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TANGENT SPACES OF THE TEICHMÜLLER SPACE OF THE TORUS WITH THURSTON'S WEAK METRIC

HIDEKI MIYACHI, KEN'ICHI OHSHIKA, AND ATHANASE PAPADOPOULOS

ABSTRACT. In this paper, we show that the analogue of Thurston's asymmetric metric on the Teichmüller space of flat structures on the torus is weak Finsler and we give a geometric description of its unit sphere at each point in the tangent space to Teichmüller space. We then introduce a family of weak Finsler metrics which interpolate between Thurston's asymmetric metric and the Teichmüller metric of the torus (which coincides with the hyperbolic metric). We describe the infinitesimal unit spheres of the metrics in this family.

RÉSUMÉ EN FRANÇAIS. Dans cet article, on montre que l'analogue de la métrique asymétrique de Thurston sur l'espace de Teichmüller des structures plates sur le tore est une métrique Finslérienne faible, et on donne une description de sa sphère unité dans chaque espace tangent à cet espace de Teichmüller. On introduit ensuite une famille de métriques Finslériennes faibles qui interpolent entre la métrique asymétrique de Thurston et la métrique de Teichmüller du tore (qui concide avec la métrique hyperbolique). On décrit les sphères unité infinitésimales des métriques de cette famille.

1. PRELIMINARIES

We shall use the following identification between the Teichmüller space of the torus and the upper half-plane model of the hyperbolic plane \mathbb{H} :

Let T^2 be a two-dimensional torus and fix a pair of generators a, b of $\pi_1(T^2)$ represented by two simple closed curves on this surface intersecting at one point. The Teichmüller space of T^2 , denoted by $\mathcal{T}(T^2)$, is the set of equivalence classes of pairs (Σ, f) , where Σ is a Riemann surface and $f : T^2 \rightarrow \Sigma$ a homeomorphism, and where two pairs $(\Sigma_1, f_1), (\Sigma_2, f_2)$ are defined to be equivalent when $f_1 \circ f_2^{-1}$ is isotopic to a bi-holomorphic homeomorphism. By the uniformisation theorem, for every point x in $\mathcal{T}(T^2)$, there is a unique complex number ζ with $\text{Im}(\zeta) > 0$ such that x is represented by the pair $(\mathbb{C}/(\mathbb{Z} + \zeta\mathbb{Z}), f)$, where f is a homeomorphism taking the

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homotopy class of a, b to $1, \zeta \in \pi_1(\mathbb{C}/(\mathbb{Z} + \zeta\mathbb{Z}))$ respectively. In this way, $\mathcal{T}(T^2)$ is identified with $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$.

We also need the following notion:

A *weak metric* δ on a set X is a map $\delta : X \times X \rightarrow \mathbb{R}$ satisfying the following:

- (1) $\delta(x, x) = 0$ for every x in X ;
- (2) $\delta(x, y) \geq 0$ for every x and y in X ;
- (3) $\delta(x, y) + \delta(y, z) \geq \delta(x, z)$ for every x, y and z in X .

In the paper [1], the following a weak metric was introduced on \mathbb{H} : First, for $\zeta_1, \zeta_2 \in \mathbb{H}$, we let

$$(1.1) \quad M(\zeta_1, \zeta_2) = \sup_{x \in \mathbb{R}} \left| \frac{\zeta_2 - z}{\zeta_1 - x} \right|.$$

The weak metric δ is then defined by setting $\delta(\zeta_1, \zeta_2) = \log M(\zeta_1, \zeta_2)$.

In the same paper, the following explicit expression of δ was obtained:

$$(1.2) \quad \delta(\zeta_1, \zeta_2) = \log \left(\frac{|\zeta_2 - \bar{\zeta}_1| + |\zeta_2 - \zeta_1|}{|\zeta_1 - \bar{\zeta}_1|} \right).$$

It was also shown that this weak metric has the following two properties:

- (1) The arithmetic symmetrization of the weak metric δ , that is, the weak metric $S\delta$ defined by

$$S\delta(\zeta_1, \zeta_2) = \frac{1}{2} (\delta(\zeta_1, \zeta_2) + \delta(\zeta_2, \zeta_1))$$

is a genuine metric and coincides with the Poincaré metric of the upper half-plane;

- (2) The weak metric δ is an analogue for the torus of Thurston's asymmetric metric on Teichmüller space.

The last statement needs some explanation, and we give it now.

