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A new formulation of Economic Model Predictive Control without terminal constraint

Mazen Alamir ^a, Gabriele Pannochia ^b,

^a*Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-lab, 38000 Grenoble, France.*

^b*Dept. of Chemical Engineering, University of Pisa, 56126 Pisa, Italy*

Abstract

In this paper, it is shown that a simple formulation of Economic Model Predictive Control can be used which possesses two features that are generally viewed as mutually exclusive, namely, a rather short prediction horizon (reachability-compatible) on one side, and the absence of final constraint on the other side. The result holds for discrete-time models that are originated from continuous-time models by means of discretization. Practical stability at a small neighborhood of the optimal unknown steady-state pair is shown when some parameters increase and sampling time decreases. A commonly used example is given to illustrate the result.

1 Introduction

Economic Model Predictive Control (EMPC) formulations are MPC formulations in which the cost function is not expressed as a distance to some beforehand known targeted steady pairs. In a typical non economic settings, such pairs are computed in a higher stage of a two stage formulation in which the lower stage is a standard regulation-based MPC -with preassigned targeted steady state- while the higher stage performs economic static optimization in order to deliver an optimal steady pair to the lower stage.

The advantage of EMPC stems from the fact that sometimes, the computation of the static optimization results is even harder than the MPC itself or because one is interested in a non necessarily steady behavior as far as this induces a higher performance in average. In both cases however, it is necessary to be able to tune the level of actuator dynamics on the long term for obvious reasons with the possibility to force a quasi-optimal steady regime as a limit case.

The discussion above suggests that penalizing the state increment should be effective in deriving a tunable EMPC that addresses the above concerns. Strangely enough this simple and intuitive idea never showed up in any of the yet developed provable EMPC schemes

to the best of the authors' knowledge. Instead, the non intuitive concepts of (strict) passivity, stage cost rotation and so, dominated the scene. This paper fills this astonishing gap by giving a simple provably stable formulation that does not need these counter intuitive technicalities.

On the other hand, existing provably stable formulations are exclusively of two kinds: in the first, a terminal constraint on the state is added that is a distance to the desired state which undermines one of the attractive features of EMPC mentioned above (the non availability of the steady-optimal pair) while in the second, the stability argument relies on the prediction horizon being sufficiently high which can be computationally expensive. The framework proposed in the present contribution gathers the nice properties of both, namely, the possibility to use a moderate prediction horizon (the one linked to the reachability assumption) while being free of any final constraint to enforce the stability of the resulting closed-loop system.

This paper is organized as follows: first of all, the problem is stated and notation is introduced in Section 2. The working assumptions that are needed to derive the main result are given in Section 3. Section 4 gives the statement and the proofs of the main results before an illustrative example is proposed in Section 5.

Email addresses: mazen.alamir@grenoble-inp.fr
(Mazen Alamir), mazen.alamir@grenoble-inp.fr
(Gabriele Pannochia).

2 Problem statement and notation

Consider general nonlinear systems governed by the following discrete-time dynamics:

$$x^+ = f(x, u) \quad (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \quad (1)$$

where x and $u \in \mathbb{U} \subset \mathbb{R}^m$ stand for the state and the control input vectors respectively. \mathbb{U} is a compact set of admissible control values. The dynamics (1) is supposed to be obtained by time discretization of some continuous-time dynamics with a sampling period $\tau > 0$. The dynamics is supposed to admit a set $\mathcal{Z} \subset \mathbb{R}^n \times \mathbb{U}$ of equilibrium pairs (x, u) , namely:

$$\mathcal{Z} := \left\{ (x, u) \in \mathbb{R}^n \times \mathbb{U} \mid \Delta(x, u) = 0 \right\} \quad (2)$$

where $\Delta(x, u) := \frac{1}{\tau} \|f(x, u) - x\|$. Note that although Δ depends on τ , the reference to τ is omitted for the sake of simplification. This dependence will be recalled when appropriate.

Finally, it is assumed that there exists at least a steady optimal equilibrium pair $z_s = (x_s, u_s) \in \mathcal{Z}$ that minimizes a given cost function ℓ over the set of steady pairs \mathcal{Z} , namely:

$$\ell_s := \ell(z_s) \leq \ell(z) \quad \forall z \in \mathcal{Z} \quad (3)$$

More generally the set of such z_s is denoted by $\mathcal{Z}_s \subset \mathcal{Z}$.

