WELL-POSEDNESS AND EXPONENTIAL DECAY FOR THE EULER-BERNOULLI BEAM CONVEYING FLUID EQUATION WITH NON-CONSTANT VELOCITY AND DYNAMICAL BOUNDARY CONDITIONS

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Abstract. In this paper, we consider an Euler-Bernoulli beam equation with time-varying internal fluid. We assume that the fluid is moving with non-constant velocity and dynamical boundary conditions are satisfied. We prove the existence and uniqueness of global solution under suitable assumptions on the tension of beam and on the parameters of the problem. Afterwards, we establish the exponential stability of the solution by introducing a suitable Lyapunov functional.

1. Introduction

In this paper, we study the dynamics of a tubular structure, a flexible pipe, which conveys a moving fluid. We assume that the pipe diameter is negligible compared to its length and we model the system by a one dimensional problem. In this context, the motion of the structure is driven by the following Euler-Bernoulli beam conveying fluid equation

$$\begin{cases} (m_p + 2m_f)\, w_{tt}(t) + EIw_{xxxx}(t) - (T - 2m_f V(t)^2)\, w_{xx}(t) \\ +cw_t(t) + 2m_f V_t(t) w_x(t) + 4m_f V(t) w_{xt}(t) = 0, \qquad 0 < x < L,\ t > 0, \end{cases} \tag{1.1}$$

endowed with the boundary conditions

$$\begin{cases} w_{xx}(0,t) = w_{xx}(L,t) = w(0,t) = 0, & t > 0, \\ EIw_{xxx}(L,t) - (T - 2m_f V(t)^2) w_x(L,t) + 2m_f V(t) w_t(L,t) = 0, & t > 0. \end{cases} \tag{1.2}$$

The parameters L and m_p are the length and the mass per unit length of the flexible pipe, while EI and T are its bending stiffness and its tension. The terms m_f and $V(t)$ are the mass per unit length and the velocity of the internal fluid. We assume that the velocity $V(t)$ is strictly positive or strictly negative (we refer to Section 4 for further details on the general situation). The solution of the system (1.1) with boundary conditions (1.2) represents the displacement of the flexible pipe at the position x and time t (see Figure 1). We study the well-posedness of (1.1)-(1.2) and the exponential stability of its solutions.

The equation (1.1)-(1.2) has been deduced when the velocity $V(t)$ is constant by Liu et al in [13] in presence of the environmental disturbances and a boundary control. The

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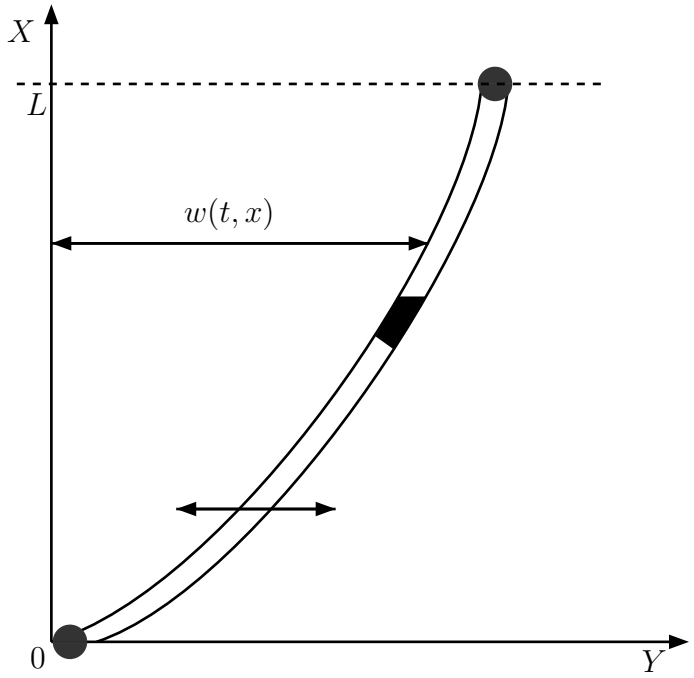

FIGURE 1. The figure represents the displacement of a flexible pipe which conveys a moving fluid.

authors consider the problem of a flexible marine riser and they compute the equation via a suitable energy functional. Finally, they prove the exponential decay of the solutions by Lyapunov method in presence of an additional boundary condition $w_x(0,t) = 0$.

In our work, we assume that the tension T is larger than a specific value T^* depending on the parameters of the problems. By exploiting such hypotheses, we firstly ensure the well-posedness of the (1.1)-(1.2). Secondly, we show the exponential stability of the energy of the solutions of (1.1)-(1.2) without considering additional boundary conditions.

Our stability result is obtained by exploiting a different energy functional from the one considered in [13] which validity is due to the assumption imposed on T. Such condition is not surprising from a practical point of view. It is reasonable to assume that the tension of the pipe has to be sufficiently strong, compared to the density and the velocity of the conveyed fluid, in order to have the stability. Otherwise, the fluid inside the pipe may dynamically interacts with its motion, possibly causing the flexible pipe to vibrate.

Euler-Bernoulli beam conveying fluid equations are found in many practical applications. They are used to model for instance risers of offshore platforms, pipes carrying chemical fluids, exhaust pipes in the engines, flue-gases stacks, air-conditioning ducts, tubes in heat exchangers and power plants, etc. A similar dynamics to (1.1) is studied in [8] by Khemmoudj where the internal damping $cw_t(t,x)$ is replaced by a viscoelastic term. There, the author considers suitable boundary conditions and he proves the exponential stability by Lyapunov method. In [4], Conrad et al. consider the equation $w_{tt} + w_{xxxx} = 0$ and

they prove exponential stability in presence of specific dynamical boundary conditions. For other similar results, we refer to [2, 15].

