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TANGENT SPACES OF THE TEICHMÜLLER SPACE OF THE TORUS WITH THURSTON'S WEAK METRIC

HIDEKI MIYACHI, KEN'ICHI OHSHIKA, AND ATHANASE PAPADOPOULOS

ABSTRACT. In this paper, we show that the analogue of Thurston's asymmetric metric on the Teichmüller space of flat structures on the torus is weak Finsler and we give a geometric description of its unit circle at each point in the tangent space to Teichmüller space. We then introduce a family of weak Finsler metrics which interpolate between Thurston's asymmetric metric and the Teichmüller metric of the torus (which coincides with the hyperbolic metric). We describe the unit tangent circles of the metrics in this family.

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1. PRELIMINARIES

We shall use the following identification between the Teichmüller space of the torus and the upper half-plane model of the hyperbolic plane \mathbb{H} :

Let T^2 be a two-dimensional torus and fix a pair of generators a, b of $\pi_1(T^2)$ represented by two simple closed curves on this surface intersecting at one point. The Teichmüller space of T^2 , denoted by $\mathcal{T}(T^2)$, is the set of equivalence classes of pairs (Σ, f) , where Σ is a Riemann surface and $f : T^2 \rightarrow \Sigma$ a homeomorphism, and where two pairs $(\Sigma_1, f_1), (\Sigma_2, f_2)$ are defined to be equivalent when $f_1 \circ f_2^{-1}$ is isotopic to a biholomorphism. From the uniformisation theorem, for every point x in $\mathcal{T}(T^2)$, there is a unique complex number ζ with $\text{Im}(\zeta) > 0$ such that x is represented by the pair $(\mathbb{C}/(\mathbb{Z} + \zeta\mathbb{Z}), f)$, where f is a homeomorphism taking the homotopy class of a, b to $1, \zeta \in \mathbb{Z} + \zeta\mathbb{Z} = \pi_1(\mathbb{C}/(\mathbb{Z} + \zeta\mathbb{Z}))$ respectively. In this way, $\mathcal{T}(T^2)$ is identified with $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. This identification induces an isometry when the Teichmüller space $\mathcal{T}(T^2)$ is equipped with the Teichmüller metric and \mathbb{H} is equipped with the metric of constant curvature -4 . In the sequel, we shall refer to this metric on \mathbb{H} as the *hyperbolic metric* d_{hyp} . The isometry between the space $\mathcal{T}(T^2)$ equipped with the so-called Teichmüller metric and the space \mathbb{H} equipped with the hyperbolic metric is

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a result of Teichmüller, see [4, Section 9] and [5, Section 9] for an English translation of Teichmüller's paper.

We also need the following notion:

A *weak metric* δ on a set X is a map $\delta : X \times X \rightarrow \mathbb{R}$ satisfying the following:

- (1) $\delta(x, x) = 0$ for every x in X ;
- (2) $\delta(x, y) \geq 0$ for every x and y in X ;
- (3) $\delta(x, y) + \delta(y, z) \geq \delta(x, z)$ for every x, y and z in X .

In the paper [1], the following weak metric was introduced on \mathbb{H} : First, for $\zeta_1, \zeta_2 \in \mathbb{H}$, we let

$$(1.1) \quad M(\zeta_1, \zeta_2) = \sup_{x \in \mathbb{R}} \left| \frac{\zeta_2 - x}{\zeta_1 - x} \right|.$$

The weak metric δ is then defined by setting $\delta(\zeta_1, \zeta_2) = \log M(\zeta_1, \zeta_2)$.

In the same paper, the following explicit expression of δ was obtained:

$$(1.2) \quad \delta(\zeta_1, \zeta_2) = \log \left(\frac{|\zeta_2 - \bar{\zeta}_1| + |\zeta_2 - \zeta_1|}{|\zeta_1 - \bar{\zeta}_1|} \right).$$

We note that this implies that

$$(1.3) \quad \delta(\zeta_2, \zeta_1) = \delta(\zeta_1, \zeta_2) + \log \frac{\operatorname{Im}(\zeta_1)}{\operatorname{Im}(\zeta_2)}.$$

It was also shown that this weak metric has the following two properties:

- (1) The arithmetic symmetrisation of the weak metric δ , that is, the weak metric $S\delta$ defined by

$$S\delta(\zeta_1, \zeta_2) = \frac{1}{2} (\delta(\zeta_1, \zeta_2) + \delta(\zeta_2, \zeta_1))$$

is a genuine metric and coincides with the hyperbolic metric of the upper half-plane.

