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SIMON RICHE AND GEORDIE WILLIAMSON

Dedicated to Roman Bezrukavnikov, in admiration.

Abstract. In this paper we apply Treumann's "Smith theory for sheaves" in the context of the Iwahori–Whittaker model of the Satake category. We deduce two results in the representation theory of reductive algebraic groups over fields of positive characteristic: (a) a geometric proof of the linkage principle; (b) a character formula for tilting modules in terms of the p-canonical basis, valid in all blocks and in all characteristics.

1. Introduction

1.1. **Geometric representation theory of reductive algebraic groups.** Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic $\ell > 0$, and consider the category $\mathrm{Rep}(\mathbf{G})$ of finite-dimensional algebraic representations of $\mathbf{G}$. The study of this category has led to significant progress in modular representation theory over the last fifty years; however several fundamental questions (e.g. dimensions and characters of simple and tilting modules) remain only partially understood.

One tempting avenue of pursuit is to find relationships to $\mathscr{D}$-modules or constructible sheaves, and hence bring sheaf theory into play. The archetypal example of the success of such an approach is the Beĭlinson–Bernstein localization theorem, which establishes such an link for modules over complex semi-simple Lie algebras. Beĭlinson–Bernstein localization is an indispensable tool in modern Geometric Representation Theory, leading to proofs of the Kazhdan–Lusztig conjectures, character formulas for real reductive groups, etc.

Back in the setting of $\mathrm{Rep}(\mathbf{G})$, the *geometric Satake equivalence* provides such a connection to constructible sheaves. To $\mathbf{G}$ we can associate the affine Grassmannian $\mathcal{G}r$ of its complex Langlands dual group (an infinite-dimensional algebraic variety), and one has an equivalence of tensor categories between $\mathrm{Rep}(\mathbf{G})$ and a certain category of perverse sheaves with $\Bbbk$-coefficients on $\mathcal{G}r$. The geometric Satake equivalence is central to modern approaches to the Langlands program, and has become a cornerstone of Geometric Representation Theory.

However, in contrast to Beĭlinson–Bernstein localization, the geometric Satake equivalence has been surprisingly ineffective at answering questions about $\mathrm{Rep}(\mathbf{G})$. For example, several basic statements and constructions involving $\mathrm{Rep}(\mathbf{G})$ (e.g. the linkage principle, or Frobenius twist) have no geometric explanation. This is the more surprising, as several known or conjectured formulas (e.g. Lusztig's character formula) involve Kazhdan–Lusztig polynomials or their ℓ-counterparts, which encode dimensions of stalks of sheaves on the affine Grassmannian and flag variety.

Nowadays we have several proofs of Lusztig's character formula for large ℓ, however none of them pass through the geometric Satake equivalence!

1.2. **Overview.** The main result of the present paper is a proof of the linkage principle via the geometric Satake equivalence. Our proof also explains that each "block" in the linkage principle is controlled by a partial affine flag variety for the Langlands dual group, which gives us new proofs of Lusztig's conjecture on simple characters (for large ℓ) and of a conjecture of the authors on tilting characters (for all ℓ). The techniques of this paper provide a powerful new tool in the study of representations of $\mathbf{G}$.

Our argument is a simple application of two new tools in Geometric Representation Theory. The first one is Smith–Treumann theory, which is a variant of equivariant localization for tori. In this theory the circle action is replaced by the action of a cyclic group of order ℓ, and the coefficients must be of characterstic ℓ. We apply this theory to the loop rotation action on the affine Grassmannian. Whilst the fixed points under the full loop rotation action are rather boring (infinitely many finite partial flag varieties), the fixed points under the subgroup of ℓ-th roots of unity are rich (finitely many partial affine flag varieties).

The second ingredient is the Iwahori–Whittaker realisation of the Satake category. This replaces the category of perverse sheaves in the Satake equivalence with an equivalent category satisfying a certain equivariance condition with respect to the pro-unipotent radical of the Iwahori subgroup. (This condition is inspired by "Whittaker conditions" in the representation theory of p-adic groups, hence the name.) It turns out that in the Iwahori–Whittaker realisation, the components of the fixed points discussed above match precisely the decomposition of $\mathrm{Rep}(\mathbf{G})$ given by the linkage principle. Our main theorem asserts that the Smith restriction functor gives an equivalence between tilting sheaves in the Iwahori–Whittaker realisation and a certain category of parity complexes on the fixed points. It is then straightforward to deduce the linkage principle. The character formulas for simple and tilting modules alluded to above are also an immediate consequence.

In the rest of the introduction, we give a more detailed overview of the techniques and results of this paper.

1.3. **The linkage principle.** As above, let $\mathbf{G}$ be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic $\ell > 0$, and let $\mathrm{Rep}(\mathbf{G})$ be its category of finite-dimensional algebraic representations. Fix a maximal torus and Borel subgroup $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$, and let $\mathfrak{R}^+ \subset \mathfrak{R} \subset \mathbf{X}$ denote the (positive) roots inside the lattice of characters of $\mathbf{T}$.[1] The simple objects in $\mathrm{Rep}(\mathbf{G})$ are classified by dominant weights $\mathbf{X}^+ \subset \mathbf{X}$; given $\lambda \in \mathbf{X}^+$ we denote by $\nabla(\lambda)$ the induced $\mathbf{G}$-module of highest weight λ, and by $\mathsf{L}(\lambda)$ its simple socle.

Let W_{f} denote the Weyl group of $(\mathbf{G}, \mathbf{T})$, and consider the affine Weyl group

$$W_{\mathrm{aff}} := W_{\mathrm{f}} \ltimes \mathbb{Z}\mathfrak{R}$$

which acts naturally on $\mathbf{X}$. The *linkage principle* [Ve, Hu, J1, A1] states that we have a decomposition

$$\mathrm{Rep}(\mathbf{G}) = \bigoplus_{\gamma \in \mathbf{X}/(W_{\mathrm{aff}}, \bullet_\ell)} \mathrm{Rep}_\gamma(\mathbf{G}), \tag{1.1}$$

[1]We warn the reader that in the body of the paper we switch to a Langlands dual notation.

where each summand is the Serre subcategory

$$\mathrm{Rep}_\gamma\, \mathbf{G} = \langle \mathsf{L}(\lambda) : \lambda \in \gamma \cap \mathbf{X}^+ \rangle.$$

Notice that we do not consider the standard action of W_{aff} on $\mathbf{X}$, but rather the "dot" action (denoted $\bullet_\ell$); that is, if $\rho := \frac{1}{2}\sum_{\alpha \in \mathfrak{R}^+} \alpha$, then

$$(w\mathsf{t}_\mu) \bullet_\ell \lambda := w(\lambda + \ell\mu + \rho) - \rho$$

for $w \in W_{\mathrm{f}}$, $\mu \in \mathbb{Z}\mathfrak{R}$ and $\lambda \in \mathbf{X}$.

Remark 1.1. The subcategory $\mathrm{Rep}_\gamma(\mathbf{G})$ will be called the *block* of γ. This is an abuse since this subcategory might be decomposable, hence is not a "block" in the strict sense, but is convenient. In fact the blocks of $\mathrm{Rep}(\mathbf{G})$ in the strict sense have been described by Donkin [Do]. We believe our methods should allow to shed some light on this description, and hope to come back to this question in a future publication.

1.4. **The geometric Satake equivalence.** Let H be the complex[2] reductive group which is Langlands dual to $\mathbf{G}$, and denote its maximal torus by T (so that $X_*(T) = \mathbf{X} = X^*(\mathbf{T})$). Let LH and L^+H denote the "loop" group (ind-)schemes whose R points are $H(R((z)))$ and $H(R[[z]])$ respectively, for any $\mathbb{C}$-algebra R; and let $\mathcal{G}r_H := LH/L^+H$ denote the affine Grassmannian. The affine Grassmannian is an ind-projective ind-scheme whose T-fixed points (resp. L^+H-orbits) are in bijection with $\mathbf{X}$ (resp. $\mathbf{X}^+$).

The *geometric Satake equivalence* [L3, Gi, BD, MV1] gives an equivalence of Tannakian categories

$$(\mathsf{Perv}_{L^+H}(\mathcal{G}r_H, \Bbbk), \star) \cong (\mathrm{Rep}(\mathbf{G}), \otimes) \tag{1.2}$$

where $\mathsf{Perv}_{L^+H}(\mathcal{G}r_H, \Bbbk)$ denotes the category of L^+H-equivariant perverse sheaves on $\mathcal{G}r_H$ with coefficients in $\Bbbk$, with its natural convolution product $\star$.

1.5. **Smith–Treumann theory.** A fundamental role in our proof is played by Treumann's "Smith theory for sheaves" [Tr]. The basic idea is that, when dealing with coefficients of characteristic ℓ, one should be able localize to fixed points of actions of cyclic groups of order ℓ. (This theory should be compared with Borel's "localization theorem" for manifolds equipped with an action of the circle group; in this analogy, finite cyclic groups become "discrete circles". See [W2] for more comments on this analogy)

More precisely, let X be a variety endowed with an action of the group $\boldsymbol{\mu}_\ell$ of ℓ-th roots of unity. One has two (Verdier dual) restriction functors

$$D^{\mathrm{b}}_{\boldsymbol{\mu}_\ell}(X, \Bbbk) \underset{i^*}{\overset{i^!}{\rightrightarrows}} D^{\mathrm{b}}_{\boldsymbol{\mu}_\ell}(X^{\boldsymbol{\mu}_\ell}, \Bbbk)$$

between the $\boldsymbol{\mu}_\ell$-equivariant derived categories of constructible $\Bbbk$-sheaves on X and on $X^{\boldsymbol{\mu}_\ell}$.

[2]In a few paragraphs we will instead assume that H is defined over a field of characteristic p where $p \neq 0, \ell$.

A fundamental observation of Treumann is that the compositions of these functors with the quotient functor to the *Smith category*

$$\mathsf{Sm}_{\mathrm{Treu}}(X^{\boldsymbol{\mu}_\ell}, \Bbbk) := D^{\mathrm{b}}_{\boldsymbol{\mu}_\ell}(X^{\boldsymbol{\mu}_\ell}, \Bbbk)/\langle \boldsymbol{\mu}_\ell\text{-perfect complexes}\rangle$$

become canonically isomorphic. Here an object in $D^{\mathrm{b}}_{\boldsymbol{\mu}_\ell}(X^{\boldsymbol{\mu}_\ell}, \Bbbk)$ is $\boldsymbol{\mu}_\ell$*-perfect* if its stalks (naturally complexes of $\boldsymbol{\mu}_\ell$-modules) may be represented by a bounded complex of free $\Bbbk[\boldsymbol{\mu}_\ell]$-modules. The resulting *Smith restriction* functor

$$D^{\mathrm{b}}_{\boldsymbol{\mu}_\ell}(X, \Bbbk) \xrightarrow{i^{!*}} \mathsf{Sm}_{\mathrm{Treu}}(X^{\boldsymbol{\mu}_\ell}, \Bbbk)$$

has remarkable properties. For examples, it commutes with essentially all sheaf theoretic functors [Tr]. It can be thought of as an analogue of hyperbolic localization for μ_ℓ-actions.

For technical reasons (namely, to ensure that the Smith category of a point satisfies appropriate parity vanishing properties), we use a variant of Treumann's construction, proposed by the second author in [W2]. Namely, we assume that the action of $\boldsymbol{\mu}_\ell$ can be extended to an action of the multiplicative group $\mathbb{G}_{\mathrm{m}}$ on X and consider the *equivariant Smith category*

$$\mathsf{Sm}(X^{\boldsymbol{\mu}_\ell}, \Bbbk) := D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X^{\boldsymbol{\mu}_\ell}, \Bbbk)/\left\langle \begin{array}{c}\text{complexes whose restriction}\\ \text{to } \boldsymbol{\mu}_\ell \subset \mathbb{G}_{\mathrm{m}} \text{ are } \boldsymbol{\mu}_\ell\text{-perfect}\end{array}\right\rangle.$$

With this definition, the theory of parity complexes from [JMW] applies in the Smith quotient, which will be crucial for our arguments.[3]

1.6. **Fixed points.** To apply this idea in our setting, note that $\mathcal{G}r_H$ has a natural action of $\mathbb{G}_{\mathrm{m}}$ via "loop rotation," induced by the action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{C}((z))$ which "rescales" z. A beautiful fact (that we first learned from R. Bezrukavnikov) is that, if $\boldsymbol{\mu}_\ell \subset \mathbb{G}_{\mathrm{m}}$ denotes (as above) the subgroup of ℓ-th roots of unity, we have a decomposition

$$(\mathcal{G}r_H)^{\mu_\ell} = \bigsqcup_{\gamma \in \mathbf{X}/(W_{\mathrm{aff}}, \square_\ell)} \mathcal{G}r_{H,\gamma}, \tag{1.3}$$

where the action $\square_\ell$ is defined by $(w\mathsf{t}_\mu) \square_\ell \lambda = w(\lambda + \ell\mu)$ for $w \in W_{\mathrm{f}}$, $\mu \in \mathbb{Z}\mathfrak{R}$ and $\lambda \in \mathbf{X}$. Moreover, each component on the right-hand side is a partial affine flag variety for the loop group $L_\ell H$ representing $R \mapsto H(R((z^\ell)))$, whose "partiality" is governed by the stabilizer of an element in γ. For example, for $\gamma = W_{\mathrm{aff}} \square_\ell 0$ we obtain the "thin affine Grassmannian" (defined as above for $\mathcal{G}r_H$, but now with z replaced by z^ℓ); and if γ has trivial stabiliser under W_{aff} then $\mathcal{G}r_{H,\gamma}$ is the full affine flag variety for H.

The similarity between (1.3) and (1.1) is rather striking; for example there are as many components in the right-hand side of (1.3) as summands in the decomposition (1.1). However there is a fundamental difference: (1.1) involves the dot action (with W_{f} fixing $-\rho$); whereas (1.3) involves the unshifted action (with W_{f} fixing 0). Thus we do not expect the Smith restriction functor to realise the linkage principle in this setting.[4].

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[3]The fact that Smith–Treumann theory can be made to accommodate the theory of parity sheaves was first pointed out by Leslie–Lonergan [LL]. The version they use is however different, and—from our point of view—technically more involved.

[4]The effect of Smith restriction in this setting is investigated in [LL]. The authors show that it realises the "Frobenius contraction" functor of Gros–Kaneda [GK].

1.7. **The Iwahori-Whittaker model.** To get around this issue, we replace the "traditional" Satake category $\mathsf{Perv}_{L^+H}(\mathcal{G}r_H, \Bbbk)$ with the "Iwahori–Whittaker model" considered in [BGMRR]. There it is proved that (under a mild assumption, satisfied e.g. if H is of adjoint type, which we assume from now on for simplicity) one has an equivalence of abelian categories

$$\mathsf{Perv}_{L^+H}(\mathcal{G}r_H, \Bbbk) \xrightarrow{\sim} \mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_H, \Bbbk) \tag{1.4}$$

where the right-hand side denotes a category of perverse sheaves of $\mathcal{G}r_H$ which satisfy a certain equivariance condition with respect to the pro-unipotent radical $\mathrm{Iw}_{\mathrm{u}}^+$ of an Iwahori subgroup; such perverse sheaves are called "Iwahori–Whittaker."[5]

A crucial point for us is that on simple objects the equivalence (1.4) sends the intersection cohomology complex associated with the L^+H-orbit parametrized by $\lambda \in \mathbf{X}^+$ (which corresponds to $\mathsf{L}(\lambda)$ under (1.2)) to the Iwahori–Whittaker intersection cohomology complex associated with the $\mathrm{Iw}_{\mathrm{u}}^+$-orbit parametrized by $\lambda + \rho$. Thus, after passage to the Iwahori–Whittaker model, our issue with the two distinct actions goes away, and the linkage principle is reflected perfectly in geometry of the $\boldsymbol{\mu}_\ell$-fixed points. In particular, two simple Iwahori–Whittaker perverse sheaves, parametrized by some weights λ and μ, lie in the same summand in the linkage principle if and only if the corresponding fixed points L_λ and L_μ lie in the same component of the fixed points!

Another favorable property of the $\mathrm{Iw}_{\mathrm{u}}^+$-action on $\mathcal{G}r_H$ is that each orbit is isomorphic to an affine space. This setting is known to imply nice properties for categories of perverse sheaves (see e.g. [BGS]), and in particular that this category admits a transparent structure of highest weight category. The situation is even more favorable here in that the "relevant" orbits (i.e. those which support a nonzero Iwahori–Whittaker local system) have dimensions of constant parity in each connected component of $\mathcal{G}r_H$. This implies that the tilting objects in $\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_H, \Bbbk)$ are parity in the sense of [JMW]; in particular the indecomposable tilting perverse sheaves coincide with the self-dual indecomposable parity objects.

1.8. **Main theorems.** Recall that we let $\mathbb{G}_{\mathrm{m}}$ act on $\mathcal{G}r_H$ via loop rotation. The Iwahori–Whittaker condition and the loop rotation equivariance are compatible; we thus obtain a Smith restriction functor

$$i^{!*} : D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(\mathcal{G}r_H, \Bbbk) \to \mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_H)^{\boldsymbol{\mu}_\ell}, \Bbbk).$$

We write $\mathsf{Parity}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(\mathcal{G}r_H, \Bbbk)$ (resp. $\mathsf{SmParity}_{\mathcal{IW}}((\mathcal{G}r_H)^{\boldsymbol{\mu}_\ell}, \Bbbk)$) for the additive category of parity sheaves in the source (resp. target) of this functor. Our first main result is the following.

Theorem 1.2. *Smith restriction yields a fully faithful functor*

$$i^{!*} : \mathsf{Parity}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(\mathcal{G}r_H, \Bbbk) \xrightarrow{\sim} \mathsf{SmParity}_{\mathcal{IW}}((\mathcal{G}r_H)^{\boldsymbol{\mu}_\ell}, \Bbbk).$$

Remarkably, the proof of this theorem is a few lines once one has the appropriate technology in place. It is an easy consequence of Beĭlinson's lemma, once one knows that $i^{!*}$ preserves standard and costandard objects; this in turn follows because $i^{!*}$ commutes with $*$- and !-extension.

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[5] One can make sense of this condition in various ways. In this work (following Bezrukavnikov) we use étale sheaves and the Artin–Schreier covering, which necessitates passing to $\mathcal{G}r_H$ defined over a field of characteristic $p > 0$ (with $p \neq \ell$). The geometric Satake equivalence is also available in this setting.

Recall from §1.7 that the self-dual indecomposable Iwahori–Whittaker parity complexes on $\mathcal{G}r_H$ coincide with the indecomposable tilting perverse sheaves in $\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G)$ (which in turn correspond to tilting modules under the geometric Satake equivalence). Given $\lambda \in \rho + \mathbf{X}^+$, we denote by $\mathscr{E}_\lambda^{\mathcal{IW}}$ the corresponding indecomposable parity complex.

By Theorem 1.2, the image of any $\mathscr{E}_\lambda^{\mathcal{IW}}$ under $i^{!*}$ has to be supported on a single component. This has the following consequence, from which one easily obtains the promised proof of the linkage principle.

Corollary 1.3. *If* $\mathrm{Hom}(\mathscr{E}_\lambda^{\mathcal{IW}}, \mathscr{E}_\mu^{\mathcal{IW}}) \neq 0$ *then* $W_{\mathrm{aff}} \,\square_\ell\, \lambda = W_{\mathrm{aff}} \,\square_\ell\, \mu$.

Theorem 1.2 implies that many questions about tilting Iwahori–Whittaker perverse sheaves on $\mathcal{G}r_G$ (and hence about tilting $\mathbf{G}$-modules) may be answered after applying Smith restriction. However, in order for this to be useful, one needs another way of understanding the Smith parity complexes on $(\mathcal{G}r_H)^{\boldsymbol{\mu}_\ell}$, which is the subject of our second main result.

Consider the following diagram of quotient and forgetful functors:

$$\mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_H)^{\boldsymbol{\mu}_\ell}, \Bbbk) \xleftarrow{\;Q\;} D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}((\mathcal{G}r_H)^{\boldsymbol{\mu}_\ell}, \Bbbk) \xrightarrow{\;\mathsf{For}\;} D^{\mathrm{b}}_{\mathcal{IW}_\ell}((\mathcal{G}r_H)^{\boldsymbol{\mu}_\ell}, \Bbbk).$$

(Here the subscript in $\mathcal{IW}_\ell$ indicates that the Iwahori–Whittaker condition is imposed with respect to the action of an Iwahori subgroup in $L_\ell H$ now.) Our second main theorem is the following.

Theorem 1.4. *The functors Q and For preserve indecomposable parity complexes.*

The first step towards the proof of Theorem 1.4 is the observation that given parity complexes $\mathscr{E}, \mathscr{F} \in D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}((\mathcal{G}r_H)^{\boldsymbol{\mu}_\ell}, \Bbbk)$ of the same parity we have canonical isomorphisms:

$$\mathrm{Hom}^\bullet(\mathsf{For}(\mathscr{E}), \mathsf{For}(\mathscr{F})) = \mathrm{Hom}^\bullet(\mathscr{E}, \mathscr{F}) \otimes_{x \mapsto 0} \Bbbk. \tag{1.5}$$

$$\mathrm{Hom}(Q(\mathscr{E}), Q(\mathscr{F})) = \mathrm{Hom}^\bullet(\mathscr{E}, \mathscr{F}) \otimes_{x \mapsto 1} \Bbbk. \tag{1.6}$$

(Here, $x \in \mathsf{H}^2_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \Bbbk)$ denotes the equivariant parameter, and $\mathrm{Hom}^\bullet(\mathscr{E}, \mathscr{F})$ is viewed as an $\mathsf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \Bbbk)$-module in the natural way. The tensor products are taken over $\mathsf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \Bbbk)$, with the indicated module structure on $\Bbbk$.) The isomorphism (1.5) is simply the equivariant formality of homomorphisms between parity complexes, which follows from a standard parity argument. The isomorphism (1.6) essentially follows from the analysis of the Smith category of a point.

The statement for For in Theorem 1.4 is immediate from (1.5), and is a basic ingredient in the theory of parity complexes. The statement for Q is potentially more surprising, as inverting the equivariant parameter rarely preserves indecomposability. The key point is that the $\mathbb{G}_{\mathrm{m}}$-action on $(\mathcal{G}r_H)^{\boldsymbol{\mu}_\ell}$ factors through an action of $\mathbb{G}_{\mathrm{m}}/\boldsymbol{\mu}_\ell = \mathbb{G}_{\mathrm{m}}$. As a consequence, this action is indistinguishable from the trivial action when we take $\Bbbk$-coefficients. Hence we obtain canonical isomorphisms:

$$\mathrm{Hom}^\bullet(\mathscr{E}, \mathscr{F}) = \mathsf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \Bbbk) \otimes_\Bbbk \mathrm{Hom}^\bullet(\mathsf{For}(\mathscr{E}), \mathsf{For}(\mathscr{F})), \tag{1.7}$$

$$\mathrm{Hom}(Q(\mathscr{E}), Q(\mathscr{F})) = \mathrm{Hom}^\bullet(\mathsf{For}(\mathscr{E}), \mathsf{For}(\mathscr{F})). \tag{1.8}$$

Now the statement for Q in Theorem 1.4 follows from the statement for For, as a finite-dimensional graded algebra is local if and only if its degree-0 part is.

Remark 1.5. The isomorphism (1.8) shows that "$Q \circ \mathsf{For}^{-1}$" behaves like a degrading functor. Degrading functors are ubiquitous in modern Geometric Representation Theory. In algebra, they are often realised by forgetting the grading on a graded module; in geometry they are often associated with forgetting a mixed structure. The above shows that Smith–Treumann theory provides another possible geometric realisation of degrading functors.

1.9. **Tilting characters.** A fundamental question in the representation theory of $\mathbf{G}$ is to determine the characters of the indecomposable tilting modules. In [RW1] we started advocating the idea that character formulas for $\mathbf{G}$-modules should be expressed in terms of the ℓ-canonical basis of W_{aff}, and illustrated this idea by a conjectural formula for characters of indecomposable tilting modules in the principal block, under the assumption that ℓ is bigger than the Coxeter number h of $\mathbf{G}$. This formula was proved in case $\mathbf{G} = \mathrm{GL}_n(\Bbbk)$ in [RW1], and then for a general reductive group in a joint work with P. Achar and S. Makisumi, see [AMRW]. A simple consequence of the results of §1.8 is a new and much simpler proof of this character formula, along with an extension to a formula valid in all blocks of $\mathrm{Rep}(\mathbf{G})$, without any restriction of ℓ.

To state this formula, recall that the summands on the right-hand side of (1.1) can be parametrized by the weights in the intersection $\mathcal{A}$ of the weight lattice with the closure of the fundamental alcove for the dot action of W_{aff}. For $\lambda \in \mathcal{A}$ we denote by $W_\lambda \subset W_{\mathrm{aff}}$ the stabilizer of λ for $\bullet_\ell$ (a standard parabolic subgroup), and by $W_{\mathrm{aff}}^{(\lambda)}$ the subset of W_{aff} consisting of elements w which are both maximal in wW_λ and minimal in $W_{\mathrm{f}}w$. Then the indecomposable tilting $\mathbf{G}$-modules in the block of $W_{\mathrm{aff}} \bullet_\ell \lambda$ are in a natural bijection with $W_{\mathrm{ext}}^{(\lambda)}$, and we denote by $\mathsf{T}(w \bullet_\ell \lambda)$ the module of highest weight $w \bullet_\ell \lambda$.

The tilting character formula alluded to above can be stated as follows.

Theorem 1.6. *For any $\lambda \in \mathcal{A}$ and $w \in W_{\mathrm{aff}}^{\lambda}$ we have*

$$\mathsf{ch}(\mathsf{T}(w \bullet_\ell \lambda)) = \sum_{y \in W_{\mathrm{aff}}^{(\lambda)}} {}^\ell n_{y,w}(1) \cdot \chi_{y \bullet_\ell 0},$$

where χ_μ is the Weyl character formula attached to the dominant weight μ, and ${}^\ell n_{y,w}$ is the antispherical ℓ-Kazhdan–Lusztig polynomial attached to (y, w).

See [RW1] for a comparison of this formula with an earlier formula conjectured by Andersen [A2], which was one of our sources of inspiration.

1.10. **Simple characters.** Using ideas of Andersen [A2] recently refined by Sobaje [Sob], from the formula in Theorem 1.6 one can in theory deduce a character formula for simple $\mathbf{G}$-modules, in all blocks and all characteristics. This can be done in at least two ways. The first possibility is to use a "reciprocity formula" due to Andersen [A3] (based on earlier work of Jantzen) and which expresses multiplicities of simple modules in Weyl modules in terms of multiplicities of induced modules in indecomposable tilting modules. This method has the advantage of allowing to deduce Lusztig's conjectural formula [L1] in case the relevant ℓ-Kazhdan–Lusztig polynomials coincide with the corresponding standard Kazhdan–Lusztig polynomials, but it requires the assumption that $p \geqslant 2h - 2$, and does not produce a very natural formula in general, since it involves a certain "twist" (denoted $y \mapsto \hat{y}$ below) on indices.

To explain this in more detail, let us assume that $p \geqslant 2h-2$ and that $\mathbf{G}$ is quasi-simple, and denote by $\alpha_0^\vee$ the highest coroot. We then set

$$Y := \{w \in W_{\mathrm{aff}} \mid w \text{ is minimal in } W_{\mathrm{f}}w \text{ and } \langle w \square_\ell \rho, \alpha_0^\vee \rangle < p(h-1)\}.$$

This subset does not depend on p, and is an ideal in the Bruhat order on the set of elements w minimal in $W_{\mathrm{f}}w$; in fact, in terms of the notation of [RW2], it consists of the elements w sending the fundamental alcove A_{fund} inside the portion of the dominant region delimited by the hyperplane orthogonal to $\alpha_0^\vee$ and passing through ρ. Consider also the operation $y \mapsto \hat{y}$ on W_{aff} corresponding to the operation $A \mapsto \hat{A}$ on alcoves considered in [Soe] or [RW2]. Then, in view of [RW1, Proposition 1.8.1], from Theorem 1.6 we obtain that for any $w \in Y$ we have

$$[\nabla(w \bullet_\ell 0)] = \sum_{y \in Y} {}^\ell n_{w,\hat{y}}(1) \cdot [\mathsf{L}(y \bullet_\ell 0)]$$

in the Grothendieck group of $\mathrm{Rep}(\mathbf{G})$. In order to compare this formula with that in Lusztig's conjecture, one needs to invert these equations. In general we do not know how to do that explicitly. However, if we assume that each polynomial ${}^\ell n_{w,\hat{y}}$ in these formulas coincides with the corresponding "standard" Kazhdan–Lusztig polynomial $n_{w,\hat{y}}$ (as considered e.g. in [Soe]), then the inverse matrix is computed in [Soe, Theorem 5.1]; from this result we obtain that

$$[\mathsf{L}(w \bullet_\ell 0)] = \sum_{y \in Y} (-1)^{\ell(w)+\ell(y)} h_{y,w}(1) \cdot [\nabla(y \bullet_\ell 0)]$$

for any $w \in Y$, as predicted by Lusztig in [L1]. This property is well known to hold in large characteristic (without any explicit bound), which explains why Theorem 1.6 provides a new proof of Lusztig's conjecture in large characteristics.