The paper is organized as follows. In Section 2, we present the well-posedness of the equation (1.1)-(1.2) in Theorem 2.2. In Section 3, we ensure our stability result in Theorem 3.1 by introducing a suitable Lyapunov functional.

2. Well-posedness of the problem

The aim of this section is to prove the existence and uniqueness of solutions for (1.1)-(1.2). To this purpose, we define the space

$$\mathcal{V} = \Big\{ w \in H^2\big((0,L),\mathbb{R}\big) \;\Big|\; w(0) = 0 \Big\}$$

and the Hilbert space

$$\mathcal{H} = \Big\{ (w,v) \;\Big|\; w \in \mathcal{V},\ v \in L^2\big((0,L),\mathbb{R}\big) \Big\}$$

equipped with the norm $\|\cdot\|_{\mathcal{H}}$ induced by the scalar product

$$\big\langle f_1, f_2 \big\rangle_{\mathcal{H}} = \int_0^L \Big(\partial_x^2 w_1 \partial_x^2 w_2 + \partial_x w_1 \partial_x w_2 + v_1 v_2 \Big) dx, \quad f_1 = \big(w_1, v_1\big),\ f_2 = \big(w_2, v_2\big) \in \mathcal{H}.$$

We define the family $\{\mathcal{A}_0(t)\}_{t\geq 0}$ of operators in $\mathcal{H}$ such that

$$\begin{aligned} \mathcal{A}_0(t) f &= \mathcal{A}_0(t) \begin{pmatrix} w \\ v \end{pmatrix} \\ &= \begin{pmatrix} v \\ -\frac{EI}{m_p+2m_f} w_{xxxx} + \frac{T-2m_f V^2(t)}{m_p+2m_f} w_{xx} - \frac{4m_f V(t)}{m_p+2m_f} v_x \end{pmatrix}, \end{aligned}$$

for every $f = (w,v)$ in the domain $D(\mathcal{A}_0(t))$ defined by

$$\begin{aligned} D(\mathcal{A}_0(t)) = \Big\{ (w,v) \in \mathcal{H} \mid\ & w \in \mathcal{V} \cap H^4,\ v \in \mathcal{V},\ w_{xx}(0) = w_{xx}(L) = 0, \\ & 2m_f V(t) v(L) = -EI w_{xxx}(L) + (T - 2m_f V^2(t)) w_x(L) \Big\}. \end{aligned}$$

Let $\{B(t)\}_{t\geq 0}$ be the family of bounded operators in $\mathcal{H}$ such that

$$B(t) f = \begin{pmatrix} 0 \\ \frac{c}{m_p+2m_f} v + \frac{2m_f V_t(t)}{m_p+2m_f} w_x \end{pmatrix}, \qquad \forall f \in \mathcal{H},$$

By imposing $v = w_t$, the Cauchy problem (1.1)-(1.2) becomes the following one in $\mathcal{H}$

$$\begin{pmatrix} w_t \\ v_t \end{pmatrix} = \mathcal{A}(t) \begin{pmatrix} w \\ v \end{pmatrix}, \qquad \mathcal{A}(t) := \big(\mathcal{A}_0(t) + B(t)\big), \tag{2.1}$$

endowed with the boundary conditions (1.2). The main result of this section is the well-posedness of the dynamics of (2.1)-(1.2) in $\mathcal{H}$ which is presented by the following theorem.

Definition 2.1. A two parameter family of bounded linear operators $U(t,s)$ with $0 \leq s \leq t$ on $\mathcal{H}$ is called an evolution system if the following conditions are satisfied:

- $U(s,s) = I$ for every $s \geq 0$;
- $U(t,r)U(r,s) = U(t,s)$ for every $0 \leq s \leq r \leq t$;
- for every $f \in \mathcal{H}$, the function $(t,s) \to U(t,s)f$ in $\mathcal{H}$ is continuous with respect to the norm of $\mathcal{H}$ for $0 \leq s \leq t$.

Theorem 2.2. *Let $V \in C^2\big([0,+\infty),\mathbb{R}\big)$ be a strictly positive or strictly negative function. Let $T > 0$ be such that $T > 2m_f \sup_{t\geq 0} V^2(t)$. There exists a unique evolution system $U(t,s)$ with $0 \leq s \leq t$ in $\mathcal{H}$ satisfying*

- *$U(t,s)D(\mathcal{A}_0(s)) = D(\mathcal{A}_0(t))$ for every $0 \leq s \leq t$;*
- *for every $s \geq 0$ and $f \in D(\mathcal{A}_0(s))$, the function $t \to U(t,s)f$ in $\mathcal{H}$ is continuously derivable with respect to the norm of $\mathcal{H}$ for $s \leq t$ and*

$$\frac{d}{dt}U(t,s)f = \mathcal{A}(t)U(t,s)f.$$

Theorem 2.2 ensures the well-posedness of (2.1) endowed with the boundary conditions (1.2), and then of (1.1)-(1.2). The result is guaranteed when the tension T is sufficiently large with respect to the velocity $V(t)$ and to the mass m_f.