In what follows the following notation is used:

- let $N \in \mathbb{N}$ be some finite prediction horizon
- boldfaced \mathbf{u} denotes a sequence of $N+1$ control actions over a prediction horizon of length N , namely:

$$\mathbf{u} := (u_0, u_1, \dots, u_{N-1}, u_N) \in \mathbb{U}^{N+1} \quad (4)$$

- For any control sequence \mathbf{u} given by (4), the following notation is used to denote the corresponding *warm start* sequence:

$$\mathbf{u}^+ = (u_1, u_2, \dots, u_N, u_N) \in \mathbb{U}^{N+1} \quad (5)$$

- Given a control sequence $\mathbf{u} \in \mathbb{U}^{N+1}$ and an initial state x , $\mathbf{x}^{\mathbf{u}}(x) := \{x_k^{\mathbf{u}}(x)\}_{k=0}^N$ denotes the sequence of states on the system's trajectory starting at x under the control sequence \mathbf{u} , namely¹:

$$x_0^{\mathbf{u}}(x) = x \quad , (\forall k) \quad x_{k+1}^{\mathbf{u}}(x) = f(x_k^{\mathbf{u}}(x), u_k) \quad (6)$$

¹ Note that the subscript k in $x_k^{\mathbf{u}}(x)$ denotes time increment and not state vector-related component index.

- Given any function h defined on $\mathbb{R}^n \times \mathbb{U}$, the following short notation is used:

$$h_k^{\mathbf{u}}(x) := h(x_k^{\mathbf{u}}(x), u_k) \quad (7)$$

This holds in particular for ℓ and Δ invoked earlier. Moreover, when there is no ambiguity regarding the initial state x , the argument x is omitted leading to the notation $x_k^{\mathbf{u}}, h_k^{\mathbf{u}}$ instead of $x_k^{\mathbf{u}}(x), h_k^{\mathbf{u}}(x)$.

- For any set \mathbb{V} in some euclidian space, an ϵ -neighborhood of \mathbb{V} , denoted by $\mathcal{V}_\epsilon(\mathbb{V})$ is the set of all points ξ such that $d(\xi, \mathbb{V}) := \min_{\eta \in \mathbb{V}} \|\xi - \eta\| \leq \epsilon$.

The following definition is used to express the main convergence results:

Definition 1 (Quasi-steady optimal trajectory)

A trajectory (\mathbf{x}, \mathbf{u}) is said to be ϵ -quasi steady optimal if and only if the following conditions hold for all k :

$$|\ell(x_k, u_k) - \ell_s| \leq \epsilon \quad , \quad \Delta(x_k, u_k) \leq \epsilon \quad (8)$$

Using the above notation, the following open-loop cost function is considered in the present paper for a given initial state x and a candidate sequence of future actions \mathbf{u} over a prediction horizon of length N :

$$J(\mathbf{u}, x) := \underbrace{\gamma \left[\ell_N^{\mathbf{u}} + \alpha \Delta_N^{\mathbf{u}} \right]}_{\Psi_N(\mathbf{u}, x)} + \underbrace{\sum_{k=0}^N \left[\ell_k^{\mathbf{u}} + \alpha \Delta_k^{\mathbf{u}} \right]}_{V(\mathbf{u}, x)} \quad (9)$$

where the notation $h_k^{\mathbf{u}} := h_k^{\mathbf{u}}(x)$ is used for $h \in \{\ell, \Delta\}$.

This cost function is used to define the open-loop optimal control problem given by:

$$P(x) : \min_{\mathbf{u} \in \mathbb{U}} J(\mathbf{u}, x) \rightarrow (J^*(x), \mathbf{u}^*(x)) \quad (10)$$

leading to the MPC state feedback given by:

$$K_{\text{MPC}}(x) := \mathbf{u}_0^*(x) \quad (11)$$

We are interested in so called Economic-MPC formulations, meaning that $\ell(x, u)$ is not defined as a distance to some desired steady pair $(x_s, u_s) \in \mathcal{Z}_s$ that minimizes ℓ as the latter is supposed to be unknown or its a priori on-line computation is to be avoided. Consequently, no reference to such pair is included in the cost function nor in any terminal constraint.

This paper investigates the conditions under which the resulting closed-loop behavior of (1) under (11) is asymptotically ϵ -quasi steady optimal with an ϵ that can be made as small as desired by conveniently choosing

the design parameters (α, γ) and the sampling period τ .

In what follows, the following short notation is used:

$$h_k^*(x) := h_k^{\mathbf{u}^*(x)}(x) \quad h \in \{\ell, \Delta\} \quad (12)$$

Remark 1 *Note that only control constraints are explicitly considered through the subset \mathbb{U} . State constraints are assumed to be softened through some exact penalty in the definition of the map ℓ . This greatly simplifies the exposition of the main ideas and avoid disturbing technicalities regarding the recursive feasibility issue.*

3 Working assumptions

Assumption 1 (Properties of f and ℓ)

- (1) f and ℓ are continuous.
- (2) $\forall \ell > 0$, the set $B_{\bar{\ell}} := \{x \in \mathbb{R}^n \text{ s.t. } \exists u \in \mathbb{U}, \ell(x, u) \leq \bar{\ell}\}$ is compact.
- (3) The minimal steady values invoked in (3) is $\ell_s = 0$.

