Optimal inequalities between distances in convex projective domains

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Abstract

On any proper convex domain in real projective space there exists a natural Riemannian metric, the Blaschke metric. On the other hand, distances between points can be measured in the Hilbert metric. Using techniques of optimal control, we provide inequalities involving the Hilbert distance between two points of the domain and the Riemannian length of the line segment joining these points, thus strengthening a result of Tholozan. Our estimates are valid for a whole class of complete Riemannian metrics on convex projective domains, namely those induced by centro-affine hypersurface immersions which are asymptotic to the boundary of the convex cone over the domain. On this class our inequalities are optimal.

Keywords: Hilbert distance, Blaschke metric, centro-affine hypersurface immersion, affine hypersphere
MSC 2020: 53A15, 52A38, 58E10

1 Introduction

Proper convex domains $\Omega \subset \mathbb{R}P^n$ allow to define several projectively invariant distances. On of them is the Hilbert distance. For two distinct points $a, b \in \Omega$, it is defined by

$$d^H(a,b) = \frac{1}{2}\log\frac{||ya|| \cdot ||xb||}{||yb|| \cdot ||xa||} = \frac{1}{2}\log(a,b;y,x),$$

where $x, y \in \partial\Omega$ are the boundary points lying on the projective line l through a, b, such that the order of points on l is x, a, b, y, and the quantities $||ya||, \dots$ are the corresponding coordinate differences in any affine chart on l. The quantity $(a, b; x, y)$ is the projective cross-ratio of the quadruple of points.

Another possibility to define distances in the domain Ω is by a Riemannian metric. Let $K_\Omega \subset \mathbb{R}^{n+1}$ be the proper convex cone over the closure of the domain Ω. Lift Ω into the interior of K_Ω. Such a hypersurface immersion $f : \Omega \to K_\Omega^o$, smooth enough and equipped with the position vector as transversal vector field, gives rise to a centro-affine fundamental form h on Ω. If the immersion is locally strongly convex, then this form defines a Riemannian metric on Ω. We suppose throughout the paper that the immersion is of hyperbolic type, i.e., the hypersurface is bent away from the origin.

As a special case, we obtain the *Blaschke metric* (or Cheng-Yau metric) if the immersion is a proper affine hypersphere with mean curvature -1 which is asymptotic to the boundary ∂K_Ω. The Blaschke metric gives rise to the *Blaschke distance* d^B on Ω (see [6],[2]).

Along with the fundamental form, a centro-affine hypersurface immersion f defines a symmetric 3-rd order tensor field on Ω, the *cubic form* C. On affine hyperspheres the cubic form is bounded by an explicit function of the dimension n [5],

$$C(X,X,X) \le 2\frac{n-1}{\sqrt{n}}(h(X,X))^{3/2} \tag{1}$$

for all tangent vector fields X on Ω.

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In [2], Benoist and Hulin proved by a general compactness argument [3] that both distances d^B, d^H are strongly equivalent, i.e., one can be bounded by a multiple of the other, where the constants depend only on the dimension n. In [9], Tholozan used this result to prove the remarkable inequality

$$d^B(a,b) < d^H(a,b) + 1$$

for all pairs of points $a, b \in \Omega$. This inequality actually holds also for Riemannian distances d^R generated by general convex non-degenerate centro-affine hypersurface immersions of Ω.

In this paper we improve this result to optimality as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}P^n$ be a proper open convex domain, and let $f : \Omega \to K^o_\Omega \subset \mathbb{R}^{n+1}$ be a non-degenerate convex lift of class C^2 into the interior of the convex cone over Ω. Let h be the centro-affine metric induced on Ω by f, and let d^R be the corresponding geodesic distance. Let further d^H be the Hilbert distance on Ω. Then for every pair of points $a, b \in \Omega$ the inequalities*

$$d^R(a,b) \le l^R(a,b) < \log\left(\exp(d^H(a,b)) + \sqrt{\exp(2d^H(a,b)) - 1}\right) < d^H(a,b) + \log 2$$

hold, where $l^R(a,b)$ is the Riemannian length of the projective line segment joining the points a, b. The inequalities cannot be improved.

In the case when Ω is an ellipsoid both the Hilbert and the Blaschke metric coincide, and Ω is isometric to hyperbolic space [9]. Note that in this case the cubic form of the affine sphere over Ω vanishes. This suggests that the deviation of the Hilbert distance from the centro-affine Riemannian distance can somehow be controlled by a bound on the cubic form of the centro-affine hypersurface immersion f. We investigate this dependence and provide optimal inequalities between $l^R(a,b)$ and $d^H(a,b)$ with the bound on the cubic form appearing as a parameter.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}P^n$ be a proper open convex domain, and let $f : \Omega \to K^o_\Omega \subset \mathbb{R}^{n+1}$ be a non-degenerate convex lift of class C^3 into the interior of the convex cone over Ω. Let C be the cubic form and h the centro-affine metric induced on Ω by f, and let d^R be the corresponding geodesic distance. Suppose that the cubic form satisfies the bound*

$$C(X,X,X) \le 2\gamma(h(X,X))^{3/2} \tag{2}$$

for all tangent vector fields X on Ω, where $\gamma \ge 0$ is some constant. Let further d^H be the Hilbert distance on Ω. Then for every pair of points $a, b \in \Omega$ the inequalities

$$\begin{aligned} d^R(a,b) \le l^R(a,b) &< \begin{cases} \frac{2\mu}{\mu^2+1}\log\frac{(\mu^2+1)(E-1)+2}{2}, & d^H(a,b) \le \log\frac{\mu^2+1}{\mu^2-1}, \\ \log\frac{(\mu-1)(\sqrt{E+1}+\sqrt{E-1})}{(\mu+1)(\sqrt{E+1}-\sqrt{E-1})} + \frac{2\mu}{\mu^2+1}\log\frac{2\mu^2}{\mu^2-1}, & d^H(a,b) \ge \log\frac{\mu^2+1}{\mu^2-1}, \end{cases} \\ &< d^H(a,b) + \log 2 + \log\frac{\mu-1}{\mu+1} + \frac{2\mu}{\mu^2+1}\log\frac{2\mu^2}{\mu^2-1}, \end{aligned}$$

hold, where $l^R(a,b)$ is the Riemannian length of the projective line segment joining the points a, b, $E = \exp(d^H(a,b))$, and $\mu = \frac{\gamma}{2} + \sqrt{1 + \frac{\gamma^2}{4}}$. The inequalities cannot be improved.

With the same techniques we can derive also a lower bound on the Riemannian length.

Theorem 1.3. *Assume the notations and conditions of Theorem 1.2. Then the inequalities*

$$l^R(a,b) > \frac{2\mu}{\mu^2+1}\log\frac{\exp(d^H(a,b))(\mu^2+1)+\mu^2-1}{2\mu^2} > \frac{2\mu}{\mu^2+1}d^H(a,b) - \frac{2\mu}{\mu^2+1}\log\frac{2\mu^2}{\mu^2+1}$$

hold for every $a, b \in \Omega$. The inequalities cannot be improved.

The upper and lower bounds on $l^R(a,b)$ are depicted in Fig. 1. It is also possible to obtain a lower bound on the geodesic distance d^R.

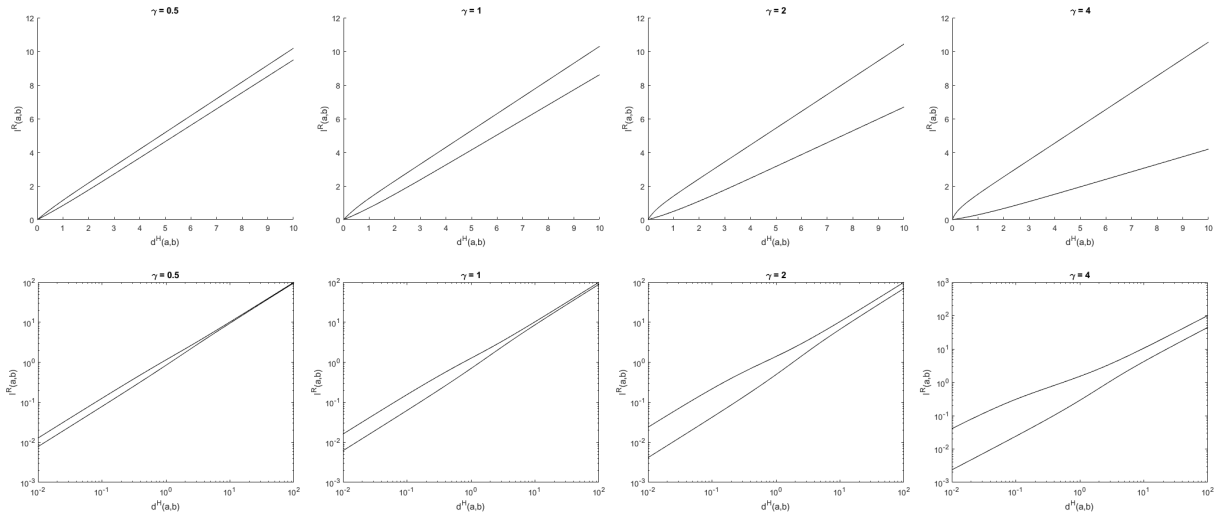

Figure 1: Upper and lower bounds on the Riemannian length of line segments as a function of the Hilbert distance between the end-points for different values of γ. Linear plot (top row) and log-log plot (bottom row).

Theorem 1.4. *Assume the notations and conditions of Theorem 1.2. Then the inequality*

$$d^R(a,b) \geq \mu^{-1} d^H(a,b)$$

holds for every $a, b \in \Omega$.