The following proposition is the core of the proof of Theorem 2.2. It ensures the existence of a unique evolution system on $\mathcal{H}$ generated by the family of operators $(\mathcal{A}_0(t), D(\mathcal{A}_0(t)))$. Such result leads to Theorem 2.2 thanks to the properties of the family of operators $B(t)$.

Proposition 2.3. *Let $V \in C^1\big([0,+\infty),\mathbb{R}\big)$ be a strictly positive or strictly negative function. Let $T > 0$ be such that $T > 2m_f \sup_{t\geq 0} V^2(t)$. There exists a unique evolution system $U(t,s)$ with $0 \leq s \leq t$ in $\mathcal{H}$ satisfying*

- *$U(t,s)D(\mathcal{A}_0(s)) = D(\mathcal{A}_0(t))$ for every $0 \leq s \leq t$;*
- *for every $s \geq 0$ and $f \in D(\mathcal{A}_0(s))$, the function $t \to U(t,s)f$ in $\mathcal{H}$ is continuously derivable with respect to the norm of $\mathcal{H}$ for $t \geq s$ and*

$$\frac{d}{dt}U(t,s)f = \mathcal{A}_0(t)U(t,s)f.$$

Proof. Thanks to the assumptions on the velocity $V(t)$, we can define the norm $\|\cdot\|_t$ for every $t \geq 0$ of $\mathcal{H}$ induced by the scalar product

$$\langle f_1, f_2\rangle_t = \int_0^L \big(\alpha(t)\partial_x^2 w_1 \partial_x^2 w_2 + \beta(t)\partial_x w_1 \partial_x w_2 + \gamma(t) v_1 v_2\big)dx,$$

for every $f_1 = (w_1, v_1)$ and $f_2 = (w_2, v_2) \in \mathcal{H}$ with

$$\alpha := \frac{EI}{2m_f}, \qquad \beta(t) = \frac{T - 2m_f V^2(t)}{2m_f}, \qquad \gamma = \frac{m_p + 2m_f}{2m_f}.$$

The assumptions on the potential V and on the tension T yield the existence of $C > 1$ such that

$$C^{-1}\|\cdot\|_{\mathcal{H}} \leq \|\cdot\|_t \leq C\|\cdot\|_{\mathcal{H}}, \qquad \forall t \geq 0. \tag{2.2}$$

The domain of $\mathcal{A}_0(t)$ can be rewritten in terms of the parameters α and $\beta(t)$ as follows

$$D(\mathcal{A}_0(t)) = \Big\{(w,v) \in \mathcal{H} \mid w \in \mathcal{V} \cap H^4,\ v \in \mathcal{V},\ w_{xx}(0) = w_{xx}(L) = 0,$$
$$V(t)v(L) = -\alpha w_{xxx}(L) + \beta(t)w_x(L)\Big\}.$$

1) Dissipative property. First, we prove that $\mathcal{A}_0(t)$ is dissipative for every $t \geq 0$ in $\mathcal{H}$ with respect to the scalar product $\langle \cdot, \cdot \rangle_t$. Let us denote

$$a_1 := -\frac{EI}{m_p + 2m_f}, \qquad a_2(t) := \frac{T - 2m_f V^2(t)}{m_p + 2m_f}, \qquad a_3(t) := -\frac{4m_f V(t)}{m_p + 2m_f},$$

For every $f \in D(\mathcal{A}_0(t))$, thanks to the relations

$$\gamma a_1 = -\alpha, \qquad \gamma a_2(t) = \beta(t), \qquad \gamma(t)a_3(t) = -2V(t),$$

there holds

$$\begin{aligned}
\langle f, \mathcal{A}_0(t)f \rangle_t &= \int_0^L \Big(\alpha w_{xx}v_{xx} + \beta(t)w_x v_x + \gamma\big(a_1 w_{xxxx} + a_2(t)w_{xx} + a_3(t)v_x\big)v\Big)dx \\
&= -\alpha w_{xxx}(L)v(L) + \beta(t)w_x(L)v(L) \\
&\quad + \int_0^L \Big(\big(\alpha + \gamma a_1\big)w_{xxxx} + \big(\beta(t) - \gamma a_2(t)\big)w_{xx} + \gamma a_3(t)v_x\big)v\Big)dx \\
&= \big(-\alpha w_{xxx}(L) + \beta(t)w_x(L)\big)v(L) - \int_0^L 2V(t)v_x v dx \\
&= V(t)v(L)^2 - V(t)v(L)^2 = 0.
\end{aligned}$$

2) Surjectivity conditions. Second, fixed $t \geq 0$, we ensure the surjectivity of the map

$$(\lambda I - \mathcal{A}_0(t)) : D(\mathcal{A}_0(t)) \subset \mathcal{H} \longrightarrow \mathcal{H}$$

for every $\lambda > 0$. The property is equivalent to prove that, for every $f^* = (w^*, v^*) \in \mathcal{H}$, there exists a unique solution $f = (w, v) \in D(\mathcal{A}_0(t))$ of the equation $(\lambda I - \mathcal{A}_0(t))f = f^*$. In other words, we need to study the existence of a unique solution of the following system of equations

$$\begin{cases}
\lambda w(x) - v(x) = w^*(x), \\
\lambda v(x) - a_1 w_{xxxx}(x) - a_2(t)w_{xx}(x) - a_3(t)v_x(x) = v^*(x), \qquad x \in [0, L], \\
V(t)v(L) = -\alpha w_{xxx}(L) + \beta(t)w_x(L), \\
w_{xx}(0) = w_{xx}(L) = w(0) = 0.
\end{cases} \tag{2.3}$$