- (2) The weak metric δ is an analogue for the torus of Thurston's asymmetric metric on Teichmüller space.

The last statement needs some explanation, and we give it now.

For any two points z_1, z_2 in the Teichmüller space $\mathcal{T}(T^2)$, we take representatives $(\Sigma_1 = \mathbb{C}/(\mathbb{Z} + \zeta_1\mathbb{Z}), f_1)$, $(\Sigma_2 = \mathbb{C}/(\mathbb{Z} + \zeta_2\mathbb{Z}), f_2)$, and we regard them as tori equipped with the quotient flat metrics induced by the flat metric of the Euclidean plane. We set $\delta(z_1, z_2) = \delta(\zeta_1, \zeta_2)$. In [1], a weak metric on $\mathcal{T}(T^2)$ was defined as follows. Let $\mathcal{S}(T^2)$ denote the set of homotopy classes of essential simple closed curves on the torus. We set,

$$(1.4) \quad \kappa(z_1, z_2) = \log \sup_{s \in \mathcal{S}(T^2)} \frac{\operatorname{length}_{\Sigma_2}(f_2(s))}{\operatorname{length}_{\Sigma_1}(f_1(s))},$$

where length denotes the length of the closed geodesic in the corresponding homotopy class. The formula for $\kappa(z_1, z_2)$ is the analogue, in this Euclidean

setting, of the formula for Thurston's metric in the hyperbolic setting given in [6, p. 8]. Theorem 3 of [1] says the following:

$$(1.5) \quad \kappa(z_1, z_2) = \delta(\zeta_1, \zeta_2)$$

for any $z_1, z_2 \in \mathcal{T}(T^2)$ and $z_i = (\Sigma_i = \mathbb{C}/(\mathbb{Z} + \zeta_i\mathbb{Z}), f_i)$ for $i = 1, 2$.

The metric δ has another characterisation which is given in [1]. For two metrics g_1, g_2 on $T_0 = \mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}$ and a homeomorphism $\varphi: T_0 \rightarrow T_0$, we define

$$\mathcal{L}(\varphi) = \sup_{x \neq y} \left(\frac{d_{g_2}(\varphi(x), \varphi(y)) / \text{length}_{g_2}(s)}{d_{g_1}(x, y) / \text{length}_{g_1}(s)} \right),$$

where s is a non-trivial simple closed curve, and set

$$\lambda(g_1, g_2) = \inf_{\varphi} (\log \mathcal{L}(\varphi)).$$

The function λ is invariant under the action of homeomorphisms on T_0 homotopic to the identity. Hence, λ defines a weak metric on $\mathcal{T}(T^2)$. The metric is called the *normalised weak Lipschitz distance*. In [1], it was shown that $\kappa(\zeta_1, \zeta_2) = \lambda(\zeta_1, \zeta_2)$ for any $\zeta_1, \zeta_2 \in \mathbb{H}$.

In the rest of this paper, we investigate further properties of the weak metric $\kappa = \delta$. We first show that the geodesics of the hyperbolic metric of \mathbb{H} are geodesics with respect to this weak metric. We then show that this metric is weak Finsler (in a sense we shall make precise) and we give a geometric description of its unit circle at each point in the tangent space to Teichmüller space. We then introduce a family of weak Finsler metrics which interpolates between the weak metric δ and the hyperbolic metric (which coincides with the Teichmüller metric) which arises naturally from the construction given in this paper. We describe the unit tangent circle at each point for each weak metric in this family.

2. GEODESICS FOR THE WEAK METRIC δ

In this section, we give an explicit expression for the point where the supremum of (1.1) is attained for given $\zeta_1, \zeta_2 \in \mathbb{H}$ and show its geometric meaning.

First we note the following, which can be shown easily from the definition of δ :

Lemma 2.1. *For $\lambda > 0$ and $\tau \in \mathbb{R}$, we have*

$$(2.1) \quad \delta(\lambda\zeta_1 + \tau, \lambda\zeta_2 + \tau) = \delta(\zeta_1, \zeta_2),$$

$$(2.2) \quad \delta(-\bar{\zeta}_1, -\bar{\zeta}_2) = \delta(\zeta_1, \zeta_2).$$

