 the element u is regular unipotent, so that by Lemma 2.16 we have

$$\dim(V^u) = \dim(V_0).$$

Finally, by Proposition 8.6 we have

$$\dim(V_0) = \sum_{n \geq 0} (-1)^n \cdot \dim \operatorname{Hom}_{D_{\mathcal{I}\mathcal{W}}^b(\mathbf{F}l_{G,\mathbb{k}})}(\Delta_0^{\mathcal{I}\mathcal{W}}, \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)[n]).$$

Combining all these remarks, we obtain that

$$\begin{aligned} \dim \operatorname{Hom}_{\mathbf{P}_{\mathcal{I}\mathcal{W}}}(\Delta_0^{\mathcal{I}\mathcal{W}}, \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)) \\ \leq \sum_{n \geq 0} (-1)^n \cdot \dim \operatorname{Hom}_{D_{\mathcal{I}\mathcal{W}}^b(\mathbf{F}l_{G,\mathbb{k}})}(\Delta_0^{\mathcal{I}\mathcal{W}}, \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)[n]). \end{aligned}$$

The remarks in the preceding paragraph imply that the right-hand side equals

$$\dim \operatorname{Hom}_{\mathbf{P}_{\mathcal{I}\mathcal{W}}}(\Delta_0^{\mathcal{I}\mathcal{W}}, \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)) - \dim \operatorname{Hom}_{D_{\mathcal{I}\mathcal{W}}^b(\mathbf{F}l_{G,\mathbb{k}})}(\Delta_0^{\mathcal{I}\mathcal{W}}, \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)[1]),$$

so that we necessarily have $\dim \operatorname{Hom}_{D_{\mathcal{I}\mathcal{W}}^b(\mathbf{F}l_{G,\mathbb{k}})}(\Delta_0^{\mathcal{I}\mathcal{W}}, \mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)[1]) = 0$; in other words, the corestriction of $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ to $\mathbf{F}l_{G,0}^{\mathcal{I}\mathcal{W}}$ vanishes in perverse degree 1, hence is perverse. Similar arguments show that the restriction of this perverse sheaf to $\mathbf{F}l_{G,0}^{\mathcal{I}\mathcal{W}}$ is perverse, which completes the proof.

9. PROOF OF THEOREM 5.4

In this section we finally give the proof of Theorem 5.4 (assuming that (2) and the first condition in (1) hold). As in the preceding sections we assume that $X^*(T)/\mathbb{Z}\mathfrak{A}$ is free. All our results will be obtained as consequences of Theorem 8.1; we therefore also assume that for any indecomposable factor of the root system \mathfrak{A} , the prime ℓ is strictly bigger than the corresponding value in the table of Figure 5.1.

9.1. Comparison of $\tilde{\mathbf{P}}_{\mathcal{I}\mathcal{W}}^0$ and $\mathbf{P}_{\mathcal{I}\mathcal{W}}^0$. We start with the following consequence of Theorem 8.1.

Proposition 9.1. *Any object in $\mathbf{P}_{\mathcal{I}\mathcal{W}}$ is isomorphic to a subquotient of an object of the form $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ with V in $\operatorname{Rep}(G_{\mathbb{k}}^{\vee})$.*

Proof. In view of Corollary 7.8, to prove the proposition it suffices to prove that each object $\mathcal{T}_{\mu}^{\mathcal{I}\mathcal{W}}$ with $\mu \in \mathbf{X}_{+}^{\vee}$ is a subquotient of an object $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$. This property follows from Theorem 8.1, see Remark 8.7. \square

Using this proposition we obtain the following.

Corollary 9.2. *The functor $\operatorname{Av}_{\mathcal{I}\mathcal{W}}^{\operatorname{asp}}$ of Theorem 7.2(2) is an equivalence of categories.*

Proof. By Theorem 7.2(2) we already know that $\operatorname{Av}_{\mathcal{I}\mathcal{W}}^{\operatorname{asp}}$ is fully faithful, and that its essential image is stable under subquotients. Now this essential image contains the objects $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ with V in $\operatorname{Rep}(G_{\mathbb{k}}^{\vee})$; hence Proposition 9.1 implies that it contains all objects, or in other words that $\operatorname{Av}_{\mathcal{I}\mathcal{W}}^{\operatorname{asp}}$ is essentially surjective. \square

Corollary 9.2 allows us to restate Proposition 9.1 as saying that any object in $\mathbf{P}_{\text{Iw}}^{\text{asp}}$ is a subquotient of an object $\Pi^{\text{asp}}(\mathcal{Z}(V))$ with V in $\text{Rep}(G_{\mathbb{k}}^{\vee})$. Now, recall the quotient functor $\mathbf{P}_{\text{Iw}}^{\text{asp}} \rightarrow \mathbf{P}_{\text{Iw}}^0$ considered in the proof of Lemma 8.10. Applying this functor we deduce the following, which establishes Property (a) in §5.2.