For any two points z_1, z_2 in the Teichmüller space $\mathcal{T}(T^2)$, we take representatives $(\Sigma_1 = \mathbb{C}/(\mathbb{Z} + \zeta_1\mathbb{Z}), f_1), (\Sigma_2 = \mathbb{C}/(\mathbb{Z} + \zeta_2\mathbb{Z}), f_2)$, and we regard them as tori equipped with the quotient flat metrics induced by the flat metric of the Euclidean plane. We set $\delta(z_1, z_2) = \delta(\zeta_1, \zeta_2)$. In [1], a weak metric on $\mathcal{T}(T^2)$ was defined as follows. Let $\mathcal{S}(T^2)$ denote the set of homotopy classes of essential simple closed curves on the torus. We set

$$(1.3) \quad \kappa(z_1, z_2) = \log \sup_{s \in \mathcal{S}(T^2)} \frac{\text{length}_{\Sigma_2}(f_2(s))}{\text{length}_{\Sigma_1}(f_1(s))}$$

where length denotes the length of the closed geodesic in the corresponding homotopy class. The formula for $\kappa(z_1, z_2)$ is the analogue, in this Euclidean setting, of the formula for Thurston's metric in the hyperbolic setting given in [3, p. 8]. Theorem 3 of [1] says the following:

$$(1.4) \quad \kappa(z_1, z_2) = \delta(\zeta_1, \zeta_2)$$

for any $z_1, z_2 \in \mathcal{T}(T^2)$ and $z_i = (\Sigma_i = \mathbb{C}/(\mathbb{Z} + \zeta_i\mathbb{Z}), f_i)$ for $i = 1, 2$.

In the rest of this paper, we investigate further properties of the weak metric $\kappa = \delta$. We first show that the geodesics of the Poincaré metric in \mathbb{H} are geodesics with respect to this weak metric. We then show that this metric is weak Finsler (in a sense we shall make precise) and we give a geometric description of its unit sphere at each point in the tangent space to Teichmüller space. We then introduce a family of weak Finsler metrics which interpolates between the weak metric δ and the Poincaré metric (which coincides with the Teichmüller metric) and we describe the infinitesimal unit sphere at each point for each weak metric in this family.

2. GEODESICS FOR THE WEAK METRIC δ

In this section, we give an explicit expression for the point where the supremum of (1.1) is attained for given $\zeta_1, \zeta_2 \in \mathbb{H}$ and show its geometric meaning.

First we note the following, which can be shown easily from the definition of δ :

Lemma 2.1. *For $\lambda > 0$ and $\tau \in \mathbb{R}$, we have*

$$(2.1) \quad \delta(\lambda\zeta_1 + \tau, \lambda\zeta_2 + \tau) = \delta(\zeta_1, \zeta_2)$$

$$(2.2) \quad \delta(-\bar{\zeta}_1, -\bar{\zeta}_2) = \delta(\zeta_1, \zeta_2).$$

The following is an explicit expression for the supremum in (1.1):

Proposition 2.1. *For $\operatorname{Re}(\zeta_1) < \operatorname{Re}(\zeta_2)$, the supremum in (1.1) is attained at*

$$x_+ = \frac{(\alpha - a)^2 + \beta^2 - b^2}{2(\alpha - a)} - \frac{\sqrt{(\alpha - a)^2 + (\beta - b)^2} \sqrt{(\alpha - a)^2 + (\beta + b)^2}}{2(\alpha - a)}$$

where $\zeta_1 = a + ib$, $\zeta_2 = \alpha + i\beta$.

For $\operatorname{Re}(\zeta_2) < \operatorname{Re}(\zeta_1)$, from (2.2), by considering $-\bar{\zeta}_1$ and $-\bar{\zeta}_2$ instead of ζ_1 and ζ_2 respectively, we get a similar expression.

Proof. From the invariance (2.1), we may assume that $a = 0$ after we consider $\zeta_1 - a$ and $\zeta_2 - a$ instead of ζ_1 and ζ_2 . By assumption, we have $\alpha > 0$. Set

$$f(x) = \left| \frac{\zeta_2 - x}{\zeta_1 - x} \right|^2 = \frac{(x - \alpha)^2 + \beta^2}{x^2 + b^2} = 1 + \frac{A + Bx}{x^2 + b^2},$$