Assumption 1-(2) is typically enforced by the soft constraints-related penalty [such as $\max\{0, \underline{x} - x, x - \bar{x}\} \leq 0$] that is included in the definition of ℓ to express constraints on the state evolution. As for the last assumption 1-(3), it is a standard assumption that is commonly introduced without loss of generality in order to simplify the discussion.

Assumption 2 [*N-reachability of steady optimal pair*] *There exists a set $\mathbb{X}_0 \subset \mathbb{R}^n$ such that for any $x \in \mathbb{X}_0$, $\exists \mathbf{u}^\#(x) \in \mathbb{U}$ s.t. $(x_N^{\mathbf{u}^\#(x)}, u_N^\#(x)) \in \mathcal{Z}_s$.*

This is a standard assumption that is used in the convergence proof of MPC schemes. Note that the knowledge of the control sequence $\mathbf{u}^\#(x)$ is not required. Only its existence is needed for the analysis of optimal solution properties.

Assumption 3 [*Optimal stationarity condition*] *There exists a continuous strictly increasing map ψ such that:*

$$\psi(0) = 0 \quad \text{and} \quad \ell_s \leq \ell(z) + \psi(\Delta(z)) \quad (13)$$

Moreover, ψ is Lipschitz-continuous.

Note that this last assumption simply states that for any stationary pair z ($\Delta(z) = 0$), $\ell_s \leq \ell(z)$. This is nothing but a simple technical rewriting of condition (3). It also means that $\ell(z)$ might be lower than ℓ_s provided that z is not a stationary pair. Recall that since ℓ_s is supposed to be 0 (Assumption 1), the inequality (13) becomes $0 \leq \ell(z) + \psi(\Delta(z))$

Assumption 4 [*Local properties*] *There is a continuous function K_s , vanishing at zero, such that the following implication holds:*

$$\left\{d(z, \mathcal{Z}_s) \leq \epsilon\right\} \Rightarrow V(\mathbf{u}^\#(x), x) \leq K_s(\epsilon) \quad (14)$$

This is a rather weak technical assumption which can be expressed in simple terms as follows: when the state is in the neighborhood of \mathcal{Z}_s , it can be steered inside \mathcal{Z}_s with low cost which vanishes with the distance to \mathcal{Z}_s . Note that the knowledge of $K_s(\cdot)$ is not required, only its existence is needed to prove the main result.

The last assumption that is needed to derive the main result is the following:

Assumption 5 *The map defined by*

$$F(\mathbf{u}, x) := [\ell + \alpha\Delta](x_N^{\mathbf{u}}(x), u_N) \in \mathbb{R} \quad (15)$$

satisfies the following implication for sufficiently small $|\eta|$ and $\|\delta\|$:

$$\left\{F(\mathbf{u}, x) = \eta\right\} \Rightarrow \left\{\exists \mathbf{u}^\dagger \mid F(\mathbf{u}^\dagger, x + \delta) = 0\right\} \quad (16)$$

Remark 2 *This Assumption is obviously very difficult to check. nevertheless, we believe that it is not so restrictive in practice. The following comments can shed some light on the relevance and the restriction it implies:*

- (1) *Note first of all that for differentiable setting and in the absence of saturation on the control, Assumption 4 can be expressed in terms of the implicit function theorem, namely, a slight modification of (x, η) into $(x + \delta, 0)$ can be compensated by a corresponding slight modification in the argument \mathbf{u} . For this to hold, it suffices that the rank of the sensitivity of F to \mathbf{u} at (x, η) be equal to 1 which is generically true.*
- (2) *When constraints are involved, the satisfaction of the condition is no more trivial. Still, assuming that the sequence $\mathbf{u} := (\mathbf{u}^{(a)}, \mathbf{u}^{(na)})$ satisfying $F(\mathbf{u}, x) = \eta$ can be split into saturated components $\mathbf{u}^{(a)}$ and non saturated components $\mathbf{u}^{(na)}$. In this context, Assumption 4 suggests that the sensitivity of F to $\mathbf{u}^{(na)}$ is of rank 1 so that one can always perturb $\mathbf{u}^{(na)}$, keeping unchanged $\mathbf{u}^{(a)}$ in order to compensate for the infinitesimal change on F induced by infinitesimal η and δ .*
- (3) *Note that the conditions of Assumption 5 could have been required only on the optimal sequences $\mathbf{u}^*(x)$ rather than on any sequence of control \mathbf{u} . This is because in the sequel, the implication in (16) is only used for such optimal sequences.*

(4) Based on the above discussion, it comes out that Assumption 5 can be replaced by several other checkable Assumptions of low level. But this might induce non necessary conditions such as differentiability while these conditions are only sufficient. It is preferred here to keep the high level condition (16) that might hold even for non differentiable settings. The above discussion helps for better understanding the underlying requirements.