Corollary 9.3. *The inclusion $\tilde{\mathbf{P}}_{\text{Iw}}^0 \rightarrow \mathbf{P}_{\text{Iw}}^0$ is an equivalence of categories.*

9.2. Description of H . Recall the subgroup $H \subset Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})$ introduced in Proposition 5.8. Corollary 9.3 allows us to restate this proposition as providing an equivalence of monoidal categories $\mathbf{P}_{\text{Iw}}^0 \cong \text{Rep}(H)$. In view of the comments in §5.2, to complete the proof of Theorem 5.4, it therefore only remains to prove the following.

Proposition 9.4. *Assume (in addition to our running assumptions) that the \mathbb{Z} -module $X_*(T)/\mathbb{Z}\mathfrak{X}^{\vee}$ has no ℓ -torsion. Then the embedding $H \subset Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})$ is an equality.*

Proof. The proof will be based on the results of §§2.5–2.7, which will be applied to $G_{\mathbb{k}}^{\vee}$. Since the conditions in Figure 5.1 imply in particular that ℓ is good for $G_{\mathbb{k}}^{\vee}$, our assumptions indeed guarantee that these results are applicable.

Recall that the connected components of Gr_G are in a canonical bijection with the quotient of \mathbf{X}^{\vee} by the coroot lattice of (G, T) , which itself identifies with $X^*(Z(G_{\mathbb{k}}^{\vee}))$. In particular, every object of $\text{Perv}_{\text{sph}}(\text{Gr}_G, \mathbb{k})$ has a canonical decomposition as a direct sum of subobjects parametrized by $X^*(Z(G_{\mathbb{k}}^{\vee}))$; under the geometric Satake equivalence (see §4.1), this decomposition corresponds to the canonical decomposition of a $G_{\mathbb{k}}^{\vee}$ -module according to the (diagonalizable) action of $Z(G_{\mathbb{k}}^{\vee})$.

The projection $\text{Fl}_G \rightarrow \text{Gr}_G$ induces a bijection between the sets of connected components of Fl_G and Gr_G . Hence any object of \mathbf{P}_{Iw}^0 also admits a canonical decomposition as a direct sum of subobjects parametrized by $X^*(Z(G_{\mathbb{k}}^{\vee}))$, and from this we see that the embedding $Z(G_{\mathbb{k}}^{\vee}) \hookrightarrow G_{\mathbb{k}}^{\vee}$ factors through H ; in other words H contains $Z(G_{\mathbb{k}}^{\vee})$.

We have proved in Theorem 6.1 that \mathfrak{u} is regular unipotent. Hence the proposition will follow from Lemma 2.12 provided we prove that for any finite-dimensional tilting $G_{\mathbb{k}}^{\vee}$ -module V the embedding

$$V^{Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})} \hookrightarrow V^H$$

is an equality, or in other words that $\dim(V^{Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})}) = \dim(V^H)$. Now, by Lemma 2.15 the left-hand side is equal to the dimension of the 0-weight space V_0 of V . On the other hand, in view of Proposition 5.8 we have

$$V^H \cong \text{Hom}_H(\mathbb{k}, V) \cong \text{Hom}_{\mathbf{P}_{\text{Iw}}^0}(\delta^0, \mathcal{Z}^0(V)).$$

Corollary 9.2 allows to identify the category \mathbf{P}_{Iw}^0 with the category \mathbf{P}_{IW}^0 of §7.5, and using Lemma 7.6 we deduce an isomorphism

$$\text{Hom}_{\mathbf{P}_{\text{Iw}}^0}(\delta^0, \mathcal{Z}^0(V)) \cong \text{Hom}_{\mathbf{P}_{\text{IW}}^0}(\Delta_0^{\text{IW}}, \mathcal{Z}^{\text{IW}}(V)).$$

Theorem 8.1 implies that the right-hand side is equal to the multiplicity $(\mathcal{Z}^{\text{IW}}(V) : \Delta_0^{\text{IW}})$. Finally, by Remark 8.7 this multiplicity is also equal to $\dim(V_0)$, which shows that $\dim(V^{Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})}) = \dim(V^H)$ and therefore finishes the proof. \square

9.3. Complement: indecomposability of the objects $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$. We finish this section by observing that Proposition 9.4 allows to prove the following complement to Theorem 8.1.