Remark 1.7. The condition on w considered above is not the same as in Lusztig's formulation of his conjecture. However, the two versions are known to be equivalent under the present assumptions, due to results of Kato; see [W1, §§1.12–1.13] for more details and references.

The other method to obtain a character formula for simple $\mathbf{G}$-modules, which works for all values of p thanks to the results of [Sob], is to express multiplicities of the simple $\mathbf{G}$-modules whose highest weight is restricted in the baby Verma $\mathbf{G}_1\mathbf{T}$-modules. In this way one obtains a formula that may be compared with the "periodic" formulation of Lusztig's conjecture, see [L2]. This formula was made explicit in [RW2], under the assumption that $p \geqslant 2h-1$. The extension of the tilting character formula in Theorem 1.6 makes it desirable to extend the validity of these results to smaller values of p, and we plan to come back to this question in a future publication.

1.11. **Acknowledgements.** It is a pleasure and an honor to dedicate this paper to Roman Bezrukavnikov, who taught us so much about our subject. His inspiring influence on our works should be clear, and cannot be overestimated.

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2. Preliminaries on equivariant sheaves

2.1. **Equivariant sheaves.** We start by recalling some generalities on étales sheaves on schemes endowed with an action of a finite group. We fix a coefficient ring $\Bbbk$ (not necessarily assumed to be commutative).

Let X be a scheme, and let A be a finite group acting on X (by scheme automorphisms). For any $g \in A$, we denote by $\alpha_g : X \xrightarrow{\sim} X$ the action on X. Recall that an A-equivariant (étale) sheaf of $\Bbbk$-modules is the datum of an étale sheaf $\mathscr{F}$ of $\Bbbk$-modules on X together with a collection $(\varphi_g)_{g\in A}$ where, for any $g \in A$,

$$\varphi_g : \alpha_g^* \mathscr{F} \xrightarrow{\sim} \mathscr{F}$$

is an isomorphism of sheaves of $\Bbbk$-modules, this collection satisfying the condition that for $g, h \in A$ we have

$$\varphi_h \circ \alpha_h^*(\varphi_g) = \varphi_{gh} \tag{2.1}$$

as morphisms from $\alpha_{gh}^* \mathscr{F}$ to $\mathscr{F}$. (We will often abuse notation, and omit the isomorphisms $(\varphi_g)_{g\in A}$ from the notation.) Morphisms of A-equivariant sheaves are defined as morphisms of sheaves compatible (in the natural way) with the isomorphisms φ_g. The (abelian) category of A-equivariant sheaves of $\Bbbk$-modules will be denoted $\mathsf{Sh}_A(X, \Bbbk)$. We have a natural "forgetful" exact functor

$$\mathsf{For}_A : \mathsf{Sh}_A(X, \Bbbk) \to \mathsf{Sh}(X, \Bbbk) \tag{2.2}$$

(which simply forgets the collection of isomorphisms $(\varphi_g)_{g\in A}$), where $\mathsf{Sh}(X, \Bbbk)$ denotes the category of sheaves of $\Bbbk$-modules on X. If A acts trivially on X, we have a canonical identification

$$\mathsf{Sh}_A(X, \Bbbk) = \mathsf{Sh}(X, \Bbbk[A]). \tag{2.3}$$

If X, Y are two schemes with actions of A (with actions denoted α_-^X and α_-^Y respectively), and $f : X \to Y$ is an A-equivariant morphism, then for any $g \in A$ we have a canonical isomorphism

$$(\alpha_g^X)^* \circ f^* \cong f^* \circ (\alpha_g^Y)^*.$$

As a consequence, the functor f^* induces an exact functor

$$\mathsf{Sh}_A(Y, \Bbbk) \to \mathsf{Sh}_A(X, \Bbbk),$$

which will also be denoted f^*. Similarly, we have a canonical isomorphism

$$(\alpha_g^Y)^* \circ f_* \cong f_* \circ (\alpha_g^X)^*.$$

(Here we use the fact that $(\alpha_g^Z)^* \cong (\alpha_{g^{-1}}^Z)_*$ for $Z = X$ or Y.) As a consequence, f_* induces a functor

$$f_* : \mathsf{Sh}_A(X, \Bbbk) \to \mathsf{Sh}_A(Y, \Bbbk).$$

2.2. **Equivariant sheaves and injective resolutions.** We consider again a scheme X endowed with an action of the finite group A. For any $\mathscr{F}$ in $\mathsf{Sh}(X, \Bbbk)$ we set

$$\mathsf{Av}_A(\mathscr{F}) := \bigoplus_{g \in A} \alpha_g^* \mathscr{F}.$$

We endow $\mathsf{Av}_A(\mathscr{F})$ with the structure of an A-equivariant sheaf by defining, for any $g \in A$, the isomorphism

$$\varphi_g : \alpha_g^* \mathsf{Av}_A(\mathscr{F}) \xrightarrow{\sim} \mathsf{Av}_A(\mathscr{F})$$

as the canonical identification

$$\alpha_g^* \left(\bigoplus_{h \in A} \alpha_h^* \mathscr{F} \right) = \bigoplus_{h \in A} \alpha_{hg}^* \mathscr{F} = \bigoplus_{a \in A} \alpha_a^* \mathscr{F}.$$

This construction extends in a natural way to an exact functor

$$\mathsf{Av}_A : \mathsf{Sh}(X, \Bbbk) \to \mathsf{Sh}_A(X, \Bbbk).$$

Lemma 2.1. *The functor Av_A is left and right adjoint to the forgetful functor* (2.2)*.*

Proof. To prove the lemma we have to define morphisms of functors

$$\mathsf{For}_A \circ \mathsf{Av}_A \to \mathrm{id}, \quad \mathrm{id} \to \mathsf{For}_A \circ \mathsf{Av}_A, \quad \mathsf{Av}_A \circ \mathsf{For}_A \to \mathrm{id}, \quad \mathrm{id} \to \mathsf{Av}_A \circ \mathsf{For}_A,$$

and check the appropriate zigzag relations. Here we have

$$\mathsf{For}_A \circ \mathsf{Av}_A = \bigoplus_{g \in A} \alpha_g^*,$$

and the first two morphisms are defined as the projection to and embedding of the factor $\alpha_e^* = \mathrm{id}$. On the other hand, for $(\mathscr{F}, (\varphi_g)_{g \in A})$ in $\mathsf{Sh}_A(X)$, the morphisms

$$\mathscr{F} \to \mathsf{Av}_A \circ \mathsf{For}_A(\mathscr{F}) \to \mathscr{F}$$

are defined as $\bigoplus_{g \in A} (\varphi_g)^{-1}$ and $\bigoplus_{g \in A} \varphi_g$ respectively. (These morphisms of sheaves are morphisms of A-equivariant sheaves thanks to the cocyle condition (2.1).) The zigzag relations are all trivial to check. □

Lemma 2.1 implies that the functor Av_A sends injective objects of $\mathsf{Sh}(X, \Bbbk)$ to injective objects of $\mathsf{Sh}_A(X, \Bbbk)$. Since the category $\mathsf{Sh}(X, \Bbbk)$ has enough injectives (see [SP, Tag 01DU]), it follows that the same property holds in $\mathsf{Sh}_A(X, \Bbbk)$. In fact, if $\mathscr{F}$ belongs to $\mathsf{Sh}_A(X, \Bbbk)$, and if $\mathscr{I}$ is an injective object of $\mathsf{Sh}(X, \Bbbk)$ such that we have an injection $\mathsf{For}_A(\mathscr{F}) \hookrightarrow \mathscr{I}$, then the map $\mathscr{F} \to \mathsf{Av}_A(\mathscr{I})$ deduced by adjunction is injective, and $\mathsf{Av}_A(\mathscr{I})$ is injective in $\mathsf{Sh}_A(X, \Bbbk)$.

In particular, if $f : X \to Y$ is an A-equivariant morphism between schemes with A-actions, recall that we have the (non derived) pushforward functor

$$f_* : \mathsf{Sh}_A(X, \Bbbk) \to \mathsf{Sh}_A(Y, \Bbbk).$$

From the considerations on injective objects above we deduce that this functor admits a derived functor

$$Rf_* : D^+\mathsf{Sh}_A(X, \Bbbk) \to D^+\mathsf{Sh}_A(Y, \Bbbk),$$

which can be computed by means of injective resolutions.

Lemma 2.2. *For any injective object $\mathscr{I}$ in $\mathsf{Sh}_A(X, \Bbbk)$, the object $\mathsf{For}_A(\mathscr{I})$ is injective in $\mathsf{Sh}(X, \Bbbk)$.*

Proof. If $\mathscr{J}$ is an injective object in $\mathsf{Sh}_A(X,\Bbbk)$ and $\mathsf{For}_A(\mathscr{I}) \hookrightarrow \mathscr{J}$ is an embedding, the injection $\mathscr{I} \hookrightarrow \mathsf{Av}_A(\mathscr{J})$ must be split since both objects are injective. Hence $\mathscr{I}$ is a direct summand in $\mathsf{Av}_A(\mathscr{J})$, so that $\mathsf{For}_A(\mathscr{I})$ is a direct summand in the injective sheaf $\mathsf{For}_A \circ \mathsf{Av}_A(\mathscr{J}) = \bigoplus_{g\in A} \alpha_g^* \mathscr{J}$, hence is injective. □

From this lemma one obtains that for an equivariant morphism $f : X \to Y$ as above we have a natural commutative diagram

$$\begin{array}{ccc} D^+\mathsf{Sh}_A(X,\Bbbk) & \xrightarrow{Rf_*} & D^+\mathsf{Sh}_A(Y,\Bbbk) \\ \downarrow{\scriptstyle \mathsf{For}_A} & & \downarrow{\scriptstyle \mathsf{For}_A} \\ D^+\mathsf{Sh}(X,\Bbbk) & \xrightarrow{Rf_*} & D^+\mathsf{Sh}(Y,\Bbbk), \end{array}$$

where the lower horizontal arrow is the standard pushforward functor. In particular, in case X and Y are of finite type over some field $\mathbb{F}$ of finite cohomological dimension (e.g. algebraically closed), and $\Bbbk$ is torsion (e.g. a field of positive characteristic), it is known that the "standard" functor Rf_* sends $D^{\mathrm{b}}\mathsf{Sh}(X,\Bbbk)$ into $D^{\mathrm{b}}\mathsf{Sh}(Y,\Bbbk)$, see [SP, Tag 0F10]. It follows that the "equivariant" functor Rf_* considered above restricts to a functor

$$Rf_* : D^{\mathrm{b}}\mathsf{Sh}_A(X,\Bbbk) \to D^{\mathrm{b}}\mathsf{Sh}_A(Y,\Bbbk).$$

Of course, since the functor

$$f^* : \mathsf{Sh}_A(Y,\Bbbk) \to \mathsf{Sh}_A(X,\Bbbk)$$

is exact, we have an induced functor

$$f^* : D\mathsf{Sh}_A(Y,\Bbbk) \to D\mathsf{Sh}_A(X,\Bbbk)$$

which maps $D^+\mathsf{Sh}_A(Y,\Bbbk)$ into $D^+\mathsf{Sh}_A(X,\Bbbk)$ and $D^{\mathrm{b}}\mathsf{Sh}_A(Y,\Bbbk)$ into $D^{\mathrm{b}}\mathsf{Sh}_A(X,\Bbbk)$ (and is compatible with the usual pullback functor f^* in the obvious way). It is easily checked that the functor

$$f^* : D^+\mathsf{Sh}_A(Y,\Bbbk) \to D^+\mathsf{Sh}_A(X,\Bbbk)$$

is left adjoint to

$$Rf_* : D^+\mathsf{Sh}_A(X,\Bbbk) \to D^+\mathsf{Sh}_A(Y,\Bbbk).$$

2.3. **Equivariant sheaves and quotient map.** From now on we assume that X is of finite type over some base scheme S, and that each α_g is an automorphism of S-schemes. We assume furthermore that the action is admissible in the sense of [SGA1, Exposé 5, Définition 1.7]. Then we have a quotient scheme X/A, and a finite quotient morphism $q : X \to X/A$, see [SGA1, Exposé V, Corollaire 1.5]. (Here, by definition X/A is the scheme which represents the functor $Z \mapsto \mathrm{Hom}(X,Z)^A$, where A acts on $\mathrm{Hom}(X,Z)$ via its action on X. It can be constructed by gluing affine schemes of the form $\mathrm{Spec}(R^A)$ where $\mathrm{Spec}(R) \subset X$ is an A-stable affine open subscheme of X.)

By finiteness the functor q_* is then exact (see [SP, Tag 03QP]), and we therefore obtain a functor

$$q_* : \mathsf{Sh}_A(X,\Bbbk) \to \mathsf{Sh}_A(X/A,\Bbbk) \overset{(2.3)}{\cong} \mathsf{Sh}(X/A,\Bbbk[A]).$$

Remark 2.3. As explained in [SGA1, Exposé V, Proposition 1.8], the action of A on X is admissible iff each orbit of A is included in an affine open subset of X. This condition is automatic if $S = \mathrm{Spec}(\mathbb{F})$ for some field $\mathbb{F}$ and X is quasi-projective over S, see e.g. [Se, p. 59, Exemple 1]. (This setting is the only one we will consider in practice.)

Recall that a complex $\mathscr{G} \in D\mathsf{Sh}(X/A, \Bbbk[A])$ is said to be *of finite tor dimension* if there exist $a, b \in \mathbb{Z}$ such that for any $\mathscr{G}' \in \mathsf{Sh}(X/A, \Bbbk[A]^{\mathrm{op}})$ we have

$$\mathscr{H}^n(\mathscr{G}' \overset{L}{\otimes}_{\Bbbk[A]} \mathscr{G}) = 0 \quad \text{unless } n \in [a, b],$$

see [SP, Tag 08FZ]. Recall also that $\mathscr{G}$ has finite tor dimension iff for any geometric point $\overline{x}$ of X the complex of $\Bbbk[A]$-modules $\mathscr{G}_{\overline{x}}$ has finite tor dimension, see [SP, Tag 0DJJ].

The action of A on X induces an action on geometric points. Namely, if

$$\overline{x} : \mathrm{Spec}(K) \to X$$

is a geometric point and $g \in A$, then the geometric point $g \cdot \overline{x}$ is the composition

$$\mathrm{Spec}(K) \xrightarrow{\overline{x}} X \xrightarrow{\alpha_g} X.$$

We will say that the A-action on X is *free* if each geometric point of X has trivial stabilizer for this action.

Lemma 2.4. *Assume that $\Bbbk$ is a field, and that the A-action on X is free. Then for any $\mathscr{G}$ in $\mathsf{Sh}_A(X, \Bbbk)$, the sheaf $q_* \mathscr{G} \in \mathsf{Sh}(X, \Bbbk[A])$ has finite tor dimension.*

Proof. By the comments above, to prove the lemma it suffices to prove that for any geometric point $\overline{y} : \mathrm{Spec}(K) \to X/A$ the $\Bbbk[A]$-module $\mathscr{G}_{\overline{y}}$ has finite tor dimension. Now by [SP, Tag 03QP] we have

$$(q_* \mathscr{G})_{\overline{y}} = \bigoplus_{\overline{x}} \mathscr{G}_{\overline{x}},$$

where $\overline{x}$ runs over the set $X_{\overline{y}}$ of maps $\overline{x} : \mathrm{Spec}(K) \to X$ such that $q \circ \overline{x} = \overline{y}$. However q is surjective, and its fibers are the A-orbits (see [SGA1, Exposé V, §1]). Hence $X_{\overline{y}}$ is nonempty, and A acts transitively on this set. Our assumption ensures on the other hand that this action has trivial stabilizers. Therefore, if we fix some $\overline{x} \in X_{\overline{y}}$, we deduce a bijection $A \xrightarrow{\sim} X_{\overline{y}}$ determined by $g \mapsto g \cdot \overline{x}$. The A-equivariant structure on $\mathscr{G}$ provides a canonical isomorphism

$$\mathscr{G}_{\overline{x}} \xrightarrow{\sim} \mathscr{G}_{g \cdot \overline{x}}$$

for each $g \in A$. Using these data we obtain an isomorphism

$$(q_* \mathscr{G})_{\overline{y}} = \Bbbk[A] \otimes_{\Bbbk} \mathscr{G}_{\overline{x}},$$

which is easily seen to be A-equivariant. Hence $(q_* \mathscr{G})_{\overline{y}}$ is free as a $\Bbbk[A]$-module, hence of finite tor dimension. □

2.4. **Stalks at fixed points.** We continue with the assumptions of §2.3. The closed subscheme $X^A \subset X$ is the scheme which represents the functor $Z \mapsto \mathrm{Hom}(Z, X)^A$, where A acts on $\mathrm{Hom}(Z, X)$ via its action on X. (The representability of this functor is easy in our setting: since our action is admissible it suffices to treat the case $X = \mathrm{Spec}(R)$ is affine; then X^A is the spectrum of the maximal quotient of R on which A acts trivially, i.e. the quotient of R by the ideal generated by the elements $x - g \cdot x$ for $x \in R$ and $g \in A$.) As a set, the closed subscheme

$X^A \subset X$ consists of the points $x \in X$ which are fixed by A and such that the induced action on the residue field $k(x)$ is trivial.

By definition, any geometric point of X which is stable under the A-action considered in §2.3 factors through a geometric point of X^A. In particular, if A is a simple group then the A-action on the open subset $U := X \smallsetminus X^A \subset X$ is free.

Lemma 2.5. *The A-action on U is admissible.*

Proof. Consider the closed subset $X^A \subset X$. Since the quotient morphism q is finite, hence closed, the subset $q(X^A) \subset X/A$ is closed. Now from the definition of the subset $X^A \subset X$ given above and the fact that the fibers of q are the A-orbits in X, one sees that

$$U = q^{-1}(X/A \smallsetminus q(X^A)).$$

Hence the claim follows from [SGA1, Exposé V, Corollaire 1.4]. □

From this lemma we obtain in particular that the quotient U/A exists as a scheme. In fact, in the proof of this lemma we have seen that the open embedding $j : U \hookrightarrow X$ induces an open embedding $\overline{\jmath} : U/A \to X/A$, with complement $q(X^A)$, and that the quotient morphism $q_U : U \to U/A$ is the restriction of q to U.

We now denote by $i : X^A \to X$ the closed embedding. Note that i is A-equivariant for the trivial A-action on X^A; we therefore have a functor

$$i^* : D\mathsf{Sh}_A(X, \Bbbk) \to D\mathsf{Sh}_A(X^A, \Bbbk) \overset{(2.3)}{\cong} D\mathsf{Sh}(X^A, \Bbbk[A]).$$

In other words, if $\mathscr{F}$ belongs to $D\mathsf{Sh}_A(X, \Bbbk)$, then $i^*(\mathsf{For}_A(\mathscr{F}))$ admits a canonical "lift" as a complex of sheaves of $\Bbbk[A]$-modules. In particular, for any geometric point $\overline{x}$ of X^A the complex

$$\mathscr{F}_{i(\overline{x})} = (i^*\mathscr{F})_{\overline{x}}$$

is in a natural way an object of $D(\Bbbk[A]-\mathsf{Mod})$.

From now on we assume that $S = \mathrm{Spec}(\mathbb{F})$ for some field $\mathbb{F}$ of finite cohomological dimension, and that X is of finite type over $\mathbb{F}$. (This assumption implies that U is also of finite type over $\mathbb{F}$. By [SGA1, Corollaire 1.5] we deduce that X/A and U/A also are of finite type.) We also assume that $\Bbbk$ is a field of characteristic $\ell > 0$ which is nonzero in $\mathbb{F}$. The proof of the following proposition was explained to us by L. Illusie and W. Zheng. (A different, longer and slightly less easy proof can also be deduced from [DL, Proposition 3.7].)

Proposition 2.6. *Assume that A is a simple group. If $\mathscr{F} \in D^{\mathrm{b}}\mathsf{Sh}_A(U, \Bbbk)$ is such that $\mathsf{For}_A(\mathscr{F})$ has constructible cohomology sheaves (see [SP, Tag 03RW]), then for any geometric point $\overline{x}$ of X^A the complex of $\Bbbk[A]$-modules*

$$(Rj_*\mathscr{F})_{\overline{x}}$$

is perfect.

Proof. Recall that the morphism j has finite cohomological dimension by [SP, Tag 0F10]. As a consequence the complex $Rj_*\mathscr{F}$ is bounded, so that $(Rj_*\mathscr{F})_{\overline{x}}$ is bounded. On the other hand, by [SGA4$\frac{1}{2}$, Th. finitude, Théorème 1.1], the complex $Rj_*\mathscr{F}$ is constructible; hence its stalk $(Rj_*\mathscr{F})_{\overline{x}}$ has finite-dimensional cohomology. By [SP, Tag 0658], we deduce that to prove that this complex is perfect it suffices to prove that it has finite tor dimension. But since the image x of $\underline{x}$ is fixed by A we have $q^{-1}(q(x)) = \{x\}$, which implies that

$$(Rj_*\mathscr{F})_{\overline{x}} = \big(q_*(Rj_*\mathscr{F})\big)_{q(\overline{x})}$$

by [SP, Tag 03QP]. Now we have $q \circ j = \overline{\jmath} \circ q_U$, so that

$$q_*(Rj_*\mathscr{F}) = R\overline{\jmath}_*((q_U)_*\mathscr{F}).$$

By Lemma 2.4 the sheaf $(q_U)_*\mathscr{F}$ has finite tor-dimension, hence by [SGA4, Exposé XVII, Théorème 5.2.11] and [SP, Tag 0F10] the complex $R\overline{\jmath}_*((q_U)_*\mathscr{F})$ also does, so that its stalk

$$\big(R\overline{\jmath}_*((q_U)_*\mathscr{F})\big)_{q(\overline{x})}$$

must have finite tor dimension, which finishes the proof. □

3. Smith theory for étale sheaves

3.1. **ϖ-equivariant derived categories.** The formalism of "Smith theory" that we will build will use the equivariant derived category of Bernstein–Lunts [BL]. This category is explicitly constructed only in a topological setting in [BL], but it is well known that it applies also in the setting of étale sheaves under appropriate assumptions, see [BL, §4.3]. In this subsection we briefly recall this construction in the particular case that we require.

So, from now on we fix an algebraically closed field $\mathbb{F}$ of characteristic p, and a finite field $\Bbbk$ of characteristic $\ell \neq p$. We will consider (admissible) actions of the finite $\mathbb{F}$-group scheme $\varpi = \boldsymbol{\mu}_\ell$ of ℓ-th roots of unity on $\mathbb{F}$-schemes of finite type. Here since $\mathbb{F}$ is algebraically closed, $\boldsymbol{\mu}_\ell$ is the constant group scheme associated with the finite group $\boldsymbol{\mu}_\ell(\mathbb{F})$, so that the constructions of Section 2 also apply in this setting. For simplicity, we will not explicitly distinguish the group scheme $\boldsymbol{\mu}_\ell$ and the finite group $\boldsymbol{\mu}_\ell(\mathbb{F})$.

The construction of [BL] uses some "acyclic resolutions." In this case these resolutions can be constructed explicitly as follows: for any $n \geqslant 0$ we set

$$V_n := \mathbb{F}^n \smallsetminus \{0\},$$

with the (admissible) action of ϖ induced by the dilation action of the multiplicative group $\mathbb{G}_{\mathrm{m}}$. We have

$$\mathsf{H}^m(V_n;\Bbbk) = \begin{cases} \Bbbk & \text{if } m = 0; \\ 0 & \text{if } 1 \leqslant m \leqslant 2n-2. \end{cases}$$

From this we see that for any $\mathbb{F}$-scheme X of finite type the projection

$$p_{X,n} : V_n \times X \to X$$

is $(2n-2)$-acyclic, in the sense that for any X-scheme Y of finite type the morphism $p^Y_{X,n} : Y \times_X (V_n \times X) \to Y$ is such that for any (étale) $\Bbbk$-sheaf $\mathscr{F}$ the morphism

$$\mathscr{F} \to \tau_{\leqslant 2n-2}((p^Y_{X,n})_*(p^Y_{X,n})^*\mathscr{F})$$

induced by adjunction is an isomorphism. (In fact, here by the Künneth formula [SP, Tag 0F1N] we have $(p^Y_{X,n})_*(p^Y_{X,n})^*\mathscr{F} \cong \mathsf{H}^\bullet(V_n,\Bbbk) \otimes_\Bbbk \mathscr{F}$.)

We now fix an $\mathbb{F}$-scheme X of finite type endowed with an admissible action of ϖ. For any $n \geqslant 1$ we set

$$P^X_n := V_n \times X,$$

and consider the projection $f^X_n : P^X_n \to X$ as above. Since the actions of ϖ on V_n and X are admissible, this property also holds for the product $V_n \times X$, so that we can consider the quotient

$$\overline{P}^X_n := P^X_n/\varpi.$$

With the notation of §2.4, we have $(P_n^X)^\varpi = \varnothing$; therefore, by [SGA1, Exposé V, Corollaire 2.3] the quotient map $q_n^X : P_n^X \to \overline{P}_n^X$ is étale. In fact, in view of [SGA1, Exposé V, Proposition 2.6] this map is an étale locally trivial principal homogeneous space for ϖ in the sense of [SP, Tag 049A].

For any n, we will denote by

$$D^{\mathrm{b}}(X, \varpi, n, \Bbbk)$$

the category whose

- objects are triples $(\mathscr{F}_n, \mathscr{F}_X, \beta)$ where $\mathscr{F}_n$ is an object of $D^{\mathrm{b}}\mathsf{Sh}(\overline{P}_n^X, \Bbbk)$, $\mathscr{F}_X$ is an object of $D^{\mathrm{b}}\mathsf{Sh}(X, \Bbbk)$, and
$$\beta : (q_n^X)^* \mathscr{F}_n \xrightarrow{\sim} (p_n^X)^* \mathscr{F}_X$$
is an isomorphism;
- morphisms from $(\mathscr{F}_n, \mathscr{F}_X, \beta)$ to $(\mathscr{G}_n, \mathscr{G}_X, \gamma)$ are pairs (φ_n, φ_X) with
$$\varphi_n : \mathscr{F}_n \to \mathscr{G}_n, \qquad \varphi_X : \mathscr{F}_X \to \mathscr{G}_X$$
such that $\gamma \circ ((q_n^X)^* \varphi_n) = ((p_n^X)^* \varphi_X) \circ \beta$.

For any bounded interval $I \subset \mathbb{Z}$ we denote by $D^I(X, \varpi, n, \Bbbk)$ the full subcategory of $D^{\mathrm{b}}(X, \varpi, n, \Bbbk)$ whose objects are the triples $(\mathscr{F}_n, \mathscr{F}_X, \beta)$ where $\mathscr{H}^m(\mathscr{F}_X)$ vanishes unless $m \in I$. Then the category $D^I(X, \varpi, n, \Bbbk)$ does not depend (up to canonical equivalence) on the choice of n, as long as $2n - 2 \geqslant |I|$, where $|I|$ is the length of I; in fact, by the same arguments as in [BL, §2.3.4], if n, m satisfy this condition then the natural functors from $D^I(X, \varpi, n, \Bbbk)$ and $D^I(X, \varpi, m, \Bbbk)$ to the category defined similarly with P_n^X and P_m^X replaced by

$$V_n \times V_m \times X = P_n^X \times_X P_m^X$$

(with the diagonal ϖ-action) are equivalences of categories.

We can therefore define the ϖ-equivariant derived category

$$D^{\mathrm{b}}_\varpi(X, \Bbbk)$$

as the direct limit of the categories $D^I(X, \varpi, n, \Bbbk)$ where I runs over the bounded intervals of $\mathbb{Z}$. This category admits a canonical structure of triangulated category, see [BL, §§2.5.1–2.5.2]. By construction, we have a canonical triangulated forgetful functor

$$\mathsf{For}_\varpi : D^{\mathrm{b}}_\varpi(X, \Bbbk) \to D^{\mathrm{b}}\mathsf{Sh}(X, \Bbbk) \tag{3.1}$$

which sends a triple $(\mathscr{F}_n, \mathscr{F}_X, \beta)$ to $\mathscr{F}_X$.