4 Closed-loop analysis

We start by establishing a result that builds a first bridge between the penalty $\ell + \alpha\Delta$ used in the cost function (9) and the property ($\ell = 0$ and $\Delta = 0$) of optimal steady pairs.

Lemma 1 *Given a compact set $\mathbb{X} \times \mathbb{U}$, if $\alpha > L_\psi$ the (Lypschitz constant of ψ over $\mathbb{X} \times \mathbb{U}$), then the following implication hold:*

$$\ell(z) + \alpha\Delta(z) \leq \epsilon \quad \Rightarrow \quad \begin{cases} \Delta(z) \leq \kappa_1\epsilon \\ \ell(z) \in [-\kappa_2\epsilon, \epsilon] \end{cases} \quad (17)$$

where $\kappa_1 = 1/(\alpha - L_\psi)$ and $\kappa_2 = \alpha/(\alpha - L_\psi)$.

PROOF. Using the inequality (13) of Assumption 3, it comes that (recall that $\ell_s = 0$ is used without loss of generality):

$$\ell(z) \geq -\psi(\Delta(z)) \quad (18)$$

combining this with the left hand side of (17) leads to the following inequality:

$$\alpha\Delta(z) - \psi(\Delta(z)) \leq \epsilon \quad (19)$$

and denoting by L_ψ the Lypschitz-constant of ψ over $\mathbb{X} \times \mathbb{U}$, the last inequality implies:

$$(\alpha - L_\psi)\Delta(z) \leq \epsilon \quad (20)$$

which proves the first inequality of (17) as soon as $\alpha > L_\psi$ with $\kappa_1 = 1/(\alpha - L_\psi)$. In order to prove the second inequality in (17), it is sufficient to summarize the previous discussion in the following two inequalities:

$$0 \leq (\alpha - L_\psi)\Delta(z) \leq \ell(z) + \alpha\Delta(z) \leq \epsilon \quad (21)$$

since $\Delta(z) \geq 0$, this obviously gives:

$$-\alpha\Delta(z) \leq \ell(z) \leq \epsilon \quad (22)$$

On the other hand, solving the inequality in $\Delta(z)$ gives:

$$\Delta(z) \leq \frac{\epsilon}{\alpha - L_\psi} \quad (23)$$

This together with (22) implies that:

$$-\left[\frac{\alpha}{\alpha - L_\psi}\right]\epsilon \leq \ell(z) \leq \epsilon \quad (24)$$

which proves the second inequality of (17) with $\kappa_2 := \alpha/(\alpha - L_\psi)$ \square

Corollary 1 *Given a compact set $\mathbb{Z} := \mathbb{X} \times \mathbb{U}$, if α satisfies the condition of Lemma 1, there exists a continuous map φ_s , vanishing at 0 such that for sufficiently small $\epsilon > 0$, the following implication holds for all $z \in \mathbb{Z}$:*

$$\left\{ \ell(z) + \alpha\Delta(z) \leq \epsilon \right\} \Rightarrow \left\{ d(z, \mathcal{Z}_s) \leq \varphi_s(\epsilon) \right\} \quad (25)$$

PROOF. This is a straightforward consequence of Lemma 1 and Assumption 1. Indeed, if (25) does not hold then one can construct a sequence of points $z^{(j)} \in \mathbb{Z}$ such that $\lim_{j \rightarrow \infty} \ell(z^{(j)}) = 0$ and $\lim_{j \rightarrow \infty} \Delta(z^{(j)}) = 0$ while $\lim_{j \rightarrow \infty} d(z^{(j)}, \mathcal{Z}_s) > r > 0$ which, by continuity and compactness argument leads to the existence of some $z^{(\infty)}$ that is steady optimal while lying outside \mathcal{Z}_s which is obviously a contradiction by definition of \mathcal{Z}_s . \square

Lemma 2 (Properties of terminal pairs) *Given any compact set $\mathbb{X} \subset \mathbb{X}_0$ of initial states, there exist sufficiently high $\alpha > 0$ and two positive reals $\kappa_3, \kappa_4 > 0$ such that for any $x \in \mathbb{X}$, the optimal open-loop trajectory that solves the optimization problem (10) satisfies the following two terminal inequalities [see (12) for the notation]:*

$$\Delta_N^*(x) \leq \frac{\kappa_3}{\gamma} \quad \text{and} \quad |\ell_N^*(x)| \leq \frac{\kappa_4}{\gamma} \quad (26)$$

where κ_3 and κ_4 depend on \mathbb{X} and \mathbb{U} .