In the case of affine hyperspheres we get $\mu = \sqrt{n}$ by virtue of (1). Applying Theorems 1.2, 1.3 yields the following result.

Corollary 1.5. *Let $\Omega \subset \mathbb{R}P^n$ be a proper open convex domain. Let $d^H(a,b)$ be the Hilbert distance, $d^B(a,b)$ the geodesic distance in the Blaschke metric, and $l^B(a,b)$ the Riemannian length of the straight line segment in the Blaschke metric between points $a, b \in \Omega$. Then*

$$d^B(a,b) \leq l^B(a,b) < \begin{cases} \frac{2\sqrt{n}}{n+1} \log \frac{(n+1)(E-1)+2}{2}, & d^H(a,b) \leq \log \frac{n+1}{n-1}, \\ \log \frac{(\sqrt{n}-1)(\sqrt{E+1}+\sqrt{E-1})}{(\sqrt{n}+1)(\sqrt{E+1}-\sqrt{E-1})} + \frac{2\sqrt{n}}{n+1} \log \frac{2n}{n-1}, & d^H(a,b) \geq \log \frac{n+1}{n-1}, \end{cases}$$
$$< d^H(a,b) + \log 2 + \log \frac{\sqrt{n}-1}{\sqrt{n}+1} + \frac{2\sqrt{n}}{n+1} \log \frac{2n}{n-1},$$

$$l^B(a,b) > \frac{2\sqrt{n}}{n+1} \log \frac{\exp(d^H(a,b))(n+1)+n-1}{2n} > \frac{2\sqrt{n}}{n+1} d^H(a,b) - \frac{2\sqrt{n}}{n+1} \log \frac{2n}{n+1},$$

$$d^B(a,b) \geq \frac{1}{\sqrt{n}} d^H(a,b),$$

where $E = \exp(d^H(a,b))$. □

In particular, we have $\frac{1}{\sqrt{n}} d^H(a,b) \leq d^B(a,b) \leq \sqrt{n} d^H(a,b)$, which yields an explicit estimate of the constants realizing the equivalence of the two metrics.

The remainder of the paper is structured as follows. In Section 2 we prove Theorem 1.1, in Section 3 we prove Theorem 1.2, while in Section 4 we prove Theorems 1.3 and 1.4.

2 Proof of Theorem 1.1

As in [9], is it sufficient to consider the case $n = 1$. Let $\Omega \subset \mathbb{R}P^1$ be a proper open line segment, i.e., such that its complement is also a line segment. Without loss of generality we assume that the cone $K_\Omega \subset \mathbb{R}^2$

over the closure of Ω is the positive orthant $\mathbb{R}^2_+$. Following [9], we parameterize Ω by a variable $t \in \mathbb{R}$, in a way such that the point $(e^t, e^{-t}) \in \mathbb{R}^2_+$ projects to the corresponding point in Ω. Then the Hilbert distance between t_1, t_2 is given by $|t_1 - t_2|$.

A centro-affine lift of Ω into the interior of $\mathbb{R}^2_+$ is given by a curve $f : t \mapsto e^{\alpha(t)} \cdot (e^t, e^{-t})$. We suppose the function $\alpha(t)$ to be of class $C^2(\mathbb{R})$. Then the centro-affine fundamental form of the immersion f is given by $h = \ddot{\alpha} - \dot{\alpha}^2 + 1$ [9, Lemma 2.1]. Locally strong convexity of f is equivalent to the differential inequality $\ddot{\alpha} > \dot{\alpha}^2 - 1$, and the first derivative is bounded by $|\dot{\alpha}| < 1$ [9, Lemma 2.1].

We now maximize the Riemannian distance between two given points $t_1 < t_2$. This distance is given by the integral

$$l^R(t_1, t_2) = \int_{t_1}^{t_2} \sqrt{\ddot{\alpha} - \dot{\alpha}^2 + 1}\, dt.$$

We shall prove the following estimate.

Lemma 2.1. *Set $d = t_2 - t_1$. Then $l^R(t_1, t_2) < \log\left(e^d + \sqrt{e^{2d} - 1}\right)$, and this estimate is sharp.*

The proof is by presenting an explicit Bellman function for the variational problem under consideration. Before proceeding to the proof, we shall give some clues how to arrive at this expression.

Set $x = \dot{\alpha}$. The extremals for the functional $\int L(x, \dot{x})\, dt$ are given by the solutions of the Euler-Lagrange equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$ [4]. Inserting $L(x, \dot{x}) = \sqrt{\dot{x} - x^2 + 1}$, we obtain the second order ODE $\ddot{x} = 2x(3\dot{x} - 2x^2 + 2)$. This ODE has in particular the solutions

$$x = -\frac{e^{4t}c^2 + 2(e^{4t} - 2)c + e^{4t}}{(e^{2t} + (e^{2t} - 2)c)(e^{2t}c + e^{2t} - 2)}, \tag{3}$$

where $c \in (-1, 1)$ is an integration constant. The Bellman function is then constructed by computing the value of the cost function on these trajectories.

Proof. Since the problem is invariant with respect to translations of the variable t, we may set $t_2 = 0$, $t_1 = -d$. Following the principles of dynamic programming [1], consider the Bellman function

$$B(x, t) = \frac{1}{2}\log\left(\sqrt{(1 - e^{2t})(1 - x)(1 + e^{2t} - x(1 - e^{2t}))} - x(1 - e^{2t}) + 1\right) - t, \qquad t \le 0, \quad x \in [-1, 1].$$

Let us show that the value $B(x_0, t_0)$ determines the maximal value of the integral $\int_{t_0}^0 L(x, \dot{x})\, dt$ which can be achieved when starting at $x(t_0) = x_0$. We have $B(x, 0) \equiv 0$ and by the inequality between arithmetic and geometric mean

$$\frac{dB}{dt} = \frac{\partial B}{\partial x}\dot{x} + \frac{\partial B}{\partial t} = -\frac{(1 - x)(1 + e^{2t} - x(1 - e^{2t})) + (\dot{x} - x^2 + 1)(1 - e^{2t})}{2\sqrt{(1 - e^{2t})(1 - x)(1 + e^{2t} - x(1 - e^{2t}))}} \le -\sqrt{\dot{x} - x^2 + 1},$$

with equality achieved at

$$\dot{x} = \frac{(1 - x)(1 + e^{2t} - x(1 - e^{2t}))}{1 - e^{2t}} - (1 - x^2) = \frac{2(1 - x)(e^{2t} - x(1 - e^{2t}))}{1 - e^{2t}}. \tag{4}$$

Thus indeed

$$\int_{t_0}^0 \sqrt{\dot{x} - x^2 + 1}\, dt \le -\int_{t_0}^0 \frac{dB}{dt}\, dt = B(x(t_0), t_0),$$

with equality achieved at the solutions of ODE (4), easily verified to be given by (3).

The supremum of $l^R(t_1, t_2)$ is then given by $\sup_{x \in (-1,1)} B(x, -d)$, which in view of $\frac{\partial B}{\partial x} < 0$ is achieved at $x = -1$ and given by $\log(1 + \sqrt{1 - e^{-2d}}) + d = \log(e^d + \sqrt{e^{2d} - 1})$.

This value is actually not achieved on any valid trajectory, because it requires that $|x| = 1$ at the end-points of the interval, but can be approached arbitrarily closely. □

Let us now return to Theorem 1.1. Let l be the projective line passing through the points $a, b \in \Omega$, and $L \subset \mathbb{R}^{n+1}$ the two-dimensional linear subspace over l. Then the centro-affine metric h_l of the immersion $f|_{\Omega \cap l}$ into L is given by the restriction of the centro-affine metric h to $\Omega \cap l$, because the position vector field

on $\Omega \cap l$ is contained in L. Therefore the Riemannian length $l^R(a,b)$ of the line segment between a and b is equal in both metrics h and h_l. Moreover, by definition the Hilbert distance $d^H(a,b)$ is equal in Ω and in $\Omega \cap l$. The assertion of Lemma 2.1 is an inequality between these quantities as defined by the immersion of $\Omega \cap l$ into $L \sim \mathbb{R}^2$. But then the lemma proves also the second inequality in Theorem 1.1, which is between the same quantities defined by the immersion f of Ω into $\mathbb{R}^{n+1}$.

The first inequality follows from the fact that the geodesic distance between two points in a Riemannian manifold is never exceeding the length of any curve between these two points. For $n = 1$ it turns into an equality, because the straight line segment is the only path linking the two points.

The third inequality is obvious. Moreover, it is easily seen that it is sharp by taking the limit of the difference between the right-hand and the left-hand side as $d^H(a,b) \to +\infty$.

This proves Theorem 1.1 and shows that the estimates actually cannot be improved.

3 Proof of Theorem 1.2

Let us again set $n = 1$ and assume the notations at the beginning of Section 2. However, this time we consider immersions $f : \Omega \to \mathbb{R}^2_+$ defined by functions $\alpha(t)$ of class C^3.