Remark 3.1. As explained in [BL, Lemma 2.3.2], if $2n - 2 \geqslant |I|$ the functor

$$D^I(X, \varpi, n, \Bbbk) \to D^{\mathrm{b}}\mathsf{Sh}(\overline{P}_n^X, \Bbbk)$$

sending $(\mathscr{F}_n, \mathscr{F}_X, \beta)$ to $\mathscr{F}_n$ is fully faithful. In particular, morphisms between objects in $D^{\mathrm{b}}_\varpi(X, \Bbbk)$ can always be computed as morphisms in derived categories of quotients of "sufficiently acyclic" resolutions.

This construction is of course functorial in X. Namely, consider $\mathbb{F}$-schemes of finite type X, Y endowed with admissible actions of ϖ, and a ϖ-equivariant morphism of $\mathbb{F}$-schemes $f : X \to Y$. (Note that f is automatically quasi-compact since X is Noetherian. It is also locally of finite type by [SP, Tag 01T8], hence of finite type.)

(1) We have a ($*$-)pullback functor
$$f^* : D^{\mathrm{b}}_{\varpi}(Y, \Bbbk) \to D^{\mathrm{b}}_{\varpi}(X, \Bbbk),$$
which can be explicitly described in terms of the pullback functors associated with f and the induced morphism $\overline{P}^X_n \to \overline{P}^Y_n$.

(2) If f is quasi-separated we have a ($*$-)pushforward functor
$$Rf_* : D^{\mathrm{b}}_{\varpi}(X, \Bbbk) \to D^{\mathrm{b}}_{\varpi}(Y, \Bbbk).$$
(Here we use the fact that the usual pushforward functors respect the bounded derived categories, see [SP, Tag 0F10], and also that the morphisms p^X_n, p^Y_n, q^X_n, q^Y_n are smooth and that the induced morphism $\overline{P}^X_n \to \overline{P}^Y_n$ is quasi-compact and quasi-separated, which allows to use the smooth base change theorem [SP, Tag 0EYU] to "transport" the isomorphism β.)

(3) If we assume that f is separated and that Y is quasi-separated, then we also have a !-pushforward functor
$$Rf_! : D^{\mathrm{b}}_{\varpi}(X, \Bbbk) \to D^{\mathrm{b}}_{\varpi}(Y, \Bbbk),$$
see [SGA4, Exposé XVII]. (Here, the fact that the !-pushforward functors respect bounded derived categories follows from [SGA4, Exposé XVII, Corollaire 5.2.8.1], and we use the base change theorem [SGA4, Exposé XVII, Théorème 5.2.6] to transport β.)

(4) Under these assumptions we also have a !-pullback functor
$$f^! : D^{\mathrm{b}}_{\varpi}(Y, \Bbbk) \to D^{\mathrm{b}}_{\varpi}(X, \Bbbk),$$
see [SGA4, Exposé XVIII]. (The fact that the !-pullback functors respect bounded derived categories is explained in [SGA4$\frac{1}{2}$, Th. finitude, comments after Corollaire 1.5]. And once again we use the smoothness of q^X_n, q^Y_n, p^X_n, p^Y_n, and the fact that for smooth maps the $*$- and !-pullback functors coincide up to shift, see [SGA4, Exposé XVIII, Théorème 3.2.5], to transport the isomorphisms β.)

By construction, all of these functors are compatible with the forgetful functor (3.1) in the obvious way.

Remark 3.2. In practice all the schemes we will consider will be quasi-projective, hence separated, so that any morphism between them will automatically be separated.

3.2. **Equivariant derived categories and equivariant sheaves.** We continue with the setting of §3.1. In Section 2 we have studied equivariant sheaves on schemes, and in §3.1 we have considered the equivariant derived category. It is now time to explain the relation between these two constructions. This relation is based on the observation that (for any $\mathbb{F}$-scheme X of finite type with an admissible ϖ-action, and any $n \geqslant 1$) the natural pullback functor
$$\mathsf{Sh}(\overline{P}^X_n, \Bbbk) \to \mathsf{Sh}_{\varpi}(P^X_n, \Bbbk)$$
is an equivalence of categories, by the sheaf condition applied to the étale covering $q^X_n : P^X_n \to \overline{P}^X_n$. Therefore, for any ϖ-equivariant sheaf $\mathscr{F}$ on X the pullback $(p^X_n)^*\mathscr{F}$ admits a natural structure of ϖ-equivariant sheaf on P^X_n, hence descends

to a sheaf $\mathscr{F}_n$ on $\overline{P}_n^X$. Using this construction we define a canonical triangulated functor

$$D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(X,\Bbbk) \to D^{\mathrm{b}}_{\varpi}(X,\Bbbk). \tag{3.2}$$

Proposition 3.3. *The functor* (3.2) *is an equivalence of categories.*

Proof. Let us first show that our functor is fully faithful. For this we need to show that for any ϖ-equivariant sheaves $\mathscr{F},\mathscr{F}'$ on X and any $m \in \mathbb{Z}_{\geqslant 0}$, for $n \gg 0$ the natural map

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(X,\Bbbk)}(\mathscr{F},\mathscr{F}'[m]) \to \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Sh}(\overline{P}_n^X,\Bbbk)}(\mathscr{F}_n,\mathscr{F}'_n[m])$$

is an isomorphism, see Remark 3.1. By construction, this amounts to proving that for $n \gg 0$ the pullback functor induces an isomorphism

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(X,\Bbbk)}(\mathscr{F},\mathscr{F}'[m]) \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(P_n^X,\Bbbk)}((p_n^X)^*\mathscr{F},(p_n^X)^*\mathscr{F}'[m]).$$

However by adjunction we have

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(P_n^X,\Bbbk)}((p_n^X)^*\mathscr{F},(p_n^X)^*\mathscr{F}'[m]) \cong \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(X,\Bbbk)}(\mathscr{F},R(p_n^X)_*(p_n^X)^*\mathscr{F}'[m]).$$

Now the right-hand side can be replaced by

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(X,\Bbbk)}(\mathscr{F},\tau_{\leqslant 0}(R(p_n^X)_*(p_n^X)^*\mathscr{F}'[m])) \cong \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(X,\Bbbk)}(\mathscr{F},\tau_{\leqslant m}(R(p_n^X)_*(p_n^X)^*\mathscr{F}')[m]).$$

If $2n-2 \geqslant m$ the morphism

$$\mathscr{F}' \to \tau_{\leqslant m}(R(p_n^X)_*(p_n^X)^*\mathscr{F}')$$

induced by adjunction is an isomorphism, which concludes the proof of fully faithfulness.

Once fully faithfulness is established, to conclude the proof it suffices to prove that images of ϖ-equivariant sheaves on X generate $D^{\mathrm{b}}_{\varpi}(X,\Bbbk)$ as a triangulated category. This is however clear from the construction of the "standard" t-structure in [BL, §§2.5.1–2.5.2]. □

In particular, in the special case where the ϖ-action on X is trivial, using Proposition 3.3 combined with (2.3), we obtain a canonical equivalence of triangulated categories

$$D^{\mathrm{b}}_{\varpi}(X,\Bbbk) \cong D^{\mathrm{b}}\mathsf{Sh}(X,\Bbbk[\varpi]). \tag{3.3}$$

We now consider two $\mathbb{F}$-schemes X and Y of finite type, with admissible actions of ϖ, and a quasi-separated ϖ-equivariant morphism $f : X \to Y$. We have considered functors

$$D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(X,\Bbbk) \underset{f^*}{\overset{Rf_*}{\rightleftarrows}} D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(Y,\Bbbk)$$

in §2.2, and functors

$$D^{\mathrm{b}}_{\varpi}(X,\Bbbk) \underset{f^*}{\overset{Rf_*}{\rightleftarrows}} D^{\mathrm{b}}_{\varpi}(Y,\Bbbk)$$

in §3.1. These functors are related in the natural way, as explained in the following lemma.

Lemma 3.4. *The diagrams*

$$\begin{array}{ccc} D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(Y,\Bbbk) & \xrightarrow{f^*} & D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(X,\Bbbk) \\ \wr\downarrow & & \downarrow\wr \\ D^{\mathrm{b}}_{\varpi}(Y,\Bbbk) & \xrightarrow{f^*} & D^{\mathrm{b}}_{\varpi}(X,\Bbbk) \end{array} \quad \text{and} \quad \begin{array}{ccc} D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(X,\Bbbk) & \xrightarrow{Rf_*} & D^{\mathrm{b}}\mathsf{Sh}_{\varpi}(Y,\Bbbk) \\ \wr\downarrow & & \downarrow\wr \\ D^{\mathrm{b}}_{\varpi}(X,\Bbbk) & \xrightarrow{Rf_*} & D^{\mathrm{b}}_{\varpi}(Y,\Bbbk) \end{array}$$

are commutative, where the vertical arrows are the equivalences of Proposition 3.3.

Proof. The commutativity of the left diagram can be seen on the definitions; the commutativity of the right diagram follows by adjunction. □

3.3. **The crucial lemma.** We can now prove the lemma that will allow us to develop the "Smith theory for sheaves" from [Tr] in our setting of étale sheaves.

We consider again an $\mathbb{F}$-scheme X of finite type, with an admissible action of ϖ. As in §2.4 we consider the fixed points subscheme X^{ϖ} and the closed, resp. open, embedding

$$i : X^{\varpi} \hookrightarrow X, \quad \text{resp.} \quad j : X \smallsetminus X^{\varpi} \hookrightarrow X.$$

(Here i and j are automatically separated, see [SP, Tag 01L7].) For any $\mathscr{F}$ in $D^{\mathrm{b}}_{\varpi}(X,\Bbbk)$ we have a canonical morphism

$$i^! \mathscr{F} \to i^* \mathscr{F} \tag{3.4}$$

in the category $D^{\mathrm{b}}_{\varpi}(X^{\varpi},\Bbbk)$, which can be obtained by applying the functor i^* to the adjunction morphism $i_! i^! \mathscr{F} \to \mathscr{F}$.

From now on we will not consider the entire ϖ-equivariant derived category $D^{\mathrm{b}}_{\varpi}(X,\Bbbk)$, but only the full triangulated subcategory $D^{\mathrm{b}}_{\varpi,c}(X,\Bbbk)$ whose objects are those $\mathscr{F} \in D^{\mathrm{b}}_{\varpi}(X,\Bbbk)$ such that the complex $\mathsf{For}_{\varpi}(\mathscr{F})$ has constructible cohomology objects, where For_{ϖ} is as in (3.1). For any $\mathbb{F}$-scheme Y of finite type with trivial action of ϖ, we will say that an object $\mathscr{F}$ in $D^{\mathrm{b}}_{\varpi,c}(Y,\Bbbk)$ has perfect geometric stalks if, denoting by $\mathscr{F}'$ the image of $\mathscr{F}$ under the equivalence $D^{\mathrm{b}}_{\varpi}(Y,\Bbbk) \cong D^{\mathrm{b}}\mathsf{Sh}(Y,\Bbbk[\varpi])$ from (3.3), for any geometric point $\overline{y}$ of Y the complex $\mathscr{F}'_{\overline{y}}$ is a perfect complex of $\Bbbk[\varpi]$-modules.

Lemma 3.5. *For any $\mathscr{F}$ in $D^{\mathrm{b}}_{\varpi,c}(X,\Bbbk)$, the cone of (3.4) has perfect geometric stalks.*

Proof. From the standard distinguished triangle in the "recollement" formalism we see that the cone of (3.4) is isomorphic to $i^* Rj_* j^*(\mathscr{F})$. The complex we want to consider is therefore

$$\big(Rj_* j^*(\mathscr{F})\big)_{\overline{x}},$$

where $\overline{x}$ is a geometric point of X^{ϖ}. In these terms, the desired claim follows from Proposition 2.6 and Lemma 3.4. □

Later we will also need the following lemma, whose proof is close to that of Lemma 3.5. Here we consider two $\mathbb{F}$-schemes of finite type Z and Y with trivial actions of ϖ, and a quasi-separated morphism of $\mathbb{F}$-schemes $f : Z \to Y$. Then we have a derived functor $Rf_* : D^{\mathrm{b}}_{\varpi}(Z,\Bbbk) \to D^{\mathrm{b}}_{\varpi}(Y,\Bbbk)$, see §3.1, which sends the subcategory $D^{\mathrm{b}}_{\varpi,c}(Z,\Bbbk)$ into $D^{\mathrm{b}}_{\varpi,c}(Y,\Bbbk)$ by [SGA4$\frac{1}{2}$, Th. finitude, Théorème 1.1].

Lemma 3.6. *The functor*

$$Rf_* : D^{\mathrm{b}}_{\varpi,c}(Z,\Bbbk) \to D^{\mathrm{b}}_{\varpi,c}(Y,\Bbbk)$$

transforms objects with perfect geometric stalks into objects with perfect geometric stalks.

Proof. As in the proof of Proposition 2.6 it suffices to check that Rf_* transforms objects of finite tor dimension into objects of finite tor dimension, which follows from [SGA4, Exposé XVII, Théorème 5.2.11] and [SP, Tag 0F10]. □

3.4. $\mathbb{G}_{\mathrm{m}}$**-equivariant derived categories.** Later we will also need to consider equivariant derived categories for actions of the multiplicative group $\mathbb{G}_{\mathrm{m}}$ over $\mathbb{F}$. The construction of this category is similar to, and in fact simpler than, the construction in §3.1. Namely, for any $n \geqslant 1$ the ϖ-action on V_n is obtained by restriction from a natural $\mathbb{G}_{\mathrm{m}}$-action, and moreover we have a canonical map $V_n \to \mathbb{P}^{n-1}_{\mathbb{F}}$ which is a Zariski locally trivial principal $\mathbb{G}_{\mathrm{m}}$-bundle. Therefore, given any $\mathbb{F}$-scheme X endowed with an action of $\mathbb{G}_{\mathrm{m}}$, we consider the diagonal action on $V_n \times X$, and we have a Zariski locally trivial principal $\mathbb{G}_{\mathrm{m}}$-bundle

$$V_n \times X \to V_n \times^{\mathbb{G}_{\mathrm{m}}} X$$

for some scheme $V_n \times^{\mathbb{G}_{\mathrm{m}}} X$ which can be constructed by (Zariski) gluing over the natural cover which trivializes the map $V_n \to \mathbb{P}^{n-1}_{\mathbb{F}}$. If we assume X to be of finite type, then as in §3.1 the map $p_{X,n} : V_n \times X \to X$ is $(2n-2)$-acyclic, which allows to define a category $D^{\mathrm{b}}(X, \mathbb{G}_{\mathrm{m}}, n, \Bbbk)$ in terms similar to those for $D^{\mathrm{b}}(-, \varpi, n, \Bbbk)$, and check that the subcategory $D^I(X, \mathbb{G}_{\mathrm{m}}, n, \Bbbk)$ does not depend on the choice of n as long as $2n-2 \geqslant |I|$. We can finally define the equivariant derived category $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X, \Bbbk)$ as the direct limit of the categories $D^I(X, \mathbb{G}_{\mathrm{m}}, n, \Bbbk)$ (with $n \gg 0$) over the finite intervals $I \subset \mathbb{Z}$. These categories have the same functoriality properties as the categories $D^{\mathrm{b}}_{\varpi}(X, \Bbbk)$; in particular we have a natural (triangulated) forgetful functor

$$\mathsf{For}_{\mathbb{G}_{\mathrm{m}}} : D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X, \Bbbk) \to D^{\mathrm{b}}\mathsf{Sh}(X, \Bbbk).$$

As in the ϖ-equivariant setting, we will denote by $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(X, \Bbbk)$ the full subcategory of $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X, \Bbbk)$ whose objects are those $\mathscr{F}$ such that $\mathsf{For}_{\mathbb{G}_{\mathrm{m}}}(\mathscr{F})$ has constructible cohomology sheaves.

The category $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X, \Bbbk)$ has a canonical object whose image under $\mathsf{For}_{\mathbb{G}_{\mathrm{m}}}$ is the constant sheaf $\underline{\Bbbk}_X$; it will also be denoted $\underline{\Bbbk}_X$. In these terms, the $\mathbb{G}_{\mathrm{m}}$-equivariant cohomology of a complex $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X, \Bbbk)$ is defined as

$$\mathsf{H}^{\bullet}_{\mathbb{G}_{\mathrm{m}}}(X, \mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X,\Bbbk)}(\underline{\Bbbk}_X, \mathscr{F}[n]).$$

In the case $X = \mathrm{Spec}(\mathbb{F}) =: \mathrm{pt}$, it is well known (and easy to see) that we have a graded algebra isomorphism[6]

$$\mathsf{H}^{\bullet}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \underline{\Bbbk}_{\mathrm{pt}}) = \Bbbk[x], \tag{3.5}$$

where x has degree 2. If n is even, we will denote by

$$\mathrm{can}^n_{\mathrm{pt}} : \underline{\Bbbk}_{\mathrm{pt}} \to \underline{\Bbbk}_{\mathrm{pt}}[n]$$

[6]To be more precise, to get the isomorphism (3.5) one needs to fix a trivialization of the Tate sheaf on pt, see e.g. the proof of Lemma 3.9 below. This is possible—though not canonical—since $\mathbb{F}$ is algebraically closed; we fix such a trivialization once and for all.

the morphism obtained as the inverse image of $x^{n/2}$.

By definition we have an embedding $\varpi \subset \mathbb{G}_{\mathrm{m}}$, which provides a ϖ-action on X by restriction, and we have a canonical "restriction" triangulated functor

$$\mathrm{Res}^{\mathbb{G}_{\mathrm{m}}}_{\varpi} : D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X, \Bbbk) \to D^{\mathrm{b}}_{\varpi}(X, \Bbbk).$$

This functor is compatible (in the obvious sense) with the pushforward and pullback functors when they are defined, and moreover satisfies

$$\mathsf{For}_{\varpi} \circ \mathrm{Res}^{\mathbb{G}_{\mathrm{m}}}_{\varpi} \cong \mathsf{For}_{\mathbb{G}_{\mathrm{m}}}.$$

As a consequence, it must send $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(X, \Bbbk)$ into $D^{\mathrm{b}}_{\varpi,c}(X, \Bbbk)$.

3.5. **The "Smith category" of a point.** In this subsection we consider the special case of the constructions of §3.4 where $X = \mathrm{pt}$. In this case, in view of (3.3) we have an equivalence of triangulated categories

$$D^{\mathrm{b}}_{\varpi}(\mathrm{pt}, \Bbbk) \cong D^{\mathrm{b}}(\Bbbk[\varpi]\text{−}\mathsf{Mod}). \tag{3.6}$$

Under this equivalence, the full subcategory $D^{\mathrm{b}}_{\varpi,c}(\mathrm{pt}, \Bbbk)$ corresponds to the full subcategory of $D^{\mathrm{b}}(\Bbbk[\varpi]\text{−}\mathsf{Mod})$ whose objects are the complexes whose cohomology is finite dimensional (or equivalently finitely generated over $\Bbbk[\varpi]$), which itself is canonically equivalent to the category $D^{\mathrm{b}}(\Bbbk[\varpi]\text{−}\mathsf{Mof})$, where $\Bbbk[\varpi]\text{−}\mathsf{Mof}$ is the category of finite-dimensional $\Bbbk[\varpi]$-modules.

We will denote by

$$D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt}, \Bbbk)_{\varpi-\mathsf{perf}} \subset D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt}, \Bbbk)$$

the full triangulated subcategory whose objects are the complexes $\mathscr{F}$ such that $\mathrm{Res}^{\mathbb{G}_{\mathrm{m}}}_{\varpi}(\mathscr{F})$, considered as a complex of $\Bbbk[\varpi]$-modules through (3.6), is perfect (i.e. isomorphic in $D^{\mathrm{b}}(\Bbbk[\varpi]\text{−}\mathsf{Mod})$ to a bounded complex of finitely generated projective modules).

We then set

$$\mathsf{Sm}(\mathrm{pt}, \Bbbk) := D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt}, \Bbbk) / D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt}, \Bbbk)_{\varpi-\mathsf{perf}},$$

where we consider the Verdier quotient category.

The following lemma will be crucial for us below, in that it will allow to use some parity vanishing arguments in various variants of the category $\mathsf{Sm}(\mathrm{pt}, \Bbbk)$.

Lemma 3.7. *For any $n \in \mathbb{Z}$ we have*

$$\mathrm{Hom}_{\mathsf{Sm}(\mathrm{pt},\Bbbk)}(\underline{\Bbbk}_{\mathrm{pt}}, \underline{\Bbbk}_{\mathrm{pt}}[n]) = \begin{cases} \Bbbk & \text{if } n \text{ is even;} \\ 0 & \text{otherwise,} \end{cases}$$

where we omit the quotient functor $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt}, \Bbbk) \to \mathsf{Sm}(\mathrm{pt}, \Bbbk)$.

The proof of Lemma 3.7 will require some preparation. We start with the following claim.

Lemma 3.8. *For any $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt}, \Bbbk)$, there exists a canonical isomorphism of graded $\Bbbk$-vector spaces*

$$\bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}_{D^{\mathrm{b}}_{\varpi}(\mathrm{pt},\Bbbk)}\big(\underline{\Bbbk}_{\mathrm{pt}}, \mathrm{Res}^{\mathbb{G}_{\mathrm{m}}}_{\varpi}(\mathscr{F})[m]\big) \cong \mathsf{H}^{\bullet}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \mathscr{F}) \oplus \mathsf{H}^{\bullet}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \mathscr{F})[1].$$

Proof. We fix $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)$ and $m \in \mathbb{Z}$. Then for $n \gg 0$ the object $\mathscr{F}$ is represented by a triple $(\mathscr{F}_n, \mathscr{F}_X, \beta)$ in $D^{\mathrm{b}}(\mathrm{pt}, \mathbb{G}_{\mathrm{m}}, n, \Bbbk)$, and by an analogue of Remark 3.1 we have

$$\mathsf{H}^m_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \mathscr{F}) = \mathrm{Hom}_{D^{\mathrm{b}}(\mathbb{P}^{n-1}_{\mathbb{F}},\Bbbk)}(\underline{\Bbbk}_{\mathbb{P}^{n-1}_{\mathbb{F}}}, \mathscr{F}_n[m]),$$

and similarly for $\mathsf{H}^{m+1}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt}, \mathscr{F})$. If we denote by

$$\pi_n : V_n/\varpi \to \mathbb{P}^{n-1}_{\mathbb{F}}$$

the natural map, then $\mathrm{Res}^{\mathbb{G}_{\mathrm{m}}}_{\varpi}(\mathscr{F})$ is represented by the object $((\pi_n)^*\mathscr{F}_n, \mathscr{F}_X, \beta)$ in $D^{\mathrm{b}}(\mathrm{pt}, \varpi, n, \Bbbk)$, so that we have

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\varpi}(\mathrm{pt},\Bbbk)}\big(\underline{\Bbbk}_{\mathrm{pt}}, \mathrm{Res}^{\mathbb{G}_{\mathrm{m}}}_{\varpi}(\mathscr{F})[m]\big) = \mathrm{Hom}_{D^{\mathrm{b}}(V_n/\varpi,\Bbbk)}(\underline{\Bbbk}_{V_n/\varpi}, (\pi_n)^*\mathscr{F}_n[m]).$$

To prove the lemma, it therefore suffices to prove that for any $\mathscr{G}$ in $D^{\mathrm{b}}(\mathbb{P}^{n-1}_{\mathbb{F}}, \Bbbk)$ we have a canonical isomorphism

$$\begin{aligned}\mathrm{Hom}_{D^{\mathrm{b}}(V_n/\varpi,\Bbbk)}(\underline{\Bbbk}_{V_n/\varpi}, (\pi_n)^*\mathscr{G}) \cong \\ \mathrm{Hom}_{D^{\mathrm{b}}(\mathbb{P}^{n-1}_{\mathbb{F}},\Bbbk)}(\underline{\Bbbk}_{\mathbb{P}^{n-1}_{\mathbb{F}}}, \mathscr{G}) \oplus \mathrm{Hom}_{D^{\mathrm{b}}(\mathbb{P}^{n-1}_{\mathbb{F}},\Bbbk)}(\underline{\Bbbk}_{\mathbb{P}^{n-1}_{\mathbb{F}}}, \mathscr{G}[1]).\end{aligned} \tag{3.7}$$

We start by proving that

$$(\pi_n)_*\underline{\Bbbk}_{V_n/\varpi} = \underline{\Bbbk}_{\mathbb{P}^{n-1}_{\mathbb{F}}} \oplus \underline{\Bbbk}_{\mathbb{P}^{n-1}_{\mathbb{F}}}[-1]. \tag{3.8}$$

In fact, V_n/ϖ is the complement of the zero section in the line bundle $\tilde{\pi}_n : \mathscr{O}(\ell) \to \mathbb{P}^{n-1}_{\mathbb{F}}$. If $i_n : \mathbb{P}^{n-1}_{\mathbb{F}} \hookrightarrow \mathscr{O}(\ell)$ is the embedding of the zero section, and $j_n : V_n/\varpi \hookrightarrow \mathscr{O}(\ell)$ is the complementary open embedding, then we have a distinguished triangle

$$(i_n)_*\underline{\Bbbk}_{\mathbb{P}^{n-1}_{\mathbb{F}}}[-2] \to \underline{\Bbbk}_{\mathscr{O}(\ell)} \to j_*\underline{\Bbbk}_{V_n/\varpi} \xrightarrow{[1]} .$$

Applying the functor $(\tilde{\pi}_n)_*$ we deduce a distinguished triangle

$$\underline{\Bbbk}_{\mathbb{P}^{n-1}_{\mathbb{F}}}[-2] \to \underline{\Bbbk}_{\mathbb{P}^{n-1}_{\mathbb{F}}} \to (\pi_n)_*\underline{\Bbbk}_{V_n/\varpi} \xrightarrow{[1]},$$

in which is the first map is (by definition) the (shift by -2 of the) Euler class of $\mathscr{O}(\ell)$. Since $\Bbbk$ has characteristic ℓ this Euler class vanishes, and we deduce the desired isomorphism (3.8).

Next, we claim that for any $\mathscr{G}$ in $D^{\mathrm{b}}\mathsf{Sh}(\mathbb{P}^{n-1}_{\mathbb{F}}, \Bbbk)$ we have a canonical isomorphism

$$\mathscr{G} \otimes_{\Bbbk} (\pi_n)_*\underline{\Bbbk}_{V_n/\varpi} \xrightarrow{\sim} (\pi_n)_*(\pi_n)^*\mathscr{G}. \tag{3.9}$$

In fact adjunction provides a canonical morphism from the left-hand side to the right-hand side. To prove that this morphism is invertible it suffices to check this property after pullback under the surjective morphism π_n. However π_n is a principal $\mathbb{G}_{\mathrm{m}}/\varpi = \mathbb{G}_{\mathrm{m}}$-bundle, so that we have a Cartesian diagram

$$\begin{array}{ccc} V_n/\varpi \times \mathbb{G}_{\mathrm{m}} & \longrightarrow & V_n/\varpi \\ \downarrow & & \downarrow{\scriptstyle \pi_n} \\ V_n/\varpi & \xrightarrow{\pi_n} & \mathbb{P}^{n-1}_{\mathbb{F}}, \end{array}$$

hence the claim follows from the smooth base change theorem [SP, Tag 0EYU].

Combining (3.8) and (3.9) we obtain, for any $\mathscr{G}$ in $D^{\mathrm{b}}\mathsf{Sh}(\mathbb{P}^{n-1}_{\mathbb{F}}, \Bbbk)$, an isomorphism

$$(\pi_n)_*(\pi_n)^*\mathscr{G} \cong \mathscr{G} \oplus \mathscr{G}[-1].$$

In view of the isomorphism

$$\mathrm{Hom}_{D^{\mathrm{b}}(V_n/\varpi,\Bbbk)}(\underline{\Bbbk}_{V_n/\varpi},(\pi_n)^*\mathscr{G}) = \mathrm{Hom}_{D^{\mathrm{b}}(V_n/\varpi,\Bbbk)}((\pi_n)^*\underline{\Bbbk}_{\mathbb{P}^{n-1}_{\mathbb{F}}},(\pi_n)^*\mathscr{G})$$
$$\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathbb{P}^{n-1}_{\mathbb{F}},\Bbbk)}(\underline{\Bbbk}_{\mathbb{P}^{n-1}_{\mathbb{F}}},(\pi_n)_*(\pi_n)^*\mathscr{G}),$$

this implies (3.7), hence finishes the proof of the lemma. □

Using Lemma 3.8 we will be able to give a more explicit description of the category $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)_{\varpi-\mathsf{perf}}$, as follows.