For $\zeta_1 = a + ib$ and $\zeta_2 = \alpha + i\beta$ in \mathbb{H} with $\zeta_1 \neq \zeta_2$, we define

$$(2.3) \quad x_{\pm} = \frac{\alpha^2 + \beta^2 - a^2 - b^2}{2(\alpha - a)} \mp \frac{\sqrt{(\alpha - a)^2 + (\beta - b)^2} \sqrt{(\alpha - a)^2 + (\beta + b)^2}}{2(\alpha - a)}$$

if $a \neq \alpha$. When $a = \alpha$, we define

$$x_+ = \begin{cases} 0 & (\beta > b) \\ \infty & (\beta < b), \end{cases}$$

$$x_- = \begin{cases} \infty & (\beta > b) \\ 0 & (\beta < b). \end{cases}$$

The following is an explicit expression for the supremum in (1.1):

Proposition 2.1. *For $\zeta_1, \zeta_2 \in \mathbb{H}$ with $\zeta_1 \neq \zeta_2$, the supremum in (1.1) is attained at x_+ .*

Proof. Let $\zeta_1 = a + ib$ and $\zeta_2 = \alpha + i\beta$. The case where $a = \alpha$ can be easily dealt with. The case where $\alpha < a$, from (2.2), by considering $-\overline{\zeta_1}$ and $-\overline{\zeta_2}$ instead of ζ_1 and ζ_2 respectively, is reduced to the case where $a < \alpha$. Hence we only consider the case where $a < \alpha$.

We first assume that $a = 0$. By assumption, we have $\alpha > 0$. Set

$$f(x) = \left| \frac{\zeta_2 - x}{\zeta_1 - x} \right|^2 = \frac{(x - \alpha)^2 + \beta^2}{x^2 + b^2} = 1 + \frac{A + Bx}{x^2 + b^2},$$

where $A = \alpha^2 + \beta^2 - b^2$ and $B = -2\alpha$. Then,

$$f'(x) = -\frac{Bx^2 + 2Ax - Bb^2}{(x^2 + b^2)^2},$$

and the critical points of $f'(x)$ are

$$x_{\pm} = -\frac{A}{B} \pm \frac{\sqrt{A^2 + b^2 B^2}}{B}.$$

Since $\operatorname{Re}(\zeta_2) = \alpha > 0$ and $B = -2\alpha < 0$, we have $x_+ < -A/B < x_-$. Therefore, $f(x)$ attains its maximum at

$$x_+ = -\frac{A}{B} + \frac{\sqrt{A^2 + b^2 B^2}}{B}$$

$$= \frac{\alpha^2 + \beta^2 - b^2}{2\alpha} - \frac{\sqrt{\alpha^2 + (b - \beta)^2} \sqrt{\alpha^2 + (b + \beta)^2}}{2\alpha}.$$

Suppose next that $a \neq 0$ and $\alpha > a$. From the invariance (2.1) and the above calculation, by considering $\zeta_1 - a = ib$ and $\zeta_2 - a = (\alpha - a) + i\beta$ instead of ζ_1 and ζ_2 , we see that the maximum is attained at

$$a + \frac{(\alpha - a)^2 + \beta^2 - b^2}{2(\alpha - a)} - \frac{\sqrt{(\alpha - a)^2 + (b - \beta)^2} \sqrt{(\alpha - a)^2 + (b + \beta)^2}}{2(\alpha - a)}$$

which is equal to x_+ in (2.3). \square

Proposition 2.2. *The points x_+ and x_- in Proposition 2.1 are the endpoints at infinity of the hyperbolic geodesic line in \mathbb{H} passing through ζ_2 and ζ_1 . The point x_+ lies on the side of ζ_1 , and x_- lies on the side of ζ_2 .*

Proof. Set $\zeta_1 = a + ib$ and $\zeta_2 = \alpha + i\beta$ again. The case where $a = \alpha$ can be easily dealt with. Hence, as before, we may assume that $a = 0$ and $\alpha > 0$, and set $A = \alpha^2 + \beta^2 - b^2$ and $B = -2\alpha$ as in the proof of Proposition 2.1. Then

$$\begin{aligned} \left| \zeta_1 - \left(-\frac{A}{B}\right) \right|^2 &= \left| ib - \left(-\frac{A}{B}\right) \right|^2 = \frac{A^2}{B^2} + b^2 = \frac{A^2 + b^2 B^2}{B^2}, \\ \left| \zeta_2 - \left(-\frac{A}{B}\right) \right|^2 &= \left| (\alpha + i\beta) - \left(-\frac{A}{B}\right) \right|^2 \\ &= \left(\frac{A}{B} + \alpha \right)^2 + \beta^2 = \left(\frac{\alpha^2 + \beta^2 - b^2}{-2\alpha} + \alpha \right)^2 + \beta^2 \\ &= \frac{(\alpha^2 - \beta^2 + b^2)^2 + 4\alpha^2 \beta^2}{4\alpha^2} \\ &= \frac{(\alpha^2 + (\beta - b)^2)(\alpha^2 + (\beta + b)^2)}{4\alpha^2} = \frac{A^2 + b^2 B^2}{B^2}. \end{aligned}$$