Let us compute the cubic form C of the immersion f. It is given by the derivative ∇h, where h is the affine metric and ∇ is the induced affine connection. The latter is defined by the decomposition of the canonical flat affine connection D of the ambient space $\mathbb{R}^2$ into a tangential part ∇ and a transversal part $h \cdot f$ [8, p. 28]. Let $X = \dot f = \dot\alpha f + Jf$, where $J = \mathrm{diag}(1,-1)$, be the basis tangent vector field. Then we have the decomposition

$$D_X X = \ddot f = \ddot\alpha f + \dot\alpha \dot f + J\dot f = \ddot\alpha f + \dot\alpha^2 f + 2\dot\alpha J f + f = (\ddot\alpha - \dot\alpha^2 + 1) f + 2\dot\alpha(\dot\alpha f + Jf) = h(X,X)\cdot f + 2\dot\alpha \cdot X.$$

Therefore $\nabla_X X = 2\dot\alpha \cdot X$. Hence we obtain

$$C(X,X,X) = Xh(X,X) - 2h(\nabla_X X, X) = \frac{d}{dt}(\ddot\alpha - \dot\alpha^2 + 1) - 4\dot\alpha(\ddot\alpha - \dot\alpha^2 + 1) = \dddot\alpha - 6\dot\alpha\ddot\alpha + 4\dot\alpha^3 - 4\dot\alpha.$$

Condition (2) can then be written as

$$|\dddot\alpha - 6\dot\alpha\ddot\alpha + 4\dot\alpha^3 - 4\dot\alpha| \le 2\gamma(\ddot\alpha - \dot\alpha^2 + 1)^{3/2}. \tag{5}$$

We shall now show that solutions $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying this differential inequality must satisfy certain bounds on the values of $(\ddot\alpha, \dot\alpha)$ of their derivatives.

Lemma 3.1. *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a C^3 function satisfying (5) for some $\gamma \ge 0$. Then for every $t \in \mathbb{R}$ we have*

$$\sqrt{\ddot\alpha - \dot\alpha^2 + 1} \le \mu(1 - |\dot\alpha|),$$

where $\mu \ge 1$ depends on γ by the relations $\mu = \frac{\gamma}{2} + \sqrt{1 + \frac{\gamma^2}{4}}$, $\gamma = \frac{\mu^2 - 1}{\mu}$.

Proof. Set $h = \ddot\alpha - \dot\alpha^2 + 1$. Then (5) can be written as

$$-2\gamma h^{3/2} + 4\dot\alpha h \le \dot h \le 2\gamma h^{3/2} + 4\dot\alpha h \tag{6}$$

Since $h \ge 0$, we have $\ddot\alpha \ge \dot\alpha^2 - 1$. Therefore, if for some t_0 we have $\dot\alpha(t_0) > 1$, then $\dot\alpha(t)$ blows up before t reaches $+\infty$. Likewise, if $\dot\alpha(t_0) < -1$, then $\dot\alpha(t)$ blows up before t reaches $-\infty$. Thus $|\dot\alpha| \le 1$.

Define functions $\xi_\pm = \dot\alpha \pm \mu^{-1}\sqrt{h}$. By (6) we then have

$$\dot\xi_\pm = \ddot\alpha \pm \mu^{-1}\frac{\dot h}{2\sqrt{h}} \ge \ddot\alpha + \mu^{-1}\frac{-2\gamma h^{3/2} \pm 4\dot\alpha h}{2\sqrt{h}} = h + \dot\alpha^2 - 1 + \mu^{-1}(-\gamma h \pm 2\dot\alpha\sqrt{h}) = \xi_\pm^2 - 1.$$

Here we used that $1 - \mu^{-1}\gamma = \mu^{-2}$.

As above we have $|\xi_\pm| \le 1$ and hence $\dot\alpha + \mu^{-1}\sqrt{h} \le 1$, $-1 \le \dot\alpha - \mu^{-1}\sqrt{h}$. It follows that $\sqrt{h} \le \mu(1 \pm \dot\alpha)$. □

Lemma 3.2. *Suppose in addition to the conditions in Lemma 3.1 that the function α defines a non-degenerate centro-affine immersion $f : t \mapsto e^{\alpha(t)} \cdot (e^t, e^{-t})$ which is asymptotic to the boundary of the orthant $\mathbb{R}^2_+$. Then for every $t \in \mathbb{R}$ we have that*

$$\mu^{-1}(1 + |\dot\alpha|) \le \sqrt{\ddot\alpha - \dot\alpha^2 + 1}.$$

Proof. Let $h = \ddot\alpha - \dot\alpha^2 + 1 > 0$ be the affine metric. Define functions $\psi_\pm = \dot\alpha \pm \mu\sqrt{h}$. By (6) we then have

$$\dot\psi_\pm = \ddot\alpha \pm \mu\frac{\dot h}{2\sqrt{h}} \le \ddot\alpha + \mu\frac{2\gamma h^{3/2} \pm 4\dot\alpha h}{2\sqrt{h}} = h + \dot\alpha^2 - 1 + \mu(\gamma h \pm 2\dot\alpha\sqrt{h}) = \psi_\pm^2 - 1.$$

Here we used that $1 + \mu\gamma = \mu^2$.

Now suppose for the sake of contradiction that for some t_0 we have $\psi_+(t_0) < 1$. Choose η_0 such that $\psi_+(t_0) < -\tanh(t_0 + \eta_0)$. Then we have $\kappa(t) = \psi_+(t) + \tanh(t + \eta_0) < 0$ for all $t > t_0$. Indeed, let $t_1 > t_0$ be the smallest point such that $\kappa(t_1) = 0$. For all $t < t_1$ close enough to t_1 we have $\kappa(t) < 0$, $\psi_+(t) \in (-1, 1)$ and

$$\frac{d\kappa}{dt} \le \psi_+(t)^2 - 1 + \frac{1}{\cosh(t + \eta_0)^2} = (\psi_+(t) - \tanh(t + \eta_0)) \cdot \kappa(t) < -2\kappa(t).$$

But then κ cannot reach zero in finite time, a contradiction. Hence such a point t_1 cannot exist. We obtain $\dot\alpha(t) + \mu\sqrt{h(t)} < -\tanh(t + \eta_0)$ for all $t \ge t_0$ and by integration

$$0 < \mu\int_{t_0}^{t} \sqrt{h(s)}\, ds < [-\alpha(s) - \log\cosh(s + \eta_0)]_{t_0}^{t} = \left[-s - \alpha(s) - \log\frac{e^{\eta_0} + e^{-(2s+\eta_0)}}{2}\right]_{t_0}^{t}.$$

But $t + \alpha(t) \to +\infty$ for $t \to +\infty$, because $e^{t+\alpha}$ is the first coordinate of the immersion f, which by assumption is asymptotic to the boundary of $\mathbb{R}^2_+$. Therefore the right-most expression tends to $-\infty$ as $t \to +\infty$, a contradiction. As a consequence, we have $\psi_+ \ge 1$ and $\mu\sqrt{h} \ge 1 - \dot\alpha$.

The inequality $\mu\sqrt{h} \ge 1 + \dot\alpha$ is proven similarly by deducing a contradiction from the assumption $\psi_-(t_0) > -1$ for some t_0. □

Corollary 3.3. *Assume the conditions of Lemmas 3.1,3.2. Then the Riemannian length $l^R(t_1, t_2)$ is bounded by $\mu^{-1}d^H(t_1, t_2) \le l^R(t_1, t_2) \le \mu d^H(t_1, t_2)$.*

Proof. By Lemmas 3.1,3.2 we have $\sqrt{\ddot\alpha - \dot\alpha^2 + 1} \in [\mu^{-1}, \mu]$. The claim now immediately follows. □

Corollary 3.4. *Assume the conditions of Lemmas 3.1,3.2. If $\gamma = 0$, then $\alpha \equiv const$, and the image of f is a hyperbola. In this case the Riemannian length $l^R(t_1, t_2)$ coincides with the Hilbert distance $d^H(t_1, t_2)$.*

Proof. From $\gamma = 0$ we obtain $\mu = 1$, and hence by Lemmas 3.1,3.2 we have $1 + |\dot\alpha| \le 1 - |\dot\alpha|$. It follows that $\alpha \equiv const$, $h \equiv 1$, and $l^R(t_1, t_2) = |t_1 - t_2| = d^H(t_1, t_2)$. □

In the sequel we shall assume that $t_1 < t_2$ and $\gamma > 0$, hence $\mu > 1$.

We may write the problem of maximizing the Riemannian length $l^R(t_1, t_2)$ under constraint (2) on the cubic form of the centro-affine immersion as an optimal control problem. Introduce variables $x = \dot\alpha$, $y = \sqrt{h}$. Then the dynamics of the system can be written as

$$\dot x = y^2 + x^2 - 1, \quad \dot y = 2xy + u\gamma y^2, \qquad u \in [-1, 1]. \tag{7}$$

Here the first equation comes from the definition of y, and the second equation is equivalent to (6). The variable u is a scalar control. The objective is to maximize

$$l^R(t_1, t_2) = \int_{t_1}^{t_2} y(t)\, dt \to \sup. \tag{8}$$

In addition we have the state constraints

$$\mu^{-1}(1 + |x|) \le y \le \mu(1 - |x|) \tag{9}$$

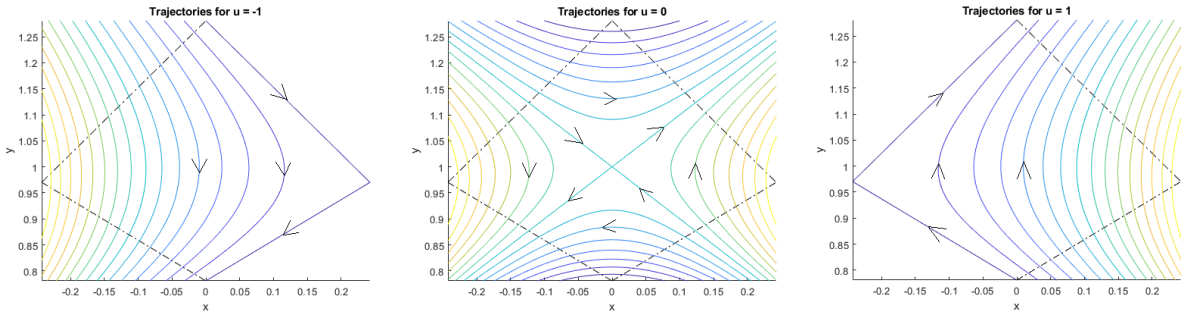

Figure 2: Trajectories of system (7) for different constant values of u. The parameter γ equals 0.5. The dash-dotted lines delimit the feasible region given by state constraints (9).

from Lemmas 3.1, 3.2.