Lemma 3.9. *The full subcategory $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)_{\varpi-\mathsf{perf}} \subset D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)$ consists of the complexes $\mathscr{F}$ such that*

$$\dim_{\Bbbk}\big(\mathsf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\mathscr{F})\big) < \infty.$$

As a consequence, in $\mathsf{Sm}(\mathrm{pt},\Bbbk)$ we have a canonical isomorphism of functors

$$\mathrm{id} \cong [2].$$

Proof. Recall from (3.5) that there exists a canonical morphism

$$\mathrm{can}^2_{\mathrm{pt}} : \underline{\Bbbk}_{\mathrm{pt}} \to \underline{\Bbbk}_{\mathrm{pt}}[2]$$

in $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\Bbbk)$. More explicitly, this morphism can be constructed as follows: consider the natural dilation action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{A}^1_{\mathbb{F}}$. If we denote by $i : \mathrm{pt} = \{0\} \hookrightarrow \mathbb{A}^1_{\mathbb{F}}$ the embedding, then we have $i^!(\underline{\Bbbk}_{\mathbb{A}^1_{\mathbb{F}}}) = \underline{\Bbbk}_{\mathrm{pt}}[-2]$. Using adjunction, we deduce a canonical map

$$i_*(\underline{\Bbbk}_{\mathrm{pt}}[-2]) \to \underline{\Bbbk}_{\mathbb{A}^1_{\mathbb{F}}}.$$

(Note that we have ignored a Tate twist here; see Footnote 6.) Applying i^* and shifting by 2, we obtain the morphism $\mathrm{can}^2_{\mathrm{pt}}$. Since $(\mathbb{A}^1_{\mathbb{F}})^{\varpi} = \{0\}$, from this description and Lemma 3.5 we obtain that the cone C of $\mathrm{can}^2_{\mathrm{pt}}$ belongs to $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)_{\varpi-\mathsf{perf}}$. In particular, since the tensor product of any bounded complex with a perfect complex is perfect, this implies that for any $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\Bbbk)$ we have a canonical isomorphism

$$\mathscr{F} \xrightarrow{\sim} \mathscr{F}[2],$$

providing the desired isomorphism of functors $\mathrm{id} \cong [2]$.

Now we claim that the triangulated subcategory $\langle C\rangle_\Delta$ of $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)$ generated by C is exactly the subcategory whose objects are the complexes $\mathscr{F}$ such that

$$\dim_{\Bbbk}\big(\mathsf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\mathscr{F})\big) < \infty.$$

Indeed we have $\mathsf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},C) = \Bbbk[2]$, so that C belongs to this subcategory. To prove the opposite inclusion, we prove by induction that for any $n \in \mathbb{Z}_{\geqslant 0}$, any complex $\mathscr{F}$ such that $\dim_{\Bbbk}\big(\mathsf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\mathscr{F})\big) = n$ belongs to $\langle C\rangle_\Delta$. In fact, if $n = 0$ then using Lemma 3.8 we see that any object $\mathscr{F}$ such that $\mathsf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\mathscr{F}) = 0$ satisfies $\mathrm{Res}^{\mathbb{G}_{\mathrm{m}}}_{\varpi}(\mathscr{F}) = 0$, hence $\mathsf{For}_{\mathbb{G}_{\mathrm{m}}}(\mathscr{F}) = 0$. From the definition, we see that this implies that $\mathscr{F} = 0$. Fix now $n \geqslant 0$, and assume the result is known for n. If $\dim_{\Bbbk}\big(\mathsf{H}^\bullet_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\mathscr{F})\big) = n+1$, and if m is maximal such that

$$\mathsf{H}^m_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\mathscr{F}) \neq 0,$$

then any choice of a nonzero vector in this space provides a morphism

$$\underline{\Bbbk}_{\mathrm{pt}}[-m] \to \mathscr{F}$$

in $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X,\Bbbk)$. By maximality the composition of this map with $\mathrm{can}^2_{\mathrm{pt}}[-m-2]$ vanishes, so that this map must factor through a morphism

$$C[-m-2] \to \mathscr{F}.$$

From the long exact sequence in equivariant cohomology we see that the cone $\mathscr{G}$ of this map satisfies $\dim_{\Bbbk}\big(\mathsf{H}^{\bullet}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\mathscr{G})\big) = n$, which allows to conclude by induction.

The two claims we have proved so far show that the subcategory with objects those complexes $\mathscr{F}$ such that $\mathsf{H}^{\bullet}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\mathscr{F})$ is finite dimensional is included in $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)_{\varpi-\mathsf{perf}}$. On the other hand, if $\mathscr{F}$ is an object of $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)$ such that $\mathrm{Res}^{\mathbb{G}_{\mathrm{m}}}_{\varpi}(\mathscr{F})$ is perfect, then

$$\dim_{\Bbbk}\left(\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^{\mathrm{b}}_{\varpi}(\mathrm{pt},\Bbbk)}(\underline{\Bbbk}_{\mathrm{pt}}, \mathrm{Res}^{\mathbb{G}_{\mathrm{m}}}_{\varpi}(\mathscr{F})[n])\right) < \infty.$$

From Lemma 3.8 we deduce that in this case $\mathsf{H}^{\bullet}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\mathscr{F})$ is finite dimensional, which concludes the proof. □

We can finally give the proof of Lemma 3.7.

Proof of Lemma 3.7. Lemma 3.9 shows in particular that $\underline{\Bbbk}_{\mathrm{pt}}$ does not belong to $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)_{\varpi-\mathsf{perf}}$, hence has nonzero image in $\mathsf{Sm}(\mathrm{pt},\Bbbk)$. In view of the isomorphism $\mathrm{id} \cong [2]$, this shows that $\mathrm{Hom}_{\mathsf{Sm}(\mathrm{pt},\Bbbk)}(\underline{\Bbbk}_{\mathrm{pt}}, \underline{\Bbbk}_{\mathrm{pt}}[n]) \neq 0$ for any even n. Hence to conclude it only remains to prove that

$$\dim \mathrm{Hom}_{\mathsf{Sm}(\mathrm{pt},\Bbbk)}(\underline{\Bbbk}_{\mathrm{pt}}, \underline{\Bbbk}_{\mathrm{pt}}[n]) \leqslant \begin{cases} 1 & \text{if } n \text{ is even;} \\ 0 & \text{otherwise} \end{cases}$$

A morphism ϖ from $\underline{\Bbbk}_{\mathrm{pt}}$ to $\underline{\Bbbk}_{\mathrm{pt}}$ in $\mathsf{Sm}(\mathrm{pt},\Bbbk)$ is represented by a diagram

$$\underline{\Bbbk}_{\mathrm{pt}} \xleftarrow{f} \mathscr{F} \xrightarrow{g} \underline{\Bbbk}_{\mathrm{pt}}[n]$$

in which $\mathscr{F}$ belongs to $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)$, f and g are morphisms in $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)$, and the cone of f belongs to $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)_{\varpi-\mathsf{perf}}$, i.e. has finite-dimensional cohomology (see Lemma 3.9). In particular, from the long exact sequence in equivariant cohomology and (3.5) we obtain that there exists $N \in 2\mathbb{Z}$ (which, for later use, we will assume to be at least $-n$) such that for $m \geqslant N$ we have

$$\mathsf{H}^{m}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\mathscr{F}) = \begin{cases} \Bbbk & \text{if } m \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

If we choose a nonzero element in $\mathsf{H}^{N}_{\mathbb{G}_{\mathrm{m}}}(\mathrm{pt},\mathscr{F})$, considered as a morphism $h : \underline{\Bbbk}_{\mathrm{pt}}[-N] \to \mathscr{F}$ in $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)$, then the cone of h has finite-dimensional equivariant cohomology, i.e. belongs to $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)_{\varpi-\mathsf{perf}}$ by Lemma 3.9. As a consequence, ϖ can also be represented by the diagram

$$\underline{\Bbbk}_{\mathrm{pt}} \xleftarrow{f\circ h} \underline{\Bbbk}_{\mathrm{pt}}[-N] \xrightarrow{g\circ h} \underline{\Bbbk}_{\mathrm{pt}}[n].$$

In case n is odd we have

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)}(\underline{\Bbbk}_{\mathrm{pt}}[-N], \underline{\Bbbk}_{\mathrm{pt}}[n]) = 0,$$

so that $g \circ h$ must be zero, which finishes the proof in this case.

On the other hand, if n is even both spaces $\mathrm{Hom}_{D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)}(\underline{\Bbbk}_{\mathrm{pt}}[-N], \underline{\Bbbk}_{\mathrm{pt}})$ and $\mathrm{Hom}_{D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)}(\underline{\Bbbk}_{\mathrm{pt}}[-N], \underline{\Bbbk}_{\mathrm{pt}}[n])$ are 1-dimensional, with a basis given by $\mathrm{can}^N_{\mathrm{pt}}[-N]$

and $\mathrm{can}^{n+N}_{\mathrm{pt}}[-N]$ respectively. Hence to conclude, it only remains to prove that for $M, M' \geqslant n$ even, the diagrams

$$\underline{\Bbbk}_{\mathrm{pt}} \xleftarrow{\mathrm{can}^{M}_{\mathrm{pt}}[-M]} \underline{\Bbbk}_{\mathrm{pt}}[-M] \xrightarrow{\mathrm{can}^{n+M}_{\mathrm{pt}}[-M]} \underline{\Bbbk}_{\mathrm{pt}}[n]$$

and

$$\underline{\Bbbk}_{\mathrm{pt}} \xleftarrow{\mathrm{can}^{M'}_{\mathrm{pt}}[-M']} \underline{\Bbbk}_{\mathrm{pt}}[-M'] \xrightarrow{\mathrm{can}^{n+M'}_{\mathrm{pt}}[-M']} \underline{\Bbbk}_{\mathrm{pt}}[n]$$

represent the same morphism in $\mathsf{Sm}(\mathrm{pt}, \Bbbk)$. However we can assume that $M' \geqslant M$; then the morphism $\mathrm{can}^{M'-M}_{\mathrm{pt}}[-M'] : \underline{\Bbbk}_{\mathrm{pt}}[-M'] \to \underline{\Bbbk}_{\mathrm{pt}}[-M]$ has a cone which belongs to $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt}, \Bbbk)_{\varpi-\mathsf{perf}}$, and satisfies

$$\mathrm{can}^{M'}_{\mathrm{pt}}[-M'] = (\mathrm{can}^{M}_{\mathrm{pt}}[-M]) \circ (\mathrm{can}^{M'-M}_{\mathrm{pt}}[-M']),$$
$$\mathrm{can}^{n+M'}_{\mathrm{pt}}[-M'] = (\mathrm{can}^{n+M}_{\mathrm{pt}}[-M]) \circ (\mathrm{can}^{M'-M}_{\mathrm{pt}}[-M']).$$

The desired claim follows. □

4. Fixed points of roots of unity on the affine Grassmannian

As in Section 3 we let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$.

4.1. **Affine Weyl group.** Let G be a connected reductive algebraic group over $\mathbb{F}$, and choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. The Weyl group of (G, T) will be denoted W_{f}. (Here, the subscript stands for "finite," and is here to avoid any confusion with the affine Weyl group introduced below.) We will also denote by U the unipotent radical of B, by B^+ the Borel subgroup opposite to B with respect to T, and by U^+ the unipotent radical of B^+.

We will denote by $\mathbf{X} := X^*(T)$ the character lattice of T, and by $\mathbf{X}^\vee := X_*(T)$ its cocharacter lattice. Let $\mathfrak{R} \subset \mathbf{X}$ be the root system of (G, T), and let $\mathfrak{R}^+ \subset \mathfrak{R}$ be the system of positive roots consisting of the T-weights in $\mathrm{Lie}(U^+)$. Let also $\mathfrak{R}^{\mathrm{s}}$ be the associated basis of $\mathfrak{R}$ (the "simple roots"). These data define a set S_{f} of Coxeter generators for W_{f}, consisting of the reflections s_α with $\alpha \in \mathfrak{R}^{\mathrm{s}}$.

The *affine Weyl group* is the semi-direct product

$$W_{\mathrm{aff}} := W_{\mathrm{f}} \ltimes \mathbb{Z}\mathfrak{R}^\vee,$$

where $\mathfrak{R}^\vee \subset \mathbf{X}^\vee$ is the coroot system, and $\mathbb{Z}\mathfrak{R}^\vee$ is the coroot lattice. For $\lambda \in \mathbb{Z}\mathfrak{R}^\vee$ we will denote by t_λ the image of λ in W_{aff}. The group W_{aff} admits a natural structure of Coxeter group extending that of W_{f}; the corresponding simple reflections $S_{\mathrm{aff}} \subset W_{\mathrm{aff}}$ consist of S_{f} together with the elements $\mathsf{t}_{\beta^\vee} s_\beta$ with β a maximal root in $\mathfrak{R}$.

Given $n \in \mathbb{Z}$, we will consider two actions of W_{aff} on $V := \mathbf{X}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ defined, for $w \in W_{\mathrm{f}}$ and $\lambda \in \mathbb{Z}\mathfrak{R}^\vee$, by

$$(w\mathsf{t}_\lambda) \cdot_n \mu = w(\mu - n\lambda), \qquad (w\mathsf{t}_\lambda) \mathbin{\square_n} \mu = w(\mu + n\lambda)$$

for $\mu \in \mathbf{X}^\vee$, where in the right-hand side we consider the natural action of W_{f} on $\mathbf{X}^\vee$. (Here the action $\cdot_n$ appears due to the sign conventions in Bruhat–Tits theory; but the action $\square_n$ is closer to the action which will be relevant when considering Representation Theory.) Of course, these actions are related via

$$w \mathbin{\square_n} \mu = -(w \cdot_n (-\mu))$$

for any $w \in W_{\mathrm{aff}}$ and $\mu \in V$.

We set

$$\mathbf{a}_n := \{\lambda \in V \mid \forall \alpha \in \mathfrak{R}^+,\ -n < \langle \lambda, \alpha \rangle < 0\}.$$

Then the closure $\overline{\mathbf{a}}_n$ of $\mathbf{a}_n$ is a fundamental domain for the action of W_{aff} on V via $\cdot_n$ and via $\square_n$. These actions stabilize $\mathbf{X}^\vee$, and a fundamental domain for the action of W_{aff} on $\mathbf{X}^\vee$ (for each of these actions) is therefore $\overline{\mathbf{a}}_n \cap \mathbf{X}^\vee$.

The *affine roots* are the formal linear combinations $\alpha+m\hbar$ with $\alpha \in \mathfrak{R}$ and $m \in \mathbb{Z}$. To such a combination we attach an affine function $f^n_{\alpha+m\hbar}$ on V, determined by

$$f^n_{\alpha+m\hbar}(v) = \langle \alpha, v \rangle + nm,$$

and an element $s_{\alpha+m\hbar} \in W_{\mathrm{aff}}$ determined by

$$s_{\alpha+m\hbar} = \mathsf{t}_{m\alpha^\vee} s_\alpha.$$

We then have

$$s_{\alpha+m\hbar} \cdot_n v = v - f^n_{\alpha+m\hbar}(v)\alpha^\vee$$

for any $v \in V$.

Remark 4.1. In practice, when considering these constructions in later sections, the integer n will be either 1 or a prime number different from p. As this assumption does not simplify the discussion in any way, we will not impose any restriction on n in this section.

4.2. **Some Bruhat–Tits theory.** For any positive integer n, we set $\mathscr{K}_n := \mathbb{F}((z^n))$. We will consider $\mathscr{K} := \mathscr{K}_1$ as a valued field with its natural valuation (such that z has valuation 1), and endow each $\mathscr{K}_n$ with the valuation obtained by restriction. (In this way, all the fields $\mathscr{K}_n$ are canonically isomorphic, but their valuations differ.) We will denote by $\mathscr{O}_n$ the valuation ring of $\mathscr{K}_n$, so that $\mathscr{O}_n := \mathbb{F}[[z^n]]$. For any $\lambda \in \mathbf{X}^\vee$ we have a point $z^\lambda \in G(\mathscr{K})$, defined as the image of z under the map $(\mathscr{K})^\times \to G(\mathscr{K})$ induced by λ. If $\lambda \in n\mathbf{X}^\vee$, then z^λ belongs to $G(\mathscr{K}_n)$.

The group scheme $G \times_{\mathrm{Spec}(\mathbb{F})} \mathrm{Spec}(\mathscr{K}_n)$ is a (split) connected reductive group scheme over $\mathscr{K}_n$, so that one can consider the associated (enlarged) Bruhat–Tits building $\mathfrak{B}_n$. Our choice of maximal torus in G provides a split maximal torus $T \times_{\mathrm{Spec}(\mathbb{F})} \mathrm{Spec}(\mathscr{K}_n) \subset G \times_{\mathrm{Spec}(\mathbb{F})} \mathrm{Spec}(\mathscr{K}_n)$, which itself defines an apartment A_n in $\mathfrak{B}_n$. This apartment is an affine space with underlying vector space V, and it is endowed with a canonical action of $N_G(T)(\mathscr{K}_n)$ whose vectorial part factors through the natural action of $N_G(T)(\mathscr{K}_n)/T(\mathscr{K}_n) = W_{\mathrm{f}}$ on V, and such that for $\lambda \in \mathbf{X}^\vee$ the element $z^{n\lambda} \in T(\mathscr{K}_n)$ acts by translation by $-n\lambda$. Let us choose, for any $w \in W_{\mathrm{f}}$, a lift $\dot{w}$ of w in $N_G(T)$. Then we will consider the map

$$\iota_n : W_{\mathrm{aff}} \to N_G(T)(\mathscr{K}_n)$$

defined by $\iota_n(\mathsf{t}_\lambda w) = z^{n\lambda}\dot{w}$ for $w \in W_{\mathrm{f}}$ and $\lambda \in \mathbb{Z}\mathfrak{R}^\vee$.

If we choose a W_{f}-fixed point in A_n, then the action of V on this point defines an identification

$$V \xrightarrow{\sim} A_n, \tag{4.1}$$

under which the action of $N_G(T)(\mathscr{K}_n)$ on A_n identifies with the action of W_{aff} on V provided by $\cdot_n$. We will fix such an identification once and for all, and use it to identify all the data considered above related to V as data related to A_n. (None of our considerations below will depend on the choice of identification (4.1).)

The collection of fixed points of the reflections $s_{\alpha+m\hbar}$ (or in other words of kernels of the functions $f^n_{\alpha+m\hbar}$) defines a hyperplane arrangement in A_n, hence a collection of *facets* (whose closures are the nonempty intersections of such hyperplanes). In particular, $\mathbf{a}_n$ is a facet of maximal dimension, i.e. an alcove. Another example of facet is the intersection $\mathbf{o}_n$ of the reflection hyperplanes associated with all the reflections s_β with $\beta \in \mathfrak{R}$, i.e. the set of W_{f}-fixed points. The facets we will mainly be interested in are those contained in the closure $\overline{\mathbf{a}}_n$ of $\mathbf{a}_n$.

To any facet $\mathbf{f}$ in A_n, Bruhat–Tits theory associates a "parahoric group scheme" $P_{\mathbf{f}}$ over $\mathrm{Spec}(\mathscr{O}_n)$, characterized as the unique (up to isomorphism) smooth affine group scheme over $\mathrm{Spec}(\mathscr{O}_n)$ with connected geometric fibers such that

$$P_{\mathbf{f}} \times_{\mathrm{Spec}(\mathscr{O}_n)} \mathrm{Spec}(\mathscr{K}_n) = G \times_{\mathrm{Spec}(\mathbb{F})} \mathrm{Spec}(\mathscr{K}_n), \tag{4.2}$$

and whose $\mathscr{O}_n$-points identify (via this isomorphism) with the pointwise stabilizer of $\mathbf{f}$ in $G(\mathscr{K}_n)$. In particular we have

$$P_{\mathbf{o}_n} = G \times_{\mathrm{Spec}(\mathbb{F})} \mathrm{Spec}(\mathscr{O}_n),$$

and $P_{\mathbf{a}_n}$ is an Iwahori group scheme, whose group of $\mathscr{O}_n$-points is the inverse image Iw_n of B under the map $G(\mathscr{O}_n) \to G$ of evaluation at $z^n = 0$. This construction is compatible with inclusions of closures of facets in a natural way; in particular for any facet $\mathbf{f}$ contained in $\overline{\mathbf{a}}_n$ we have an inclusion

$$P_{\mathbf{a}_n} \subset P_{\mathbf{f}}. \tag{4.3}$$

4.3. **Loop groups and partial affine flag varieties.** As above we fix a positive integer n. The n-th *loop group* associated with G is the ind-affine group ind-scheme L_nG over $\mathbb{F}$ which represents the functor sending an $\mathbb{F}$-algebra R to $G(R((z^n)))$. The associated *arc group* (or positive loop group) is the affine group scheme L_n^+G over $\mathbb{F}$ which represents the functor sending R to $G(R[[z^n]])$.

The case we are mostly interested in is when $n = 1$. In this case (here and in later related notation), we will usually omit the subscript from the notation. The case of a more general n however naturally appears when considering the action of n-th roots of unity by loop rotation. Namely, we have a natural action of the multiplicative group $\mathbb{G}_{\mathrm{m}}$ over $\mathbb{F}$ on LG by loop rotation. This action stabilizes the subgroup L^+G. Denote now by $\boldsymbol{\mu}_n \subset \mathbb{G}_{\mathrm{m}}$ the subgroup scheme of n-th roots of unity; we can then consider the fixed-points ind-scheme $(LG)^{\boldsymbol{\mu}_n}$ and the fixed-points scheme $(L^+G)^{\boldsymbol{\mu}_n}$ in the sense of §2.4.

Lemma 4.2. *We have identifications*

$$(LG)^{\boldsymbol{\mu}_n} = L_nG, \quad (L^+G)^{\boldsymbol{\mu}_n} = L_n^+G.$$

Proof. For any $\mathbb{F}$-algebra R, the R-points $(LG)(R)$ consist of the $\mathbb{F}$-schemes morphisms $\mathrm{Spec}(R((z))) \to G$. Therefore, the R-points of $(LG)^{\boldsymbol{\mu}_n}$ consist of the $\boldsymbol{\mu}_n$-invariant morphisms $\mathrm{Spec}(R((z))) \to G$, i.e. the morphisms which factor through the quotient $\mathrm{Spec}(R((z))) \to \mathrm{Spec}(R((z)))/\boldsymbol{\mu}_n = \mathrm{Spec}(R((z^n)))$. This proves the first identification. The proof of the second one is similar. □

It is well known that the fppf sheafification of the functor

$$R \mapsto (LG)(R)/(L^+G)(R)$$

is represented by an ind-projective ind-scheme of ind-finite type over $\mathbb{F}$, which is called the *affine Grassmannian* of G, and will be denoted $\mathcal{G}r_G$. The main goal of this section is to describe the ind-scheme $(\mathcal{G}r_G)^{\boldsymbol{\mu}_n}$, see Proposition 4.6 below. This will require discussing more general "partial affine flag varieties" attached to L_nG, as follows.

If $\mathbf{f} \subset \overline{\mathbf{a}}_n$ is a facet, we can consider the affine group scheme $L_n^+ P_{\mathbf{f}}$ over $\mathbb{F}$ which represents the functor sending R to $P_{\mathbf{f}}(R[[z^n]])$. In view of (4.2), $L_n^+ P_{\mathbf{f}}$ is a subgroup of $L_n G$. The partial affine flag variety

$$\mathcal{F}l_{\mathbf{f}}^n$$

associated with $\mathbf{f}$ is the ind-projective ind-scheme of ind-finite type over $\mathbb{F}$ which represents the fppf sheafification of the presheaf

$$R \mapsto L_n G(R)/L_n^+ P_{\mathbf{f}}(R).$$

These ind-schemes are the main object of study of [PR]. In particular, the connected components of $\mathcal{F}l_{\mathbf{f}}^n$ are in a natural bijection with the algebraic fundamental group of G (see [PR, Theorem 0.1]); the component corresponding to the neutral element will be denoted $\mathcal{F}l_{\mathbf{f}}^{n,\circ}$.

If $\alpha \in \mathfrak{R}$, we will denote by U_α the root subgroup of G attached to α. Then, for an affine root $\alpha + m\hbar$, we will denote by $U_{\alpha+m\hbar}$ the subgroup of LG which, for any isomorphism $u_\alpha : \mathbb{G}_{\mathrm{a}} \xrightarrow{\sim} U_\alpha$, identifies with the image of the morphism $x \mapsto u_\alpha(xz^m)$.

The following statements are easily checked.

Lemma 4.3. *Let $\alpha \in \mathfrak{R}$ and $m \in \mathbb{Z}$.*

(1) *The subgroup $U_{\alpha+m\hbar}$ is stable under the action of $\boldsymbol{\mu}_n$, and we have*

$$(U_{\alpha+m\hbar})^{\boldsymbol{\mu}_n} = \begin{cases} U_{\alpha+m\hbar} & \text{if } m \in n\mathbb{Z}; \\ \{1\} & \text{otherwise.} \end{cases}$$

(2) *If $\lambda \in \mathbf{X}^\vee$, we have*

$$z^\lambda \cdot U_{\alpha+m\hbar} \cdot z^{-\lambda} = U_{\alpha+(m+\langle\lambda,\alpha\rangle)\hbar}.$$

(3) *If $\mathbf{f} \subset \overline{\mathbf{a}}_n$ is a facet, we have $U_{\alpha+mn\hbar} \subset L_n^+ P_{\mathbf{f}}$ iff $f_{\alpha+m\hbar}^n$ takes nonnegative values on $\mathbf{f}$.*

4.4. **Big cells in partial affine flag varieties.** Our arguments below will make use of the "big cell" in $\mathcal{F}l_{\mathbf{f}}^n$, whose construction we now recall following de Cataldo–Haines–Li [dCHL]. We first consider the affine group ind-scheme $L_n^{(-1)}G$ which represents the functor sending R to the kernel of the morphism

$$G(R[z^{-n}]) \to G(R)$$

of evaluation at $z^{-n} = 0$. Then $L_n^{(-1)}G$ is a subgroup ind-scheme of $L_n G$, and we set

$$L_n^{--} P_{\mathbf{a}_n} = L_n^{(-1)} G \cdot U^+.$$

With this definition, it is well known (see e.g. [Fa, §2]) that the action on the base point induces an open embedding

$$L_n^{--} P_{\mathbf{a}_n} \to \mathcal{F}l_{\mathbf{a}_n}^{n,\circ},$$

and that from this one can obtain an open cover of $\mathcal{F}l_{\mathbf{a}_n}^{n,\circ}$ parametrized by W_{aff}, where the open subset corresponding to w is the image of

$$\iota_n(w) \cdot L_n^{--} P_{\mathbf{a}_n} \cdot \iota_n(w)^{-1}$$

under the map $g \mapsto g \cdot [\iota_n(w)]$. (Here, $[\iota_n(w)]$ is the image of $\iota_n(w)$ in $\mathcal{F}l_{\mathbf{a}_n}^{n,\circ}$.)

For a general facet $\mathbf{f} \subset \overline{\mathbf{a}}_n$, we denote by $W_{\mathrm{aff}}^{\mathbf{f}}$ the pointwise stabilizer of $\mathbf{f}$ in W_{aff} (a finite parabolic subgroup) and set

$$L_n^{--}P_{\mathbf{f}} = \bigcap_{w \in W_{\mathrm{aff}}^{\mathbf{f}}} \iota_n(w) \cdot L_n^{--}P_{\mathbf{a}_n} \cdot \iota_n(w)^{-1}.$$

The following claim is easily checked.

Lemma 4.4. *For $\alpha \in \mathfrak{R}$ and $m \in \mathbb{Z}$, we have $U_{\alpha+nm\hbar} \subset L_n^{--}P_{\mathbf{f}}$ iff $f^n_{\alpha+m\hbar}$ takes negative values on $\mathbf{f}$.*

With this definition, as explained in [dCHL, §3.8.1], the action on the base point defines an open embedding

$$L_n^{--}P_{\mathbf{f}} \to \mathcal{F}l_{\mathbf{f}}^{n,\circ}.$$

One can obtain from this an open cover of $\mathcal{F}l_{\mathbf{f}}^{n,\circ}$ parametrized by the quotient $W_{\mathrm{aff}}/W_{\mathrm{aff}}^{\mathbf{f}}$, where the open subset attached to a coset $wW_{\mathrm{aff}}^{\mathbf{f}}$ is the image of the subgroup

$$\iota_n(w) \cdot L_n^{--}P_{\mathbf{f}} \cdot \iota_n(w)^{-1}$$

under the morphism of action on the image of $\iota_n(w)$. (These data do not depend on the choice of w in its coset, and this claim can be deduced from the corresponding fact for $\mathbf{a}_n$ by using the morphism $\mathcal{F}l_{\mathbf{a}_n}^{n,\circ} \to \mathcal{F}l_{\mathbf{f}}^{n,\circ}$ induced by (4.3).)