Proposition 9.5. *Assume that the conditions in Proposition 9.4 are satisfied. Then if V is an indecomposable tilting $G_{\mathbb{k}}^{\vee}$ -module, $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ is indecomposable.*

Proof. We assume that V is an indecomposable tilting $G_{\mathbb{k}}^{\vee}$ -module. Let $\Pi_{\mathcal{I}\mathcal{W}}^0 : \mathcal{P}_{\mathcal{I}\mathcal{W}} \rightarrow \mathcal{P}_{\mathcal{I}\mathcal{W}}^0$ be the quotient functor. In view of Lemma 7.6, to prove that $\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V)$ is indecomposable it suffices to prove that $\Pi_{\mathcal{I}\mathcal{W}}^0(\mathcal{Z}^{\mathcal{I}\mathcal{W}}(V))$ is indecomposable. As observed in the course of the proof of Proposition 9.4, Corollary 9.2 allows us to identify $\mathcal{P}_{\mathcal{I}\mathcal{W}}^0$ with $\mathcal{P}_{\mathcal{I}\mathcal{W}}^0$, so that to conclude it suffices to prove that $\mathcal{Z}^0(V)$ is indecomposable. Now Proposition 5.8 and Proposition 9.4 imply that we have an algebra isomorphism

$$\mathrm{End}_{\mathcal{P}_{\mathcal{I}\mathcal{W}}^0}(\mathcal{Z}^0(V)) \cong \mathrm{End}_{Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})}(V);$$

hence to conclude we have to show that the algebra $\mathrm{End}_{Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})}(V)$ is local. By Frobenius reciprocity we have an algebra isomorphism

$$\mathrm{End}_{Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})}(V) \cong \mathrm{Hom}_{G_{\mathbb{k}}^{\vee}}(V, V \otimes \mathcal{O}(G_{\mathbb{k}}^{\vee}/Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})))$$

(where the description of the algebra structure on the right-hand side involves the multiplication in the ring $\mathcal{O}(G_{\mathbb{k}}^{\vee}/Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u}))$), and as seen in the course of the proof of Lemma 2.14 we have an algebra isomorphism $\mathcal{O}(G_{\mathbb{k}}^{\vee}/Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})) \cong \mathcal{O}(\mathcal{N}_{G_{\mathbb{k}}^{\vee}})$, where $\mathcal{N}_{G_{\mathbb{k}}^{\vee}}$ is the nilpotent cone of $G_{\mathbb{k}}^{\vee}$. The dilation action of \mathbb{G}_m on $\mathcal{N}_{G_{\mathbb{k}}^{\vee}}$ induces a grading on $\mathcal{O}(\mathcal{N}_{G_{\mathbb{k}}^{\vee}})$, such that $G_{\mathbb{k}}^{\vee}$ acts by graded algebra automorphisms, and whose degree-0 part is \mathbb{k} . This structure induces a grading on $\mathrm{End}_{Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})}(V)$, whose degree-0 part is the algebra $\mathrm{End}_{G_{\mathbb{k}}^{\vee}}(V)$, which is local since V is indecomposable. We deduce that $\mathrm{End}_{Z_{G_{\mathbb{k}}^{\vee}}(\mathfrak{u})}(V)$ is local, see [GG, Theorem 3.1]. \square