It is easily checked that for a constant control u the trajectories of system (7) are given by the level curves of the first integral

$$I_u = \frac{(\mu_u^2+1)y}{\mu_u(y^2-x^2+1)-(\mu_u^2-1)xy},$$

where $\mu_u = \frac{u\gamma}{2} + \sqrt{1+\frac{u^2\gamma^2}{4}}$.

Theorem 1.2 will be proven by means of Lemmas 3.5, 3.6 below. The proofs of the lemmas give no clue how the assertions of these lemmas have been obtained, however. Before we state the lemmas, we shall therefore sketch how one can arrive at these statements by optimal control techniques.

Let $K \subset \mathbb{R}^2$ be the feasible region in the (x,y)-plane which is delimited by state constraints (9). The boundary segments of K are trajectories of the system with constant control $u = \pm 1$ (see Fig. 2). Moreover, no feasible trajectory of the system can leave the upper right and the lower left boundary segment after hitting it. Likewise, no feasible trajectory can leave the upper left and the lower right boundary segment after hitting it in backward time. If a trajectory of (7) leaves K, then it cannot return to K anymore. Therefore, if the initial point $(x(t_1), y(t_1))$ and the terminal point $(x(t_2), y(t_2))$ of a trajectory of (7) satisfy (9), then all intermediate points do so too. Thus we do not need to take the state constraints into account when considering the first order optimality conditions for a trajectory with fixed initial and terminal point.

Let us apply the Pontryagin Maximum Principle to control problem (7),(8) with fixed initial and terminal point [7]. According to this principle, if $(x(t), y(t))$, $t \in [t_1, t_2]$, is a maximizer of (8) under the additional constraints $(x(t_i), y(t_i)) = (x_i, y_i)$, $i = 1, 2$, then there exist a nonnegative constant λ and differentiable functions $\psi(t), \phi(t)$ (the so-called adjoint variables), not all equal to zero, such that at every $t \in [t_1, t_2]$ the control u maximizes the *Pontryagin function*

$$\mathcal{H} = 2\lambda y + \psi \cdot (y^2 + x^2 - 1) + \phi \cdot (2xy + u\gamma y^2),$$

and the adjoint variables are solutions of the differential equations

$$\dot{\psi} = -\frac{\partial \mathcal{H}}{\partial x} = -2(x\psi + y\phi), \quad \dot{\phi} = -\frac{\partial \mathcal{H}}{\partial y} = -2(\lambda + y\psi + x\phi + u\gamma y\phi). \tag{10}$$

It follows that the control u is given by the sign of ϕ whenever this variable does not vanish.

If the partial derivative $\frac{\partial \mathcal{H}}{\partial u}$ vanishes identically on some interval, then the corresponding trajectory is called a *singular arc*. Let us compute the corresponding control u. If $\phi \equiv 0$, then also $\dot{\phi} \equiv 0$, which entails $\lambda + y\psi \equiv 0$. If $\lambda = 0$, then also $\psi = 0$, and by (10) the adjoint variables vanish identically. Thus the presence of a singular arc entails $\lambda > 0$. Differentiating further, we obtain $\dot{y}\psi + y\dot{\psi} = u\gamma y^2\psi = -\lambda u\gamma y \equiv 0$, entailing $u \equiv 0$.

Thus on optimal trajectories the control u is piece-wise constant and taking values in $\{-1, 0, +1\}$. If some trajectory is optimal in the larger class of trajectories with free end-points, then the necessary optimality conditions obtained for trajectories with fixed end-points still apply. After some calculations one obtains the following solution.

Lemma 3.5. *Consider control problem (7) — (9) with fixed initial point $(x(t_1), y(t_1)) = (x_1, y_1)$ and free terminal point $(x(t_2), y(t_2))$. Let $T = t_2 - t_1$ be the time horizon. Then the optimal value of the problem is given by $B(x_1, y_1, -T)$, where the function $B : K \times \mathbb{R}_- \to \mathbb{R}_+$ is defined as follows.*

Let $(x, y) \in K$ and $t \in \mathbb{R}_-$. If

$$-t \le t_{+1}(x,y) = \begin{cases} \frac{1}{2}\log\frac{\mu(y^2+1-x^2)-(\mu^2-1)xy+(\mu^2-3)y}{(\mu y-(1-x))(\mu(1-x)+y)}, & \mu y > 1-x, \\ +\infty, & \mu y = 1-x, \end{cases}$$

then

$$B(x,y,t) = \frac{\mu}{\mu^2+1}\log\frac{(y+\mu(1-x))e^{-2t} - y + \mu(1+x)}{\mu(-(\mu y-(1-x))e^{-2t}+1+x+\mu y)}.$$

If

$$t_{+1}(x,y) < -t \le \hat{t}(x,y) = \begin{cases} \frac{1}{2}\log\frac{\mu y+1+x}{\mu y-1+x}, & \mu y > 1-x, \\ +\infty, & \mu y = 1-x, \end{cases}$$

then

$$B(x,y,t) = \frac{\mu}{\mu^2+1}\log\left(\frac{(y+\mu-\mu x)(\mu^4 xy - \mu^4 y + \mu^3 x^2 - \mu^3 y^2 - \mu^3 + 6\mu^2 y + \mu x^2 - \mu y^2 - \mu - xy - y)}{8\mu y(\mu y - 1 + x)} + e^{-2t}\frac{(\mu^2+1)(y+\mu-\mu x)^2}{8\mu y}\right).$$

If

$$\hat{t}(x,y) < -t \le t^*(x,y) = \begin{cases} \frac{1}{2}\log\frac{1-x^2+y^2}{(y+1-x)(x+y-1)}, & x+y > 1, \\ +\infty, & x+y \le 1, \end{cases}$$

then

$$B(x,y,t) = \frac{1}{2}\log\frac{W + (x+\mu y-1)(\mu+y-\mu x)e^{-2t} + \mu^2 xy + \mu x^2 - \mu y^2 - \mu - xy}{(\mu+1)^2 y} + \\ + \frac{\mu}{\mu^2+1}\log\frac{2\mu(W\mu(1+\mu^2) + 2\mu^2((x+\mu y-1)(\mu x - y - \mu)e^{-2t} - \mu^2 xy - \mu x^2 + \mu y^2 + \mu + xy))}{(\mu^2-1)^2(x+\mu y-1)((x+\mu y-1)e^{-2t} - x - \mu y - 1)}$$

with

$$W = \sqrt{[(y-\mu x+\mu)(x+\mu y-1)e^{-2t} + \mu^2 xy + \mu x^2 - \mu y^2 - \mu - xy]^2 - [\mu^2 y - y]^2}.$$

If $t^*(x,y) < -t$, then

$$B(x,y,t) = \frac{1}{2}\log\frac{W + (-y+\mu-\mu x)(\mu y - x + 1)e^{-2t} + \mu^2 xy - \mu x^2 + \mu y^2 + \mu - xy}{(\mu+1)^2 y} + \\ + \frac{\mu}{\mu^2+1}\log\frac{2\mu^2((\mu^2+1)W + 2\mu(-(-y+\mu-\mu x)(\mu y - x + 1)e^{-2t} - \mu^2 xy + \mu x^2 - \mu y^2 - \mu + xy))}{(\mu^2-1)^2((y-\mu+\mu x)e^{-2t} - \mu - y - \mu x)(y - \mu + \mu x)}$$

with

$$W = \sqrt{[(-y+\mu-\mu x)(\mu y - x + 1)e^{-2t} + \mu^2 xy - \mu x^2 + \mu y^2 + \mu - xy]^2 - [\mu^2 y - y]^2}.$$

Proof. The proof involves many calculations, which cannot all be included. We shall give only intermediate results which serve as certificates of the claimed inequalities.

Let us show that $B(x,y,t)$ is the Bellman function of the problem. Denote the four expressions defining B in the lemma by $B_I, B_{II}, B_{III}, B_{IV}$. Denote the expressions for W in the lemma by W_{III}, W_{IV}. For brevity we define the quantities $a = \mu y - (1-x)$, $b = \mu y - (1+x)$, $c = \mu(1-x) - y$, $d = \mu(1+x) - y$. These quantities are nonnegative by (9). Recall also that $y > 0$, $1 \pm x > 0$, $\mu > 1$.

Consistency: For fixed $(x,y) \in K$ the values $0, t_{+1}(x,y), \hat{t}(x,y), t^*(x,y)$ form an increasing sequence. Indeed, straightforward calculation yields

$$t_{+1}(x,y) = \frac{1}{2}\log\left(1 + \frac{2c}{a(y+\mu(1-x))}\right) \ge 0,$$

$$\hat{t}(x,y) - t_{+1}(x,y) = -\frac{1}{2}\log\left(1 - \frac{4y}{(\mu y + 1 + x)(\mu(1-x)+y)}\right) > 0,$$

and for $x + y > 1$ we get

$$t^*(x,y) - \hat{t}(x,y) = \frac{1}{2}\log\left(1 + \frac{2yc}{(\mu y + 1 + x)(y + 1 - x)(x + y - 1)}\right) \geq 0.$$

According to the decomposition $\alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$ the expression W_{III} is the square root of the product of two functions which are both linear in e^{-2t} with nonnegative leading coefficient $(y + \mu(1-x))a$. Inserting $t = -\hat{t}$ into these linear factors, one easily calculates that these evaluate to $2y$ and $2\mu^2 y$, respectively. For general $-t \geq \hat{t}$ these factors are hence nonnegative, and W_{III} is well-defined. Likewise, the linear factors involved in the definition of W_{IV} have leading coefficient $(\mu y + 1 - x)c \geq 0$. Inserting $t = -t^*$ yields the positive values $\frac{2y(1-x)^2(\mu^2-1)}{(y+1-x)(x+y-1)}$, $\frac{2y^3(\mu^2-1)}{(y+1-x)(x+y-1)}$. Therefore W_{IV} is well-defined for $-t \geq t^*$.