where $A = \alpha^2 + \beta^2 - b^2$ and $B = -2\alpha$. Then,

$$f'(x) = -\frac{Bx^2 + 2Ax - Bb^2}{(x^2 + b^2)^2},$$

and the critical points of $f'(x)$ are

$$x_{\pm} = -\frac{A}{B} \pm \frac{\sqrt{A^2 + b^2 B^2}}{B}.$$

Since $\operatorname{Re}(\zeta_1) = \alpha > 0$ and $B = -2\alpha < 0$, $x_+ < -A/B < x_-$. Therefore, $f(x)$ attains its maximum at

$$\begin{aligned} x_+ &= -\frac{A}{B} \pm \frac{\sqrt{A^2 + b^2 B^2}}{B} \\ &= \frac{\alpha^2 + \beta^2 - b^2}{2\alpha} - \frac{\sqrt{\alpha^2 + (b - \beta)^2} \sqrt{\alpha^2 + (b + \beta)^2}}{2\alpha}. \end{aligned}$$

□

Theorem 2.1. *The point x_+ in Proposition 2.1 is the endpoint at infinity of the hyperbolic geodesic ray in \mathbb{H} starting from ζ_2 and passing through ζ_1 .*

Proof. Let $\zeta_1 = a + ib$ and $\zeta_2 = \alpha + i\beta$. We may assume that $a = 0$ and $\alpha > 0$, and set $A = \alpha^2 + \beta^2 - b^2$ and $B = -2\alpha$ as in Proof of Proposition 2.1. Then

$$\begin{aligned} \left| \zeta_1 - \left(-\frac{A}{B}\right) \right|^2 &= \left| ib - \left(-\frac{A}{B}\right) \right|^2 = \frac{A^2}{B^2} + b^2 = \frac{A^2 + b^2 B^2}{B^2}, \\ \left| \zeta_2 - \left(-\frac{A}{B}\right) \right|^2 &= \left| (\alpha + i\beta) - \left(-\frac{A}{B}\right) \right|^2 \\ &= \left(\frac{A}{B} + \alpha \right)^2 + \beta^2 = \left(\frac{\alpha^2 + \beta^2 - b^2}{-2\alpha} + \alpha \right)^2 + \beta^2 \\ &= \frac{(\alpha^2 - \beta^2 + b^2)^2 + 4\alpha^2 \beta^2}{4\alpha^2} \\ &= \frac{(\alpha^2 + (\beta - b)^2)(\alpha^2 + (\beta + b)^2)}{4\alpha^2} = \frac{A^2 + b^2 B^2}{B^2}. \end{aligned}$$

This means that $-A/B$ is the centre of the Euclidean half-circle centered on the real axis and joining the points ζ_1 and ζ_2 and that x_+ is an endpoint of that half-circle. Since $x_+ < -A/B < x_-$, x_+ , ζ_1 , ζ_2 lie the half-circle in this order. Since such a half-circle is a hyperbolic geodesic, we have completed the proof. □

Theorem 2.2. *Hyperbolic geodesics in \mathbb{H} are geodesic with respect to the weak metric δ .*

Proof. Suppose that ζ_1, ζ_2 and ζ_3 lie on a hyperbolic geodesic γ in this order. By Theorem 2.1, the endpoint at infinity x of γ which lies on the side of ζ_1 not containing ζ_2, ζ_3 attains the supremum of (1.1) for $M(\zeta_1, \zeta_2)$, $M(\zeta_2, \zeta_3)$ and $M(\zeta_1, \zeta_3)$. Then by Section 1, we have

$$M(\zeta_1, \zeta_2) = \frac{|\zeta_2 - x|}{|\zeta_1 - x|}, \quad M(\zeta_2, \zeta_3) = \frac{|\zeta_3 - x|}{|\zeta_2 - x|},$$

and

$$M(\zeta_1, \zeta_3) = \frac{|\zeta_3 - x|}{|\zeta_1 - x|}.$$

This implies $M(\zeta_1, \zeta_3) = M(\zeta_1, \zeta_2)M(\zeta_2, \zeta_3)$, and hence $\delta(\zeta_1, \zeta_3) = \delta(\zeta_1, \zeta_2) + \delta(\zeta_2, \zeta_3)$. This means that γ is a geodesic with respect to δ . □

Since this point x_+ is the slope of the horizontal foliation of the Teichmüller map from ζ_2 to ζ_1 , combining Theorem 2.2 with Eq. (1.4), we have the following.