This means that $-A/B$ is the centre of the Euclidean semicircle perpendicular to the real axis passing through the points ζ_1 and ζ_2 , and that x_+, x_0 are the endpoints of this semicircle. Since $x_+ < -A/B < x_-$, x_+, ζ_1, ζ_2 and x_- lie on the semicircle in this order. Since such a semicircle is a hyperbolic geodesic, we have completed the proof. \square

Theorem 2.1. *Hyperbolic geodesics in \mathbb{H} are geodesic with respect to the weak metric δ . Conversely, every geodesic with respect to δ is a hyperbolic geodesic.*

Proof. Suppose that ζ_1, ζ_2 and ζ_3 lie on a hyperbolic geodesic γ in this order. By Proposition 2.2, the endpoint at infinity x of γ which lies on the side of ζ_1 not containing ζ_2, ζ_3 attains the supremum of (1.1) for $M(\zeta_1, \zeta_2), M(\zeta_2, \zeta_3)$ and $M(\zeta_1, \zeta_3)$, provided that $x \neq \infty$. Then by Section 1, we have

$$M(\zeta_1, \zeta_2) = \frac{|\zeta_2 - x|}{|\zeta_1 - x|}, \quad M(\zeta_2, \zeta_3) = \frac{|\zeta_3 - x|}{|\zeta_2 - x|},$$

and

$$M(\zeta_1, \zeta_3) = \frac{|\zeta_3 - x|}{|\zeta_1 - x|}.$$

This implies that $M(\zeta_1, \zeta_3) = M(\zeta_1, \zeta_2)M(\zeta_2, \zeta_3)$, hence $\delta(\zeta_1, \zeta_3) = \delta(\zeta_1, \zeta_2) + \delta(\zeta_2, \zeta_3)$. This means that γ is a geodesic with respect to δ .

If $x = \infty$, then in the same setting, we have $M(\zeta_1, \zeta_2) = M(\zeta_2, \zeta_3) = M(\zeta_1, \zeta_3) = 1$, and again γ is a geodesic with respect to δ .

Conversely, suppose that γ is a geodesic with respect to δ , and let $\zeta_1, \zeta_2, \zeta_3$ be arbitrary three points lying on γ in this order. Then we have $\delta(\zeta_1, \zeta_3) = \delta(\zeta_1, \zeta_2) + \delta(\zeta_2, \zeta_3)$. By Eq. (1.3), this implies that $\delta(\zeta_3, \zeta_1) = \delta(\zeta_3, \zeta_2) + \delta(\zeta_2, \zeta_1)$, hence $S\delta(\zeta_1, \zeta_3) = S\delta(\zeta_1, \zeta_2) + S\delta(\zeta_2, \zeta_3)$ for the arithmetic symmetrisation $S\delta$. Since $S\delta$ coincides with the hyperbolic metric, we see that γ is also a hyperbolic geodesic. \square

Before discussing the connection to Teichmüller theory, we shall give a brief comment on this theory. The ideal boundary $\partial\mathbb{H} = \partial\mathcal{T}(T^2)$ is canonically identified with the Thurston compactification of $\mathcal{T}(T^2)$. Recall that the Thurston compactification of $\mathcal{T}(T^2)$ consists of the projective classes of measured foliations on the base surface (torus) $T_0 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. A *measured foliation* on T_0 is an equivalence class of a pair consisting of a foliation on T_0 together with a transverse measure. (Note in the general Thurston theory, the foliations may have singular points, whereas in the case of the torus that we are discussing, the foliations are without singularities). Two such pairs are equivalent if either they are isotopic. (In the general case, one has to include Whitehead moves in the equivalence relation, but in the case of the torus, there are no such moves.) For $\alpha \in \hat{\mathbb{R}}$, we define a measured foliation associated with α to be the pair consisting of the foliation obtained as integral curves of unit vectors satisfying $(1 + |\alpha|)^{-1}(dx + \alpha dy) = 0$ on T_0 and the transverse measure defined by $(1 + |\alpha|)^{-1}|dx + \alpha dy|$. α is called the *slope* of the foliation. Notice that when $\alpha = \infty$, the associated measured foliation consists of the integral curves of the unit lines satisfying $dy = 0$ equipped with the transverse measure $|dy|$.