PROOF. Let us consider the following definitions:

$$V^*(x) := \sum_{k=0}^N \left[\ell_k^*(x) + \alpha\Delta_k^*(x) \right] \quad (27)$$

$$V^\#(x) := \sum_{k=0}^N \left[\ell_k^\#(x) + \alpha\Delta_k^\#(x) \right] \quad (28)$$

corresponding to the integrals of the stage costs over the trajectories starting from x under the optimal control $\mathbf{u}^*(x)$ and the control $\mathbf{u}^\#(x)$ invoked in reachability Assumption 2. It comes by definition that:

$$J(\mathbf{u}^\#, x) = V^\#(x) \quad (\text{vanishing terminal costs}) \quad (29)$$

$$J^*(x) := V^*(x) + \gamma(\ell_N^*(x) + \alpha\Delta_N^*(x)) \quad (30)$$

This gives by optimality of $J^*(x)$:

$$\ell_N^*(x) + \alpha \Delta_N^*(x) \leq \frac{V^\#(x) - V^*(x)}{\gamma} \leq \frac{K_0}{\gamma} \quad (31)$$

for some constant $K_0 := 2 \max_{(x,u) \in U^{N+1} \times \mathbb{X}} V(\mathbf{u}, x)$ which exists by virtue of the continuity of the involved maps (Assumption 1). Using inequality (31) together with Lemma 1 obviously gives the results for with $\kappa_3 := K_0 \kappa_1$ and $\kappa_4 = K_0 \max\{1, \kappa_2\}$. \square

The following straightforward corollary is used in the proof of the main result:

Corollary 2 *Under the Assumptions and notation of Lemma 2, the following inequality holds for all $x \in \mathbb{X}$:*

$$V^*(x) \leq V^\#(x) + \kappa_4 \quad (32)$$

PROOF. This is a direct consequence of the inequality:

$$V^*(x) + \gamma(\ell_N^*(x) + \alpha \Delta_N^*(x)) \leq V^\#(x) \quad (33)$$

which, by virtue of (26), obviously implies that:

$$V^*(x) \leq V^\#(x) - \gamma \ell_N^*(x) \leq V^\#(x) + \kappa_4 \quad (34)$$

which proves the corollary. \square

The following is another consequence of Lemma 2 that is crucial in the proof of the main result:

Corollary 3 *[Recursive satisfaction of terminal properties] For any state on the closed-loop trajectory starting at \mathbb{X}_0 , the inequalities (26) hold provided that α and γ are taken sufficiently high and the sampling period is taken sufficiently small.*

PROOF. Indeed since α is sufficiently high and the initial state lies in \mathbb{X}_0 , the inequalities (26) hold for x_0 . On the other hand, if γ is taken sufficiently high these inequalities imply that $F(\mathbf{u}^*(x_0), x_0) = \eta := (\kappa_4 + \alpha \kappa_3)/\gamma$ is satisfied for sufficiently small η . This means that if τ is taken sufficiently small so that $x_1^*(x_0) - x_0 = O(\tau)$ becomes sufficiently small to make the implication (16) valid, then it can be deduced that there is $\mathbf{u}^\# := \mathbf{u}^\dagger$ that steers the next state x_1 on the closed loop trajectory to \mathcal{Z}_s . This means that the arguments used in the proof of Lemma 2 can be reused to show that (26) hold for the next state. By induction, the argument can now be reiterated to prove that these inequalities hold for all the states on the closed-loop trajectory. \square

Proposition 3 (Main result) *For any desired precision $\epsilon > 0$, provided that α is high enough, there exist sufficiently high $\gamma > 0$ and sufficiently small sampling time*

τ such that any resulting closed-loop trajectory starting at $x_0 \in \mathbb{X}_0$ asymptotically becomes quasi-steady optimal in the sense of Definition 1.

PROOF. Consider the following definition:

$$\bar{V} := \sup_{x_0 \in \mathbb{X}_0} V(\mathbf{u}^\#(x_0), x_0) \quad (35)$$

and the resulting set:

$$\mathbb{X} := \left\{ x \in \mathbb{R}^n \mid V^*(x) \leq \kappa_4 + \max\{\bar{V}, K_s(1)\} \right\} \quad (36)$$

where $K_s(\cdot)$ is the map invoked in Assumption 4. The subset \mathbb{X} is bounded below by virtue of (17). Now since by Corollary 2, $V^*(x_0) \leq \kappa_4 + V^\#(x_0) \leq \kappa_4 + \bar{V}$, it comes out by induction that if it can be proved that when the $(x_k, u_0^*(x_k))$ lies inside $\mathbb{Z} := \mathbb{X} \times \mathbb{U}$ so is the next pair on the closed-loop trajectory, then the closed-loop trajectory remains inside \mathbb{Z} . Therefore, \mathbb{Z} -related Lipschitz constants can be invoked recursively.