Solving the previous system is equivalent to study

$$\begin{cases}
\lambda^2 w - a_1 w_{xxxx} - a_2(t)w_{xx} - a_3(t)\lambda w_x = v^* - a_3(t)w_x^* + \lambda w^*, \\
\lambda V(t)w(L) - V(t)w^*(L) = -\alpha w_{xxx}(L) + \beta(t)w_x(L), \\
w_{xx}(0) = w_{xx}(L) = w(0) = 0.
\end{cases}$$

Let $g^* = v^* - a_3(t)w_x^* + \lambda w^*$. The weak formulation of the previous system is given by

$$\begin{aligned}&\int_0^L \Big(\lambda^2 w\phi - a_1 w_{xx}\phi_{xx} + a_2(t)w_x\phi_x - a_3(t)\lambda w_x(x)\phi\Big)dx\\ &- a_1 w_{xxx}(L)\phi(L) - a_2(t)w_x(L)\phi(L) = \int_0^L g^*\phi dx\end{aligned} \tag{2.4}$$

with $\phi \in \mathcal{V}$. The identity (2.4) can be rewritten as

$$\begin{aligned}&\int_0^L \Big(\lambda^2 w\phi - a_1 w_{xx}\phi_{xx} + a_2(t)w_x\phi_x - a_3(t)\lambda w_x(x)\phi\Big)dx\\ &+ \gamma^{-1}\big(\alpha w_{xxx}(L) - \beta(t)w_x(L)\big)\phi(L) = \int_0^L g^*\phi dx,\end{aligned}$$

Thanks to the validity of the boundary conditions in L, we have

$$\begin{aligned}&\int_0^L \Big(\lambda^2 w\phi - a_1 w_{xx}\phi_{xx} + a_2(t)w_x\phi_x - a_3(t)\lambda w_x\phi\Big)dx - \gamma^{-1}\lambda V(t)w(L)\phi(L)\\ &= -\gamma^{-1}V(t)w^*(L)\phi(L) + \int_0^L g^*\phi dx.\end{aligned}$$

We consider the bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{A}_0(t)}$ in $\mathcal{V}$ such that, for every $w, \phi \in \mathcal{V}$,

$$\begin{aligned}\langle w,\phi\rangle_{\mathcal{A}_0(t)} = &\int_0^L \Big(\lambda^2 w\phi - a_1 w_{xx}\phi_{xx} + a_2(t)w_x\phi_x - a_3(t)\lambda w_x\phi\Big)dx\\ &- \gamma^{-1}\lambda V(t)w(L)\phi(L).\end{aligned}$$

We notice that, for every $w \in \mathcal{V}$,

$$\langle w,w\rangle_{\mathcal{A}_0(t)} = \int_0^L \big(\lambda^2 w^2 - a_1 w_{xx}^2 + a_2(t)w_x^2\big)dx - \Big(\frac{a_3(t)}{2} + \gamma^{-1}V(t)\Big)\lambda w(L)^2.$$

Thanks to the identity

$$\frac{a_3(t)}{2} + \gamma^{-1}V(t) = 0,$$

$\langle\cdot,\cdot\rangle_{\mathcal{A}_0(t)}$ is a coercive bilinear form as $-a_1 > 0$ and $a_2(t) > 0$. From the Lax-Milgram theorem, the weak formulation admits an unique solution. Thanks to the regularity of $w \in H^4((0,L),\mathbb{R})$ and by using particular ϕ, it is possible to recover the boundary conditions in w and v is defined by (2.3) which is unique. Now, $f = (w,v) \in D(\mathcal{A}_0(t))$. This shows the surjectivity, for every $\lambda > 0$, of the map

$$(\lambda I - \mathcal{A}_0(t)) : D(\mathcal{A}_0(t)) \subset \mathcal{H} \longrightarrow \mathcal{H}.$$

3) Conclusion. Thanks to the point **1)**, the operator $\mathcal{A}(t)$ is dissipative for $t \geq 0$ with respect to the scalar product $\langle\cdot,\cdot\rangle_t$ and then, for every $\lambda > 0$,

$$\|(\lambda I - \mathcal{A}_0(t))\psi\|_t \geq \lambda\|\psi\|_t, \qquad \psi \in D(\mathcal{A}_0(t)).$$

Now, we define a new norm of $\mathcal{H}$

$$\|\cdot\|'_t := m(t)\|\cdot\|_t, \qquad \forall t \geq 0$$

by choosing a suitable strictly positive function $m : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that there exists $C > 1$ such that

$$C^{-1}\|\cdot\|_{\mathcal{H}} \leq \|\cdot\|'_t \leq \|\cdot\|'_s \leq C\|\cdot\|_{\mathcal{H}}, \qquad \forall 0 \leq s \leq t$$

(by keeping in mind the validity of the inequality (2.2)). Thus, we have

$$\|(\lambda I - \mathcal{A}_0(t))\psi\|'_t \geq \lambda\|\psi\|'_t, \qquad \forall \lambda > 0,\ \psi \in D(\mathcal{A}_0(t)).$$