REFERENCES

- [AR1] P. Achar and S. Riche, *Modular perverse sheaves on flag varieties I: tilting and parity sheaves* (with a joint appendix with G. Williamson), *Ann. Sci. Éc. Norm. Supér.* **49** (2016), 325–370.
- [AR2] P. Achar and S. Riche, *Reductive groups, the loop Grassmannian, and the Springer resolution*, *Invent. Math.* **214** (2018), 289–436.
- [AR3] P. Achar and S. Riche, *Central sheaves on affine flag varieties*, book in preparation.
- [AB] S. Arkhipov and R. Bezrukavnikov, *Perverse sheaves on affine flags and Langlands dual group* (with an appendix by R. Bezrukavnikov and I. Mirković), *Israel J. Math.* **170** (2009), 135–183.
- [Ba] M. Barry, *Bases for fixed points of unipotent elements acting on the tensor square and the spaces of alternating and symmetric 2-tensors*, *J. Algebra* **251** (2002), 395–412.
- [BR] P. Baumann and S. Riche, *Notes on the geometric Satake equivalence*, in *Relative Aspects in Representation Theory, Langlands Functoriality and Automorphic Forms, CIRM Jean-Morlet Chair, Spring 2016* (V. Heiermann, D. Prasad, Eds.), 1–134, *Lecture Notes in Math.* 2221, Springer, 2018.
- [BBD] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, in *Analyse et topologie sur les espaces singuliers, I (Luminy, 1981)*, 5–172, *Astérisque* **100** (1982).
- [BBM] A. Beilinson, R. Bezrukavnikov, and I. Mirković, *Tilting exercises*, *Mosc. Math. J.* **4** (2004), 547–557, 782.
- [BD] A. Beilinson and V. Drinfeld, *Quantization of Hitchin’s integrable system and Hecke eigensheaves*, unpublished preprint available at <http://www.math.uchicago.edu/~mitya/langlands.html>.

- [Be1] R. Bezrukavnikov, *On tensor categories attached to cells in affine Weyl groups*, with an appendix by D. Gaitsgory, in *Representation theory of algebraic groups and quantum groups*, 69–100, Adv. Stud. Pure Math. 40, Math. Soc. Japan, 2004.
- [Be2] R. Bezrukavnikov, *On two geometric realizations of an affine Hecke algebra*, Publ. Math. IHES **123** (2016), 1–67.
- [BM] R. Bezrukavnikov and I. Mirković, appendix to [AB].
- [BMR] R. Bezrukavnikov, I. Mirković, and D. Rumynin, *Localization of modules for a semisimple Lie algebra in prime characteristic*, with an appendix by R. Bezrukavnikov and S. Riche, Ann. of Math. **167** (2008), 945–991.
- [BeR] R. Bezrukavnikov and S. Riche, *A topological approach to Soergel theory*, preprint [arXiv:1807.07614](https://arxiv.org/abs/1807.07614).
- [BY] R. Bezrukavnikov and Z. Yun, *On Koszul duality for Kac–Moody groups*, Represent. Theory **17** (2013), 1–98.
- [Bo] A. Borel, *Linear algebraic groups, second edition*, Springer, 1991.
- [CG] N. Chriss and V. Ginzburg, *Representation theory and complex geometry* Birkhäuser Boston, Inc., 1997.
- [DM] P. Deligne and J. S. Milne, *Tannakian categories*, in *Hodge cycles, motives, and Shimura varieties*, 101–228, Lecture Notes in Mathematics 900, Springer-Verlag, 1982.
- [DG] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, with an appendix *Corps de classes local* by Michiel Hazewinkel, Masson & Cie, North-Holland Publishing Co., 1970.
- [Dod] C. Dodd, *Equivariant coherent sheaves, Soergel bimodules, and categorification of affine Hecke algebras*, preprint [arXiv:1108.4028](https://arxiv.org/abs/1108.4028).
- [Dou] J. M. Douglass, *An inversion formula for relative Kazhdan–Lusztig polynomials*, Comm. Algebra **18** (1990), 371–387.
- [FM] M. Finkelberg and I. Mirković, *Semi-infinite flags I. Case of global curve \mathbb{P}^1* , in *Differential topology, infinite-dimensional Lie algebras, and applications*, 81–112, Amer. Math. Soc. Transl. Ser. 2, vol. 194, Amer. Math. Soc., 1999.
- [Gab] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.
- [G1] D. Gaitsgory, *Construction of central elements in the affine Hecke algebra via nearby cycles*, Invent. Math. **144** (2001), 253–280.
- [G2] D. Gaitsgory, *Braiding compatibilities*, appendix to [Be1].
- [Gi] V. Ginzburg, *Perverse sheaves on a loop group and Langlands’ duality*, preprint [arXiv:alg-geom/9511007](https://arxiv.org/abs/alg-geom/9511007) (1995).
- [GG] R. Gordon and E. Green, *Graded Artin algebras*, J. Algebra **76** (1982), 111–137.
- [Hei] J. Heinloth, *Uniformization of \mathcal{G} -bundles*, Math. Ann. **347** (2010), 499–528.
- [Her] S. Herpel, *On the smoothness of centralizers in reductive groups*, Trans. Amer. Math. Soc. **365** (2013), 3753–3774.
- [Hu] J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, American Mathematical Society, 1995.
- [IM] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of \mathfrak{p} -adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 5–48.
- [J1] J. C. Jantzen, *Representations of Algebraic Groups, second edition*, Mathematical Surveys and Monographs 107, Amer. Math. Soc., 2003.
- [J2] J. C. Jantzen, *Nilpotent orbits in representation theory*, in *Lie theory*, 1–211, Progr. Math. 228, Birkhäuser Boston, 2004.
- [JMW] D. Juteau, C. Mautner, and G. Williamson, *Parity sheaves and tilting modules*, Ann. Sci. Éc. Norm. Supér. **49** (2016), 257–275.
- [KL] D. Kazhdan and G. Lusztig, *Proof of the Deligne–Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), 153–215.
- [Ko] M. Korhonen, *Unipotent elements forcing irreducibility in linear algebraic groups*, J. Group Theory **21** (2018), 365–396.
- [KLT] S. Kumar, N. Lauritzen, and J. F. Thomsen, *Frobenius splitting of cotangent bundles of flag varieties*, Invent. Math. **136** (1999), 603–621.
- [La] R. Lawther, *Jordan block sizes of unipotent elements in exceptional algebraic groups*, Comm. Algebra **23** (1995), 4125–4156.
- [Lin] J. Lindsey, II, *Groups with a t . i . cyclic Sylow subgroup*, J. Algebra **30** (1974), 181–235.