Continuity: Inserting $t = -t_{+1}(x,y)$ into B_I and B_{II}, we obtain the same expression

$$\frac{\mu}{\mu^2+1}\log\frac{(\mu^2-1)(y+\mu(1-x))}{2\mu a}.$$

Inserting $t = -\hat{t}(x,y)$ into B_{II} and B_{III}, we obtain the same expression

$$\frac{\mu}{\mu^2+1}\log\frac{\mu(y+\mu(1-x))}{a}.$$

Inserting $t = -t^*(x,y)$ into B_{III} and B_{IV}, we obtain for $x + y > 1$ the same expression

$$\frac{1}{2}\log\frac{(\mu-1)(y-x+1)}{(\mu+1)(y+x-1)} + \frac{\mu}{\mu^2+1}\log\frac{2\mu^2}{\mu^2-1}.$$

Hence B is continuous.

Initial value: Inserting $t = 0$ into B_I, we obtain $B(x,y,0) \equiv 0$.

Bellman inequality: Let us show that on every trajectory of (7) we have

$$\frac{d}{dt}B(x,y,t) + y = \frac{\partial B}{\partial x}\cdot\dot{x} + \frac{\partial B}{\partial y}\cdot\dot{y} + \frac{\partial B}{\partial t} + y \leq 0, \tag{11}$$

with equality if $u = \hat{u}(x,y,t)$, where

$$\hat{u}(x,y,t) = \begin{cases} +1, & -t < t^*(x,y) \text{ and } c > 0, \\ 0, & -t = t^*(x,y) \text{ and } c > 0, \\ -1, & -t > t^*(x,y) \text{ or } c = 0. \end{cases}$$

For $0 \leq -t < t_{+1}$ we have

$$\frac{dB_I(x,y,t)}{dt} + y = -\frac{y^2(\mu^2-1)(e^{-2t}-1)(1-u)(e^{-2t}(1-x)+1+x)(\mu(1-x)+y)}{(a(e^{2t_{+1}}-e^{-2t})(\mu(1-x)+y)+4y)((\mu(1-x)+y)e^{-2t}+d)} \leq 0,$$

with equality if $u = 1$. Note that if $c = 0$, then $t_{+1} = 0$, and hence the inequality $-t < t_{+1}$ cannot hold.

For $t_{+1} \leq -t \leq \hat{t}$ we have

$$\frac{dB_{II}(x,y,t)}{dt} + y = -\frac{y(\mu^2-1)(1-u)c((\mu(1-x)+y)a^2(e^{2\hat{t}}-e^{-2t})+2\mu yc)}{a(\mu(1-x)+y)((\mu^2+1)a(\mu(1-x)+y)(e^{-2t}-e^{2t_{+1}})+4y(\mu^2-1))} \leq 0,$$

with equality if $u = 1$ or $c = 0$.

For $\hat{t} < -t \leq t^*$ we have

$$\frac{dB_{III}(x,y,t)}{dt} + y = -\frac{y(\mu^2-1)(1-u)A}{2a(e^{-2t}-e^{2\hat{t}})(y+\mu(1-x))((y+\mu(1-x))a(e^{-2t}-e^{2\hat{t}})+2y(\mu^2+1))},$$

where $A = ((y^2 + (1-x)^2)(e^{-2t} - e^{2\hat{t}}) + \frac{2y(y+\mu(1-x))}{a})W_{III} + A'$ and

$$\left((y^2 + (1-x)^2)(e^{-2t} - e^{2\hat{t}}) + \frac{2y(y+\mu(1-x))}{a}\right)^2 W_{III}^2 - (A')^2 =$$
$$= ((x-y-1)(x+y-1)e^{-2t} + y^2 + 1 - x^2)^2 (y+\mu(1-x))a((y+\mu(1-x))e^{-2t} + d)a(e^{-2t} - e^{2\hat{t}}) \geq 0.$$

Since the coefficient at W_{III} in A is nonnegative, we obtain $A \geq 0$ and hence $\frac{dB_{III}(x,y,t)}{dt} + y \leq 0$, with equality if $u = 1$. Note that if $c = 0$, then $\hat{t} = t^*$, and the condition $\hat{t} < -t \leq t^*$ cannot hold.

For $-t > t^*$ we have

$$\frac{dB_{IV}(x,y,t)}{dt} + y = -\frac{y(\mu^2-1)(1+u)aA}{2c(ce^{-2t} + y + \mu(1+x))(\mu y + 1 - x)((\mu y + 1 - x)a(e^{-2t} - e^{2\hat{t}}) + 4\mu y)},$$

where $A = ((y^2 + (1-x)^2)(e^{-2t} - e^{2t^*}) + \frac{4y^2(1-x)}{(x+y-1)(y+1-x)})W_{IV} + A'$ and

$$\left((y^2 + (1-x)^2)(e^{-2t} - e^{2t^*}) + \frac{4y^2(1-x)}{(x+y-1)(y+1-x)}\right)^2 W_{IV}^2 - (A')^2 =$$
$$= (x-y-1)^2(x+y-1)^2(e^{-2t} - e^{2t^*})^2 c(\mu y + 1 - x)(ce^{-2t} + y + \mu(1+x))((\mu y + 1 - x)(e^{-2t} - 1) + 2)$$
$$\geq 0.$$

Since the coefficient at W_{IV} in A is nonnegative, we obtain $A \geq 0$ and hence $\frac{dB_{IV}(x,y,t)}{dt} + y \leq 0$, with equality if $u = -1$.

Feasibility: Let us show that application of the optimal control $\hat{u}$ guarantees that the trajectory does not leave the feasible region K. The only boundary segments through which a trajectory can escape K are the upper right and the lower left one. On the upper right segment we have $c = 0$ and hence the optimal control is $\hat{u} = -1$. The trajectory then moves along the boundary segment. On the lower left segment we have $t^* = +\infty$ and $c > 0$. Therefore the optimal control is $\hat{u} = +1$ and the trajectory again moves along the boundary segment.

Let now $(x_1, y_1) \in K$ and $t_1 < t_2$. Let further $(x(t), y(t))$ be a trajectory of (7) satisfying (9) for $t \in [t_1, t_2]$ and with initial conditions $(x(t_1), y(t_1)) = (x_1, y_1)$. Let $u(t)$ be the control on this trajectory, and set $T = t_2 - t_1$. Then we have

$$\int_{t_1}^{t_2} y(t)\,dt \leq -\int_{t_1}^{t_2} \left(\frac{d}{dt}B(x(t), y(t), t - t_2)\right) dt = B(x_1, y_1, t_1 - t_2) - B(x(t_2), y(t_2), 0) = B(x_1, y_1, -T).$$

Here the first inequality holds by virtue of (11), and the last equality by virtue of the initial condition $B(x, y, 0) = 0$.

If the control is chosen such that $u(t) = \hat{u}(x(t), y(t), t - t_2)$, then the inequality is actually an equality. On the corresponding trajectory the value of the cost function (8) then equals the upper bound $B(x_1, y_1, -T)$. Thus $B(x_1, y_1, -T)$ is indeed the optimal value of the problem. □

The optimal solutions obtained by application of control $\hat{u}$ are structured as follows.

If $T \leq t_{+1}(x_1, y_1)$, where t_{+1} is the time needed to reach the upper right boundary segment of the feasible set with control $u \equiv +1$, then the optimal control is given by $\hat{u} \equiv +1$ on the whole trajectory (see Fig. 3, left-most).

If $t_{+1}(x_1, y_1) < T \leq \hat{t}(x_1, y_1)$, then on the optimal trajectory the control $u \equiv +1$ is optimal for $t \in [t_1, t_1 + t_{+1}(x_1, y_1))$, that is up to the point when the trajectory reaches the boundary of the feasible set. For $t \in (t_1 + t_{+1}(x_1, y_1), t_2]$ the control $u \equiv -1$ is optimal and the trajectory moves along the boundary of the feasible set (see Fig. 3, left-center).

If $\hat{t}(x_1, y_1) < T \leq t^*(x_1, y_1)$, then between the arcs with controls $u = \pm 1$ there appears a singular arc with control $u \equiv 0$ (see Fig. 3, right-center).

Finally, for $T > t^*(x_1, y_1)$ the optimal trajectory consists of three arcs, on which the control equals $-1, 0, -1$, respectively, with the third arc located on the upper right boundary segment (see Fig. 3, right-most).

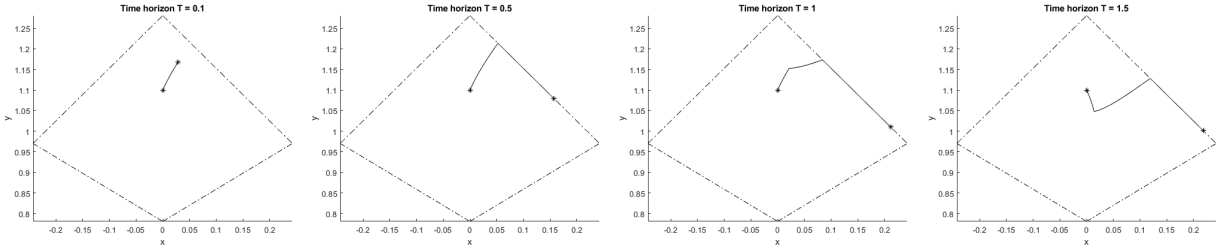

Figure 3: Optimal trajectories of system (7) with initial point $(x, y) = (0, 1.1)$ for different time horizons $T = t_2 - t_1$. The initial and terminal points are marked with stars. The parameter γ equals 0.5. The dash-dotted lines delimit the feasible region given by state constraints (9).