Note that in the special case $n = 1$ and $\mathbf{f} = \mathbf{o}_1$, we have

$$L_1^{--}P_{\mathbf{o}_1} = L^{(-1)}G.$$

For $m \in \mathbb{Z}_{\geqslant 1}$, we will also denote by $L^{(-1)}G(m)$, resp. $L_n^{(-1)}G(m)$, the subgroup of $L^{(-1)}G$, resp. $L_n^{(-1)}G$ which represents the functor sending R to the preimage of $T(R[t^{-1}]/t^{-m})$ under the composition

$$L^{(-1)}G(R) \hookrightarrow G(R[t^{-1}]) \to G(R[t^{-1}]/t^{-m}),$$

resp. the preimage of $T(R[t^{-n}]/t^{-mn})$ under the composition

$$L_n^{(-1)}G(R) \hookrightarrow G(R[t^{-n}]) \to G(R[t^{-n}]/t^{-nm}).$$

Below we will require the following properties of these subgroups:

(1) for fixed $\lambda \in \mathbf{X}^\vee$ and $m \in \mathbb{Z}_{>0}$, for $m' \gg 0$ we have $z^\lambda L^{(-1)}G(m')z^{-\lambda} \subset L^{(-1)}G(m)$;
(2) for any facet $\mathbf{f} \subset \overline{\mathbf{a}}_n$, for $m \gg 0$ we have $L_n^{(-1)}G(m) \subset L_n^{--}P_{\mathbf{f}}$.

(Here, (2) follows from the fact that for any given $w \in W_{\mathrm{aff}}$, for $m \gg 0$ we have $L_n^{(-1)}G(m) \subset \iota_n(w) \cdot L_n^{--}P_{\mathbf{a}_n} \cdot \iota_n(w)^{-1}$, see [dCHL]. For (1), we can use [dCHL, Remark 3.1.1] to reduce the claim to the case $G = \mathrm{GL}_n(\mathbb{F})$, which is clear from a matrix calculation.)

Lemma 4.5. *For any $\lambda \in (-\overline{\mathbf{a}}_n) \cap \mathbf{X}^\vee$, we have*

$$(z^\lambda \cdot L_1^{--}P_{\mathbf{o}_1} \cdot z^{-\lambda})^{\boldsymbol{\mu}_n} = L_n^{--}P_{\mathbf{f}_\lambda},$$

where $\mathbf{f}_\lambda \subset \overline{\mathbf{a}}_n$ is the facet containing $-\lambda$.

Proof. For any $\alpha \in \mathfrak{R}$, let us denote by i_α the largest integer such that the function $f^n_{\alpha+i_\alpha\hbar}$ takes negative values on $\mathbf{f}_\lambda$. In fact, since $\mathbf{f}_\lambda \subset \overline{\mathbf{a}}_n$ we can describe this integer very explicitly; namely:

- if $\alpha \in \mathfrak{R}^+$ then $\langle \lambda, \alpha \rangle \in \{0, \cdots, n\}$, and

$$i_\alpha = \begin{cases} 0 & \text{if } \langle \lambda, \alpha \rangle > 0; \\ -1 & \text{if } \langle \lambda, \alpha \rangle = 0; \end{cases} \tag{4.4}$$

- if $\alpha \in -\mathfrak{R}^+$ then $\langle \lambda, \alpha \rangle \in \{-n, \cdots, 0\}$, and

$$i_\alpha = \begin{cases} -1 & \text{if } \langle \lambda, \alpha \rangle > -n; \\ -2 & \text{if } \langle \lambda, \alpha \rangle = -n. \end{cases} \tag{4.5}$$

Recall from (1)–(2) above that we can choose m large enough such that

$$L_n^{(-1)}G(m) \subset L_n^{--}P_{\mathbf{f}_\lambda} \quad \text{and} \quad z^{-\lambda} \cdot L_n^{(-1)}G(m) \cdot z^\lambda \subset L^{(-1)}G.$$

Then the arguments in [dCHL, Proofs of Lemma 3.6.3 and Proposition 3.6.4] show that we have a direct product decomposition

$$L_n^{--}P_{\mathbf{f}_\lambda} = L_n^{(-1)}G(m) \cdot \prod_{\alpha \in \mathfrak{R}^+} \prod_{j=-m}^{i_\alpha} U_{\alpha+jn\hbar} \cdot \prod_{\alpha \in -\mathfrak{R}^+} \prod_{j=-m}^{i_\alpha} U_{\alpha+jn\hbar}.$$

(Here we use an arbitrary order on $\mathfrak{R}^+$ and on $-\mathfrak{R}^+$.) Our choice of m guarantees that $z^{-\lambda} \cdot L_n^{(-1)}G(m) \cdot z^\lambda \subset L^{(-1)}G$, and in view of Lemma 4.3(2), for any affine root $\alpha + jn\hbar$ appearing in the decomposition above, the fact that $f^n_{\alpha+j\hbar}(-\lambda) < 0$ implies that

$$z^{-\lambda} \cdot U_{\alpha+jn\hbar} \cdot z^\lambda \subset L^{(-1)}G.$$

These considerations show that

$$L_n^{--}P_{\mathbf{f}_\lambda} \subset z^\lambda \cdot L_1^{--}P_{\mathbf{o}_1} \cdot z^{-\lambda},$$

so that

$$L_n^{--}P_{\mathbf{f}_\lambda} \subset (z^\lambda \cdot L_1^{--}P_{\mathbf{o}_1} \cdot z^{-\lambda})^{\boldsymbol{\mu}_n}.$$

To prove the reverse inclusion, we continue with some m as above, and choose $m' \gg 0$ such that

$$z^\lambda L^{(-1)}G(m')z^{-\lambda} \subset L^{(-1)}G(nm)$$

(see (1) above). We then have

$$(z^\lambda L^{(-1)}G(m')z^{-\lambda})^{\boldsymbol{\mu}_n} \subset (L^{(-1)}G(nm))^{\boldsymbol{\mu}_n} = L_n^{(-1)}G(m).$$

As above we have a direct product decomposition

$$L^{(-1)}G = L^{(-1)}G(m') \cdot \prod_{\alpha \in \mathfrak{R}^+} \prod_{j=-m'}^{-1} U_{\alpha+j\hbar} \cdot \prod_{\alpha \in -\mathfrak{R}^+} \prod_{j=-m'}^{-1} U_{\alpha+j\hbar},$$

where now (for notational convenience) we choose the order on $\mathfrak{R}^+$ such that all the roots such that $\langle \lambda, \alpha \rangle = 0$ are bigger than the other ones, and the order on $-\mathfrak{R}^+$ such that all the roots such that $\langle \lambda, \alpha \rangle = -n$ are bigger that the other ones. From this decomposition we see that $(z^\lambda \cdot L^{(-1)}G \cdot z^{-\lambda})^{\boldsymbol{\mu}_n}$ is the product of $(z^\lambda L^{(-1)}G(m')z^{-\lambda})^{\boldsymbol{\mu}_n}$, which is included in $L_n^{--}P_{\mathbf{f}_\lambda}$ by the choices of m and m', and of

$$\left(\prod_{\alpha \in \mathfrak{R}^+} \prod_{j=-m'+\langle \lambda, \alpha \rangle}^{-1+\langle \lambda, \alpha \rangle} U_{\alpha+j\hbar} \cdot \prod_{\alpha \in -\mathfrak{R}^+} \prod_{j=-m'+\langle \lambda, \alpha \rangle}^{-1+\langle \lambda, \alpha \rangle} U_{\alpha+j\hbar} \right)^{\boldsymbol{\mu}_n},$$

which by Lemma 4.3 is included in

$$\prod_{\substack{\alpha\in\mathfrak{R}^+\\ \langle\lambda,\alpha\rangle>0}} \prod_{j=-N}^{0} U_{\alpha+jn\hbar} \cdot \prod_{\substack{\alpha\in\mathfrak{R}^+\\ \langle\lambda,\alpha\rangle=0}} \prod_{j=-N}^{-1} U_{\alpha+jn\hbar} \cdot \prod_{\substack{\alpha\in-\mathfrak{R}^+\\ \langle\lambda,\alpha\rangle>-n}} \prod_{j=-N}^{-1} U_{\alpha+jn\hbar} \cdot \prod_{\substack{\alpha\in-\mathfrak{R}^+\\ \langle\lambda,\alpha\rangle=-n}} \prod_{j=-N}^{-2} U_{\alpha+jn\hbar}$$

for $N \gg 0$. Here all the affine root subgroups are included in $L_n^{--}P_{\mathbf{f}_\lambda}$ by (4.4)–(4.5) and Lemma 4.4, which finally proves that

$$(z^\lambda \cdot L^{(-1)}G \cdot z^{-\lambda})^{\boldsymbol{\mu}_n} \subset L_n^{--}P_{\mathbf{f}_\lambda}$$

and concludes the proof. □

4.5. **Fixed points on the affine Grassmannian.** The main result of the present section is the following claim. (Here, for $\lambda \in \mathbf{X}^\vee$ we denote by L_λ the coset of z^λ in $\mathcal{G}r_G$.)

Proposition 4.6. *For any $\lambda \in (-\overline{\mathbf{a}}_n) \cap \mathbf{X}^\vee$, the map $g \mapsto g \cdot L_\lambda$ factors through an open and closed embedding*

$$\mathcal{F}l^{n,\circ}_{\mathbf{f}_\lambda} \to (\mathcal{G}r_G)^{\boldsymbol{\mu}_n},$$

where $\mathbf{f}_\lambda$ is as in Lemma 4.5. Moreover, the induced map

$$\bigsqcup_{\lambda\in(-\overline{\mathbf{a}}_n)\cap\mathbf{X}^\vee} \mathcal{F}l^{n,\circ}_{\mathbf{f}_\lambda} \to (\mathcal{G}r_G)^{\boldsymbol{\mu}_n}$$

is an isomorphism of ind-schemes.

Proof. Arguments similar to those for Lemma 4.5 show that for $\lambda \in (-\overline{\mathbf{a}}_n) \cap \mathbf{X}^\vee$, the point L_λ is fixed under the action of $L_n^+P_{\mathbf{a}_n}$. Since this point is also stable under the action of lifts of elements in $W_{\mathrm{aff}}^{\mathbf{f}_\lambda}$, it is stabilized by $L_n^+P_{\mathbf{f}_\lambda}$. Since $\boldsymbol{\mu}_n$ acts trivially on L_nG, our morphism therefore indeed factors through a morphism

$$\mathcal{F}l^{n}_{\mathbf{f}_\lambda} \to (\mathcal{G}r_G)^{\boldsymbol{\mu}_n},$$

which we can then restrict to the wished-for morphism $\mathcal{F}l^{n,\circ}_{\mathbf{f}_\lambda} \to (\mathcal{G}r_G)^{\boldsymbol{\mu}_n}$.

We will now prove that the induced morphism

$$\bigsqcup_{\lambda\in(-\overline{\mathbf{a}}_n)\cap\mathbf{X}^\vee} \mathcal{F}l^{n,\circ}_{\mathbf{f}_\lambda} \to (\mathcal{G}r_G)^{\boldsymbol{\mu}_n} \tag{4.6}$$

is an isomorphism, which will conclude the proof. For this we use the "big cell" theory recalled in §4.4. Namely, recall that for $\nu \in \mathbf{X}^\vee$ the morphism $g \mapsto g \cdot L_\nu$ defines an open embedding

$$z^\nu \cdot L_1^{--}P_{\mathbf{o}_1} \cdot z^{-\nu} \to \mathcal{G}r_G,$$

and that the images of these maps constitute an open cover of $\mathcal{G}r_G$. These open subsets are stable under the $\mathbb{G}_{\mathrm{m}}$-action by loop rotation, hence also under the $\boldsymbol{\mu}_n$-action we consider here; it follows that $(\mathcal{G}r_G)^{\boldsymbol{\mu}_n}$ has an open cover parametrized by $\mathbf{X}^\vee$, with the subset corresponding to ν naturally isomorphic to $(z^\nu L_1^{--}P_{\mathbf{o}_1} z^{-\nu})^{\boldsymbol{\mu}_n}$.

Similarly, for any $\lambda \in (-\overline{\mathbf{a}}_n) \cap \mathbf{X}^\vee$ and any coset $wW_{\mathrm{aff}}^{\mathbf{f}_\lambda}$ in $W_{\mathrm{aff}}/W_{\mathrm{aff}}^{\mathbf{f}_\lambda}$, we have considered in §4.4 an open subset of $\mathcal{F}l^{n,\circ}_{\mathbf{f}_\lambda}$ naturally isomorphic to $\iota_n(w) \cdot L_n^{--}P_{\mathbf{f}_\lambda} \cdot$

$\iota_n(w)^{-1}$. Now since $(-\overline{\mathbf{a}}_n) \cap \mathbf{X}^\vee$ is a fundamental domain for the action of W_{aff} on $\mathbf{X}^\vee$ (via $\square_n$), we have a bijection

$$\bigsqcup_{\lambda \in (-\overline{\mathbf{a}}_n) \cap \mathbf{X}^\vee} W_{\mathrm{aff}}/W_{\mathrm{aff}}^{\mathbf{f}_\lambda} \xrightarrow{\sim} \mathbf{X}^\vee$$

sending $wW_{\mathrm{aff}}^{\mathbf{f}_\lambda} \in W_{\mathrm{aff}}/W_{\mathrm{aff}}^{\mathbf{f}_\lambda}$ to $-(w \cdot_n (-\lambda)) = w \square_n \lambda$.

To conclude the proof, we will show that the map (4.6) identifies the open subset of $\mathcal{F}l_{\mathbf{f}_\lambda}^{n,\circ}$ associated with the coset $wW_{\mathrm{aff}}^{\mathbf{f}_\lambda}$ with the open subset of $(\mathcal{G}r_G)^{\boldsymbol{\mu}_n}$ corresponding to $w \square_n \lambda$. For this it suffices to prove the equality

$$(z^{w \square_n \lambda} \cdot L_1^{--} P_{\mathbf{o}_1} \cdot z^{-w \square_n \lambda})^{\boldsymbol{\mu}_n} = \iota_n(w) \cdot L_n^{--} P_{\mathbf{f}_\lambda} \cdot \iota_n(w)^{-1}. \tag{4.7}$$

In case $w = 1$, the equality (4.7) was checked in Lemma 4.5. The deduce the general case, write $w = \mathsf{t}_\mu v$ with $\mu \in \mathbf{X}^\vee$ and $v \in W_{\mathrm{f}}$. Then we have

$$\begin{aligned}(z^{w \square_n \lambda} \cdot L_1^{--} P_{\mathbf{o}_1} \cdot z^{-w \square_n \lambda})^{\boldsymbol{\mu}_n} &= (z^{v(\lambda)+n\mu} \cdot L_1^{--} P_{\mathbf{o}_1} \cdot z^{-v(\lambda)-n\mu})^{\boldsymbol{\mu}_n} \\ &= z^{n\mu} \dot{v} \cdot (z^\lambda \cdot L_1^{--} P_{\mathbf{o}_1} \cdot z^{-\lambda})^{\boldsymbol{\mu}_n} \cdot \dot{v}^{-1} z^{-n\mu} = z^{n\mu} \dot{v} \cdot L_n^{--} P_{\mathbf{f}_\lambda} \cdot \dot{v}^{-1} z^{-n\mu},\end{aligned}$$

which concludes the proof. □

For $\lambda \in (-\overline{\mathbf{a}}_n) \cap \mathbf{X}^\vee$, we will denote by $\mathcal{G}r_{G,(\lambda)}$ the image of $\mathcal{F}l_{\mathbf{f}_\lambda}^{n,\circ}$ in $(\mathcal{G}r_G)^{\boldsymbol{\mu}_n}$ under the map of Proposition 4.6. We then have

$$(\mathcal{G}r_G)^{\boldsymbol{\mu}_n} = \bigsqcup_{\lambda \in (-\overline{\mathbf{a}}_n) \cap \mathbf{X}^\vee} \mathcal{G}r_{G,(\lambda)},$$

which describes $(\mathcal{G}r_G)^{\boldsymbol{\mu}_n}$ as the union of its connected components.

Remark 4.7. The action of $L_n G$ on L_0 induces an embedding

$$L_n G / L_n^+ G \hookrightarrow (\mathcal{G}r_G)^{\varpi}.$$

Here $L_n G / L_n^+ G$ is of course isomorphic to $\mathcal{G}r_G$. In terms of the decomposition in Proposition 4.6, this embedding identifies $L_n G / L_n^+ G$ with the union of the components $\mathcal{G}r_{G,(\lambda)}$ where λ runs over $(-\overline{\mathbf{a}}_n) \cap n\mathbf{X}^\vee$.

4.6. **Orbits on the affine Grassmannian.** A crucial role in our discussion will be played by the following Iwahori subgroups of L^+G, for which we introduce special notation:

$$\mathrm{Iw} := L_1^+ P_{\mathbf{a}_1}, \quad \mathrm{Iw}^+ := \dot{w}_0 \cdot \mathrm{Iw} \cdot (\dot{w}_0)^{-1}.$$

(Here, w_0 is the longest element in W_{f}.) More concretely, Iw, resp. Iw^+, is the inverse image of B, resp. B^+, under the map $\mathrm{ev}_0 : L^+G \to G$ sending z to 0. We also denote by Iw_{u} and $\mathrm{Iw}_{\mathrm{u}}^+$ the pro-unipotent radicals of Iw and Iw^+, i.e. the inverse images of U and U^+ under ev_0.

The group scheme L^+G acts on the affine Grassmannian $\mathcal{G}r_G$ (see §4.3), and the orbits of this action are parametrized by the subsemigroup $\mathbf{X}_+^\vee \subset \mathbf{X}^\vee$ of dominant cocharacters. More precisely, we have

$$(\mathcal{G}r_G)_{\mathrm{red}} = \bigsqcup_{\lambda \in \mathbf{X}_+^\vee} \mathcal{G}r_G^\lambda \quad \text{with } \mathcal{G}r_G^\lambda := L^+G \cdot L_\lambda, \tag{4.8}$$

where the left-hand side denotes the reduced ind-scheme associated with $\mathcal{G}r_G$. Moreover, for any $\lambda \in \mathbf{X}_+^\vee$ the closure $\overline{\mathcal{G}r_G^\lambda}$ is a projective $\mathbb{F}$-scheme of finite type, on which the action of L^+G factors through an action of a quotient group scheme of finite type.

The orbits of Iw and Iw^+ can be described similarly: we have

$$(\mathcal{G}r_G)_{\mathrm{red}} = \bigsqcup_{\lambda \in \mathbf{X}^\vee} \mathcal{G}r_{G,\lambda} \quad \text{with } \mathcal{G}r_{G,\lambda} := \mathrm{Iw} \cdot L_\lambda$$

and

$$(\mathcal{G}r_G)_{\mathrm{red}} = \bigsqcup_{\lambda \in \mathbf{X}^\vee} \mathcal{G}r^+_{G,\lambda} \quad \text{with } \mathcal{G}r^+_{G,\lambda} := \mathrm{Iw}^+ \cdot L_\lambda.$$

Moreover, each Iw-orbit (resp. Iw^+-orbit) is also an Iw_{u}-orbit (resp. $\mathrm{Iw}^+_{\mathrm{u}}$-orbit), and for any $\mu \in \mathbf{X}^\vee_+$ we have

$$\mathcal{G}r^\mu_G = \bigsqcup_{\lambda \in W_{\mathrm{f}}\mu} \mathcal{G}r_{G,\lambda} = \bigsqcup_{\lambda \in W_{\mathrm{f}}\mu} \mathcal{G}r^+_{G,\lambda}.$$

For $\lambda \in \mathbf{X}^\vee$, the embedding of $\mathcal{G}r^+_{G,\lambda}$ in $\mathcal{G}r_G$ will be denoted j^+_λ.

If $n \in \mathbb{Z}_{>0}$, we can also consider the Iwahori subgroups $\mathrm{Iw}_n, \mathrm{Iw}^+_n \subset L_nG$ defined as above, and their pro-unipotent radicals $\mathrm{Iw}_{\mathrm{u},n}, \mathrm{Iw}^+_{\mathrm{u},n}$.

Lemma 4.8. *We have*

$$\mathrm{Iw}^{\boldsymbol{\mu}_n} = \mathrm{Iw}_n, \quad (\mathrm{Iw}^+)^{\boldsymbol{\mu}_n} = \mathrm{Iw}^+_n, \quad (\mathrm{Iw}_{\mathrm{u}})^{\boldsymbol{\mu}_n} = \mathrm{Iw}_{\mathrm{u},n}, \quad (\mathrm{Iw}^+_{\mathrm{u}})^{\boldsymbol{\mu}_n} = \mathrm{Iw}^+_{\mathrm{u},n}.$$

For any $\lambda \in \mathbf{X}^\vee$ we have

$$(\mathcal{G}r_{G,\lambda})^{\boldsymbol{\mu}_n} = \mathrm{Iw}_n \cdot L_\lambda, \quad (\mathcal{G}r^+_{G,\lambda})^{\boldsymbol{\mu}_n} = \mathrm{Iw}^+_n \cdot L_\lambda.$$

Proof. The identifications in the first sentence are immediate consequences of Lemma 4.2.

For the description of $(\mathcal{G}r_{G,\lambda})^{\boldsymbol{\mu}_n}$, for any $\alpha \in \mathfrak{R}$ we set $\delta_\alpha = 1$ if $\alpha \in \mathfrak{R}^+$, and $\delta_\alpha = 0$ otherwise. Using the notation introduced in §4.3, we set

$$\mathrm{Iw}^\lambda_{\mathrm{u}} := \prod_{\alpha \in \mathfrak{R}} \left(\prod_{\delta_\alpha \leqslant m < \langle \lambda, \alpha \rangle} U_{\alpha + m\hbar} \right),$$

where the products are taken in any chosen order. Then it is well known that the map $u \mapsto u \cdot L_\lambda$ induces an isomorphism $\mathrm{Iw}^\lambda_{\mathrm{u}} \xrightarrow{\sim} \mathcal{G}r_{G,\lambda}$. We deduce an isomorphism $(\mathrm{Iw}^\lambda_{\mathrm{u}})^{\boldsymbol{\mu}_n} \xrightarrow{\sim} (\mathcal{G}r_{G,\lambda})^{\boldsymbol{\mu}_n}$, and here by Lemma 4.3 we have

$$(\mathrm{Iw}^\lambda_{\mathrm{u}})^{\boldsymbol{\mu}_n} = \prod_{\alpha \in \mathfrak{R}} \left(\prod_{\substack{\delta_\alpha \leqslant m < \langle \lambda, \alpha \rangle \\ n | m}} U_{\alpha + m\hbar} \right).$$

It follows that $(\mathcal{G}r_{G,\lambda})^{\boldsymbol{\mu}_n} = \mathrm{Iw}_n \cdot L_\lambda$, as desired. The proof that $(\mathcal{G}r^+_{G,\lambda})^{\boldsymbol{\mu}_n} = \mathrm{Iw}^+_n \cdot L_\lambda$ is similar. ∎

Remark 4.9. Standard considerations show that for any facet $\mathbf{f} \subset \overline{\mathbf{a}}_n$, the Iw^+_n-orbits on $\mathcal{F}l^n_{\mathbf{f}}$ are parametrized in a natural way by the quotient $W_{\mathrm{aff}}/W^{\mathbf{f}}_{\mathrm{aff}}$. On the other hand, the Iw^+-orbits on $\mathcal{G}r_G$ are naturally parametrized by $\mathbf{X}^\vee$, so that by Lemma 4.8 the Iw^+_n-orbits on $(\mathcal{G}r_G)^{\mu_n}$ are also parametrized by $\mathbf{X}^\vee$. Under the identification of Proposition 4.6, for any $\lambda \in (-\overline{\mathbf{a}}_n) \cap \mathbf{X}^\vee$ the orbit in $\mathcal{F}l^n_{\mathbf{f}_\lambda}$ corresponding to the coset $wW^{\mathbf{f}}_{\mathrm{aff}}$ is mapped to the orbit in $(\mathcal{G}r_G)^{\mu_n}$ parametrized by $w \square_n \lambda$.

5. Iwahori–Whittaker sheaves on the affine Grassmannian

We continue with the setting of Section 4.

5.1. **Iwahori–Whittaker sheaves.** The category of sheaves on $\mathcal{G}r_G$ we will study is the *Iwahori–Whittaker* derived category, whose definition we briefly recall. (For more details, see e.g. [AR1, Appendix A].)

From now on we let $\Bbbk$ be a finite field of positive characteristic $\ell \neq p$ containing a nontrivial p-th root of unity. After choosing such a root of unity ζ, we obtain an Artin–Schreier local system $\mathscr{L}_{\mathrm{AS}}$ on $\mathbb{G}_{\mathrm{a}}$, defined as the direct summand of the local system $\mathrm{AS}_* \underline{\Bbbk}_{\mathbb{G}_{\mathrm{a}}}$ on which $\mathbb{F}_p$ acts via $n \mapsto \zeta^n$. (Here, $\mathrm{AS} : \mathbb{G}_{\mathrm{a}} \to \mathbb{G}_{\mathrm{a}}$ is the map $x \mapsto x^p - x$, a Galois cover of group $\mathbb{F}_p$.) We choose once and for all a morphism of $\mathbb{F}$-algebraic groups $\chi_0 : U^+ \to \mathbb{G}_{\mathrm{a}}$ which is nontrivial on any root subgroup of U^+ associated with a simple root, and denote by

$$\chi : \mathrm{Iw}_{\mathrm{u}}^+ \to \mathbb{G}_{\mathrm{a}}$$

the composition of χ_0 with the morphism $\mathrm{Iw}_{\mathrm{u}}^+ \to U^+$ induced by ev_0.

For $X \subset \mathcal{G}r_G$ a locally closed finite union of Iw^+-orbits, we can choose a smooth quotient J of $\mathrm{Iw}_{\mathrm{u}}^+$ of finite type such that the $\mathrm{Iw}_{\mathrm{u}}^+$-action on X factors through an action of J, and such that χ factors through a morphism $\chi_J : J \to \mathbb{G}_{\mathrm{a}}$. Then the $(J, \chi_J^* \mathscr{L}_{\mathrm{AS}})$-equivariant derived category of $\Bbbk$-sheaves on X is by definition the full subcategory of $D^{\mathrm{b}}\mathsf{Sh}(X, \Bbbk)$ whose objects are the complexes $\mathscr{F}$ whose pullback under the action map $J \times X \to X$ is isomorphic to $\chi_J^* \mathscr{L}_{\mathrm{AS}} \boxtimes \mathscr{F}$. It is well known that this subcategory is triangulated, and that it does not depend on the choice of J; it will be denoted

$$D^{\mathrm{b}}_{\mathcal{IW}}(X, \Bbbk).$$

It is known also that the perverse t-structure on $D^{\mathrm{b}}\mathsf{Sh}(X, \Bbbk)$ restricts to a t-structure on $D^{\mathrm{b}}_{\mathcal{IW}}(X, \Bbbk)$, which will also be called the perverse t-structure. (Here, "$\mathcal{IW}$" stands for "Iwahori–Whittaker." We will use this expression as a replacement for "$(J, \chi_J^* \mathscr{L}_{\mathrm{AS}})$-equivariant" where J is as above, in all circumstances where this notion does not depend on the choice of J.)

One can also define the category

$$D^{\mathrm{b}}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$$

of Iwahori–Whittaker sheaves on $\mathcal{G}r_G$ as the direct limit of the categories $D^{\mathrm{b}}_{\mathcal{IW}}(X, \Bbbk)$ where X runs over the closed finite unions of Iw^+-orbits, ordered by inclusion. (Here, the transition functors are the—fully faithful—pushforward functors.) Since, for $X \subset Y$, the pushforward functor $D^{\mathrm{b}}_{\mathcal{IW}}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathcal{IW}}(Y, \Bbbk)$ is t-exact, from the perverse t-structures on the categories $D^{\mathrm{b}}_{\mathcal{IW}}(X, \Bbbk)$ we obtain a perverse t-structure on $D^{\mathrm{b}}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$, whose heart will be denoted $\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$.