The point $x_+ \in \partial\mathbb{H} = \partial\mathcal{T}(T^2)$ discussed at the beginning of this section corresponds to the slope of the horizontal foliation of the Teichmüller map from ζ_2 to ζ_1 . Geometrically, the leaves of the horizontal foliation are stretched under the deformation along the Teichmüller geodesic segment from ζ_1 to ζ_2 (see Figure 1). Combining Theorem 2.1 with Eq. (1.5), we have the following.

Corollary 2.1. *Suppose that $z, z' \in \mathcal{T}(T^2)$ correspond to $\zeta_1, \zeta_2 \in \mathbb{H}$ respectively. Then the distance $\kappa(z, z') = \delta(\zeta_1, \zeta_2)$ is attained by the slope of the horizontal foliation for the Teichmüller map from z to z' .*

3. THE WEAK FINSLER STRUCTURE OF THE WEAK METRIC δ

We recall now the notion of weak norm and weak Finsler metric on a manifold, adapted to the case we are dealing with. We start with a *weak norm* on a finite-dimensional vector space V . This is a map $V \rightarrow [0, \infty)$, $v \mapsto \|v\|$, satisfying

- (1) $\|0\| = 0$;
- (2) $\|v\| \geq 0$ for all v in V ;
- (3) $\|v + v'\| \leq \|v\| + \|v'\|$ for all v in V .

A metric on a smooth manifold M is said to be *weak Finsler* if M is equipped with a continuous field of weak norms defined on the tangent space at each point of M such that the distance between two points in M is equal to the infimum of the lengths of piecewise C^1 -paths joining them, the length of such a path being computed as the integral over this path of the weak norms of the tangent vectors.

FIGURE 1. The Teichmüller ray (the hyperbolic geodesic ray) from $\zeta_1 = i$ to $\zeta_2 = \lambda i$ with $\lambda > 1$ is the vertical ray emanating from ζ_1 . In this case, $x_+ = 0$, and the leaves of the horizontal foliation are defined by $dx = 0$, which is stretched along the deformation from ζ_1 to ζ_2 .

In this section, we show that the weak metric δ on $\mathcal{T}(T^2)$ is weak Finsler and we give a description of its induced weak norm on the tangent space of each point in this space. We start with the following proposition.

Proposition 3.1. *Let ζ be a point in \mathbb{H} , and v a tangent vector at ζ . The weak metric δ induces on v a weak norm $\|v\|_\delta$ expressed by*

$$\|v\|_\delta = \frac{|v| + \operatorname{Im}(v)}{2\operatorname{Im}(\zeta)}.$$

The meaning of the expression “induced weak norm” will be clear from the computation done in the proof, and it acquires its complete significance in Corollary 3.1 which follows.

Proof. Set $\zeta' = \zeta + tv$ ($t > 0$). Then,

$$\begin{aligned} |\zeta' - \zeta| &= |\zeta + tv - \zeta| = t|v|; \\ |\zeta' - \bar{\zeta}| &= |\zeta - \bar{\zeta} + tv| = |\zeta - \bar{\zeta}| \left| 1 + t \frac{v}{\zeta - \bar{\zeta}} \right| \\ &= |\zeta - \bar{\zeta}| \left(1 + t \operatorname{Re} \left(\frac{v}{\zeta - \bar{\zeta}} \right) + o(t) \right) \\ &= |\zeta - \bar{\zeta}| \left(1 + t \operatorname{Re} \left(\frac{v}{2i \operatorname{Im}(\zeta)} \right) + o(t) \right) \\ &= 2 \operatorname{Im}(\zeta) \left(1 + t \frac{\operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)} + o(t) \right). \end{aligned}$$

Hence, we have

$$\begin{aligned} \delta(\zeta, \zeta + tv) &= \log \frac{1}{2 \operatorname{Im}(\zeta)} \left(t|v| + 2 \operatorname{Im}(\zeta) \left(1 + t \frac{\operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)} + o(t) \right) \right) \\ &= \log \left(1 + t \frac{|v| + \operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)} + o(t) \right) \\ &= t \frac{|v| + \operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)} + o(t). \end{aligned}$$

Thus, we obtain

$$\lim_{t \searrow 0} \frac{\delta(\zeta, \zeta + tv)}{t} = \frac{|v| + \operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)}.$$

□

Notice that as an invariant expression, the weak metric in Proposition 3.1 is presented as

$$(3.1) \quad \|\cdot\|_\delta = \frac{\sqrt{dx^2 + dy^2} + dy}{2y} = ds_{hyp} + \frac{1}{2} d \log y$$

on $T_\zeta \mathbb{H}$ and $\zeta = x + iy \in \mathbb{H}$, where ds_{hyp} is the length element of the hyperbolic metric on \mathbb{H} of constant curvature -4 .