Corollary 2.1. *Suppose that $z, z' \in \mathcal{T}(T^2)$ correspond to $\zeta_1, \zeta_2 \in \mathbb{H}$ respectively. Then the distance $\kappa(z, z') = \delta(\zeta_1, \zeta_2)$ is attained by the slope of the horizontal foliation for the Teichmüller map from z to z' .*

3. THE WEAK FINSLER STRUCTURE OF THE WEAK METRIC δ

We recall now the notion of weak norm and weak Finsler metric on a manifold, adapted to the case we are dealing with. We start with a *weak norm* on a finite-dimensional vector space V . This is a map $V \rightarrow [0, \infty)$, $v \mapsto \|v\|$, satisfying

- (1) $\|0\| = 0$
- (2) $\|v\| \geq 0$ for all v in V ;
- (3) $\|v + v'\| \leq \|v\| + \|v'\|$ for all v in V .

A metric on a smooth manifold M is said to be *weak Finsler* if M is equipped with a continuous field of weak norms defined on the tangent space at each point of M such that the distance between two points in M is equal to the infimum of the lengths of piecewise C^1 paths joining them, the length of such a path being computed as the integral over this path of the weak norms of the tangent vectors.

In this section, we show that the weak metric δ on $\mathcal{T}(T^2)$ is weak Finsler and we give a description of its induced weak norm on the tangent space of each point in this space. We start with the following proposition.

Proposition 3.1. *Let ζ be a point in \mathbb{H} , and v a tangent vector at ζ . The weak metric δ induces on v a weak norm $\|v\|_\delta$ expressed by*

$$\|v\|_\delta = \frac{|v| + \operatorname{Im}(v)}{2\operatorname{Im}(\zeta)}.$$

The meaning of the expression “induced weak norm” will be clear from the computation done in the proof, and it acquires its complete significance in Corollary 3.1 that follows.

Proof. Set $\zeta' = \zeta + tv$ ($t > 0$). Then,

$$\begin{aligned} |\zeta - \zeta'| &= |\zeta - (\zeta + tv)| = t|v| \\ |\zeta - \bar{\zeta}'| &= |\zeta - \bar{\zeta} - t\bar{v}| = |\zeta - \bar{\zeta}| \left| 1 - t \frac{\bar{v}}{\zeta - \bar{\zeta}} \right| \\ &= |\zeta - \bar{\zeta}| \left(1 - t \operatorname{Re} \left(\frac{\bar{v}}{\zeta - \bar{\zeta}} \right) + o(t) \right) \\ &= |\zeta - \bar{\zeta}| \left(1 - t \operatorname{Re} \left(\frac{\bar{v}}{2i \operatorname{Im}(\zeta)} \right) + o(t) \right) \\ &= 2 \operatorname{Im}(\zeta) \left(1 + t \frac{\operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)} + o(t) \right). \end{aligned}$$

Hence, we have

$$\begin{aligned} \delta(\zeta, \zeta + tv) &= \log \frac{1}{2 \operatorname{Im}(\zeta)} \left(t|v| + 2 \operatorname{Im}(\zeta) \left(1 + t \frac{\operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)} + o(t) \right) \right) \\ &= \log \left(1 + t \frac{|v| + \operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)} + o(t) \right) \\ &= t \frac{|v| + \operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)} + o(t). \end{aligned}$$

Thus, we obtain

$$\lim_{t \searrow 0} \frac{\delta(\zeta, \zeta + tv)}{t} = \frac{|v| + \operatorname{Im}(v)}{2 \operatorname{Im}(\zeta)}.$$

□

Notice that as an invariant expression, the weak metric in Proposition 3.1 is presented as

$$(3.1) \quad \|\cdot\|_\delta = \frac{\sqrt{dx^2 + dy^2} + dy}{2y} = ds_{hyp} + \frac{1}{2} d \log y$$

on $T_\zeta \mathbb{H}$ and $\zeta = x + iy \in \mathbb{H}$, where ds_{hyp} is the length element of the hyperbolic metric on \mathbb{H} of constant curvature -4 .

Combining this with Theorem 2.2, we have the following corollary.

Corollary 3.1. *The weak metric space $(\mathcal{T}(T^2), \delta)$ is a weak Finsler metric space with the corresponding weak norm $\|\cdot\|_\delta$ given in Proposition 3.1.*

Proof. First we show that

$$(3.2) \quad \int_{\theta_1}^{\theta_2} \|\dot{\gamma}(\theta)\|_\delta d\theta \geq \delta(\zeta_1, \zeta_2)$$

for $\zeta_1, \zeta_2 \in \mathbb{H}$ and any piecewise C^1 -path $\gamma: [\theta_1, \theta_2] \rightarrow \mathbb{H}$ connecting ζ_1 to ζ_2 . Indeed, from the invariant expression (3.1), the integration in the left-hand side of (3.2) is at least equal to the hyperbolic distance (of constant curvature -4) between ζ_1 and ζ_2 minus the Busemann function at ζ_1, ζ_2 (see (4.1)), which is the right-hand side of (3.1) (cf. §4). This observation

also implies that the integration in the left-hand side of (3.2) is minimized only when it is done along the hyperbolic geodesic from ζ_1 to ζ_2 .