The considerations on stabilizers from the proof of Lemma 4.8 can be used to see that for $\lambda \in \mathbf{X}^\vee$, the orbit $\mathcal{G}r^+_{G,\lambda}$ supports a nonzero Iwahori–Whittaker local system iff λ belongs to the subset

$$\mathbf{X}^\vee_{++} := \{\mu \in \mathbf{X}^\vee \mid \forall \alpha \in \mathfrak{R}^+, \langle \mu, \alpha \rangle > 0\}.$$

Moreover, in this case there exists (up to isomorphism) exactly one such local system of rank 1; it will be denoted $\mathscr{L}^\lambda_{\mathrm{AS}}$. This remark implies that for any $\mu \in \mathbf{X}^\vee \smallsetminus \mathbf{X}^\vee_{++}$ the category $D^{\mathrm{b}}_{\mathcal{IW}}(\mathcal{G}r^+_{G,\mu}, \Bbbk)$ is 0; in particular, the restriction and co-restriction of any object in $D^{\mathrm{b}}_{\mathcal{IW}}(X, \Bbbk)$ (where X is any locally closed finite union of Iw^+-orbits containing $\mathcal{G}r^+_{G,\mu}$) vanishes.

Remark 5.1. If we assume that there exists an element $\varsigma \in \mathbf{X}^\vee$ such that $\langle \varsigma, \alpha \rangle = 1$ for all $\alpha \in \mathfrak{R}^{\mathrm{s}}$, then we have $\mathbf{X}^\vee_{++} = \varsigma + \mathbf{X}^\vee_{+}$.

For $\lambda \in \mathbf{X}^\vee_{++}$, we set

$$\Delta^{\mathcal{IW}}_\lambda := (j^+_\lambda)_! \mathscr{L}^\lambda_{\mathrm{AS}}[\dim(\mathcal{G}r^+_{G,\lambda})], \quad \nabla^{\mathcal{IW}}_\lambda := (j^+_\lambda)_* \mathscr{L}^\lambda_{\mathrm{AS}}[\dim(\mathcal{G}r^+_{G,\lambda})].$$

Since j^+_λ is an affine embedding, these objects are perverse sheaves by [BBDG, Corollaire 4.1.3]. Standard arguments (going back to [BGS]) show that the category $\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$ admits a structure of highest weight category (in the sense of [Ri, §7]) with weight set $\mathbf{X}^\vee_{++}$, standard objects the objects $(\Delta^{\mathcal{IW}}_\lambda : \lambda \in \mathbf{X}^\vee_{++})$, and costandard objects the objects $(\nabla^{\mathcal{IW}}_\lambda : \lambda \in \mathbf{X}^\vee_{++})$. In particular, one can consider the tilting objects in this category, i.e. the objects which admit both a filtration with standard subquotients, and a filtration with costandard subquotients. Recall that, as remarked in [BBM], this notion can also be characterized topologically: a perverse sheaf $\mathscr{F}$ is tilting iff the complexes $(j^+_\lambda)^* \mathscr{F}$ and $(j^+_\lambda)^! \mathscr{F}$ are perverse (i.e. are direct sums of copies of $\mathscr{L}^\lambda_{\mathrm{AS}}[\dim(\mathcal{G}r^+_{G,\lambda})]$) for any $\lambda \in \mathbf{X}^\vee_{++}$.

The full subcategory of $\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$ whose objects are the tilting objects will be denoted $\mathsf{Tilt}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$. The general theory of highest weight categories (reviewed e.g. in [Ri]) guarantees that the indecomposable objects in this category are parametrized in a natural way by $\mathbf{X}^\vee_{++}$. More precisely, for any $\lambda \in \mathbf{X}^\vee_{++}$ there exists a unique (up to isomorphism) indecomposable object $\mathscr{T}^{\mathcal{IW}}_\lambda$ in $\mathsf{Tilt}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$ which is supported on $\overline{\mathcal{G}r^+_{G,\lambda}}$, and whose restriction to $\mathcal{G}r^+_{G,\lambda}$ is $\mathscr{L}^\lambda_{\mathrm{AS}}[\dim(\mathcal{G}r^+_{G,\lambda})]$; then the assignment $\lambda \mapsto \mathscr{T}^{\mathcal{IW}}_\lambda$ induces a bijection between $\mathbf{X}^\vee_{++}$ and the set of isomorphism classes of indecomposable objects in $\mathsf{Tilt}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$.

5.2. **Loop rotation equivariant Iwahori–Whittaker sheaves.** We will need to "add" the (loop rotation) $\mathbb{G}_{\mathrm{m}}$-equivariance in the construction of §5.1. We therefore consider a locally closed finite union of Iw^+-orbits $X \subset \mathcal{G}r_G$ as above. The $\mathbb{G}_{\mathrm{m}}$-action by loop rotation on $\mathcal{G}r_G$ stabilizes each Iw^+-orbit, hence also X, so that we can consider the $\mathbb{G}_{\mathrm{m}}$-equivariant derived category of étale $\Bbbk$-sheaves $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X, \Bbbk)$. The quotient J of $\mathrm{Iw}^+_{\mathrm{u}}$ as in §5.1 can be chosen in such a way that the $\mathbb{G}_{\mathrm{m}}$-action on $\mathrm{Iw}^+_{\mathrm{u}}$ induces an action on J. Since the morphism $\chi : \mathrm{Iw}^+_{\mathrm{u}} \to \mathbb{G}_{\mathrm{a}}$ is $\mathbb{G}_{\mathrm{m}}$-equivariant (for the trivial $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{G}_{\mathrm{a}}$), so is χ_J, and the local system $\chi_J^* \mathscr{L}_{\mathrm{AS}}$ is therefore $\mathbb{G}_{\mathrm{m}}$-equivariant. We define the category

$$D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(X, \Bbbk)$$

as the full subcategory of $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X, \Bbbk)$ whose objects are the complexes $\mathscr{F}$ such that

$$a_J^* \mathscr{F} \cong \chi_J^* \mathscr{L}_{\mathrm{AS}} \boxtimes \mathscr{F} \quad \text{in } D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(J \times X, \Bbbk),$$

where $a_J : J \times X \to X$ is the action morphism. (Here, $\mathbb{G}_{\mathrm{m}}$ acts diagonally on $J \times X$.) Arguments similar to those for the case when the $\mathbb{G}_{\mathrm{m}}$-action is dropped show that $D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(X, \Bbbk)$ is a triangulated subcategory of $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X, \Bbbk)$; in fact this category is the essential image of the fully faithful functor

$$D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(X, \Bbbk)$$

sending a complex $\mathscr{F}$ to $(a_J)_!\big(\chi_J^* \mathscr{L}_{\mathrm{AS}} \boxtimes \mathscr{F}\big)$. It is also easily checked that this category does not depend on the choice of J, and that the perverse t-structure on $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}}}(X, \Bbbk)$ restricts to a t-structure on $D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(X, \Bbbk)$.

Taking the direct limit of the categories $D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(X,\Bbbk)$ where X runs over the closed finite unions of Iw^+-orbits, we obtain a triangulated category

$$D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(\mathcal{G}r_G,\Bbbk)$$

with a natural perverse t-structure, whose heart will be denoted $\mathsf{Perv}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(\mathcal{G}r_G,\Bbbk)$. We have a natural t-exact forgetful functor

$$D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(\mathcal{G}r_G,\Bbbk) \to D^{\mathrm{b}}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk). \tag{5.1}$$

5.3. **Parity complexes.** Let $X \subset \mathcal{G}r_G$ be a locally closed finite union of Iw^+-orbits. Recall (see [JMW, RW1]) that an object $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathcal{IW}}(X,\Bbbk)$ is called $*$-even, resp. !-even, if for any $\lambda \in \mathbf{X}^\vee_{++}$ such that $\mathcal{G}r^+_{G,\lambda} \subset X$ the complex $(j^+_\lambda)^*\mathscr{F}$, resp. $(j^+_\lambda)^!\mathscr{F}$, is concentrated in even degrees, i.e. is a direct sum of objects of the form $\mathscr{L}^\lambda_{\mathrm{AS}}[n]$ with $n \in 2\mathbb{Z}$. (Here, by abuse we still denote by j^+_λ the embedding of $\mathcal{G}r^+_{G,\lambda}$ in X. Note also that if $\lambda \in \mathbf{X}^\vee \smallsetminus \mathbf{X}^\vee_{++}$ is such that $\mathcal{G}r^+_{G,\lambda} \subset X$, then as explained in §5.1 we have $(j^+_\lambda)^*\mathscr{F} = (j^+_\lambda)^!\mathscr{F} = 0$ for any $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathcal{IW}}(X,\Bbbk)$, so that no condition is required for these strata.) We define similarly the $*$-odd and !-odd objects (requiring that n is odd in this case), and we say that $\mathscr{F}$ is even, resp. odd, if it is both $*$-even and !-even, resp. $*$-odd and !-odd.

These notions can also be considered in $D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(X,\Bbbk)$; more precisely an object $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(X,\Bbbk)$ is said to be $*$-even, resp. !-even, etc., if its image in $D^{\mathrm{b}}_{\mathcal{IW}}(X,\Bbbk)$ (under the forgetful functor) is $*$-even, resp. !-even, etc. If $\mathscr{F} \in D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(X,\Bbbk)$ is $*$-even, for any $\lambda \in \mathbf{X}^\vee_{++}$ such that $\mathcal{G}r^+_{G,\lambda} \subset X$ the complex $(j^+_\lambda)^*\mathscr{F}$ is a direct sum of objects of the form $\mathscr{L}^\lambda_{\mathrm{AS}}[n]$ with $n \in 2\mathbb{Z}$ in $D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(\mathcal{G}r^+_{G,\lambda},\Bbbk)$. A similar comment applies to !-even objects (with respect to !-restriction), and to $*$-odd and !-odd objects.

By definition, the category $D^{\mathrm{b}}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)$, resp. $D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(\mathcal{G}r_G,\Bbbk)$, is the direct limit of the categories $D^{\mathrm{b}}_{\mathcal{IW}}(X,\Bbbk)$, resp. $D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(X,\Bbbk)$, where X runs over the closed finite unions of Iw^+-orbits in $\mathcal{G}r_G$. Hence it makes sense to consider even and odd complexes in these categories. The general theory of [JMW] (see also [RW1, ACR] for some comments on the Iwahori–Whittaker case) guarantees that for any $\lambda \in \mathbf{X}^\vee_{++}$ there exists a unique (up to isomorphism) indecomposable object in $D^{\mathrm{b}}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)$, resp. $D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(\mathcal{G}r_G,\Bbbk)$, which has the same parity as $\dim(\mathcal{G}r^+_{G,\lambda})$, which is supported on $\overline{\mathcal{G}r^+_{G,\lambda}}$, and whose restriction to $\mathcal{G}r^+_{G,\lambda}$ is $\mathscr{L}^\lambda_{\mathrm{AS}}[\dim(\mathcal{G}r^+_{G,\lambda})]$. This object will be denoted

$$\mathscr{E}^{\mathcal{IW}}_\lambda, \quad \text{resp.} \quad \mathscr{E}^{\mathcal{IW}}_{\lambda,\mathbb{G}_{\mathrm{m}}}.$$

It is known also that the image of $\mathscr{E}^{\mathcal{IW}}_{\lambda,\mathbb{G}_{\mathrm{m}}}$ under the forgetful functor (5.1) is $\mathscr{E}^{\mathcal{IW}}_\lambda$, see e.g. [MR, Lemma 2.4].

As remarked already in [BGMRR], these objects have an alternative description, as follows. It is known that the parity of $\dim(\mathcal{G}r^+_{G,\lambda})$ (with $\lambda \in \mathbf{X}^\vee_{++}$) is constant on each connected component of $\mathcal{G}r_G$. As a consequence, a tilting object supported on a component where these dimensions are even, resp. odd, is even, resp. odd. In particular, by unicity, for any $\lambda \in \mathbf{X}^\vee_{++}$ we must have

$$\mathscr{T}^{\mathcal{IW}}_\lambda \cong \mathscr{E}^{\mathcal{IW}}_\lambda. \tag{5.2}$$

Using these considerations we prove the following lemma, to be used later.

Lemma 5.2. *The forgetful functor induces an equivalence of categories*

$$\mathsf{Perv}_{\mathcal{IW},\mathbb{G}_\mathrm{m}}(\mathcal{G}r_G,\Bbbk) \xrightarrow{\sim} \mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk).$$

Proof. It follows from the general theory of equivariant perverse sheaves (recalled e.g. in [BR, §1.16]) that the forgetful functor $\mathsf{Perv}_{\mathbb{G}_\mathrm{m}}(\mathcal{G}r_G,\Bbbk) \to \mathsf{Perv}(\mathcal{G}r_G,\Bbbk)$ is fully-faithful; therefore, so is its restriction $\mathsf{Perv}_{\mathcal{IW},\mathbb{G}_\mathrm{m}}(\mathcal{G}r_G,\Bbbk) \to \mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)$. This general theory also implies that the essential image of this functor is stable under subquotients (see e.g. [J3, §12.19]). Now from (5.2) and the fact that each object $\mathscr{E}^{\mathcal{IW}}_\lambda$ belongs to the essential image of the forgetful functor $D^\mathrm{b}_{\mathcal{IW},\mathbb{G}_\mathrm{m}}(\mathcal{G}r_G,\Bbbk) \to D^\mathrm{b}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)$ (see the remarks above), we see that the essential image of our functor contains all the tilting objects. By the general theory of highest weight categories (see [Ri, Proposition 7.17]), the canonical functor provides an equivalence of triangulated categories

$$K^\mathrm{b}\mathsf{Tilt}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk) \xrightarrow{\sim} D^\mathrm{b}\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk). \tag{5.3}$$

In particular, it follows that any object of $\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)$ is a subquotient of a tilting object, hence that it belongs to this essential image, which finishes the proof. □

6. Smith theory for Iwahori–Whittaker sheaves on $\mathcal{G}r_G$

We continue with the setting of Sections 4–5. Our goal in this section is build a "Smith theory" for the category $D^\mathrm{b}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)$, following Treumann [Tr] and Leslie–Lonergan [LL].

6.1. **The Iwahori–Whittaker Smith category.** As in Section 3 we consider the subgroup scheme $\varpi = \boldsymbol{\mu}_\ell \subset \mathbb{G}_\mathrm{m}$, and the fixed points $(\mathcal{G}r_G)^\varpi \subset \mathcal{G}r_G$ with respect to the loop rotation action. This subscheme is described in §4.5; in particular since $(\mathrm{Iw}^+)^\varpi = \mathrm{Iw}^+_\ell$ (see Lemma 4.8), this group acts on $(\mathcal{G}r_G)^\varpi$, and each Iw^+_ℓ-orbit is also an $\mathrm{Iw}^+_{\mathrm{u},\ell}$-orbit.

The $\mathbb{G}_\mathrm{m}$-action on $\mathcal{G}r_G$ stabilizes $(\mathcal{G}r_G)^\varpi$, hence induces an action on this sub-ind-scheme. On the other hand, as explained above we also have an action of $\mathrm{Iw}^+_{\mathrm{u},\ell}$ on $(\mathcal{G}r_G)^\varpi$. The analysis in §4.6 shows that the orbits of the latter action are naturally parametrized by $\mathbf{X}^\vee$, and that each orbit is stable under the action of $\mathbb{G}_\mathrm{m}$. Repeating the construction in §5.2, now with the morphism $\mathrm{Iw}^+_{\mathrm{u},\ell} \to \mathbb{G}_\mathrm{a}$ obtained by restricting χ one can define for any locally closed finite union of Iw^+_ℓ-orbits $Y \subset (\mathcal{G}r_G)^\varpi$ the Iwahori–Whittaker loop rotation equivariant derived category

$$D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}(Y,\Bbbk).$$

We define $D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}(Y,\Bbbk)_{\varpi-\mathsf{perf}}$ as the full subcategory of $D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}(Y,\Bbbk)$ whose objects are the complexes $\mathscr{F}$ such that $\mathrm{Res}^{\mathbb{G}_\mathrm{m}}_\varpi(\mathscr{F})$ has perfect geometric stalks in the sense of §3.3. We then define the *Iwahori–Whittaker Smith category* of Y as the Verdier quotient

$$\mathsf{Sm}_{\mathcal{IW}}(Y,\Bbbk) := D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}(Y,\Bbbk)/D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}(Y,\Bbbk)_{\varpi-\mathsf{perf}}.$$

This category has a natural structure of triangulated category; the (cohomological) shift functor will be denoted [1] as usual.

We now check that this construction is functorial in the following sense.

Lemma 6.1. *Let $Y, Z \subset (\mathcal{G}r_G)^\varpi$ be two locally closed finite unions of Iw_ℓ^+-orbits such that $Z \subset Y$. Denoting by f this inclusion, for $? \in \{*, !\}$ there exist canonical functors*

$$f_?^{\mathsf{Sm}} : \mathsf{Sm}_{\mathcal{IW}}(Z, \Bbbk) \to \mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk), \quad f_{\mathsf{Sm}}^? : \mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk) \to \mathsf{Sm}_{\mathcal{IW}}(Z, \Bbbk)$$

such that the diagrams

$$\begin{array}{ccc} D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Z, \Bbbk) & \xrightarrow{Rf_?} & D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Y, \Bbbk) \\ \downarrow & & \downarrow \\ \mathsf{Sm}_{\mathcal{IW}}(Z, \Bbbk) & \xrightarrow{f_?^{\mathsf{Sm}}} & \mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk) \end{array} \quad \text{and} \quad \begin{array}{ccc} D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Y, \Bbbk) & \xrightarrow{f^?} & D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Z, \Bbbk) \\ \downarrow & & \downarrow \\ \mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk) & \xrightarrow{f_{\mathsf{Sm}}^?} & \mathsf{Sm}_{\mathcal{IW}}(Z, \Bbbk) \end{array}$$

are commutative, where the vertical arrows are the quotient functors.

Proof. By the universal property of Verdier quotients, we need to show is that the functors Rf_*, $Rf_!$, resp. f^*, $f^!$, send $D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Z, \Bbbk)_{\varpi-\mathsf{perf}}$ into $D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Y, \Bbbk)_{\varpi-\mathsf{perf}}$, resp. $D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Y, \Bbbk)_{\varpi-\mathsf{perf}}$ into $D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Z, \Bbbk)_{\varpi-\mathsf{perf}}$. For the functor f^* this claim is obvious from definition, and for Rf_* it follows from Lemma 3.6. For the functor $Rf_!$, one can argue as follows. If $\overline{y} : \mathrm{Spec}(K) \to Y$ is a geometric point of Y, then by [SGA4, Exposé XVII, Proposition 5.2.8] we have

$$(Rf_! \mathscr{F})_{\overline{y}} \cong R\Gamma(Z \times_Y \mathrm{Spec}(K), \mathscr{F}'),$$

where $\mathscr{F}'$ is the pullback of $\mathscr{F}$. Now $Z \times_Y \mathrm{Spec}(K)$ is a locally closed subscheme of $\mathrm{Spec}(K)$, hence is either $\varnothing$ or $\mathrm{Spec}(K)$. Hence $(Rf_! \mathscr{F})_{\overline{y}}$ is either equal to $\mathscr{F}_{\overline{y}}$ or to 0, which shows that $Rf_! \mathscr{F}$ must belong to $D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Y, \Bbbk)_{\varpi-\mathsf{perf}}$.

Finally we treat the case of $f^!$. For this we can assume that f is either a closed embedding or an open embedding. In the latter case we have $f^! = f^*$, hence the claim is known. In the former case, we denote by g the complementary open embedding. Then, given $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Y, \Bbbk)_{\varpi-\mathsf{perf}}$ we consider the distinguished triangle

$$f_* f^! \mathscr{F} \to \mathscr{F} \to g_* g^* \mathscr{F} \xrightarrow{[1]}.$$

Here $\mathscr{F}$ and $g_* g^* \mathscr{F}$ belong to $D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Y, \Bbbk)_{\varpi-\mathsf{perf}}$, hence so does $f_* f^! \mathscr{F}$. This implies that $f^! \mathscr{F}$ belongs to $D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Z, \Bbbk)_{\varpi-\mathsf{perf}}$, which completes the proof. □

It is easily seen that $(f^*_{\mathsf{Sm}}, f^{\mathsf{Sm}}_*)$ and $(f^{\mathsf{Sm}}_!, f^!_{\mathsf{Sm}})$ are adjoint pairs of functors. In particular, if f is a closed embedding then the functor $f^{\mathsf{Sm}}_* = f^{\mathsf{Sm}}_!$ is fully faithful, so that the category $\mathsf{Sm}_{\mathcal{IW}}(Z, \Bbbk)$ can (and will) be identified with a full triangulated subcategory in $\mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$. It is also easily checked that, given a decomposition of Y as a disjoint union of a closed (in Y) finite union of Iw_ℓ^+-orbits and its open complement, we have canonical distinguished triangles as in the "recollement" setting of [BBDG, §1.4].

The full faithfulness of pushforward under closed embeddings allows to define the category $\mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_G^+)^\varpi, \Bbbk)$ as the direct limit of the categories $\mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$ where Y runs over the closed finite unions of Iw_ℓ^+-orbits in $(\mathcal{G}r_G^+)^\varpi$.

6.2. **The Smith localization functor.** We will be particularly interested in the construction of §6.1 in the case $Y = X^\varpi$ for some locally closed finite union of Iw^+-orbits $X \subset \mathcal{G}r_G$. In this case, we denote by $i_X : X^\varpi \to X$ the embedding. For any $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(X,\Bbbk)$, we have objects $i_X^!\mathscr{F}$ and $i_X^*\mathscr{F}$ in $D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}_{\mathrm{m}}}(X^\varpi,\Bbbk)$, and a canonical morphism

$$i_X^!\mathscr{F} \to i_X^*\mathscr{F},$$

see (3.4). It follows from Lemma 3.5 that the cone of this morphism is killed by the quotient functor

$$D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}_{\mathrm{m}}}(X^\varpi,\Bbbk) \to \mathsf{Sm}_{\mathcal{IW}}(X^\varpi,\Bbbk).$$

We can therefore define the functor

$$i_X^{!*} : D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(X,\Bbbk) \to \mathsf{Sm}_{\mathcal{IW}}(X^\varpi,\Bbbk)$$

as the composition of either i_X^* or $i_X^!$ with this quotient functor.

This functor is compatible with the push/pull functors associated with locally closed embeddings, in the following sense.

Proposition 6.2. *If $X, Y \subset \mathcal{G}r_G$ are two locally closed finite unions of* Iw^+*-orbits such that $X \subset Y$, and if we denote by $f : X \to Y$ the embedding and by $f^\varpi : X^\varpi \to Y^\varpi$ its restriction to X^ϖ, then we have canonical isomorphisms of functors*

$$i_Y^{!*} \circ f_* \cong (f^\varpi)_*^{\mathsf{Sm}} \circ i_X^{!*}, \qquad i_Y^{!*} \circ f_! \cong (f^\varpi)_!^{\mathsf{Sm}} \circ i_X^{!*},$$
$$i_X^{!*} \circ f^* \cong (f^\varpi)^*_{\mathsf{Sm}} \circ i_Y^{!*}, \qquad i_X^{!*} \circ f^! \cong (f^\varpi)^!_{\mathsf{Sm}} \circ i_Y^{!*}.$$

Proof. The first, resp. second, isomorphism on the first line follows from the base change theorem (see [SGA4, Exposé XVIII, Corollaire 3.1.12.3] and [SGA4, Exposé XVII, Théorème 5.2.6] respectively) if we see $i_Y^{!*}$ and $i_X^{!*}$ as the compositions of $i_Y^!$ and $i_X^!$, resp. of i_Y^* and i_X^*, with the appropriate quotient functors. The isomorphisms on the second line follow similarly from the compatibility of pullback functors with composition. □

Taking the direct limit of the functors $i_X^{!*}$ for X a closed finite union of Iw^+-orbits in $\mathcal{G}r_G$, we also obtain a functor

$$i_{\mathcal{G}r_G}^{!*} : D^{\mathrm{b}}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(\mathcal{G}r_G,\Bbbk) \to \mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_G)^\varpi,\Bbbk).$$

6.3. **Some first properties of the Iwahori–Whittaker Smith category.** Let us fix some $\lambda \in \mathbf{X}^\vee_{++}$, and consider the $\mathrm{Iw}^+_{\mathrm{u},\ell}$-orbit $(\mathcal{G}r^+_{G,\lambda})^\varpi \subset (\mathcal{G}r_G)^\varpi$. (Once again, the Iwahori–Whittaker category associated with an orbit labelled by a weight in $\mathbf{X}^\vee \smallsetminus \mathbf{X}^\vee_{++}$ vanishes; these coweights can therefore be ignored.) We set

$$\mathscr{L}^\lambda_{\mathsf{Sm}} := i^{!*}_{\mathcal{G}r^+_{G,\lambda}}(\mathscr{L}^\lambda_{\mathrm{AS}}).$$

Lemma 6.3. *For any $n \in \mathbb{Z}$, we have*

$$\mathrm{Hom}_{\mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r^+_{G,\lambda})^\varpi,\Bbbk)}(\mathscr{L}^\lambda_{\mathsf{Sm}}, \mathscr{L}^\lambda_{\mathsf{Sm}}[n]) = \begin{cases} \Bbbk & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Since $\mathrm{Iw}^+_{\mathrm{u},\ell}$ acts transitively on $(\mathcal{G}r^+_{G,\lambda})^\varpi$ (see Lemma 4.8), we have an equivalence of triangulated categories

$$D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}_{\mathrm{m}}}((\mathcal{G}r^+_{G,\lambda})^\varpi,\Bbbk) \cong D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt},\Bbbk)$$

which matches $\mathscr{L}^\lambda_{\mathrm{AS}}$ with $\underline{\Bbbk}_{\mathrm{pt}}$. This equivalence induces an equivalence

$$\mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r^+_{G,\lambda})^\varpi,\Bbbk) \cong \mathsf{Sm}(\mathrm{pt},\Bbbk),$$

where the right-hand side is defined in §3.5. The claim then follows from Lemma 3.7. □

We consider once again a general locally closed finite union of Iw_ℓ^+-orbits $Y \subset (\mathcal{G}r_G)^\varpi$.

Lemma 6.4. *There exists a canonical isomorphism of endofunctors of* $\mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$

$$\mathrm{id} \xrightarrow{\sim} [2].$$

Proof. As explained in Lemma 3.7, there exists a canonical map $\underline{\Bbbk}_{\mathrm{pt}} \to \underline{\Bbbk}_{\mathrm{pt}}[2]$ in $D^{\mathrm{b}}_{\mathbb{G}_{\mathrm{m}},c}(\mathrm{pt}, \Bbbk)$ whose cone has perfect geometric stalks. Pulling back to Y we deduce a canonical morphism $\underline{\Bbbk}_Y \to \underline{\Bbbk}_Y[2]$ whose cone has perfect geometric stalks. Since the tensor product with $\underline{\Bbbk}_Y$, resp. $\underline{\Bbbk}_Y[2]$, defines an endofunctor of $D^{\mathrm{b}}_{\mathcal{IW}_\ell, \mathbb{G}_{\mathrm{m}}}(Y, \Bbbk)$ which is isomorphic to id, resp. to [2], the desired claim follows. □

Proposition 6.5. *For any* $\mathscr{F}, \mathscr{G}$ *in* $\mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$*, the* $\Bbbk$*-vector space*

$$\mathrm{Hom}_{\mathsf{Sm}_{\mathcal{IW}}(Y,\Bbbk)}(\mathscr{F}, \mathscr{G})$$

is finite-dimensional.

Proof. The proof proceeds by induction on the number of Iw_ℓ^+-orbits in Y. In fact the distinguished triangles from the "recollement" setting (see §6.1) reduce the proof to the case Y consists of one orbit, which follows from Lemma 6.3. □

7. Parity objects in Smith categories

We continue with the setting of Sections 4–6.