Let us denote, for every $t \geq 0$ and $x \in [0, L]$,

$$f(t,x) = \left(1 + \frac{x}{L}\Big(\frac{|V(t)|}{\alpha} - 1\Big)\right),$$

$$g(t,x) = sign(V(t))\frac{\beta(t)}{\alpha}\frac{x}{L}.$$

We introduce the family of operators Γ_t with $t \geq 0$ such that

$$\Gamma_t : (w, v) \mapsto \big(w, f(t,\cdot)v - g(t,\cdot)w_x\big).$$

We notice that

$$\Gamma_t(D(\mathcal{A}_0(t))) = \Big\{(w,v) \in \mathcal{H} \mid w \in \mathcal{V} \cap H^4, \\ v \in \mathcal{V},\ w_{xx}(0) = w_{xx}(L) = 0, \quad v(L) = -sign(V(t))w_{xxx}(L)\Big\}$$

which does not depends on time since $V(t)$ has constant sign. Each $\Gamma_t : \mathcal{H} \to \mathcal{H}$ is invertible and Γ_t^{-1} is a family of linear bounded and invertible operators satisfying [10, hypotheses (4.3) and (4.4); p. 309]. In conclusion, the validity of [10, Theorem 4.2] yields the result. □

Proof of Theorem 2.2. The statement follows from Proposition 2.3 and classical arguments of existence of evolution systems (see for instance [14, Chapter 6.1; Theorem 1.2 & Theorem 1.5]). The property is due to the the smoothness of the family of bounded operators $B(t)$ which is due to hypotheses imposed on the velocity $V(t)$. □

3. Exponential stability of the problem

Main result

In this section, we ensure the stability result for the solution of the problem (1.1)-(1.2). To the purpose, we introduce the following assumptions,

Assumptions I. Let $V \in C^2\big([0,+\infty),\mathbb{R}\big)$ be a strictly positive or strictly negative function. Let $T>0$ be such that

$$T > 2m_f \sup_{t\geq 0} V(t)^2 + \max\{T_1, T_2\},$$

where we denote

$$T_1 = \frac{L^2}{4}(m_p + 2m_f) + 2\sqrt{2}Lm_f \sup_{t\geq 0}|V(t)|$$

and

$$T_2 = \frac{c^2L^2}{8(c-m_p-2m_f)} + 2m_f \sup_{t\geq 0}|V_t(t)V(t)|.$$

Assumptions I ensure that the tension T of the beam is sufficiently strong with respect to the velocity V and to the parameters of the problem. In this framework, the quantity $\frac{T}{2} - m_f V^2(t) > 0$ for every $t \geq 0$ and it is valid the well-posedness result provided in the previous section by Theorem 2.2. In addition, the assumption on T allows us to consider the following energy functional associated to the solutions of (1.1)-(1.2)

$$\begin{aligned} E(t) =& \frac{1}{2}(m_p+2m_f)\int_0^L w_t^2(t)\,dx + \frac{EI}{2}\int_0^L w_{xx}^2(t)\,dx \\ &+ \Big(\frac{T}{2} - m_f V^2(t)\Big)\int_0^L w_x^2(t)\,dx. \end{aligned} \tag{3.1}$$

We are finally ready to state the stability result of the problem 1.1-(1.2).

Theorem 3.1. *Let the problem* (1.1)-(1.2) *satisfy Assumptions I. There exist two positive constants k_0 and k_1, such that for any solution w of* (1.1)-(1.2)*, the energy functional* (3.1) *decays as follows*

$$E(t) \leq k_0 E(s) e^{-k_1(t-s)} \quad \forall\ 0 \leq s \leq t. \tag{3.2}$$

The proof of Theorem 3.1 is established in the final part of the section by gathering different results. We firstly compute the time derivative of the energy functional $\frac{d}{dt}E(t)$. After, we introduce a suitable Lyapunov functional $\mathcal{L}(t)$ for which there exist $C_1, C_2 > 0$ so that $C_1E(t) \leq \mathcal{L}(t) \leq C_2E(t)$. Finally, we show the existence of $C_3 > 0$ such that $\frac{d}{dt}\mathcal{L}(t) \leq -C_3\mathcal{L}(t)$. This identity implies $\mathcal{L}(t) \leq \mathcal{L}(s)e^{-C_3(t-s)}$ and Theorem 3.1 is proved by gathering the previous results.

Some preliminaries

We start by recalling the following Sobolev-Poincaré inequality (see [1] for further details).

Lemma 3.2. *(Sobolev-Poincaré inequality) For every $v \in \{w \in H^1(0,L) \mid w(0)=0\}$, there holds*

$$\int_0^L v^2\,dx \leq P\int_0^L v_x^2\,dx, \qquad \text{with} \qquad P := \frac{L^2}{2}.$$

Lemma 3.3. *Let w be a solution of the problem* (1.1)-(1.2). *The energy functional defined by* (3.1) *satisfies*

$$\begin{aligned}\frac{d}{dt}E(t) =& - c\int_0^L w_t^2(t)\,dx - 2m_fV_t(t)\int_0^L w_t(t)w_x(t)\,dx\\ &- 2m_fV_t(t)V(t)\int_0^L w_x^2(t)\,dx.\end{aligned}$$

Proof. By multiplying the first equation in (1.1) with w_t and by integrating over $(0, L)$, we obtain

$$\begin{aligned}&\int_0^L w_t(t)\Big[(m_p+2m_f)w_{tt}(t) + EIw_{xxxx}(t)\\ &- (T-2m_fV^2(t))w_{xx}(t)\Big]\,dx + 4m_fV(t)\int_0^L w_t(t)w_{xt}(t)\,dx\\ &= -c\int_0^L w_t^2(t)\,dx - 2m_fV_t(t)\int_0^L w_t(t)w_x(t)\,dx.\end{aligned}\tag{3.3}$$