We now show that the distance between any two points ζ_1 and ζ_2 is given by integrating the weak norm $\|\cdot\|_\delta$ along a parametrised geodesic joining these two points. Let $\zeta_1, \zeta_2 \in \mathbb{H}$. As in the proof of Proposition 2.1, we may assume that $\zeta_1 = ib$ and $\zeta_2 = \alpha + i\beta$ with $\alpha > 0$. Let $A = \alpha^2 + \beta^2 - b^2$, $B = -2\alpha$ and $R = \sqrt{A^2 + b^2 B^2}/|B|$. Define $\theta_1, \theta_2 \in (0, 2\pi)$ by $e^{i\theta_1} = (\zeta_1 - (-A/B))/R$ and $e^{i\theta_2} = (\zeta_2 - (-A/B))/R$. The geodesic from ζ_1 to ζ_2 is parametrized as $\gamma(\theta) = (-A/B) + Re^{i\theta}$ ($\theta_1 \leq \theta \leq \theta_2$). Hence

$$\begin{aligned} \int_{\theta_2}^{\theta_1} \|\dot{\gamma}(\theta)\|_\delta d\theta &= \int_{\theta_2}^{\theta_1} \frac{|\dot{\gamma}(\theta)| + \text{Im}(\dot{\gamma}(\theta))}{2\text{Im}(\gamma(\theta))} d\theta = \int_{\pi-\theta_1}^{\theta_2} \frac{R + R \cos \theta}{2R \sin \theta} d\theta \\ &= \frac{1}{2} \int_{\theta_2}^{\theta_1} \frac{\sin \theta}{1 - \cos \theta} d\theta = \frac{1}{2} \log(1 - \cos \theta_1) - \frac{1}{2} \log(1 - \cos \theta_2) \\ &= \frac{1}{2} \log \left(1 + \frac{A}{\sqrt{A^2 + b^2 B^2}} \right) - \log \left(1 - \frac{2\alpha^2 - A}{\sqrt{A^2 + b^2 B^2}} \right). \end{aligned}$$

An easy calculation shows that this expression is equal to

$$\log \frac{\sqrt{\alpha^2 + (\beta - b)^2} + \sqrt{\alpha^2 + (\beta + b)^2}}{2\beta}$$

that is, to $\delta(\zeta_1, \zeta_2)$. This gives what we wanted. \square

Next we describe the unit sphere in the tangent space with respect to the weak norm $\|\cdot\|_\delta$.

Proposition 3.2. *The unit circle of the tangent space at $\zeta \in \mathbb{H}$ with respect to $\|\cdot\|_\delta$ is expressed as a parabola with focus at the origin and vertex at $i\text{Im}(\zeta)$.*

Proof. Let $\zeta = \alpha + i\beta$. When $v = v_1 + iv_2 \in \mathbb{C} \cong T_\zeta \mathbb{H}$ (as real vector spaces) lies on the unit circle of the tangent space at ζ , we have

$$1 = \frac{|v| + \text{Im}(v)}{2\text{Im}(\zeta)} = \frac{\sqrt{v_1^2 + v_2^2} + v_2}{2\beta},$$

which is equivalent to

$$v_1^2 + v_2^2 = (2\beta - v_2)^2 = 4\beta^2 - 4\beta v_2 + v_2^2.$$

This means that the infinitesimal unit circle at $\zeta = \alpha + i\beta \in \mathbb{H}$ is the parabola

$$v_2 = -\frac{v_1^2}{4\beta} + \beta,$$

which implies the desired result. \square

Note that the fact that the unit ball of the weak Finsler norm has an infinite direction expresses the fact that the distance function is degenerate in this direction (that is, we have, in this direction, $\delta(x, y) = 0$ for $x \neq y$).

4. DEFORMING δ TO THE TEICHMÜLLER METRIC

In this section, we consider a family of weak Finsler metrics which interpolate between δ and the hyperbolic distance (which, as is well known, coincides with the Teichmüller distance). We then describe the infinitesimal unit ball of each of these metrics.