In order to find the optimal value of problem (7) — (9) with free initial and terminal points, we have to maximize the Bellman function $B(x, y, -T)$ over $(x, y) \in K$ for fixed T.

Lemma 3.6. *The maximum* $\max_{(x,y)\in K} B(x, y, -T)$ *is attained at*

$$(x, y) = \begin{cases} \left(-\frac{(\mu^2-1)(e^T-1)}{\mu^2(e^T-1)+e^T+1}, \frac{2\mu e^T}{\mu^2(e^T-1)+e^T+1}\right), & T \le \log\frac{\mu^2+1}{\mu^2-1}, \\ \left(-1+\frac{e^T}{\mu\sqrt{e^{2T}-1}}, \frac{e^T}{\sqrt{e^{2T}-1}}\right), & T \ge \log\frac{\mu^2+1}{\mu^2-1}. \end{cases}$$

The corresponding value of the maximum is given by

$$\max_{(x,y)\in K} B(x, y, -T) = \begin{cases} \frac{2\mu}{\mu^2+1}\log\frac{(\mu^2+1)(e^T-1)+2}{2}, & T \le \log\frac{\mu^2+1}{\mu^2-1}, \\ \log\frac{(\mu-1)(\sqrt{e^T+1}+\sqrt{e^T-1})}{(\mu+1)(\sqrt{e^T+1}-\sqrt{e^T-1})} + \frac{2\mu}{\mu^2+1}\log\frac{2\mu^2}{\mu^2-1}, & T \ge \log\frac{\mu^2+1}{\mu^2-1}. \end{cases}$$

Proof. We shall parameterize K by the variables $w = \frac{y}{1-x}$, $z = \frac{y}{1+x}$, which both run through the interval $[\mu^{-1}, \mu]$. Set $t = -T$. We shall determine the maximum of $B(x, y, t)$ by examining the signs of the derivatives with respect to these variables.

Let us show that $\frac{\partial B}{\partial z} = -\frac{(1-x)(1+x)^2}{2y}\frac{\partial B}{\partial x} + \frac{(1+x)^2}{2}\frac{\partial B}{\partial y} \ge 0$. For $-t \le t_{+1}(x, y)$ we have

$$\frac{\partial B_I}{\partial z} = \frac{\mu(x+1)^2(e^{-2t}-1)}{a(e^{2\hat{t}}-e^{-2t})((y+\mu(1-x))e^{-2t}+d)} \ge 0.$$

For $t_{+1}(x, y) \le -t \le \hat{t}(x, y)$ we have

$$\frac{\partial B_{II}}{\partial z} = \frac{\mu(e^{-2t}-1)(x+1)^2 a(y+\mu(1-x))}{2y((\mu^2+1)a(y+\mu(1-x))(e^{-2t}-e^{2t_{+1}})+4y(\mu^2-1))} \ge 0.$$

For $\hat{t}(x, y) \le -t \le t^*(x, y)$ we have

$$\frac{\partial B_{III}}{\partial z} = -\frac{(x+1)^2(e^{-2t}-1)(W_{III}-2\mu y)}{4ya(e^{2\hat{t}}-e^{-2t})((y+\mu(1-x))e^{-2t}+d)} \ge 0.$$

For $-t \ge t^*(x, y)$ we have

$$\frac{\partial B_{IV}}{\partial z} = \frac{(x+1)^2(e^{-2t}-1)a(W_{IV}+2\mu y)}{4y(ce^{-2t}+y+\mu(1+x))((\mu y+1-x)a(e^{-2t}-e^{2\hat{t}})+4\mu y)} \ge 0.$$

Hence the maximum of B is achieved at $z = \mu$. This corresponds to the upper left boundary segment of K.

We now compute the derivative $\frac{\partial B}{\partial w} = \frac{(1-x)^2(1+x)}{2y}\frac{\partial B}{\partial x} + \frac{(1-x)^2}{2}\frac{\partial B}{\partial y}$ on this segment. Note that on this segment $x \in [-\frac{\mu^2-1}{\mu^2+1}, 0]$, and $e^{2t_{+1}} = \frac{(\mu^2-1)(1+x)}{(\mu^2+1)x+\mu^2-1}$, $e^{2\hat{t}} = \frac{(\mu^2+1)(1+x)}{(\mu^2+1)x+\mu^2-1}$, and $e^{2t^*} = \frac{\mu^2(1+x)^2+1-x^2}{((\mu+1)x+\mu-1)(\mu(1+x)+1-x)}$ for $x > -\frac{\mu-1}{\mu+1}$.

For $-t \le t_{+1}(x,y)$ we have

$$\frac{\partial B_I}{\partial w} = \frac{\mu e^{-2t}(e^{-2t}-1)(e^{2t_{+1}}-1)(1-x)^2}{2((y+\mu(1-x))e^{-2t}+d)((1+x)(e^{-2t}-1)+e^{2t_{+1}}-e^{-2t})} \ge 0.$$

For $t_{+1}(x,y) \le -t \le \hat{t}(x,y)$ we have

$$\frac{\partial B_{II}}{\partial w} = -\frac{\mu^2(1-x)^2(e^{2\hat{t}}-e^{2t_{+1}})(((\mu^2+1)x+\mu^2-1)e^{-t}+(\mu^2-1)(1+x))}{2y(1+x)a(y+\mu(1-x))(2\mu^2(e^{-2t}-e^{2t_{+1}})+(\mu^2-1)(e^{2\hat{t}}-e^{-2t}))}\times$$
$$\times(((\mu^2+1)(e^{-t}-1)+2)x+(\mu^2-1)(e^{-t}-1)).$$

Hence $\frac{\partial B_{II}}{\partial w} \ge 0$ if $x \le -\frac{(\mu^2-1)(e^{-t}-1)}{(\mu^2+1)(e^{-t}-1)+2}$ and $\frac{\partial B_{II}}{\partial w} \le 0$ if $x \ge -\frac{(\mu^2-1)(e^{-t}-1)}{(\mu^2+1)(e^{-t}-1)+2}$.

For $\hat{t}(x,y) \le -t \le t^*(x,y)$ we have

$$\frac{\partial B_{III}}{\partial w} = \frac{2\mu^2(1-x)^2e^{-2t}\beta^2(e^{-t}+\mu(1+x)\sqrt{e^{-2t}-1})(e^{-t}-\mu(1+x)\sqrt{e^{-2t}-1})}{ya^2(y+\mu(1-x))((y+\mu(1-x))e^{-2t}+d)}\times$$

$$\times\frac{1}{\sqrt{(\beta e^{-2t}-x-1)(\beta e^{-2t}-\mu^2(1+x))}+\mu(1+x)(\beta(e^{-2t}-e^{2\hat{t}})+1)},$$

where we wrote β for $(\mu^2+1)x+\mu^2-1$ for brevity. Hence $\frac{\partial B_{III}}{\partial w} \ge 0$ for $x \le \frac{e^{-t}}{\mu\sqrt{e^{-2t}-1}}-1$ and $\frac{\partial B_{III}}{\partial w} \le 0$ for $x \ge \frac{e^{-t}}{\mu\sqrt{e^{-2t}-1}}-1$.

For $-t \ge t^*(x,y)$ we have

$$\frac{\partial B_{IV}}{\partial w} = -\frac{\mu^2(1-x)^2A}{yc(ce^{-2t}+y+\mu(1+x))(\mu y+1-x)((\mu y+1-x)(e^{-2t}-1)+2)},$$

where

$$\begin{aligned}A =&(((\mu^2-1)(x+1)^2+2)(e^{-2t}-1)+2)\times\\ &\times\sqrt{(((\mu^2+1)x+\mu^2-1)e^{-2t}-1-x)(((\mu^2+1)x+\mu^2-1)e^{-2t}-\mu^2(1+x))}\\ &+\mu(x+1)(2x((\mu^2-1)x+\mu^2+1)e^{-4t}-(3(\mu^2-1)x^2+\mu^2(4x+1)+1)e^{-2t}+(\mu^2-1)(x+1)^2).\end{aligned}$$

Let us show that A is nonnegative. First we consider the second summand. This is a concave quadratic polynomial in e^{-2t}. If we replace e^{-2t} by 0, we obtain the positive value $\mu(\mu^2-1)(x+1)^3$. If we replace e^{-2t} by 1, we obtain the negative value $-2\mu(x+1)$. Therefore for $e^{-2t} \ge 1$ the value of the second term is negative. Since the first summand of A is positive, the difference of the two summands will also be positive. Multiplying A by the difference of the two terms we get rid of the square root, and the resulting expression equals

$$-x\beta(1+x-e^{-2t}x)(\beta(e^{-2t}-1)+2)((\mu^2-1)^2(e^{-2t}-1)^2(x+1)^4+4(x+1)^2e^{-2t}-4xe^{-4t}(2+x)),$$

which consists of nonnegative factors. Here we denoted $(\mu^2-1)x+\mu^2+1$ by β. Thus $A \ge 0$ and hence $\frac{\partial B_{IV}}{\partial w} \le 0$.

It follows that for every fixed $t<0$ the function B is unimodal on the upper left boundary segment of K. The maximum is attained at $x = -\frac{(\mu^2-1)(e^{-t}-1)}{(\mu^2+1)(e^{-t}-1)+2}$ if this value of x satisfies $t_{+1}(x,\mu(1+x)) \le -t \le \hat{t}(x,\mu(1+x))$, and by $x = \frac{e^{-t}}{\mu\sqrt{e^{-2t}-1}}-1$ if this value of x satisfies $\hat{t}(x,\mu(1+x)) \le -t \le t^*(x,\mu(1+x))$. Straightforward calculation yields the maximizer claimed in the lemma.