We study each term appearing in first integral of (3.3)

$$(m_p+2m_f)\int_0^L w_t(t)w_{tt}(t)\,dx = \frac{1}{2}(m_p+2m_f)\frac{d}{dt}\int_0^L w_t^2(t)\,dx,\tag{3.4}$$

$$EI\int_0^L w_t(t)w_{xxxx}(t)\,dx = EIw_t(L,t)w_{xxx}(L,t) + \frac{EI}{2}\frac{d}{dt}\int_0^L w_{xx}^2(t)\,dx\tag{3.5}$$

and

$$\begin{aligned}-(T-2m_fV^2(t))\int_0^L w_t(t)w_{xx}(t)\,dx =& \left(2m_fV^2(t)-T\right)w_t(L,t)w_x(L,t)\\ &+\frac{T}{2}\frac{d}{dt}\int_0^L w_x^2(t)\,dx - m_fV^2(t)\frac{d}{dt}\int_0^L w_x^2(t)\,dx.\end{aligned}\tag{3.6}$$

The right-hand side of (3.6) can be rewritten as follows

$$\begin{aligned}&(2m_fV^2(t)-T)w_t(L,t)w_x(L,t) + \frac{T}{2}\frac{d}{dt}\int_0^L w_x^2(t)\,dx\\ &- m_f\frac{d}{dt}\Big(V^2(t)\int_0^L w_x^2\,dx\Big) + 2m_fV_t(t)V(t)\int_0^L w_x^2\,dx.\end{aligned}\tag{3.7}$$

Now, we investigate the remaining term in the left-hand side of (3.3)

$$4m_fV(t)\int_0^L w_t(t)w_{xt}(t)\,dx = 2m_fV(t)w_t^2(L,t).\tag{3.8}$$

By using (3.4)-(3.8) into (3.3) yields the following expression

$$\begin{aligned}&\frac{d}{dt}E(t)+\Big(EIw_{xxx}(L,t)-(T-2m_fV^2(t))w_x(L,t)+2m_fV(t)w_t(L,t)\Big)w_t(L,t)\\&=-c\int_0^L w_t^2(t)\,dx-2m_fV_t(t)\int_0^L w_t(t)w_x(t)\,dx-2m_fV_t(t)V(t)\int_0^L w_x^2(t)\,dx.\end{aligned}$$

The boundary conditions (1.2) complete the proof. □

A suitable Lyapunov functional

Let us introduce the functionals

$$\begin{aligned}G_1(t)&=(m_p+2m_f)\int_0^L w(t)w_t(t)\,dx,\\G_2(t)&=2m_fV(t)\int_0^L w(t)w_x(t)\,dx,\end{aligned}$$

and

$$G(t) \;=\; G_1(t)+G_2(t). \tag{3.9}$$

We define the Lyapunov functional $\mathcal{L}$ such that

$$\mathcal{L}(t)=E(t)+G(t). \tag{3.10}$$

Lemma 3.4. *Let the problem* (1.1)-(1.2) *satisfy Assumptions I. There exist two positive constants* $\xi_1>0$ *and* $\xi_2>0$ *depending on* m_p, m_f, T *and* $V(t)$ *such that, for all* $t\geq 0$,

$$\xi_1E(t)\leq\mathcal{L}(t)\leq\xi_2E(t). \tag{3.11}$$

Proof. Let P be the Poincaré's constant from Lemma 3.2 and $\alpha_1>0$. By using the Young's and the Poincaré inequalities, we obtain

$$\begin{aligned}|G_1(t)|\leq&\frac{\alpha_1}{2}\big(m_p+2m_f\big)\int_0^L w_t^2(t)\,dx\\&+\frac{P}{2\alpha_1}\big(m_p+2m_f\big)\int_0^L w_x^2(t)\,dx.\end{aligned} \tag{3.12}$$

By setting $\alpha_2=\sqrt{\frac{1}{P}}$, we have

$$\begin{aligned}|G_2(t)|\leq&2m_f|V(t)|\Big(\frac{\alpha_2P}{2}+\frac{1}{2\alpha_2}\Big)\int_0^L w_x^2(t)\,dx\\=&2m_f|V(t)|\sqrt{P}\int_0^L w_x^2(t)\,dx\end{aligned} \tag{3.13}$$

In the last relation, we impose $\alpha_2 = \sqrt{\frac{1}{P}}$ in order to minimize the function $\alpha_2 \mapsto \{\frac{P}{2}\alpha_2 + \frac{1}{2\alpha_2}\}$ which attains its minimum in $\sqrt{P}$ exactly when $\alpha_2 = \sqrt{\frac{1}{P}}$. By combining (3.1), (3.12) and (3.13), we have

$$\begin{aligned}\mathcal{L}(t) \leq &\left(\frac{T}{2} - m_f V^2(t) + \frac{P}{2\alpha_1}\big(m_p + 2m_f\big) + 2m_f|V(t)|\sqrt{P}\right)\int_0^L w_x^2(t)\,dx \\ &+ (m_p + 2m_f)\Big(1 + \frac{\alpha_1}{2}\Big)\int_0^L w_t^2(t)\,dx + EI\int_0^L w_{xx}^2(t)\,dx.\end{aligned} \tag{3.14}$$