7.1. **Definition.** As remarked already in [LL] (using slightly different definitions), the theory of parity complexes from [JMW] adapts easily to the Smith category $\mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$, where $Y \subset (\mathcal{G}r_G)^\varpi$ is any locally closed union of Iw_ℓ^+-orbits. Namely, we will say that an object $\mathscr{F}$ in $\mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$ is $*$*-even*, resp. !*-even*, if for any $\lambda \in \mathbf{X}^\vee_{++}$ such that $(\mathcal{G}r^+_{G,\lambda})^\varpi \subset Y$, denoting by $j_\lambda^{+,\varpi} : (\mathcal{G}r^+_{G,\lambda})^\varpi \to Y$ the embedding, the object $(j_\lambda^{+,\varpi})^*_{\mathsf{Sm}}\mathscr{F}$, resp. $(j_\lambda^{+,\varpi})^!_{\mathsf{Sm}}\mathscr{F}$, is isomorphic to a direct sum of copies of $\mathscr{L}^\lambda_{\mathsf{Sm}}$. (In this case we do not need to consider even shifts because of Lemma 6.4.) We will then say that $\mathscr{F}$ is *even* if it is both $*$-even and !-even, and define similarly the notions of $*$-odd, !-odd, and odd objects (replacing $\mathscr{L}^\lambda_{\mathsf{Sm}}$ by its shift by 1). We will denote by $\mathsf{Sm}^0_{\mathcal{IW}}(Y, \Bbbk)$, resp. $\mathsf{Sm}^1_{\mathcal{IW}}(Y, \Bbbk)$, resp. $\mathsf{Sm}^{\mathrm{par}}_{\mathcal{IW}}(Y, \Bbbk)$, the full subcategory of $\mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$ whose objects are the even objects, resp. the odd objects, resp. the objects which are isomorphic to a direct sum of an even and an odd object.

Recall that an additive category is called *Krull–Schmidt* if any object can be written as a direct of indecomposable objects whose endomorphism rings are local.

Lemma 7.1. *The categories* $\mathsf{Sm}^0_{\mathcal{IW}}(Y, \Bbbk)$*,* $\mathsf{Sm}^1_{\mathcal{IW}}(Y, \Bbbk)$ *and* $\mathsf{Sm}^{\mathrm{par}}_{\mathcal{IW}}(Y, \Bbbk)$ *are Krull–Schmidt.*

Proof. By Proposition 6.5 and [CYZ, Corollary A.2], to prove the lemma it suffices to prove that any idempotent in the category $\mathsf{Sm}^{\mathrm{par}}_{\mathcal{IW}}(Y, \Bbbk)$ splits. We do this by induction on the number of Iw_ℓ^+-orbits in Y. If $Y = (\mathcal{G}r^+_{G,\lambda})^\varpi$ for some $\lambda \in \mathbf{X}^\vee$, and if $\mathscr{F}$ belongs to $\mathsf{Sm}^{\mathrm{par}}_{\mathcal{IW}}(Y, \Bbbk)$ then either $\mathscr{F} = 0$ (in which case there is nothing to prove) or $\lambda \in \mathbf{X}^\vee_{++}$ and $\mathscr{F} = (\mathscr{L}^\lambda_{\mathsf{Sm}})^{\oplus n} \oplus (\mathscr{L}^\lambda_{\mathsf{Sm}})^m[1]$ for some $n, m \in \mathbb{Z}_{\geqslant 0}$. In this case, by Lemma 6.3 we have

$$\mathrm{End}_{\mathsf{Sm}_{\mathcal{IW}}(Y,\Bbbk)}(\mathscr{F}) \cong \mathrm{M}_n(\Bbbk) \times \mathrm{M}_m(\Bbbk),$$

so that any idempotent in $\mathrm{End}_{\mathsf{Sm}_{\mathcal{IW}}(Y,\Bbbk)}(\mathscr{F})$ indeed splits.

To treat the induction step, we choose a closed Iw_ℓ^+-orbit $Z \subset X$, and denote by

$$i : Z \hookrightarrow X, \qquad j : X \smallsetminus Z \hookrightarrow X$$

the embeddings. For any $\mathscr{F}$ in $\mathsf{Sm}^{\mathrm{par}}_{\mathcal{IW}}(Y, \Bbbk)$ we then have a distinguished triangle

$$i^{\mathsf{Sm}}_! i^!_{\mathsf{Sm}} \mathscr{F} \to \mathscr{F} \to j^{\mathsf{Sm}}_* j^*_{\mathsf{Sm}} \mathscr{F} \xrightarrow{[1]},$$

and the objects $i^!_{\mathsf{Sm}}\mathscr{F}$ and $j^*_{\mathsf{Sm}}\mathscr{F}$ belong to $\mathsf{Sm}^{\mathrm{par}}_{\mathcal{IW}}(Z, \Bbbk)$ and to $\mathsf{Sm}^{\mathrm{par}}_{\mathcal{IW}}(Y \smallsetminus Z, \Bbbk)$ respectively. If $e \in \mathrm{End}_{\mathsf{Sm}_{\mathcal{IW}}(Y,\Bbbk)}(\mathscr{F})$ is an idempotent, then $i^!_{\mathsf{Sm}}(e)$ and $j^*_{\mathsf{Sm}}(e)$ are idempotents too, hence they split by the induction hypothesis. By [LC, Proposition 2.3], this implies that e splits. □

We will also define the categories

$$\mathsf{Sm}^0_{\mathcal{IW}}((\mathcal{G}r_G)^\varpi, \Bbbk), \quad \mathsf{Sm}^1_{\mathcal{IW}}((\mathcal{G}r_G)^\varpi, \Bbbk) \quad \text{and} \quad \mathsf{Sm}^{\mathrm{par}}_{\mathcal{IW}}((\mathcal{G}r_G)^\varpi, \Bbbk)$$

as the direct limit of their counterparts for X, where X runs over closed finite unions of Iw_ℓ^+-orbits in $(\mathcal{G}r_G)^\varpi$. (Equivalently, these categories can be defined in terms of restrictions and corestrictions to Iw_ℓ^+-orbits, as for their counterparts above.) Of course, Lemma 7.1 implies that these categories are Krull–Schmidt.

7.2. **Basic properties.** The study of parity objects in $\mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$ is very similar to its counterpart in ordinary derived categories of sheaves performed in [JMW]; its essential ingredients are the parity vanishing property for one stratum proved in Lemma 6.3, and standard distinguished triangles associated with a decomposition of a space into a closed part and its open complement. For this reason we will not give any proof in this subsection; these can be obtained by repeating the proofs of [JMW] essentially word-for-word.

The following is the analogue of [JMW, Corollary 2.8 and Proposition 2.11].

Lemma 7.2. *If $\mathscr{F}, \mathscr{G} \in \mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$ are such that $\mathscr{F}$ is $*$-even and $\mathscr{G}$ is $!$-odd, then we have*

$$\mathrm{Hom}_{\mathsf{Sm}_{\mathcal{IW}}(Y,\Bbbk)}(\mathscr{F}, \mathscr{G}) = 0.$$

As a consequence, if $Z \subset Y$ is an open union of Iw_ℓ^+-orbits, the restriction of an indecomposable even (resp. odd) object of $\mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$ to Z is either indecomposable or zero.

Next, we define the *support* of an object $\mathscr{F} \in \mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$ as the closure of the union of the strata $(\mathcal{G}r^+_{G,\lambda})^\varpi$ where $\lambda \in \mathbf{X}^\vee_{++}$ is such that $\mathcal{G}r^+_{G,\lambda} \subset Y$ and $(j^{+,\varpi}_\lambda)^*_{\mathsf{Sm}}\mathscr{F}$ or $(j^{+,\varpi}_\lambda)^!_{\mathsf{Sm}}\mathscr{F}$ is nonzero. The following claim is the analogue of [JMW, Theorem 2.12].

Proposition 7.3. *If $\mathscr{F} \in \mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$ is even (resp. odd), nonzero, and indecomposable, then there exists exactly one $\lambda \in \mathbf{X}^\vee_{++}$ such that $(\mathcal{G}r^+_{G,\lambda})^\varpi$ is open in the support of $\mathscr{F}$.*

Moreover, for any $\lambda \in \mathbf{X}^\vee_{++}$ such that $(\mathcal{G}r^+_{G,\lambda})^\varpi \subset Y$, there exists at most one indecomposable even, resp. odd, object $\mathscr{F}$ in $\mathsf{Sm}_{\mathcal{IW}}(Y, \Bbbk)$ such that $(\mathcal{G}r^+_{G,\lambda})^\varpi$ is open in the support of $\mathscr{F}$ and $(j^{+,\varpi}_\lambda)^*\mathscr{F} \cong \mathscr{L}^\lambda_{\mathsf{Sm}}$, resp. $(j^{+,\varpi}_\lambda)^*\mathscr{F} \cong \mathscr{L}^\lambda_{\mathsf{Sm}}[1]$.

7.3. **Comparison of parity objects in $D^{\mathrm{b}}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)$ and $\mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_G)^{\varpi},\Bbbk)$.** Proposition 7.3 implies that for any $\lambda \in \mathbf{X}^{\vee}_{++}$ there exists at most one indecomposable even, resp. odd, object in $\mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_G)^{\varpi},\Bbbk)$ in the support of which $(\mathcal{G}r^{+}_{G,\lambda})^{\varpi}$ is open, and whose restriction to $(\mathcal{G}r^{+}_{G,\lambda})^{\varpi}$ is $\mathscr{L}^{\lambda}_{\mathsf{Sm}}$, resp. $\mathscr{L}^{\lambda}_{\mathsf{Sm}}[1]$. If it exists (which, as we shall see very soon, is always the case), this object will be denoted $\mathscr{E}^{\mathsf{Sm},0}_{\lambda}$, resp. $\mathscr{E}^{\mathsf{Sm},1}_{\lambda}$. (Of course, as soon as one of these objects exists the other one exists also, and we have $\mathscr{E}^{\mathsf{Sm},1}_{\lambda} \cong \mathscr{E}^{\mathsf{Sm},0}_{\lambda}[1]$.) Then any indecomposable object in $\mathsf{Sm}^{\mathrm{par}}_{\mathcal{IW}}((\mathcal{G}r_G)^{\varpi},\Bbbk)$ is isomorphic to an object $\mathscr{E}^{\mathsf{Sm},0}_{\lambda}$ or $\mathscr{E}^{\mathsf{Sm},1}_{\lambda}$, and Lemma 7.1 implies that any object of $\mathsf{Sm}^{\mathrm{par}}_{\mathcal{IW}}((\mathcal{G}r_G)^{\varpi},\Bbbk)$ is a direct sum of such objects (in an essentially unique way).

Recall that the connected components of $(\mathcal{G}r_G)^{\varpi}$ are the subvarieties $\mathcal{G}r_{G,(\lambda)}$ with $\lambda \in (-\overline{\mathbf{a}}_{\ell}) \cap \mathbf{X}^{\vee}$, see Proposition 4.6. Of course, each such connected component is contained in a connected component of $\mathcal{G}r_G$. Recall also (see §5.3) that the dimensions of the orbits $\mathcal{G}r^{+}_{G,\mu}$ with $\mu \in \mathbf{X}^{\vee}_{++}$ contained in a given connected component of $\mathcal{G}r_G$ are of constant parity. We set $\mathsf{p}(\lambda) = 0$, resp. $\mathsf{p}(\lambda) = 1$, if all these orbits contained in the connected component containing $\mathcal{G}r_{G,(\lambda)}$ are even dimensional, resp. odd dimensional. We then denote by $\mathsf{Sm}^{\natural}_{\mathcal{IW}}((\mathcal{G}r_G)^{\varpi},\Bbbk)$ the full subcategory of $\mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_G)^{\varpi},\Bbbk)$ whose objects are those whose restriction to $\mathcal{G}r_{G,(\lambda)}$ is even if $\mathsf{p}(\lambda) = 0$, and odd if $\mathsf{p}(\lambda) = 1$.

The following statement is the crux of this paper.

Theorem 7.4. *The composition*

$$\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk) \xrightarrow[\sim]{\text{Lemma 5.2}} \mathsf{Perv}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(\mathcal{G}r_G,\Bbbk) \xrightarrow{i^{!*}_{\mathcal{G}r_G}} \mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_G)^{\varpi},\Bbbk)$$

restricts to an equivalence of categories

$$\mathsf{Tilt}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk) \to \mathsf{Sm}^{\natural}_{\mathcal{IW}}((\mathcal{G}r_G)^{\varpi},\Bbbk).$$

Moreover, the objects $\mathscr{E}^{\mathsf{Sm},0}_{\lambda}$ and $\mathscr{E}^{\mathsf{Sm},1}_{\lambda}$ exist for any $\lambda \in \mathbf{X}^{\vee}_{++}$.

Proof. It easily follows from Proposition 6.2 and the considerations above that the functor $i^{!*}_{\mathcal{G}r_G}$ sends even, resp. odd, objects to even, resp. odd, objects. Therefore, since any indecomposable object in $\mathsf{Tilt}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)$ is either even or odd (see (5.2)), our functor restricts to a functor

$$\mathsf{Tilt}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk) \to \mathsf{Sm}^{\natural}_{\mathcal{IW}}((\mathcal{G}r_G)^{\varpi},\Bbbk).$$

Next, standard arguments allow to prove by induction on the length of the filtrations that, for any $\mathscr{F},\mathscr{G}$ in $\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)$ such that $\mathscr{F}$ admits a standard filtration and $\mathscr{G}$ admits a costandard filtration, this functor induces an isomorphism

$$\mathrm{Hom}_{\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)}(\mathscr{F},\mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_G)^{\varpi},\Bbbk)}(i^{!*}_{\mathcal{G}r_G}(\mathscr{F}), i^{!*}_{\mathcal{G}r_G}(\mathscr{G})).$$

(Here, the crucial case when $\mathscr{F} = \Delta^{\mathcal{IW}}_{\lambda}$ and $\mathscr{G} = \nabla^{\mathcal{IW}}_{\lambda}$ for some $\lambda \in \mathbf{X}^{\vee}_{++}$ is given by Lemma 6.3.) Full faithfulness of our functor follows.

For any $\lambda \in \mathbf{X}^{\vee}_{++}$, the object $i^{!*}_{\mathcal{G}r_G}(\mathscr{E}^{\mathcal{IW}}_{\lambda,\mathbb{G}_{\mathrm{m}}})$ is indecomposable (by full faithfulness) and either even or odd. Moreover, since $\mathcal{G}r^{+}_{G,\lambda}$ is open in the support of $\mathscr{E}^{\mathcal{IW}}_{\lambda,\mathbb{G}_{\mathrm{m}}}$ we see that $(\mathcal{G}r^{+}_{G,\lambda})^{\varpi}$ is open in the support of $i^{!*}_{\mathcal{G}r_G}(\mathscr{E}^{\mathcal{IW}}_{\lambda,\mathbb{G}_{\mathrm{m}}})$. Therefore the objects

$\mathscr{E}_\lambda^{\mathsf{Sm},0}$ and $\mathscr{E}_\lambda^{\mathsf{Sm},1}$ exist, and we have

$$i^{!*}_{\mathcal{G}r_G}(\mathscr{E}^{\mathcal{IW}}_{\lambda,\mathbb{G}_\mathrm{m}}) \cong \begin{cases} \mathscr{E}_\lambda^{\mathsf{Sm},0} & \text{if } \dim(\mathcal{G}r^+_{G,\lambda}) \text{ is even;} \\ \mathscr{E}_\lambda^{\mathsf{Sm},1} & \text{if } \dim(\mathcal{G}r^+_{G,\lambda}) \text{ is odd.} \end{cases} \tag{7.1}$$

These considerations show that our functor is essentially surjective, hence an equivalence of categories. □

7.4. **Comparison of parity objects on $(\mathcal{G}r_G)^\varpi$ and in the Smith category.** Now we consider the Iwahori–Whittaker categories

$$D^{\mathrm{b}}_{\mathcal{IW}_\ell}((\mathcal{G}r_G)^\varpi, \Bbbk) \quad \text{and} \quad D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi, \Bbbk),$$

and the quotient functor

$$\mathsf{Q} : D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi, \Bbbk) \to \mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_G)^\varpi, \Bbbk).$$

The theory of parity complexes (as in §5.3) of course also applies in the categories $D^{\mathrm{b}}_{\mathcal{IW}_\ell}((\mathcal{G}r_G)^\varpi, \Bbbk)$ and $D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi, \Bbbk)$; once again the indecomposable parity objects in these categories are classified (up to cohomological shift) by the Iw_ℓ^+-orbits in $(\mathcal{G}r_G)^\varpi$ which support a nonzero Iwahori–Whittaker local system, and the forgetful functor

$$\mathsf{For}^{\mathbb{G}_\mathrm{m}} : D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi, \Bbbk) \to D^{\mathrm{b}}_{\mathcal{IW}_\ell}((\mathcal{G}r_G)^\varpi, \Bbbk)$$

sends indecomposable parity complexes to indecomposable parity complexes. In particular, this functor induces a bijection between the sets of isomorphism classes of objects in $D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi, \Bbbk)$ and in $D^{\mathrm{b}}_{\mathcal{IW}_\ell}((\mathcal{G}r_G)^\varpi, \Bbbk)$; up to replacing the category $D^{\mathrm{b}}_{\mathcal{IW}_\ell}((\mathcal{G}r_G)^\varpi, \Bbbk)$ by an equivalent category (which we will omit from notation), one can therefore consider whenever convenient that the objects in these categories are the same.

The situation in this setting is even more favorable than in that of §5.3, due to the following property. (Here, if D is a triangulated category, we write $\mathrm{Hom}^\bullet_{\mathsf{D}}(-,-)$ for $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{D}}(-,-[n])$.)

Lemma 7.5. *For any parity complexes $\mathscr{E}, \mathscr{E}'$ in $D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi, \Bbbk)$, there exists a canonical isomorphism of graded $\Bbbk$-vector spaces*

$$\mathrm{Hom}^\bullet_{D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathscr{E}, \mathscr{E}') \cong \mathsf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt};\Bbbk) \otimes_\Bbbk \mathrm{Hom}^\bullet_{D^{\mathrm{b}}_{\mathcal{IW}_\ell}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathsf{For}^{\mathbb{G}_\mathrm{m}}(\mathscr{E}), \mathsf{For}^{\mathbb{G}_\mathrm{m}}(\mathscr{E}')).$$

Moreover, these isomorphisms are compatible with composition in the obvious way.

Proof. By definition, the $\mathbb{G}_\mathrm{m}$-action on $(\mathcal{G}r_G)^\varpi$ through the quotient

$$\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}/\varpi = \mathbb{G}_\mathrm{m}, \quad t \mapsto t^\ell.$$

In other words, if we denote by $\mathbb{G}'_\mathrm{m}$ another copy of $\mathbb{G}_\mathrm{m}$, then there exists an action of $\mathbb{G}'_\mathrm{m}$ on $(\mathcal{G}r_G)^\varpi$ from which the $\mathbb{G}_\mathrm{m}$-action we want to consider is deduced via the morphism $\mathbb{G}_\mathrm{m} \to \mathbb{G}'_\mathrm{m}$ defined by $t \mapsto t^\ell$. The $\mathbb{G}_\mathrm{m}$-action on $\mathrm{Iw}^+_{\mathrm{u},\ell}$ is similarly obtained from an action of $\mathbb{G}'_\mathrm{m}$, so that one can consider the category $D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}'_\mathrm{m}}((\mathcal{G}r_G)^\varpi, \Bbbk)$ defined in the obvious way. With this notation introduced, the same considerations as in [MR, Lemma 2.2] show that for any parity complexes $\mathscr{F}, \mathscr{F}'$ in $D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}'_\mathrm{m}}((\mathcal{G}r_G)^\varpi, \Bbbk)$, the forgetful functor

$$\mathrm{Res}^{\mathbb{G}'_\mathrm{m}}_{\mathbb{G}_\mathrm{m}} : D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}'_\mathrm{m}}((\mathcal{G}r_G)^\varpi, \Bbbk) \to D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi, \Bbbk) \tag{7.2}$$

induces an isomorphism of graded $\Bbbk$-vector spaces

$$\mathsf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt};\Bbbk) \otimes_{\mathsf{H}^\bullet_{\mathbb{G}'_\mathrm{m}}(\mathrm{pt};\Bbbk)} \mathrm{Hom}^\bullet_{D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}'_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathscr{F},\mathscr{F}') \xrightarrow{\sim} \mathrm{Hom}^\bullet_{D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathrm{Res}^{\mathbb{G}'_\mathrm{m}}_{\mathbb{G}_\mathrm{m}}(\mathscr{F}),\mathrm{Res}^{\mathbb{G}'_\mathrm{m}}_{\mathbb{G}_\mathrm{m}}(\mathscr{F}')).$$

Now since $\Bbbk$ has characteristic ℓ, the morphism $\mathsf{H}^\bullet_{\mathbb{G}'_\mathrm{m}}(\mathrm{pt};\Bbbk) \to \mathsf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt};\Bbbk)$ induced by our morphism $\mathbb{G}_\mathrm{m} \to \mathbb{G}'_\mathrm{m}$ vanishes, so that we have

$$\mathsf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt};\Bbbk) \otimes_{\mathsf{H}^\bullet_{\mathbb{G}'_\mathrm{m}}(\mathrm{pt};\Bbbk)} \mathrm{Hom}^\bullet_{D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}'_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathscr{F},\mathscr{F}') \cong \mathsf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt};\Bbbk) \otimes_\Bbbk \left(\Bbbk \otimes_{\mathsf{H}^\bullet_{\mathbb{G}'_\mathrm{m}}(\mathrm{pt};\Bbbk)} \mathrm{Hom}^\bullet_{D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}'_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathscr{F},\mathscr{F}')\right).$$

As in [MR, Lemma 2.2] the forgetful functor $\mathsf{For}^{\mathbb{G}'_\mathrm{m}}$ induces an isomorphism

$$\Bbbk \otimes_{\mathsf{H}^\bullet_{\mathbb{G}'_\mathrm{m}}(\mathrm{pt};\Bbbk)} \mathrm{Hom}^\bullet_{D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}'_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathscr{F},\mathscr{F}') \xrightarrow{\sim} \mathrm{Hom}^\bullet_{D^\mathrm{b}_{\mathcal{IW}_\ell}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathsf{For}^{\mathbb{G}'_\mathrm{m}}(\mathscr{F}),\mathsf{For}^{\mathbb{G}'_\mathrm{m}}(\mathscr{F}')),$$

so that we finally obtain a canonical isomorphism

$$\mathsf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt};\Bbbk) \otimes_\Bbbk \mathrm{Hom}^\bullet_{D^\mathrm{b}_{\mathcal{IW}_\ell}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathsf{For}^{\mathbb{G}'_\mathrm{m}}(\mathscr{F}),\mathsf{For}^{\mathbb{G}'_\mathrm{m}}(\mathscr{F}')) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathrm{Res}^{\mathbb{G}'_\mathrm{m}}_{\mathbb{G}_\mathrm{m}}(\mathscr{F}),\mathrm{Res}^{\mathbb{G}'_\mathrm{m}}_{\mathbb{G}_\mathrm{m}}(\mathscr{F}')).$$

To conclude it suffices to remark that the functor $\mathrm{Res}^{\mathbb{G}'_\mathrm{m}}_{\mathbb{G}_\mathrm{m}}$ from (7.2) induces a (canonical) bijection between the isomorphism classes of parity complexes in the categories $D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}'_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)$ and $D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)$; one can therefore replace $\mathsf{For}^{\mathbb{G}'_\mathrm{m}}(\mathscr{F})$ and $\mathsf{For}^{\mathbb{G}'_\mathrm{m}}(\mathscr{F}')$ in these isomorphisms by general parity complexes in $D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)$. □

It is clear from definitions that the functor Q sends parity complexes to parity complexes. In fact this functor (when restricted to parity complexes) is close to being an equivalence, as explained in the following statement.

Proposition 7.6. *For any complexes $\mathscr{E},\mathscr{E}'$ in $D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)$ which are either both even or both odd, there exists a canonical isomorphism of $\Bbbk$-vector spaces*

$$\mathrm{Hom}^\bullet_{D^\mathrm{b}_{\mathcal{IW}_\ell}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathsf{For}^{\mathbb{G}_\mathrm{m}}(\mathscr{E}),\mathsf{For}^{\mathbb{G}_\mathrm{m}}(\mathscr{E}')) \cong \mathrm{Hom}_{\mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathsf{Q}(\mathscr{E}),\mathsf{Q}(\mathscr{E}')).$$

Moreover, these isomorphisms are compatible with composition in the obvious way.

Proof. Recall from [JMW] that since $\mathscr{E},\mathscr{E}'$ are either both even or both odd, the graded $\Bbbk$-vector space $\mathrm{Hom}^\bullet_{D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathscr{E},\mathscr{E}')$ is concentrated in even degrees. Using Lemma 6.4, we see that the functor Q induces a canonical morphism

$$\mathrm{Hom}^\bullet_{D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathscr{E},\mathscr{E}') \to \mathrm{Hom}_{\mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathsf{Q}(\mathscr{E}),\mathsf{Q}(\mathscr{E}'))$$

which factors through a morphism

$$\Bbbk' \otimes_{\mathsf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt},\Bbbk)} \mathrm{Hom}^\bullet_{D^\mathrm{b}_{\mathcal{IW}_\ell,\mathbb{G}_\mathrm{m}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathscr{E},\mathscr{E}'[n]) \to \mathrm{Hom}_{\mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathsf{Q}(\mathscr{E}),\mathsf{Q}(\mathscr{E}')),$$

where $\Bbbk'$ means $\Bbbk$ seen as an $\mathsf{H}^\bullet_{\mathbb{G}_m}(\mathrm{pt},\Bbbk)$-module where x acts by multiplication by 1 under the identification (3.5). Standard arguments based on Lemma 6.3 and the distinguished triangles in the "recollement" setting show that the latter morphism is an isomorphism. The desired isomorphism follows, in view of Lemma 7.5. □

If $\mathscr{E}$ is an indecomposable parity complex in $D^{\mathrm{b}}_{\mathcal{IW}_\ell,\mathbb{G}_m}((\mathcal{G}r_G)^\varpi,\Bbbk)$, then as explained above the complex $\mathsf{For}^{\mathbb{G}_m}(\mathscr{E})$ is indecomposable, so that the ring

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}_\ell}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathsf{For}^{\mathbb{G}_m}(\mathscr{E}),\mathsf{For}^{\mathbb{G}_m}(\mathscr{E}))$$

is local. Since a finite dimensional graded ring whose degree-0 part is local is itself local (see [GG, Theorem 3.1]), it follows that the ring

$$\bigoplus_{n\in\mathbb{Z}}\mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}_\ell}((\mathcal{G}r_G)^\varpi,\Bbbk)}(\mathsf{For}^{\mathbb{G}_m}(\mathscr{E}),\mathsf{For}^{\mathbb{G}_m}(\mathscr{E})[n])$$

is also local. In view of Proposition 7.6, this implies that $\mathsf{Q}(\mathscr{E})$ is indecomposable. In other words, we have proved that Q sends indecomposable parity complexes to indecomposable parity complexes.

8. Applications in representation theory of reductive algebraic groups

In this section, we finally use the constructions of Sections 4–7 to derive consequences on categories of representations of split connected reductive algebraic groups over $\Bbbk$.

8.1. The geometric Satake equivalence and its Iwahori–Whittaker variant. The *Satake category* is the category

$$\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G,\Bbbk)$$

of étale L^+G-equivariant perverse sheaves on $\mathcal{G}r_G$. (By definition, this category is the inductive limit of the categories $\mathsf{Perv}_{\mathrm{sph}}(X,\Bbbk)$ where X runs over the closed finite unions of L^+G-orbits in $\mathcal{G}r_G$. And given such X, the category $\mathsf{Perv}_{\mathrm{sph}}(X,\Bbbk)$ is defined as $\mathsf{Perv}_H(X,\Bbbk)$, where H is a smooth quotient of L^+G of finite type such that the L^+G-action on X factors through H, and such that the kernel of the surjection $L^+G\to H$ is contained in $\ker(\mathrm{ev}_0)$; the resulting category does not depend on the choice of H up to canonical equivalence.) The natural convolution product $\star$ on the equivariant derived category $D^{\mathrm{b}}_{L^+G}(\mathcal{G}r_G,\Bbbk)$ restricts to an exact monoidal product on the category $\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G,\Bbbk)$, see [MV1].

The classification of the simple objects in $\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G,\Bbbk)$ is given by the general theory of perverse sheaves from [BBDG]. Namely, in view of the description of the L^+G-orbits on $\mathcal{G}r_G$ (see (4.8)) and since each of these orbits is simply connected, for any $\lambda\in\mathbf{X}^\vee_+$ there exists a unique simple perverse sheaf $\mathscr{IC}^\lambda$ in $\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G,\Bbbk)$ which is supported on $\overline{\mathcal{G}r^\lambda_G}$, and whose restriction to $\mathcal{G}r^\lambda_G$ is $\underline{\Bbbk}_{\mathcal{G}r^\lambda_G}[\dim(\mathcal{G}r^\lambda_G)]$. Moreover, the assignment $\lambda\mapsto\mathscr{IC}^\lambda$ induces a bijection between $\mathbf{X}^\vee_+$ and the set of isomorphism classes of simple objects in $\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G,\Bbbk)$.