The value of the maximum is obtained by evaluating the expression B_{II} in the first case and the expression B_{III} in the second case. Again straightforward calculation yields the value claimed in the lemma. □

The optimal trajectory realizing the maximal value in Lemma 3.6 is depicted in Fig. 4. For $t_2-t_1 \le \log\frac{\mu^2+1}{\mu^2-1}$ it consists of two arcs with control $u=\pm1$, respectively, and lies entirely on the boundary of the feasible set K. For $t_2-t_1 > \log\frac{\mu^2+1}{\mu^2-1}$ the optimal trajectory consists of three arcs with controls $+1, 0, -1$,

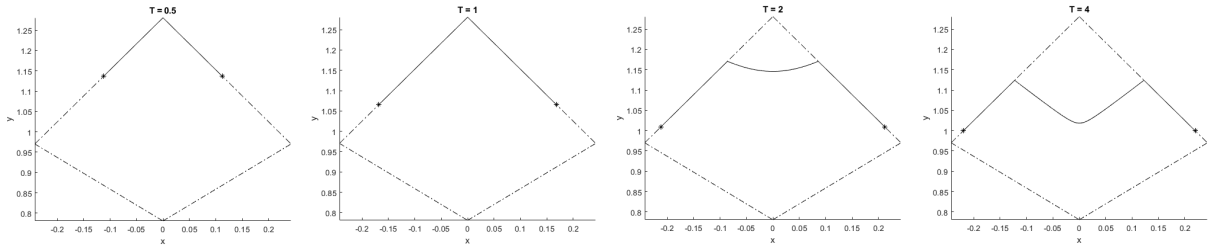

Figure 4: Optimal trajectories of system (7) with free end-points for different time horizons $T = t_2 - t_1$. The optimal initial and terminal points are marked with stars. The parameter γ equals 0.5. The dash-dotted lines delimit the feasible region given by state constraints (9).

respectively. The first and third arc lie on the boundary of K, while the central arc is singular and crosses the interior of K. The whole trajectory is symmetric about the vertical axis.

The solutions can be extended from the time interval $[t_1, t_2]$ to $\mathbb{R}$ by applying control $u = 1$ for all $t < t_1$ and control -1 for all $t > t_2$. The corresponding trajectory then tends to the left-most point $(x, y) = (-\frac{\mu^2-1}{\mu^2+1}, \frac{2\mu}{\mu^2+1})$ of K for $t \to -\infty$ and to the right-most point $(x, y) = (\frac{\mu^2-1}{\mu^2+1}, \frac{2\mu}{\mu^2+1})$ of K for $t \to +\infty$. The corresponding centro-affine immersion into $\mathbb{R}^2_+$ is of class C^2 and piece-wise analytic, but can be approximated with arbitrary precision by C^3 immersions satisfying (2). Hence the bound in Theorem 1.2 cannot be improved.

The first two inequalities in Theorem 1.2 are proven in a similar manner as for Theorem 1.1. Let us prove the last inequality. First we show that the difference between the maximal value in Lemma 3.6 and the quantity T is an increasing function of T. Indeed, the derivative of the difference with respect to T is given by

$$\frac{d}{dT}\left(\max_{(x,y)\in K} B(x, y, -T) - T\right) = \begin{cases} \frac{(\mu-1)(\mu+1-(\mu-1)e^T)}{\mu^2(e^T-1)+e^T+1}, & e^T \le \frac{\mu^2+1}{\mu^2-1}, \\ \frac{1-\sqrt{1-e^{-2T}}}{\sqrt{1-e^{-2T}}}, & e^T \ge \frac{\mu^2+1}{\mu^2-1}. \end{cases}$$

But $\mu+1-(\mu-1)e^T \le \mu+1-(\mu-1)\frac{\mu^2+1}{\mu^2-1} = \frac{2\mu}{\mu^2+1}$, and $1 > \sqrt{1-e^{-2T}}$. Hence the supremum of the difference equals its limit as $T \to +\infty$. It is not hard to prove that this limit equals $\log 2 + \log\frac{\mu-1}{\mu+1} + \frac{2\mu}{\mu^2+1}\log\frac{2\mu^2}{\mu^2-1}$. This completes the proof of Theorem 1.2.

4 Proof of Theorems 1.3 and 1.4

In order to obtain a lower bound on the Riemannian length $l^R(a, b)$ we have to consider the optimal control problem

$$\begin{aligned} \dot{x} = y^2 + x^2 - 1, \quad \dot{y} = 2xy + u\gamma y^2, \qquad & u \in [-1, 1], \\ l^R(t_1, t_2) = \int_{t_1}^{t_2} y(t)\, dt \to \inf, & \\ \mu^{-1}(1 + |x|) \le y \le \mu(1 - |x|), & \end{aligned} \tag{12}$$

which is similar to (7) — (9) with the difference that we now minimize (8). The proof is conducted along the same lines as that of Theorem 1.2, but the calculations turn out to be simpler because the optimal trajectories do not contain the singular arc and the optimal control is purely bang-bang. Assume the notations of the previous section.

Lemma 4.1. *Consider control problem* (12) *with fixed initial point* $(x(t_1), y(t_1)) = (x_1, y_1)$ *and free terminal point* $(x(t_2), y(t_2))$. *Let* $T = t_2 - t_1$ *be the time horizon. Then the optimal value of the problem is given by* $B(x_1, y_1, -T)$, *where the function* $B : K \times \mathbb{R}_- \to \mathbb{R}_+$ *is defined as follows.*

Let $(x, y) \in K$ *and* $t \in \mathbb{R}_-$. *If*

$$-t \le t_{-1}(x, y) = \begin{cases} \frac{1}{2}\log\frac{\mu(x^2-y^2-1)-(\mu^2-1)xy+(3\mu^2-1)y}{(\mu(1-x)-y)(\mu y+1-x)}, & y < \mu(1-x), \\ +\infty, & y = \mu(1-x), \end{cases}$$

then

$$B(x,y,t)=\frac{\mu}{\mu^2+1}\log\frac{\mu((\mu y+1-x)e^{-2t}+1+x-\mu y)}{(\mu(1-x)-y)e^{-2t}+\mu(1+x)+y}.$$

If $-t>t_{-1}(x,y)$, *then*

$$B(x,y,t)=\frac{\mu}{\mu^2+1}\log\left(\frac{\mu y+1-x}{8\mu^3 y(\mu(1-x)-y)}\times\right.$$
$$\left.\times\left[(\mu^2+1)(\mu(1-x)-y)(\mu y+1-x)e^{-2t}+\mu^4 y(1+x)+\mu(1+\mu^2)(y^2-x^2+1)-6\mu^2 y+y(1-x)\right]\right).$$

Proof. Let us show that $B(x,y,t)$ is the Bellman function of the problem. Denote the two expressions defining B in the lemma by B_I, B_{II}. Recall that we noted $a=\mu y-(1-x)$, $b=\mu y-(1+x)$, $c=\mu(1-x)-y$, $d=\mu(1+x)-y$.

Consistency: For $(x,y)\in K$ we have $t_{-1}(x,y)\geq 0$. Indeed, straightforward calculation yields

$$t_{-1}(x,y)=\frac{1}{2}\log\left(1+\frac{2\mu a}{c(\mu y+1-x)}\right)\geq 0.$$

Continuity: Inserting $t=-t_{-1}(x,y)$ into B_I and B_{II}, we obtain the same expression

$$\frac{\mu}{\mu^2+1}\log\frac{(\mu^2-1)(\mu y+1-x)}{2\mu c}.$$

Initial value: Inserting $t=0$ into B_I, we obtain $B(x,y,0)\equiv 0$.

Bellman inequality: Let us show that on every trajectory of (12) we have

$$\frac{d}{dt}B(x,y,t)+y=\frac{\partial B}{\partial x}\cdot\dot{x}+\frac{\partial B}{\partial y}\cdot\dot{y}+\frac{\partial B}{\partial t}+y\geq 0,$$

with equality if $u=\hat{u}(x,y,t)$, where

$$\hat{u}(x,y,t)=\begin{cases}-1, & a>0,\\ +1, & a=0.\end{cases}$$

For $0\leq -t<t_{-1}$ we have

$$\frac{dB_I(x,y,t)}{dt}+y=\frac{y^2(\mu^2-1)(e^{-2t}-1)(u+1)((1-x)e^{-2t}+1+x)}{(ce^{-2t}+\mu(1+x)+y)((\mu y+1-x)(e^{-2t}-1)+2)}\geq 0,$$

with equality if $u=-1$. Note that if $a=0$, then $t_{-1}=0$, and hence the inequality $-t<t_{-1}$ cannot hold.

For $-t\geq t_{-1}$ we have

$$\frac{dB_{II}(x,y,t)}{dt}+y=\frac{y(\mu^2-1)(u+1)a((\mu y+1-x)c^2(e^{-2t}-e^{2t_{-1}})+2\mu y((\mu^2-1)(1-x)+\mu c))}{c(\mu y+1-x)((\mu^2+1)c(\mu y+1-x)(e^{-2t}-e^{2t_{-1}})+4\mu^2 y(\mu^2-1))}\geq 0,$$

with equality if $u=-1$ or $a=0$.

Feasibility: The only boundary segments through which a trajectory can escape K are the upper right and the lower left one. On the lower left segment we have $a=0$ and hence the optimal control is $\hat{u}=+1$. The trajectory then moves along the boundary segment. On the upper right segment we have $t_{-1}=+\infty$ and $a>0$. Therefore the optimal control is $\hat{u}=-1$ and the trajectory again moves along the boundary segment.