For the lower bound, we can see that

$$\begin{aligned}\mathcal{L}(t) \geq &\left(\frac{T}{2} - m_f V^2(t) - \frac{P}{2\alpha_1}\big(m_p + 2m_f\big) - 2m_f|V(t)|\sqrt{P}\right)\int_0^L w_x^2(t)\,dx \\ &+ (m_p + 2m_f)\Big(1 - \frac{\alpha_1}{2}\Big)\int_0^L w_t^2(t)\,dx + EI\int_0^L w_{xx}^2(t)\,dx.\end{aligned} \tag{3.15}$$

We recall that $P = \frac{L^2}{2}$ and, thanks to *Assumptions I*,

$$T > 2m_f \sup_{t\geq 0} V(t)^2 + \frac{L^2}{4}(m_p + 2m_f) + 2\sqrt{2}Lm_f \sup_{t\geq 0}|V(t)|. \tag{3.16}$$

Now, we choose $\alpha_1 \in [1, 2)$ in (3.14) and (3.15) as such number such that

$$\frac{T}{2} - m_f \sup_{t\geq 0} V(t)^2 - \frac{P}{2\alpha_1}\big(m_p + 2m_f\big) - 2m_f \sup_{t\geq 0}|V(t)|\sqrt{P} > 0,$$

which existence is guaranteed by (3.16). In addition, $(m_p + 2m_f)(1 - \frac{\alpha_1}{2}) > 0$ and

$$\frac{T}{2} - m_f \inf_{t\geq 0} V^2(t) \geq \frac{T}{2} - m_f \sup_{t\geq 0} V^2(t) > 0$$

thanks to (3.16). By combining (3.14) and (3.15), we obtain (3.11). The constants ξ_1 and ξ_2 are explicitly given by

$$\begin{aligned}\xi_1 = \max\Bigg\{ &\frac{1}{\frac{T}{2} - m_f \sup_{t\geq 0} V(t)^2}\Big(\frac{T}{2} - m_f V^2(t) + \frac{P}{2\alpha_1}\Big(m_p + 2m_f\Big) \\ &+ 2m_f \sup_{t\geq 0}|V(t)|\sqrt{P},\ 2\Big(1 + \frac{\alpha_1}{2}\Big),\ 2\Bigg\},\end{aligned}$$

$$\begin{aligned}\xi_2 = \min\Bigg\{ &\frac{1}{\frac{T}{2} - m_f \inf_{t\geq 0} V^2(t)}\Big(\frac{T}{2} - m_f V^2(t) - \frac{P}{2\alpha_1}\big(m_p + 2m_f\big) \\ &- 2m_f \sup_{t\geq 0}|V(t)|\sqrt{P},\ 2\Big(1 - \frac{\alpha_1}{2}\Big),\ 2\Bigg\}.\end{aligned}$$

□

Lemma 3.5. *Let w be the solution of (1.1)-(1.2). The functional G defined by (3.9) satisfies*

$$\frac{d}{dt}G(t) = -EI\int_0^L w_{xx}^2(t)\,dx - (T-2m_fV^2(t))\int_0^L w_x^2(t)\,dx$$
$$- c\int_0^L w(t)w_t(t)\,dx + 2m_fV(t)\int_0^L w_x(t)w_t(t)\,dx + (m_p+2m_f)\int_0^L w_t^2(t)\,dx.$$

Proof. We know that $\frac{d}{dt}G(t) = \frac{d}{dt}G_1(t) + \frac{d}{dt}G_2(t)$. Now,

$$\frac{d}{dt}G_1(t) = (m_p+2m_f)\int_0^L w(t)w_{tt}(t)\,dx + (m_p+2m_f)\int_0^L w_t^2(t)\,dx$$

We use (1.1) and we obtain

$$\frac{d}{dt}G_1(t) = -EI\int_0^L w_{xxxx}(t)w(t) + (T-2m_fV^2(t))\int_0^L w_{xx}(t)w(t)\,dx$$
$$- 2m_fV_t(t)\int_0^L w_x(t)w(t)\,dx - 4m_fV(t)\int_0^L w_{xt}(t)w(t)\,dx$$
$$- c\int_0^L w(t)w_t(t)\,dx + (m_p+2m_f)\int_0^L w_t^2(t)\,dx.$$

We integrate by parts and $\frac{d}{dt}G_1(t)$ becomes

$$-\Big(EIw_{xxx}(L,t) - \big(T-2m_fV^2(t)\big)w_x(L,t) + 2m_fV(t)w_t(L,t)\Big)w(L,t)$$
$$- EI\int_0^L w_{xx}^2(t)\,dx - \big(T-2m_fV^2(t)\big)\int_0^L w_x^2(t)\,dx - 2m_fV_t(t)\int_0^L w_x(t)w(t)\,dx$$
$$- 2m_fV(t)\int_0^L w_{xt}(t)w(t)\,dx - c\int_0^L w(t)w_t(t)\,dx + (m_p+2m_f)\int_0^L w_t^2(t)\,dx.$$

By using the boundary conditions, we obtain

$$\frac{d}{dt}G_1(t) = -EI\int_0^L w_{xx}^2(t)\,dx - \big(T-2m_fV^2(t)\big)\int_0^L w_x^2(t)\,dx$$
$$- 2m_fV_t(t)\int_0^L w_x(t)w(t)\,dx - 2m_fV(t)\int_0^L w_{xt}(t)w(t)\,dx \tag{3.17}$$
$$- c\int_0^L w(t)w_t(t)\,dx + (m_p+2m_f)\int_0^L w_t^2(t)\,dx.$$

Now, we compute $\frac{d}{dt}G_2(t)$

$$\begin{aligned}\frac{d}{dt}G_2(t) &= 2m_fV(t)\int_0^L w_{xt}(t)w(t)\,dx + 2m_fV(t)\int_0^L w_x(t)w_t(t)\,dx \\ &\quad + 2m_fV_t(t)\int_0^L w(t)w_x(t)\,dx.\end{aligned} \tag{3.18}$$

By gathering (3.17) and (3.18), we achieve the claim. □

Proof of Theorem 3.1

Now, we are ready to prove our main result.