On the other hand, we denote by $G^\vee_{\mathbb{Z}}$ the unique split reductive group scheme over $\mathbb{Z}$ whose base change to $\mathbb{C}$ has root datum $(\mathbf{X}^\vee,\mathbf{X},\mathfrak{R}^\vee,\mathfrak{R})$, and then set

$$G^\vee_\Bbbk:=\mathrm{Spec}(\Bbbk)\times_{\mathrm{Spec}(\mathbb{Z})}G^\vee_{\mathbb{Z}}.$$

We will denote by $\mathrm{Rep}(G^\vee_\Bbbk)$ the category of finite-dimensional algebraic representations of the group scheme $G^\vee_\Bbbk$.

The following theorem is due (in this generality) to Mirković–Vilonen [MV1, MV2].

Theorem 8.1. *There exists an equivalence of monoidal categories*

$$\mathsf{S} : (\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G, \Bbbk), \star) \cong (\mathrm{Rep}(G^\vee_\Bbbk), \otimes_\Bbbk).$$

Remark 8.2. (1) In fact, the proof of [MV1] gives slightly more than what is stated in Theorem 8.1: the authors construct a *canonical* $\Bbbk$-group scheme out of the category $\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G, \Bbbk)$, and then check that this group scheme is isomorphic to $G^\vee_\Bbbk$.

(2) In addition to the category $\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G, \Bbbk)$, one can also consider the category $\mathsf{Perv}_{(L^+G)}(\mathcal{G}r_G, \Bbbk)$ of $\Bbbk$-perverse sheaves on $\mathcal{G}r_G$ whose restriction to each $\mathcal{G}r^\lambda_G$ ($\lambda \in \mathbf{X}^\vee_+$) has constant cohomology sheaves. We then have a canonical forgetful functor $\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G, \Bbbk) \to \mathsf{Perv}_{(L^+G)}(\mathcal{G}r_G, \Bbbk)$, which by [MV1, Proposition 2.1] is an equivalence of categories.

Once an equivalence as in Theorem 8.1 is fixed, the constructions in [MV1] provide a canonical embedding $T^\vee_\Bbbk \hookrightarrow G^\vee_\Bbbk$, where $T^\vee_\Bbbk$ is the split $\Bbbk$-torus which is Langlands dual to T (i.e. whose character lattice is $\mathbf{X}^\vee$). We will denote by $B^\vee_\Bbbk$ the Borel subgroup of $G^\vee_\Bbbk$ containing (the image of) $T^\vee_\Bbbk$ and whose roots are the negative coroots of G. For any $\lambda \in \mathbf{X}^\vee_+$ we can then consider the "induced representation"

$$\mathsf{N}(\lambda) := \mathsf{Ind}_{B^\vee_\Bbbk}^{G^\vee_\Bbbk}(\lambda).$$

It is well known that $\mathsf{N}(\lambda)$ contains a unique simple submodule, denoted $\mathsf{L}(\lambda)$, and that the assignment $\lambda \mapsto \mathsf{L}(\lambda)$ induces a bijection between $\mathbf{X}^\vee_+$ and the set of isomorphism classes of simple $G^\vee_\Bbbk$-modules. It is well known also that for any $\lambda \in \mathbf{X}^\vee_+$ we have

$$\mathsf{S}(\mathscr{IC}^\lambda) \cong \mathsf{L}(\lambda). \tag{8.1}$$

Below we will use an alternative geometric realization of $\mathrm{Rep}(G^\vee_\Bbbk)$, in terms of the Iwahori–Whittaker derived category of §5.1, which was found in [BGMRR]. The same construction as for the convolution product on $D^{\mathrm{b}}_{L^+G}(\mathcal{G}r_G, \Bbbk)$ defines a right action of the latter monoidal category on $D^{\mathrm{b}}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$. The corresponding bifunctor will also be denoted $\star$.

We will assume that there exists (and fix) an element $\varsigma \in \mathbf{X}^\vee$ such that $\langle \varsigma, \alpha \rangle = 1$ for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$. Then there exists no orbit in $\overline{\mathcal{G}r^+_{G,\varsigma}} \smallsetminus \mathcal{G}r^+_{G,\varsigma}$ which supports a nonzero Iwahori–Whittaker local system. Therefore, the canonical map $\Delta^{\mathcal{IW}}_\varsigma \to \nabla^{\mathcal{IW}}_\varsigma$ is an isomorphism, and this object is a simple perverse sheaf.

The following theorem is the main result of [BGMRR].

Theorem 8.3. *The functor sending $\mathscr{F}$ to $\Delta^{\mathcal{IW}}_\varsigma \star \mathscr{F}$ induces an equivalence of abelian categories*

$$\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G, \Bbbk) \xrightarrow{\sim} \mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk).$$

Remark 8.4. Theorem 8.3 can also be used to give an alternative proof of Lemma 5.2. Namely, we see as in the proof of this lemma that our functor is fully faithful. If $\mathscr{F}$ belongs to $\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$, by Theorem 8.3 there exists an object $\mathscr{F}'$ in $\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G, \Bbbk)$ and an isomorphism

$$\mathscr{F} \cong \Delta^{\mathcal{IW}}_\varsigma \star \mathscr{F}'.$$

By [MV1, Proposition 2.2], the perverse sheaf $\mathscr{F}'$ is equivariant for the group $\mathbb{G}_{\mathrm{m}} \ltimes L^+G$. Therefore the perverse sheaf $\Delta_\varsigma^{\mathcal{IW}} \star \mathscr{F}'$ is the image of an object in $\mathsf{Perv}_{\mathcal{IW},\mathbb{G}_{\mathrm{m}}}(\mathcal{G}r_G, \Bbbk)$, and we deduce the same property for $\mathscr{F}$.

8.2. **The linkage principle.** We now come back to the setting where G is an arbitrary connected reductive algebraic group over $\mathbb{F}$. Recall the actions $\cdot_\ell$ and $\square_\ell$ of W_{aff} on $\mathbf{X}^\vee$ defined in §4.1. The action which is relevant in representation theory is the "dot action" defined by

$$(\mathsf{t}_\lambda v) \bullet_\ell \mu = v(\mu + \rho^\vee) - \rho^\vee + \ell\lambda$$

for $\lambda \in \mathbb{Z}\mathfrak{R}^\vee$, $v \in W_{\mathrm{f}}$ and $\mu \in \mathbf{X}^\vee$, where $\rho^\vee$ is the halfsum of the positive coroots. It is clear that for any $w \in W_{\mathrm{aff}}$ and $\mu \in \mathbf{X}^\vee$ we have

$$w \bullet_\ell \mu = w \square_\ell (\mu + \rho^\vee) - \rho^\vee = w^* \cdot_\ell (\mu + \rho^\vee) - \rho^\vee, \tag{8.2}$$

where $(\mathsf{t}_\lambda v)^* := \mathsf{t}_{-\lambda} v$ for $\lambda \in \mathbb{Z}\mathfrak{R}^\vee$ and $v \in W_{\mathrm{f}}$.

The following statement is the first main result of this paper.

Theorem 8.5. *For $\lambda, \mu \in \mathbf{X}^\vee_+$, if $\mathrm{Ext}^n_{\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G,\Bbbk)}(\mathscr{IC}^\lambda, \mathscr{IC}^\mu) \neq 0$ for some n, then $W_{\mathrm{aff}} \bullet_\ell \lambda = W_{\mathrm{aff}} \bullet_\ell \mu$.*

Proof. Note that if $\mathrm{Ext}^n_{\mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G,\Bbbk)}(\mathscr{IC}^\lambda, \mathscr{IC}^\mu) \neq 0$ for some n, then the orbits $\mathcal{G}r^\lambda_G$ and $\mathcal{G}r^\mu_G$ are contained in the same connected component of $\mathcal{G}r_G$. If Z denotes the center of G, then the natural morphism $(\mathcal{G}r_G)_{\mathrm{red}} \to (\mathcal{G}r_{G/Z})_{\mathrm{red}}$ restricts, on each connected component X of $(\mathcal{G}r_G)_{\mathrm{red}}$, to an embedding of a connected component of $(\mathcal{G}r_{G/Z})_{\mathrm{red}}$. The associated functor $\mathsf{Perv}_{\mathrm{sph}}(X, \Bbbk) \to \mathsf{Perv}_{(G/Z)_\mathscr{O}}(\mathcal{G}r_{G/Z}, \Bbbk)$ is then fully faithful by Remark 8.2(2), which reduces the proof to the case G is semisimple of adjoint type, which we assume from now on.

In particular, under this assumption we can take $\varsigma = \rho^\vee$, and apply Theorem 8.3. This result implies that the simple objects in $\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$ are the perverse sheaves

$$\mathscr{IC}^{\mathcal{IW}}_\lambda := \Delta^{\mathcal{IW}}_\varsigma \star \mathscr{IC}^{\lambda-\varsigma}$$

for $\lambda \in \mathbf{X}^\vee_{++} = \varsigma + \mathbf{X}^\vee_+$. In view of (8.2), this shows that to prove the lemma it suffices to prove that for $\lambda, \mu \in \mathbf{X}^\vee_{++}$, if $\mathrm{Ext}^n_{\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)}(\mathscr{IC}^{\mathcal{IW}}_\lambda, \mathscr{IC}^{\mathcal{IW}}_\mu) \neq 0$ for some n, then $W_{\mathrm{aff}} \cdot_\ell \lambda = W_{\mathrm{aff}} \cdot_\ell \mu$.

In view of Theorem 7.4 (see also (7.1)) and the decomposition of $(\mathcal{G}r_G)^\varpi$ into its connected components (see Proposition 4.6), if $\lambda, \mu \in \mathbf{X}^\vee_{++}$ satisfy $W_{\mathrm{aff}} \cdot_\ell \lambda \neq W_{\mathrm{aff}} \cdot_\ell \mu$ then we have

$$\mathrm{Hom}_{\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)}(\mathscr{T}^{\mathcal{IW}}_\lambda, \mathscr{T}^{\mathcal{IW}}_\mu) = 0.$$

It follows that any $\mathscr{M}$ in $\mathsf{Tilt}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$ admits a canonical decomposition

$$\mathscr{M} = \bigoplus_{\lambda \in (-\overline{\mathbf{a}}_\ell) \cap \mathbf{X}^\vee} \mathscr{M}_{(\lambda)}$$

where each $\mathscr{M}_{(\lambda)}$ is a direct sum of objects $\mathscr{T}^{\mathcal{IW}}_\mu$ with $\mu \in W_{\mathrm{aff}} \cdot_\ell \lambda$; in fact the assignment $\mathscr{M} \mapsto \mathscr{M}_{(\lambda)}$ defines an endofunctor $\Pi_{(\lambda)}$ of $\mathsf{Tilt}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$, and we have a canonical isomorphism of functors

$$\mathrm{id} \cong \bigoplus_{\lambda \in (-\overline{\mathbf{a}}_\ell) \cap \mathbf{X}^\vee} \Pi_{(\lambda)}.$$

Let us still denote by $\Pi_{(\lambda)}$ the endofunctor of $D^{\mathrm{b}}\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$ obtained by conjugating $K^{\mathrm{b}}(\Pi_{(\lambda)})$ by the equivalence (5.3). Then we have

$$\mathrm{id}_{D^{\mathrm{b}}\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)} \cong \bigoplus_{\lambda\in(-\overline{\mathbf{a}}_\ell)\cap\mathbf{X}^\vee} \Pi_{(\lambda)},$$

and for $\lambda \neq \mu$ in $(-\overline{\mathbf{a}}_\ell) \cap \mathbf{X}^\vee$ there exists no nonzero morphism between objects in the essential images of $\Pi_{(\lambda)}$ and $\Pi_{(\mu)}$. In particular, given $\mu \in \mathbf{X}^\vee_{++}$, there exists a unique $\lambda \in (-\overline{\mathbf{a}}_\ell) \cap \mathbf{X}^\vee$ such that $\Pi_{(\lambda)}(\mathscr{I}\mathscr{C}^{\mathcal{IW}}_\mu) \neq 0$, and this element satisfies

$$\Pi_{(\lambda)}(\mathscr{I}\mathscr{C}^{\mathcal{IW}}_\mu) = \mathscr{I}\mathscr{C}^{\mathcal{IW}}_\mu.$$

The existence of the nonzero maps

$$\mathscr{I}\mathscr{C}^{\mathcal{IW}}_\mu \twoheadleftarrow \Delta^{\mathcal{IW}}_\mu \hookrightarrow \mathscr{T}^{\mathcal{IW}}_\mu$$

shows that in fact λ is the unique element in $(-\overline{\mathbf{a}}_\ell) \cap \mathbf{X}^\vee$ such that $W_{\mathrm{aff}} \cdot_\ell \lambda = W_{\mathrm{aff}} \cdot_\ell \mu$.

Finally, if $\lambda, \mu \in \mathbf{X}^\vee_{++}$ satisfy $W_{\mathrm{aff}} \cdot_\ell \lambda \neq W_{\mathrm{aff}} \cdot_\ell \mu$, we denote by ν, η the only elements in $(-\overline{\mathbf{a}}_\ell) \cap \mathbf{X}^\vee$ such that

$$W_{\mathrm{aff}} \cdot_\ell \lambda = W_{\mathrm{aff}} \cdot_\ell \nu, \qquad W_{\mathrm{aff}} \cdot_\ell \mu = W_{\mathrm{aff}} \cdot_\ell \eta,$$

and observe that for any $n \in \mathbb{Z}$ we have

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)}(\mathscr{I}\mathscr{C}^{\mathcal{IW}}_\lambda, \mathscr{I}\mathscr{C}^{\mathcal{IW}}_\mu[n]) = \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)}(\Pi_{(\nu)}(\mathscr{I}\mathscr{C}^{\mathcal{IW}}_\lambda), \Pi_{(\eta)}(\mathscr{I}\mathscr{C}^{\mathcal{IW}}_\mu[n])) = 0$$

since $\nu \neq \eta$. □

In view of Theorem 8.1 and (8.1), Theorem 8.5 is equivalent to the statement that if $\mathrm{Ext}^n_{\mathrm{Rep}(G^\vee_\Bbbk)}(\mathsf{L}(\lambda), \mathsf{L}(\mu)) \neq 0$ for some $n \in \mathbb{Z}$, then $W_{\mathrm{aff}} \bullet_\ell \lambda = W_{\mathrm{aff}} \bullet_\ell \mu$. This property is of course well known, and called the *Linkage Principle*, see [J2, §II.6]. (This statement was first conjectured by Verma, and proved by Andersen in full generality, after partial results of Humphreys, Kac–Weisfeiler and Carter–Lusztig; see [A1] for more details.)

The same considerations as in §4.1 show that a fundamental domain for the action of W_{aff} on $\mathbf{X}^\vee$ via $\bullet_\ell$ is given by the subset

$$\overline{C}_\ell := \{\lambda \in \mathbf{X}^\vee \mid \forall \alpha \in \mathfrak{R}^+,\, 0 \leqslant \langle \lambda + \rho^\vee, \alpha \rangle \leqslant \ell\}.$$

Below we will need to describe the subset $(W_{\mathrm{aff}} \bullet_\ell \lambda) \cap \mathbf{X}^\vee_+$ more explicitly for $\lambda \in \overline{C}_\ell$. For this we set $I_\lambda := \{s \in S_{\mathrm{aff}} \mid s \bullet_\ell \lambda = \lambda\}$, so that the stabilizer in W_{aff} of λ (for $\bullet_\ell$) is the parabolic subgroup W_λ of W_{aff} generated by I_λ. We set

$$W^{(\lambda)}_{\mathrm{aff}} := \{w \in W_{\mathrm{aff}} \mid w \text{ is maximal in } wW_\lambda \text{ and minimal in } W_{\mathrm{f}}w\}.$$

Then it is known that the assignment $w \mapsto w \bullet_\ell \lambda$ induces a bijection

$$W^{(\lambda)}_{\mathrm{aff}} \xrightarrow{\sim} (W_{\mathrm{aff}} \bullet_\ell \lambda) \cap \mathbf{X}^\vee_+; \tag{8.3}$$

see [AR2, §10.1] for similar considerations.

8.3. **The tilting character formula.** Let $\mathcal{H}_{\mathrm{aff}}$ be the Hecke algebra of $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$, and let $\mathcal{M}^{\mathrm{asph}}$ be its antispherical module, with "standard" basis $(N_w : w \in {}^{\mathrm{f}}W_{\mathrm{aff}})$ parametrized by the subset ${}^{\mathrm{f}}W_{\mathrm{aff}} \subset W_{\mathrm{aff}}$ of elements w which are minimal in $W_{\mathrm{f}}w$. (Here we follow the conventions of [Soe].) Let us consider

$$\mathcal{F}l_G^\circ := \mathcal{F}l_{\mathbf{a}_1}^{1,\circ},$$

the connected component of the base point in the affine flag variety associated with LG. We can then define, as for $\mathcal{G}r_G$, the Iwahori–Whittaker derived category $D^{\mathrm{b}}_{\mathcal{IW}}(\mathcal{F}l_G^\circ, \Bbbk)$, and its full subcategory $\mathsf{Parity}_{\mathcal{IW}}(\mathcal{F}l_G^\circ, \Bbbk)$ of parity complexes. The $\mathrm{Iw}_{\mathrm{u}}^+$-orbits on $\mathcal{F}l_G^\circ$ are naturally parametrized by W_{aff}, and those which support a nonzero Iwahori–Whittaker local system are the ones corresponding to elements in ${}^{\mathrm{f}}W_{\mathrm{aff}}$; we will denote by $\nabla_w^{\mathcal{IW}}$ and $\mathscr{E}_w^{\mathcal{IW}}$ the costandard perverse sheaf and indecomposable parity complex attached to $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, respectively. We then have a canonical isomorphism

$$\mathrm{ch} : [\mathsf{Parity}_{\mathcal{IW}}(\mathcal{F}l_G^\circ, \Bbbk)] \xrightarrow{\sim} \mathcal{M}^{\mathrm{asph}}$$

determined by

$$\mathrm{ch}([\mathscr{F}]) = \sum_{w \in {}^{\mathrm{f}}W_{\mathrm{aff}}} \dim_{\Bbbk} \mathrm{Hom}_{D^{\mathrm{b}}_{\mathcal{IW}}(\mathcal{F}l_G^\circ, \Bbbk)}(\mathscr{F}, \nabla_w^{\mathcal{IW}}[n]) \cdot v^n N_w,$$

where $[\mathsf{Parity}_{\mathcal{IW}}(\mathcal{F}l_G^\circ, \Bbbk)]$ is the split Grothendieck group of the additive category $\mathsf{Parity}_{\mathcal{IW}}(\mathcal{F}l_G^\circ, \Bbbk)$.

In terms of this isomorphism, the *ℓ-canonical basis* $({}^\ell\underline{N}_w : w \in {}^{\mathrm{f}}W_{\mathrm{aff}})$ of $\mathcal{M}^{\mathrm{asph}}$ (see [RW1, AR3]) can be characterized by

$$ {}^\ell\underline{N}_w := \mathrm{ch}(\mathscr{E}_w^{\mathcal{IW}}). \tag{8.4}$$

The associated *ℓ-Kazhdan–Lusztig polynomials* $({}^\ell n_{y,w} : y, w \in {}^{\mathrm{f}}W_{\mathrm{aff}})$ are characterized by the equality

$${}^\ell\underline{N}_w = \sum_{y \in {}^{\mathrm{f}}W_{\mathrm{aff}}} {}^\ell n_{y,w} \cdot N_y.$$

Remark 8.6. It is easily seen that the computation of the ℓ-canonical basis and ℓ-Kazhdan–Lusztig polynomials can be reduced to the case G is quasi-simple. In this case, the results of [RW1, Part III] show that this basis coincides with the basis with the same name studied in [JW], for the Coxeter system $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ and the realization considered in [RW1, Remark 10.7.2(2)]. In particular, these data can be computed algorithmically using the procedure described in [JW].

These considerations have been stated for the ind-variety $\mathcal{F}l_{\mathbf{a}_1}^{1,\circ}$, but in practice we will rather use them for the isomorphic variety $\mathcal{F}l_{\mathbf{a}_\ell}^{\ell,\circ}$, with respect to the action of $\mathrm{Iw}_{\mathrm{u},\ell}^+$. More generally we can consider a facet $\mathbf{f} \subset \overline{\mathbf{a}}_\ell$, and the basic component in the associated partial affine flag variety $\mathcal{F}l_{\mathbf{f}}^{\ell,\circ}$; see §4.3. Here again the Iwahori–Whittaker derived category (with respect to the action of $\mathrm{Iw}_{\mathrm{u},\ell}^+$) makes sense, and so does the notion of parity complexes. The indecomposable such objects can be described in terms of those on $\mathcal{F}l_{\mathbf{a}_\ell}^{\ell,\circ}$ as follows.

As usual, the general theory of parity complexes ensures that there exists at most one indecomposable parity complex on $\mathcal{F}l_{\mathbf{f}}^{\ell,\circ}$ associated with each $\mathrm{Iw}_{\mathrm{u},\ell}^+$-orbit which supports a nonzero Iwahori–Whittaker local system, and that each indecomposable parity complex is isomorphic (up to cohomological shift) to such an object. Now as usual also the $\mathrm{Iw}_{\mathrm{u},\ell}^+$-orbits on $\mathcal{F}l_{\mathbf{f}}^{\ell,\circ}$ are parametrized in the natural way by the quotient $W_{\mathrm{aff}}/W_{\mathrm{aff}}^{\mathbf{f}}$, or in other words by the elements $w \in W_{\mathrm{aff}}$ which are maximal in $wW_{\mathrm{aff}}^{\mathbf{f}}$.

In the following statement, the morphism

$$\mathcal{F}l_{\mathbf{a}_\ell}^{\ell,\circ} \to \mathcal{F}l_{\mathbf{f}}^{\ell,\circ}$$

induced by (4.3) will be denoted $\pi_{\mathbf{f}}$. We will also denote by $N_{\mathbf{f}}$ the length of the longest element in $W_{\mathrm{aff}}^{\mathbf{f}}$.

Lemma 8.7. *If $w \in W_{\mathrm{aff}}$ is maximal in $wW_{\mathrm{aff}}^{\mathbf{f}}$, then the $\mathrm{Iw}_{\mathrm{u},\ell}^{+}$-orbit on $\mathcal{F}l_{\mathbf{f}}^{\ell,\circ}$ associated with w supports a nonzero Iwahori–Whittaker local system iff w is minimal in $W_{\mathrm{f}}w$. Moreover, in this case the indecomposable Iwahori–Whittaker parity complex on $\mathcal{F}l_{\mathbf{f}}^{\ell,\circ}$ associated with w exists, and its image under $\pi_{\mathbf{f}}^{*}[N_{\mathbf{f}}]$ coincides with the indecomposable Iwahori–Whittaker parity complex on $\mathcal{F}l_{\mathbf{a}_\ell}^{\ell,\circ}$ associated with w.*

Proof. The proof is similar to that of its counterpart in the setting of Kac–Moody flag varieties considered in [ACR, Appendix A]. □

Now we return to Representation Theory. Recall that a $G_{\Bbbk}^{\vee}$-module M in $\mathrm{Rep}(G_{\Bbbk}^{\vee})$ is called *tilting* if both M and M^* admit filtrations with subquotients of the form $\mathsf{N}(\lambda)$ with $\lambda \in \mathbf{X}_{+}^{\vee}$. It is well known (see [J2, §II.E]) that the indecomposable tilting $G_{\Bbbk}^{\vee}$-modules are classified by their highest weight (a dominant weight), and that any tilting module is a direct sum of indecomposable tilting modules. The indecomposable tilting module of highest weight $\lambda \in \mathbf{X}_{+}^{\vee}$ will be denoted $\mathsf{T}(\lambda)$.

In view of the comments at the end of §8.2 (see in particular the bijection (8.3)), the following result gives a complete answer to the question of describing characters of indecomposable tilting $G_{\Bbbk}^{\vee}$-modules.

Theorem 8.8. *Let $\lambda \in \overline{C}_\ell$. Then for any $w \in W_{\mathrm{aff}}^{(\lambda)}$ we have*

$$[\mathsf{T}(w \bullet_\ell \lambda)] = \sum_{y \in W_{\mathrm{aff}}^{(\lambda)}} {}^{\ell}n_{y,w}(1) \cdot [\mathsf{N}(y \bullet_\ell \lambda)]$$

in the Grothendieck group of $\mathrm{Rep}(G_{\Bbbk}^{\vee})$.

Proof. Recall that if we denote (as in the proof of Theorem 8.5) by Z the center of G, then the group $(G/Z)_{\Bbbk}^{\vee}$ identifies with the simply-connected cover of the derived subgroup of $G_{\Bbbk}^{\vee}$. In view of the results recalled in [J2, §II.E.7], this reduces the proof to the case G is semisimple of adjoint type, which we will assume from now on. In this case we can take $\varsigma = \rho^{\vee}$ and apply Theorem 8.3.

Standard arguments show that the formula will follow provided we prove that for any $w, w' \in W_{\mathrm{aff}}^{(\lambda)}$ we have

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathrm{Rep}(G_{\Bbbk}^{\vee})}\big(\mathsf{T}(w \bullet_\ell \lambda), \mathsf{T}(w' \bullet_\ell \lambda)\big) = \sum_{y \in W_{\mathrm{aff}}^{(\lambda)}} {}^{\ell}n_{y,w}(1) \cdot {}^{\ell}n_{y,w'}(1). \tag{8.5}$$

Let $\mu := \lambda + \varsigma$, so that $\mu \in (-\overline{\mathbf{a}}_\ell) \cap \mathbf{X}^{\vee}$ in the notation of §4.1. Since the composition

$$\mathrm{Rep}(G_{\Bbbk}^{\vee}) \xrightarrow[\sim]{\text{Thm. 8.1}} \mathsf{Perv}_{\mathrm{sph}}(\mathcal{G}r_G, \Bbbk) \xrightarrow[\sim]{\text{Thm. 8.3}} \mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G, \Bbbk)$$

is an equivalence of highest weight categories, for any $w \in W_{\mathrm{aff}}^{(\lambda)}$ it sends the $G_{\Bbbk}^{\vee}$-module $\mathsf{T}(w \bullet_\ell \lambda)$ to $\mathscr{T}_{w \square_\ell \mu}^{\mathcal{IW}}$. Therefore, to prove (8.5) it suffices to prove that for any $w, w' \in W_{\mathrm{aff}}^{(\lambda)}$ we have

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathsf{Perv}_{\mathcal{IW}}(\mathcal{G}r_G,\Bbbk)}\big(\mathscr{T}^{\mathcal{IW}}_{w\square_\ell\mu}, \mathscr{T}^{\mathcal{IW}}_{w'\square_\ell\mu}\big) = \sum_{y \in W_{\mathrm{aff}}^{(\lambda)}} {}^\ell n_{y,w}(1) \cdot {}^\ell n_{y,w'}(1).$$

Then, using Theorem 7.4 and (7.1), this equality is reduced to proving that for any $w, w' \in W_{\mathrm{aff}}^{(\lambda)}$ we have

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathsf{Sm}_{\mathcal{IW}}((\mathcal{G}r_G)^\varpi,\Bbbk)}\big(\mathscr{E}^{\mathsf{Sm},0}_{w\square_\ell\mu}, \mathscr{E}^{\mathsf{Sm},0}_{w'\square_\ell\mu}\big) = \sum_{y \in W_{\mathrm{aff}}^{(\lambda)}} {}^\ell n_{y,w}(1) \cdot {}^\ell n_{y,w'}(1).$$

Now by Proposition 7.6 the Hom-space in the left-hand side can be computed in $D^{\mathrm{b}}_{\mathcal{IW}_\ell}((\mathcal{G}r_G)^\varpi, \Bbbk)$, where its dimension can be expressed in terms of the (co)stalks of the parity complexes using [JMW, Proposition 2.6]. The fixed points $(\mathcal{G}r_G)^\varpi$ are a union of partial affine flag varieties by Proposition 4.6, so that these dimensions can be computed using Lemma 8.7 and (8.4). This provides the desired formula, in view of Remark 4.9 and the fact that $W_\lambda = W_{\mathrm{aff}}^{\mathbf{f}_\mu}$ (see (8.2)). □

Remark 8.9. In the special case when ℓ is bigger than the Coxeter number h of G, we have $W_0 = \{1\}$. In this case, the formula in Theorem 8.8 was conjectured, and proved in the case of the group $G = \mathrm{GL}(n)$, in [RW1]. A proof of this formula (again for $\ell > h$ and $\lambda = 0$, but for a general reductive group) was later given in [AMRW]. It was noticed in [RW1] that a similar formula could be stated for any block of $\mathrm{Rep}(G^\vee_\Bbbk)$, see [RW1, Conjecture 1.4.3]. Theorem 8.8 confirms this formula in full generality.

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