- [Li] J. Lipman, *Notes on derived functors and Grothendieck duality*, in *Foundations of Grothendieck duality for diagrams of schemes*, Lecture Notes in Math. 1960, Springer, 2009, 1–259.
- [Lüb] F. Lübeck, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math. **4** (2001), 135–169.
- [Lu] G. Lusztig, *Singularities, character formulas, and a q -analog of weight multiplicities*, in *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, 208–229, Astérisque **101–102**, Soc. Math. France, 1983.
- [MR1] C. Mautner and S. Riche, *On the exotic t -structure in positive characteristic*, Int. Math. Res. Not. IMRN **2016**, no. 18, 5727–5774.
- [MR2] C. Mautner and S. Riche, *Exotic tilting sheaves, parity sheaves on affine Grassmannians, and the Mirković–Vilonen conjecture*, J. Eur. Math. Soc. (JEMS) **20** (2018), 2259–2332.
- [McN] G. J. McNinch, *Optimal $SL(2)$ -homomorphisms*, Comment. Math. Helv. **80** (2005), 391–426.
- [MV] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. **166** (2007), 95–143.
- [PR] G. Pappas and M. Rapoport, *Twisted loop groups and their affine flag varieties*, with an appendix by T. Haines and M. Rapoport, Adv. Math. **219** (2008), 118–198.
- [PS] A. Premet and I. Suprunenko, *The Weyl modules and the irreducible representations of the symplectic group with the fundamental highest weights*, Comm. Algebra **11** (1983), 1309–1342.
- [Ri] S. Riche, *Geometric Representation Theory in positive characteristic*, habilitation thesis, available on <https://tel.archives-ouvertes.fr/tel-01431526>.
- [So] W. Soergel, *Kategorie \mathcal{O} , perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, J. Amer. Math. Soc. **3** (1990), 421–445.
- [SR] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics 265, Springer-Verlag, 1972.
- [Sp] T. A. Springer, *Some arithmetical results on semi-simple Lie algebras*, Inst. Hautes Études Sci. Publ. Math. **30** (1966), 115–141.
- [St] R. Steinberg, *Regular elements of semisimple algebraic groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 49–80.
- [Ve] J.-L. Verdier, *Spécialisation de faisceaux et monodromie modérée*, in *Analyse et topologie sur les espaces singuliers, II, III (Luminy, 1981)*, 332–364, Astérisque **101–102** (1983).
- [Zh] X. Zhu, *On the coherence conjecture of Pappas and Rapoport*, Ann. of Math. **180** (2014), 1–85.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA, 02139, USA.

Email address: bezrukav@math.mit.edu

UNIVERSITÉ CLERMONT AUVERGNE, CNRS, LMBP, F-63000 CLERMONT-FERRAND, FRANCE.

Email address: simon.riche@uca.fr

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS GEORGIA 30602, USA.

Email address: laurajoy@uga.edu