Proof of Theorem 3.1. By the definition (3.10), we have $\frac{d}{dt}\mathcal{L}(t) = \frac{d}{dt}E(t) + \frac{d}{dt}G(t)$. Thanks to Lemma 3.3 and Lemma 3.5, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &= -c\int_0^L w_t^2(t)\,dx - 2m_fV_t(t)V(t)\int_0^L w_x^2(t)\,dx \\ &\quad - EI\int_0^L w_{xx}^2(t)\,dx - (T - 2m_fV^2(t))\int_0^L w_x^2(t)\,dx \\ &\quad - c\int_0^L w(t)w_t(t)\,dx + (m_p + 2m_f)\int_0^L w_t^2(t)\,dx.\end{aligned} \tag{3.19}$$

By using Young's inequality and Poincaré inequality, for every $\alpha_1 > 0$, we have

$$\begin{aligned}-c\int_0^L w(t)w_t(t)\,dx &\le c\frac{\alpha_1}{2}\int_0^L w_t(t)^2\,dx + \frac{1}{2\alpha_1}c\int_0^L w(t)^2\,dx \\ &\le c\frac{\alpha_1}{2}\int_0^L w_t(t)^2\,dx + \frac{P}{2\alpha_1}c\int_0^L w_x(t)^2\,dx.\end{aligned} \tag{3.20}$$

Now, thanks to *Assumptions I*, there exists $\delta \in (0, c - m_p - 2m_f)$ such that

$$T > 2m_f\sup_{t\ge 0}V(t)^2 + \frac{c^2P}{4(c - m_p - 2m_f - \delta)} + 2m_f\sup_{t\ge 0}|V_t(t)V(t)|. \tag{3.21}$$

We set $\alpha_1 = 2\frac{c - m_p - 2m_f - \delta}{c}$ in (3.20) that we use in (3.19) in order to obtain

$$\frac{d}{dt}\mathcal{L}(t) \le -\gamma_0\int_0^L w_t^2(t)\,dx - \gamma_1\int_0^L w_x^2(t)\,dx - EI\int_0^L w_{xx}^2(t)\,dx$$

where

$$\gamma_0 = c - c\frac{\alpha_1}{2} - m_p - 2m_f,$$

$$\gamma_1 = T - 2m_f\sup_{t\ge 0}|V_t(t)V(t)| - 2m_f\sup_{t\ge 0}V(t)^2 - c\frac{P}{2\alpha_1}.$$

We notice that $\gamma_0 > 0$ thanks to the choice of α_1 and $\gamma_1 > 0$ thanks to (3.21). Finally, we see that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\vartheta E(t) \tag{3.22}$$

with

$$\vartheta := \min\left\{\frac{2\gamma_0}{m_p + 2m_f}, \frac{2\gamma_1}{T - 2m_f \inf_{t\geq 0} V(t)^2}, 2\right\}.$$

A combination of (3.11) and (3.22) gives

$$\frac{d}{dt}\mathcal{L}(t) \leq -k_1\mathcal{L}(t), \quad \forall t \geq 0, \tag{3.23}$$

where $k_1 = \frac{\vartheta}{\xi_2}$. We integrate (3.23) over (s,t) and

$$\mathcal{L}(t) \leq \mathcal{L}(s)e^{-k_1(t-s)}, \quad \forall t \geq s. \tag{3.24}$$

Finally, by combining (3.11) and (3.24), we obtain (3.2) with $k_0 = \frac{\xi_2}{\xi_1}$, which completes the proof. □

4. Conclusions

In Section 2 and Section 3, we studied the well-posedness and the stability of the solutions of (1.1)-(1.2). We assumed that the velocity $V(t)$ is a sufficiently smooth function with constant sign, while the tension T is larger than a specific value T^* depending on the parameters of the problems. By exploiting such hypotheses, we proved the well-posedness of (1.1)-(1.2). Secondly, we ensured the exponential stability by introducing a suitable Lyapunov functional.

The choice of considering V with constant sign is due to the following reason. When V vanishes for some time, the dynamical boundary condition in (1.2) becomes a statical boundary condition. In this case, the problem lacks of a boundary condition on $w_t(t,x)$ and it is not clear which "natural" boundary condition appears in such a context. Nevertheless, the stability result from Section 3 could still be valid, at least from a formal point of view.

Finally, the assumption on T is not so surprising when we think to the nature of the problem modeled by (1.1)-(1.2). It is reasonable to assume that the tension of the pipe has to be sufficiently strong, compared to the density and the velocity of the conveyed internal fluid, in order to have the stability. From this perspective, it could be interesting to explore this phenomenon further, at least from a numerical point of view. One could seek for evidences of instability phenomena when the tension T is too low.

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