Using a standard receding horizon argument together with the definition of the warm start control sequence (5), it comes that at any state x_k on the closed-loop trajectory

$$\begin{aligned} V^*(x_{k+1}) &\leq V^*(x_k) - (\ell_0^*(x_k) + \alpha \Delta_0^*(x_k)) \\ &\quad + [\ell + \alpha \Delta](f(x_N^*, u_N^*), u_N^*) + \\ &\quad + \gamma [\Psi(\mathbf{u}^{*+}, x_{k+1}) - \Psi_N^*(x_k)] \end{aligned} \quad (37)$$

where

$$\begin{aligned} \Psi(\mathbf{u}^{*+}, x_{k+1}) &:= [\ell + \alpha \Delta](f(x_N^*, u_N^*), u_N^*) \\ &\leq \Psi_N^*(x_k) + (L_\ell + \alpha L_\Delta)[\tau \Delta_N^*(x_k)] \end{aligned}$$

where L_ℓ and L_Δ are the lipschitz constants of ℓ and Δ over $\mathbb{X} \times \mathbb{U}$. Using the first inequality (26) of Lemma 2, the last inequality can be rearranged to give:

$$\Psi(\mathbf{u}^{*+}, x_{k+1}) \leq \Psi_N^*(x_k) + (L_\ell + \alpha L_\Delta) \frac{\kappa_3 \tau}{\gamma} \quad (38)$$

Now using this last inequality in (37) gives:

$$\begin{aligned} V^*(x_{k+1}) &\leq V^*(x_k) - (\ell_0^*(x_k) + \alpha \Delta_0^*(x_k)) \\ &\quad + [\ell + \alpha \Delta](f(x_N^*, u_N^*), u_N^*) + \kappa_5 \tau \end{aligned} \quad (39)$$

where $\kappa_5 := (L_\ell + \alpha L_\Delta) \kappa_3$.

Regarding the term $[\ell + \alpha \Delta](f(x_N^*, u_N^*), u_N^*)$, we have by definition:

$$\begin{aligned} [\ell + \alpha \Delta](f(x_N^*, u_N^*), u_N^*) &\leq \Psi_N^*(x_k) + \\ &\quad + (L_\ell + \alpha L_\Delta) \Delta_N^*(x_k) \end{aligned} \quad (40)$$

but we have by virtue of Lemma 2:

$$\Psi_N^*(x_k) \leq \frac{\kappa_3 + \alpha\kappa_4}{\gamma} \quad ; \quad \Delta_N^*(x_k) \leq \frac{\kappa_3}{\gamma} \quad (41)$$

Using this in (40) gives:

$$[\ell + \alpha\Delta](f(x_N^*, u_N^*), u_N^*) \leq \frac{\kappa_6}{\gamma} \quad (42)$$

Therefore, inequality (39) becomes:

$$V^*(x_{k+1}) \leq V^*(x_k) - (\ell_0^*(x_k) + \alpha\Delta_0^*(x_k)) + \overbrace{\frac{\kappa_6}{\gamma} + \kappa_5\tau}^{\varphi(\gamma, \tau)} \quad (43)$$

Consider the following set:

$$\mathcal{A} := \left\{ x \mid \ell_0^*(x) + \alpha\Delta_0^*(x) \leq 2\varphi(\gamma, \tau) \right\} \quad (44)$$

inequality (43) clearly shows that as long as the state x_k on the closed-loop trajectory remains outside \mathcal{A} , V^* decreases at the next step keeping the closed loop trajectory inside \mathbb{X} . Now since V over the compact set \mathbb{X} , this cannot occur indefinitely. Therefore, there exists a finite \bar{k} such that:

$$\ell_0^*(x_{\bar{k}}) + \alpha\Delta_0^*(x_{\bar{k}}) \leq 2\varphi(\gamma, \tau) \quad \text{and} \quad x_{\bar{k}-1} \in \mathbb{X} \quad (45)$$

and using Assumption 4, this simply means that for sufficiently high γ and small τ , $x_{\bar{k}}$ satisfies $V^*(x_{\bar{k}}) \leq \kappa_4 + V^\#(x_{\bar{k}}) \leq \kappa_4 + K_s(\varphi_s \circ \varphi(\gamma, \tau)) \leq \kappa_4 + K_s(1)$. This means that $x_{\bar{k}} \in \mathbb{X}$ [see (36)]. This clearly shows that the closed-loop state trajectory lies constantly inside \mathbb{X} and the use of the Lypschitz constants is relevant and hence the resulting inequality (41) always holds.