Consider the family of weak metrics δ_p ($0 \leq p \leq 1$) defined by

$$\begin{aligned} \delta_p(\zeta_1, \zeta_2) &= \log \sup_{x \in \mathbb{R}} \frac{(\operatorname{Im}(\zeta_1))^{p/2}}{(\operatorname{Im}(\zeta_2))^{p/2}} \left| \frac{\zeta_2 - x}{\zeta_1 - x} \right| \\ &= \frac{p}{2} \log \frac{\operatorname{Im}(\zeta_1)}{\operatorname{Im}(\zeta_2)} + \log \sup_{x \in \mathbb{R}} \left| \frac{\zeta_2 - x}{\zeta_1 - x} \right|. \end{aligned}$$

Note that the function

$$(4.1) \quad \mathbb{H} \times \mathbb{H} \ni (\zeta_1, \zeta_2) \mapsto \frac{1}{2} \log \frac{\operatorname{Im}(\zeta_1)}{\operatorname{Im}(\zeta_2)} = -\frac{1}{2} \int_{\zeta_1}^{\zeta_2} d \log y$$

is the Buseman function (associated to $x = \infty \in \partial\mathbb{H}$) of the hyperbolic metric of curvature -4 , which is the Teichmüller distance. Using the same proof as in Theorem 2.2, the hyperbolic geodesic from ζ_1 to ζ_2 is the geodesic of the metric δ_p . The arithmetic symmetrisation of δ_p is the hyperbolic metric of curvature -4 , like for $\delta = \delta_0$ (cf. (1) in §1).

As we did in Proposition 3.1, we can calculate the infinitesimal form of the metric δ_p :

Let $\zeta \in \mathbb{H}$. For $v \in T_\zeta \mathbb{H} \cong \mathbb{C}$,

$$\log \frac{\operatorname{Im}(\zeta)}{\operatorname{Im}(\zeta + tv)} = -\log \left(1 + t \frac{\operatorname{Im}(v)}{\operatorname{Im}(\zeta)} \right) = -t \frac{\operatorname{Im}(v)}{\operatorname{Im}(\zeta)} + o(t)$$

as $t \searrow +0$. We obtain

$$\begin{aligned} \|v\|_{\delta_p} &:= \frac{\delta_p(\zeta, \zeta + tv)}{t} = -\frac{p}{2} \frac{\operatorname{Im}(v)}{\operatorname{Im}(\zeta)} + \frac{|v| + \operatorname{Im}(v)}{2\operatorname{Im}(\zeta)} \\ &= \frac{|v| + (1-p)\operatorname{Im}(v)}{2\operatorname{Im}(\zeta)} \\ &= \|v\|_\delta - p \frac{\operatorname{Im}(v)}{2\operatorname{Im}(\zeta)}. \end{aligned}$$

Notice that $\|v\|_{\delta_p} > 0$ when $v \neq 0$ and $p > 0$. The infinitesimal unit circle with respect to the weak norm $\|\cdot\|_{\delta_p}$ in the tangent space $T_\zeta \mathbb{H}$ is the ellipse with foci 0 and $4\operatorname{Im}(\zeta)(1-p)/(p(2-p))$. As an invariant expression, the weak metric $\|\cdot\|_{\delta_p}$ is presented as

$$(4.2) \quad \|\cdot\|_\delta = \frac{\sqrt{dx^2 + dy^2} + (1-p)dy}{2y} = ds_{hyp} + \frac{1-p}{2} d \log y$$

A discussion similar to that of the proof of Corollary 3.1 and (4.2) shows that the weak metric space $(\mathcal{T}(T^2), \delta_p)$ is a weak Finsler metric with associated

weak norm $\|\cdot\|_{\delta_p}$ and that the hyperbolic geodesic from ζ_1 to $\zeta_2 \in \mathbb{H}$ is a unique geodesic for δ_p .

It follows that $\{\delta_p\}_{0 \leq p \leq 1}$ is a continuous family of weak Finsler metrics giving a deformation from $\delta = \delta_0$ to the hyperbolic metric δ_1 (which is Teichmüller metric κ). Notice that

$$\frac{|\zeta - x|^2}{\text{Im}(\zeta)}$$

coincides with the extremal length of the measured lamination corresponding to $x \in \mathbb{R}$ up to a constant factor (depending only on x). Hence, δ_1 coincides with Kerckhoff's formula for the Teichmüller distance [2] adapted to the case of the torus.

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