Let now $(x_1,y_1)\in K$ and $t_1<t_2$. Let further $(x(t),y(t))$ be a trajectory of (12) satisfying the state constraints for $t\in[t_1,t_2]$ and with initial conditions $(x(t_1),y(t_1))=(x_1,y_1)$. Let $u(t)$ be the control on this trajectory, and set $T=t_2-t_1$. Then we have

$$\int_{t_1}^{t_2}y(t)\,dt\geq-\int_{t_1}^{t_2}\left(\frac{d}{dt}B(x(t),y(t),t-t_2)\right)dt=B(x_1,y_1,t_1-t_2)-B(x(t_2),y(t_2),0)=B(x_1,y_1,-T).$$

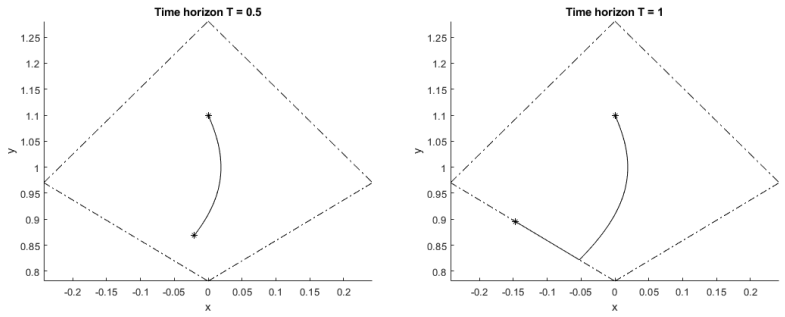

Figure 5: Optimal trajectories of system (12) with initial point $(x, y) = (0, 1.1)$ for different time horizons $T = t_2 - t_1$. The initial and terminal points are marked with stars. The parameter γ equals 0.5. The dash-dotted lines delimit the feasible region given by state constraints in (12).

If the control is chosen such that $u(t) = \hat{u}(x(t), y(t), t-t_2)$, then the inequality is actually an equality. On the corresponding trajectory the value of the cost function in (12) then equals the lower bound $B(x_1, y_1, -T)$. Thus $B(x_1, y_1, -T)$ is indeed the optimal value of the problem. □

The optimal solutions obtained by application of control $\hat{u}$ are structured as follows.

If $T \le t_{-1}(x_1, y_1)$, where t_{-1} is the time needed to reach the lower left boundary segment of the feasible set with control $u \equiv -1$, then the optimal control is given by $\hat{u} \equiv -1$ on the whole trajectory (see Fig. 5, left).

If $T > t_{-1}(x_1, y_1)$, then on the optimal trajectory the control $u \equiv -1$ is optimal for $t \in [t_1, t_1 + t_{-1}(x_1, y_1))$, that is up to the point when the trajectory reaches the boundary of the feasible set. For $t \in (t_1 + t_{-1}(x_1, y_1), t_2]$ the control $u \equiv +1$ is optimal and the trajectory moves along the boundary of the feasible set (see Fig. 5, right).

In order to find the optimal value of problem (12) with free initial and terminal points, we have to minimize the Bellman function $B(x, y, -T)$ over $(x, y) \in K$ for fixed T.

Lemma 4.2. *The minimum* $\min_{(x,y)\in K} B(x, y, -T)$ *is attained at*

$$(x, y) = \left(\frac{(e^T - 1)(\mu^2 - 1)}{e^T(\mu^2 + 1) + \mu^2 - 1}, \frac{2\mu e^T}{e^T(\mu^2 + 1) + \mu^2 - 1} \right).$$

The corresponding value of the maximum is given by

$$\max_{(x,y)\in K} B(x, y, -T) = \frac{2\mu}{\mu^2 + 1} \log \frac{e^T(\mu^2 + 1) + \mu^2 - 1}{2\mu^2}.$$

Proof. We shall again parameterize K by the variables $w = \frac{y}{1-x}$, $z = \frac{y}{1+x}$. Set $t = -T$.

Let us show that $\frac{\partial B}{\partial z} \ge 0$. For $-t \le t_{-1}(x, y)$ we have

$$\frac{\partial B_I}{\partial z} = \frac{\mu(x+1)^2(e^{-2t} - 1)}{(ce^{-2t} + \mu(1+x) + y)((\mu y + 1 - x)(e^{-2t} - 1) + 2)} \ge 0.$$

For $-t \ge t_{-1}(x, y)$ we have

$$\frac{\partial B_{II}}{\partial z} = \frac{\mu(e^{-2t} - 1)(x+1)^2 c(\mu y + 1 - x)}{2y((\mu^2 + 1)c(\mu y + 1 - x)(e^{-2t} - e^{2t_{-1}}) + 4\mu^2 y(\mu^2 - 1))} \ge 0.$$

Hence the minimum of B is achieved at $z = \mu^{-1}$. This corresponds to the lower right boundary segment of K.

We now compute the derivative $\frac{\partial B}{\partial w}$ on this segment. Note that on this segment $x \in [0, \frac{\mu^2-1}{\mu^2+1}]$ and $e^{2t_{-1}} = \frac{(\mu^2-1)(x+1)}{\mu^2-1-x(\mu^2+1)}$.

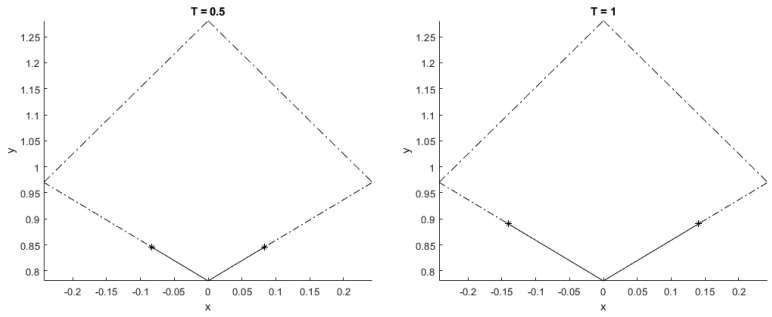

Figure 6: Optimal trajectories of system (12) with free end-points for different time horizons $T = t_2 - t_1$. The optimal initial and terminal points are marked with stars. The parameter γ equals 0.5. The dash-dotted lines delimit the feasible region given by state constraints in (12).

For $-t < t_{-1}(x, y)$ we have

$$\frac{\partial B_I}{\partial w} = \frac{\mu(e^{-2t} - 1)}{(\mu^2 - 1 - x(\mu^2 + 1))e^{-2t} + (\mu^2 + 1)(x + 1)} \geq 0.$$

At $x = 0$ we have $t_{-1} = 0$, and hence the minimum cannot be attained for $-t < t_{-1}(x, y)$.

For $-t \geq t_{-1}(x, y)$ we have

$$\frac{\partial B_{II}}{\partial w} = \frac{\mu((\mu^2 - 1)^2(x + 1)^2 - (\mu^2 - 1 - x(\mu^2 + 1))^2 e^{-2t})}{(x + 1)(\mu^2 - 1 - x(\mu^2 + 1))((\mu^2 + 1)(\mu^2 - 1 - x(\mu^2 + 1))(e^{-2t} - e^{2t_{-1}}) + 2\mu^2(\mu^2 - 1)(x + 1))}.$$

Hence $\frac{\partial B_{II}}{\partial w} \geq 0$ if $x \geq \frac{(e^{-t}-1)(\mu^2-1)}{e^{-t}(\mu^2+1)+\mu^2-1}$ and $\frac{\partial B_{II}}{\partial w} \leq 0$ if $x \leq \frac{(e^{-t}-1)(\mu^2-1)}{e^{-t}(\mu^2+1)+\mu^2-1}$.

It follows that the minimum is attained at $x = \frac{(e^{-t}-1)(\mu^2-1)}{e^{-t}(\mu^2+1)+\mu^2-1}$, $y = \mu^{-1}(1+x)$. The value of the minimum is obtained by evaluating the expression B_{II} at this point. □

The optimal trajectory realizing the minimal value in Lemma 4.2 is depicted in Fig. 6. It consists of two arcs with control $u = \mp 1$, respectively, and lies entirely on the boundary of the feasible set K. It is symmetric about the vertical axis.

The solutions can be extended from the time interval $[t_1, t_2]$ to $\mathbb{R}$ by applying control $u = -1$ for all $t < t_1$ and control $u = +1$ for all $t > t_2$. The corresponding trajectory then tends to the right-most point $(x, y) = (\frac{\mu^2-1}{\mu^2+1}, \frac{2\mu}{\mu^2+1})$ of K for $t \to -\infty$ and to the left-most point $(x, y) = (-\frac{\mu^2-1}{\mu^2+1}, \frac{2\mu}{\mu^2+1})$ of K for $t \to +\infty$. The corresponding centro-affine immersion into $\mathbb{R}^2_+$ is of class C^2 and piece-wise analytic, but can be approximated with arbitrary precision by C^3 immersions satisfying (2). Hence the bound in Theorem 1.3 cannot be improved.

The rest of the proof is similar to that of Theorem 1.2.

Let us now prove Theorem 1.4. Let $a, b \in \Omega$ be arbitrary points, and let σ be the geodesic linking these points. The Riemannian length of the curve σ is by definition equal to $d^R(a, b)$. Let l be the length of the curve σ in the Hilbert metric. Then $l \geq d^H(a, b)$. On the other hand, by Corollary 3.3 we have $\mu^{-1} d^R(a, b) \leq l \leq \mu d^R(a, b)$. The claim of Theorem 1.4 readily follows.

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