Consider now the hyperbolic space \mathbb{H} equipped with the hyperbolic distance (of constant curvature -4) and let γ be an isometrically parametrised geodesic emanating from a point ζ_0 in \mathbb{H} and converging to a point $x_0 \in \partial \mathbb{H}$. We recall that the associated *Busemann function* associated with the geodesic ray γ is defined by

$$\mathbb{H} \ni \zeta \mapsto \lim_{t \rightarrow \infty} (d_{hyp}(\zeta, \gamma(t)) - t)$$

(cf. [3, Chapter 12]). Combining this with Theorem 2.1, we have the following corollary.

Corollary 3.1. *The weak metric space $(\mathcal{T}(T^2), \delta)$ is a weak Finsler metric space with the corresponding weak norm $\|\cdot\|_\delta$ given in Proposition 3.1.*

Proof. We first show that

$$(3.2) \quad \int_{\theta_1}^{\theta_2} \|\dot{\gamma}(\theta)\|_{\delta} d\theta \geq \delta(\zeta_1, \zeta_2)$$

for $\zeta_1, \zeta_2 \in \mathbb{H}$ and any piecewise C^1 -path $\gamma: [\theta_1, \theta_2] \rightarrow \mathbb{H}$ connecting ζ_1 to ζ_2 . Indeed, from the invariant expression (3.1), the integration in the left-hand side of (3.2) is at least equal to the hyperbolic distance between ζ_1 and ζ_2 minus the difference of the Busemann functions at ζ_1, ζ_2 (see (4.2)), which is the right-hand side of (3.1) (cf. §4). This observation also implies that the integration in the left-hand side of (3.2) is minimised only when it is done along the hyperbolic geodesic from ζ_1 to ζ_2 .

We now show that the distance between any two points ζ_1 and $\zeta_2 \in \mathbb{H}$ is given by integrating the weak norm $\|\cdot\|_{\delta}$ along a parametrised geodesic joining these two points.

To this end, we first assume that $\operatorname{Re}(\zeta_1) \neq \operatorname{Re}(\zeta_2)$. As in the proof of Proposition 2.1, we may assume that $\zeta_1 = ib$ and $\zeta_2 = \alpha + i\beta$ with $\alpha > 0$. Let $A = \alpha^2 + \beta^2 - b^2$, $B = -2\alpha$ and $R = \sqrt{A^2 + b^2 B^2}/|B|$. Define $\theta_1, \theta_2 \in (0, 2\pi)$ by $e^{i\theta_1} = (\zeta_1 - (-A/B))/R$ and $e^{i\theta_2} = (\zeta_2 - (-A/B))/R$. Note that $\theta_2 < \theta_1$. The geodesic from ζ_1 to ζ_2 is parametrised as $\gamma(\theta) = (-A/B) + Re^{i(\theta_1 + \theta_2 - \theta)}$ ($\theta_2 \leq \theta \leq \theta_1$). Hence by setting ϕ to be $\theta_1 + \theta_2 - \theta$, we have

$$\begin{aligned} \int_{\theta_2}^{\theta_1} \|\dot{\gamma}(\theta)\|_{\delta} d\theta &= \int_{\theta_2}^{\theta_1} \frac{|\dot{\gamma}(\theta)| + \operatorname{Im}(\dot{\gamma}(\theta))}{2\operatorname{Im}(\gamma(\theta))} d\theta = \int_{\theta_2}^{\theta_1} \frac{R - R \cos(\theta_1 + \theta_2 - \theta)}{2R \sin(\theta_1 + \theta_2 - \theta)} d\theta \\ &= \int_{\theta_2}^{\theta_1} \frac{1 - \cos(\phi)}{2 \sin(\phi)} d\phi = \frac{1}{2} \log(1 + \cos \theta_2) - \frac{1}{2} \log(1 + \cos \theta_1) \\ &= \frac{1}{2} \left(\log \left(1 - \frac{2\alpha^2 - A}{\sqrt{A^2 + b^2 B^2}} \right) - \log \left(1 + \frac{A}{\sqrt{A^2 + b^2 B^2}} \right) \right). \end{aligned}$$

An easy calculation shows that this expression is equal to

$$\log \frac{\sqrt{\alpha^2 + (\beta - b)^2} + \sqrt{\alpha^2 + (\beta + b)^2}}{2b}$$

that is, to $\delta(\zeta_1, \zeta_2)$. This gives what we wanted.