Now let us examine what happens for $k \geq \bar{k}$:

- Either x_{k+1} remains in \mathcal{A} in which case we have by definition of \mathcal{A} and Lemma 1 that $\Delta(z_{k+1}) \leq 2\kappa_1\varphi(\gamma, \tau)$ and $|\ell(z_{k+1})| \leq \max\{1, \kappa_2\}\varphi(\gamma, \tau)$
- Or x_{k+1} goes outside \mathcal{A} but the resulting increase in V^* is limited by the fact that $\Delta_0^*(x_{\bar{k}}) \leq 2\kappa_1\varphi(\gamma, \tau)$ thanks to (17) of Lemma 1, before V^* decreases again (since the state is outside \mathcal{A}).

therefore, for all $k \geq \bar{k}$, the following inequalities hold:

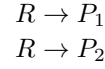
$$|\ell(x_k, u_0^*(x_k))| \leq \left[\max\{1, \kappa_2\} + 2\kappa_1 L_\ell \right] \varphi(\gamma, \tau) \quad (46)$$

$$\Delta(x_k, u^*(x_k)) \leq 2\kappa_1(1 + L_\Delta)\varphi(\gamma, \tau) \quad (47)$$

which obviously proves the result since the above inequality means that the closed-loop trajectory is asymptotically ε -quasi-steady optimal in the sense of Definition 1 with ε vanishing as γ and τ tend towards infinity and 0 respectively. \square

5 Illustrative examples

For the sake of illustration and to make an easy comparison with literature, let us consider the commonly used example of the nonlinear continuous flow stirred-tank reactor with parallel reactions [3].



that can be described by the following dimensionless energy and material balances:

$$\dot{x}_1 = 1 - 10^4 x_1^2 e^{-1/x_3} - 400 x_1 e^{-0.55/x_3} - x_1 \quad (48a)$$

$$\dot{x}_2 = 10^4 x_1^2 e^{-1/x_3} - x_2 \quad (48b)$$

$$\dot{x}_3 = u - x_3 \quad (48c)$$

where x_1 and x_2 stand for the concentrations of R and P_1 respectively while x_3 represents the temperature of the mixture in the reactor. P_2 represents the waste product. The control variable is given by the heat flow $u \in [0.049, 0.449]$. The natural stage cost would be given by $\ell(x, u) = -x_2$ since the objective is to maximize the amount of product P_1 .

Different aspects of using EMPC to address this problem have been considered in [1] where it has been recalled that without any particular care, there is a periodic optimal solution to the purely economic formulation that would include only the stage cost $\ell = -x_2$ while the use of average constraint enable to reduce the level of oscillations in the closed-loop behavior. It has been also shown that the system possesses an optimal steady pair denoted hereafter by $x_s = (0.0832, 0.0846, 0.149)$ and $u_s = 0.149$. All the Figures in the present section are centered around these values in order to better appreciate the distance to the steady-optimal **unknown** target.

In the present section, the behavior of the closed-loop under the EMPC associated to the proposed formulation is analyzed for different choices of the design parameters in order to assess the underlying theoretical development. In all the forthcoming simulations, the economic MPC design uses a prediction horizon length of $N = 20$ for a prediction horizon of 2 time units (this corresponds to a dimensionless sampling period of 0.1 used inside the predictor which is to be distinguished from the closed-loop control updating which is taken in $\{0.1, 0.02\}$) depending on the control settings (see Table 1). The optimization was done using the Casadi-Python module [2] with a single shooting implementation. The maximum number of iterations has been fixed to 100.

Four control settings are investigated in this section which are defined through the parameter sets depicted in Table 1.

Parameters	Nominal	# 1	#2	# 3
α	0	0.1	1.0	1.0
γ	0	0.1	1.0	1.0
τ	0.1	0.1	0.1	0.02

Table 1
Definition of the different investigated control settings.

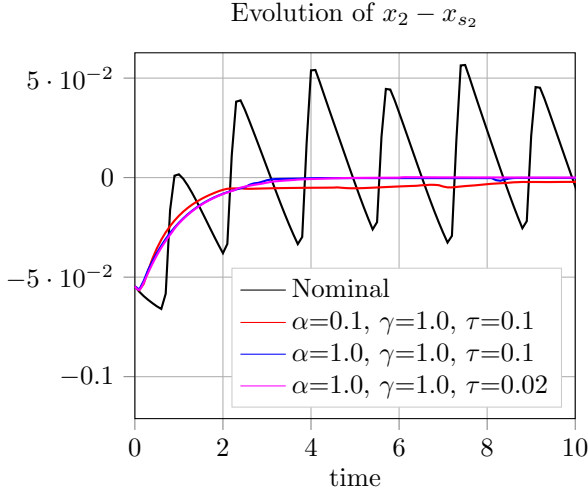


Fig. 1. Evolution of the difference $x_2 - x_{s_2} = x_2 - 0.0846$ for the different controller settings.