We now suppose that $\operatorname{Re}(\zeta_1) = \operatorname{Re}(\zeta_2)$. Then $\delta(\zeta_1, \zeta_2)$ is equal to zero when $\operatorname{Im}(\zeta_2) < \operatorname{Im}(\zeta_1)$ and to $\delta(\zeta_1, \zeta_2) = d_{hyp}(\zeta_1, \zeta_2)$ otherwise. From (3.1), $\|v\|_{\delta}$ is equal to zero when $\operatorname{Im}(v) < 0$ and to $\|v\|_{hyp} := \sqrt{ds_{hyp}^2(v, v)}$ otherwise. Hence, the integral of the δ -norm along the hyperbolic geodesic connecting from ζ_1 to ζ_1 coincides with the δ -distance from ζ_1 to ζ_2 . \square

Next we describe the unit circle in the tangent space with respect to the weak norm $\|\cdot\|_{\delta}$.

Proposition 3.2. *The unit circle of the tangent space at $\zeta \in \mathbb{H}$ with respect to $\|\cdot\|_{\delta}$ is expressed as a parabola with focus at the origin and vertex at $i\operatorname{Im}(\zeta)$.*

Proof. Let $\zeta = \alpha + i\beta$. When $v = v_1 + iv_2 \in \mathbb{C} \cong T_\zeta \mathbb{H}$ (as real vector spaces) lies on the unit circle of the tangent space at ζ , we have

$$1 = \frac{|v| + \operatorname{Im}(v)}{2\operatorname{Im}(\zeta)} = \frac{\sqrt{v_1^2 + v_2^2} + v_2}{2\beta},$$

which is equivalent to

$$v_1^2 + v_2^2 = (2\beta - v_2)^2 = 4\beta^2 - 4\beta v_2 + v_2^2.$$

This means that the unit tangent circle at $\zeta = \alpha + i\beta \in \mathbb{H}$ is the parabola

$$v_2 = -\frac{v_1^2}{4\beta} + \beta,$$

which implies the desired result. \square

Note that the fact that the unit tangent circle of the weak Finsler norm has an infinite direction expresses the fact that the distance function is degenerate in this direction (that is, we have, in this direction, $\delta(x, y) = 0$ for $x \neq y$).

4. DEFORMING δ TO THE TEICHMÜLLER METRIC

In this section, we consider a family of weak Finsler metrics which interpolate between δ and the hyperbolic distance (which, as is well known, coincides with the Teichmüller distance). We then describe the unit tangent circle of each of these metrics.

Consider the family of weak metrics δ_p ($0 \leq p \leq 1$) defined by

$$\begin{aligned} \delta_p(\zeta_1, \zeta_2) &= \log \sup_{x \in \mathbb{R}} \frac{(\operatorname{Im}(\zeta_1))^{p/2}}{(\operatorname{Im}(\zeta_2))^{p/2}} \left| \frac{\zeta_2 - x}{\zeta_1 - x} \right| \\ (4.1) \quad &= \frac{p}{2} \log \frac{\operatorname{Im}(\zeta_1)}{\operatorname{Im}(\zeta_2)} + \log \sup_{x \in \mathbb{R}} \left| \frac{\zeta_2 - x}{\zeta_1 - x} \right|. \end{aligned}$$