Figure 1 shows the closed-loop evolution of the targeted variable $x_2 = P_1$. It can be seen that the nominal settings leads to a periodic behavior as expected as this leads to higher averaged value of the product P_1 . It also shows that the proposed formulation leads to quasi-steady optimal asymptotic behavior with an error that becomes smaller as the penalty coefficients α and γ increase as well as the sampling updating period decreases as expected by the above theoretical investigation.

The effect of increasing α and γ and decreasing the sampling control period τ can be better observed on Figure 2 where the closed-loop trajectories of x_1 and x_2 are shown in the (x_1, x_2) plane (see also Figure 3 for a zoom around the terminal region). Finally, Figure 4 shows the evolution of the closed-loop feedback control for the different settings. Note that the sudden bursts in the control might be linked to the maximum number of iterations that is set arbitrarily to 100 which leads sometimes to the control delivered being not totally optimal although a warm start has been systematically used.

Trajectories in the centered (x_1, x_2) plane

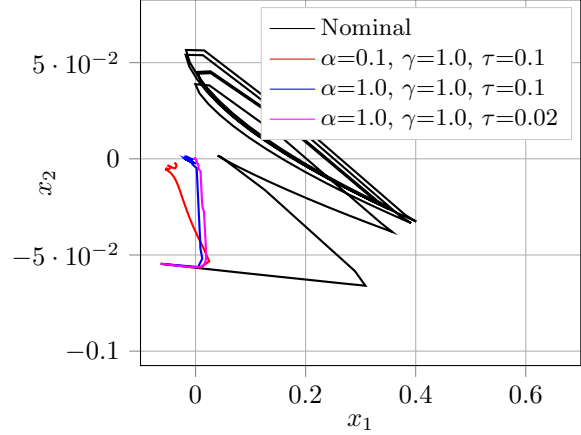


Fig. 2. Phase portrait in the (x_1, x_2) centered around the steady optimal pairs $z_s = (x_{s_1}, x_{s_2}) = (0.0832, 0.0846)$

Trajectories in the centered (x_1, x_2) plane

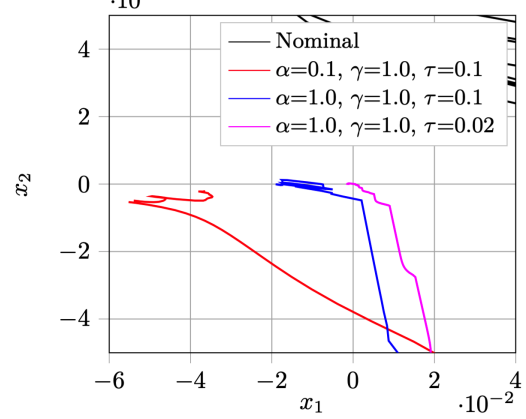


Fig. 3. Zoom on Figure 2 to underline the final error for the different controller settings. Coordinates are centered so that the $(0,0)$ corresponds to the optimal steady pair x_s .

Evolution of $u - u_s$

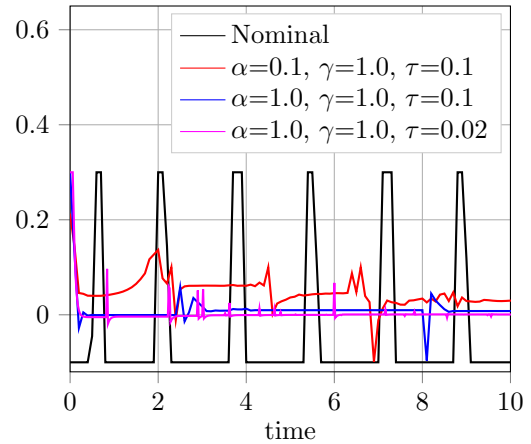


Fig. 4. Evolution of the control input u for the different control settings. The y -scale is shifted so that 0 corresponds to $u = u_s = 0.149$.

6 Conclusion

In this paper, a new formulation of economic MPC is proposed for discrete-time dynamics that are originated from the discretization of continuous dynamics. The formulation needs no terminal constraints on the state and is based on the penalization of the state increments between two successive states. Convergence to a quasi optimal steady regime has been derived using rather mild technical conditions.

References

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