Note that the function

$$(4.2) \quad \mathbb{H} \ni \zeta \mapsto \frac{1}{2} \log \frac{\operatorname{Im}(\zeta_0)}{\operatorname{Im}(\zeta)} = -\frac{1}{2} \int_{\zeta_0}^{\zeta} d \log y$$

is the Busemann function associated with the geodesic ray emanating from some fixed point $\zeta_0 \in \mathbb{H}$ converging to $x = \infty \in \partial \mathbb{H}$ of the hyperbolic metric of curvature -4 , which is the Teichmüller distance. Hence, the function

$$\frac{1}{2} \log \frac{\operatorname{Im}(\zeta_1)}{\operatorname{Im}(\zeta_2)} = \frac{1}{2} \log \frac{\operatorname{Im}(\zeta_0)}{\operatorname{Im}(\zeta_2)} - \frac{1}{2} \log \frac{\operatorname{Im}(\zeta_0)}{\operatorname{Im}(\zeta_1)}$$

appears in (4.1) is the difference of the Busemann functions. Using the same proof as in Theorem 2.1, the hyperbolic geodesic from ζ_1 to ζ_2 is the geodesic of the metric δ_p . The arithmetic symmetrisation of δ_p is the hyperbolic metric of curvature -4 , like for $\delta = \delta_0$ (cf. (1) in §1).

As we did in Proposition 3.1, we can calculate the infinitesimal form of the metric δ_p :

FIGURE 2. The infinitesimal unit circle of the weak norm $\|\cdot\|_{\delta_p}$ at $\zeta = 0$ for $p = 1, 0.3$, and 0 . Each infinitesimal unit circle has the origin as a focus, and the lower dot is another focus for the infinitesimal unit circle for $p = 0.3$.

Let $\zeta \in \mathbb{H}$. For $v \in T_\zeta \mathbb{H} \cong \mathbb{C}$,

$$\log \frac{\operatorname{Im}(\zeta)}{\operatorname{Im}(\zeta + tv)} = -\log \left(1 + t \frac{\operatorname{Im}(v)}{\operatorname{Im}(\zeta)} \right) = -t \frac{\operatorname{Im}(v)}{\operatorname{Im}(\zeta)} + o(t)$$

as $t \searrow +0$. We obtain

$$\begin{aligned} \|v\|_{\delta_p} &:= \lim_{t \searrow +0} \frac{\delta_p(\zeta, \zeta + tv)}{t} = -\frac{p \operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)} + \frac{|v| + \operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)} \\ &= \frac{|v| + (1-p)\operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)} \\ &= \|v\|_\delta - p \frac{\operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)}. \end{aligned}$$

Notice that $\|v\|_{\delta_p} > 0$ when $v \neq 0$ and $p > 0$. The unit tangent circle with respect to the weak norm $\|\cdot\|_{\delta_p}$ in the tangent space $T_\zeta \mathbb{H}$ is the ellipse with foci 0 and $-4\operatorname{Im}(\zeta)(1-p)/(p(2-p))$ (see Figure 2). As an invariant expression, the weak metric $\|\cdot\|_{\delta_p}$ is presented as

$$(4.3) \quad \|\cdot\|_{\delta_p} = \frac{\sqrt{dx^2 + dy^2} + (1-p)dy}{2y} = ds_{hyp} + \frac{1-p}{2} d \log y.$$

A discussion similar to that of the proof of Corollary 3.1 and (4.3) shows that the weak metric space $(\mathcal{T}(T^2), \delta_p)$ is a weak Finsler metric with associated weak norm $\|\cdot\|_{\delta_p}$ and that the hyperbolic geodesic from ζ_1 to $\zeta_2 \in \mathbb{H}$ is a unique geodesic for δ_p . Notice that $\|\cdot\|_{\delta_1}$ is the norm from the hyperbolic metric. From (4.3), $\|v\|_{\delta_1} \leq \|v\|_{\delta_p}^2 \leq (2-p)\|v\|_{\delta_1}$ for $0 < p \leq 1$. Hence δ_p and δ_q are bi-Lipschitz-equivalent for $0 < p, q \leq 1$. In particular, for $0 < p \leq 1$, δ_p is complete and separates two points in \mathbb{H} .

It follows that $\{\delta_p\}_{0 \leq p \leq 1}$ is a continuous family of weak Finsler metrics giving a deformation from $\delta = \delta_0$ to the hyperbolic metric δ_1 (which is

Teichmüller metric). Notice that

$$\frac{|\zeta - x|^2}{\operatorname{Im}(\zeta)}$$

coincides with the extremal length of the measured foliation corresponding to $x \in \mathbb{R}$ up to a constant factor (depending only on x). (We recall that the extremal length of a simple closed curve c on a Riemann surface is defined to be the infimum of the reciprocals of the moduli of the annuli whose core curves are homotopic to c , and the extremal length function can be extended continuously to the space of measured foliations.) Hence, δ_1 coincides with Kerckhoff's formula for the Teichmüller distance [2] adapted to the case of the torus.

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