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TOPOLOGY OPTIMIZATION FOR STEADY-STATE ANISOTHERMAL FLOW TARGETING SOLIDS WITH PIECEWISE CONSTANT THERMAL DIFFUSIVITY.

ALEXANDRE VIEIRA*, ALAIN BASTIDE*, AND PIERRE-HENRI COCQUET*†

Abstract. A Topology Optimization (TO) problem is a PDE-constrained optimization problem that aims at finding the shape of a solid inside a fluid which minimizes a given cost function. The solid is located with a penalization term added in the constraints equations that vanishes in fluid regions and becomes large in solid regions. This paper addresses a TO problem for anisothermal flows modelled by the steady-state incompressible Navier-Stokes system coupled to an energy equation, with mixed boundary conditions, under the Boussinesq assumption. We first prove the existence and uniqueness of solution to this problem as well as the convergence of finite element discretization. Next, we show that our TO problem has at least one optimal solution for cost functions that satisfy general assumptions. The convergence of discrete optimum toward the continuous one is then provided as well as necessary first order optimality conditions. Eventually, all these results let us design a numerical algorithm to solve a TO problem that looks for solid with piecewise constant thermal diffusivities. A physical problem solved numerically with this method concludes this paper.

Key words. Topology optimization, Navier-Stokes equations, anisothermal flows, Multi-materials.

AMS subject classifications. 76D55, 49Q10, 93C20, 65K99

1. Introduction. Finding the shape of a solid located inside a fluid that either minimizes or maximizes a given physical effect has several applications in engineering and the applied science (see e.g. [39, 40, 43, 44] for several different applications).

There exists various mathematical methods to deal with such problems that fall into the class of PDE-constrained optimization. The topological asymptotic expansion [4, 16, 41] consider the solid as a hole or an inhomogeneity with characteristic size ε . The so-called *topological gradient* is then defined as the first order term in the asymptotic expansion of the cost function as $\varepsilon \rightarrow 0$. The shape optimization method [25, 39, 40] computes the gradient of the cost function with respect to perturbation of the boundary of the solid also referred as *shape derivative*. We emphasize that the two aforementioned methods exactly locate the solid. Nevertheless, once the gradient of the cost function is computed, the geometry of the computational domain changes and thus these two methods usually need some specific techniques to follow the evolution of the mesh during the numerical solving.

In this paper, we choose to locate the solid thanks to a penalization term added in the Navier-Stokes equation. The latter vanishes in the fluid zone and goes to infinity in a solid region of the computational domain (see [5] for the mathematical justification of this method). Using such model as constraint in the shape optimization problem is referred as a *topology optimization* (TO) problem. It is worth noting that the major drawback of this approach is that the solid is only located when the velocity of the fluid is smaller than a given tolerance. Nevertheless, the numerical solving of a TO problem do not need specific remeshing techniques.

We refer to the review papers [2, 20] for many references that deals with numerical res-

*Physique et Ingénierie Mathématique pour l'Énergie et l'Environnement (PIMENT), Université de la Réunion, 2 rue Joseph Wetzell, 97490 Sainte-Clotilde, France. (Alexandre.Vieira@univ-reunion.fr, Alain.Bastide@univ-reunion.fr)

†Laboratoire des Sciences de l'Ingénieur Appliquées à la Mécanique et au Génie Electrique (SIAME), E2S-UPPA, Université de Pau et des Pays de l'Adour, 64000 Pau, France (Pierre-Henri.Cocquet@univ-reunion.fr, Pierre-Henri.Cocquet@univ-pau.fr)

olution of TO problems applied on several different physical settings. More precisely, we refer to [1, 3, 15, 45, 46] for numerical and physical studies on TO involving heat transfers in fluid flows (involving for instance natural, forced or mixed convection) since this is going to be the physical setting of interest of this paper.

Regarding the mathematical analysis of TO problems, we first note that they amount to find some coefficients in the PDE that minimize a given criterion and can thus be seen as parametric optimization problems [7]. They also share similarities with problems that wish to recover some unknown parameters in the PDE from measurements [17, 36, 42]. However, all the aforementioned references deal with scalar coercive problem (see also [18] that deals with Helmholtz equation which is elliptic but not coercive) and, even if they give some insight on how to mathematically tackle a TO problem for fluid flows, can not be used to study the problem of interest in this paper. We also wish to refer to [31] for several results on a general constrained optimization problem. It is however worth noting that our TO problem does not fit in the framework of [31].

To the best of our knowledge, there are only few works dealing with mathematical study of TO problems for fluid flow. We can still refer to [13] where a TO problem for incompressible Stokes equations have been studied. In [24], some existence results as well as some limitations of the shape optimization using a penalization technique are given for the incompressible Stokes equation. We finally refer to [28] where a shape optimization problem combining perimeter regularization, penalization technique and phase-field approach have been introduced.

Considering the previous literature review, there is, to the best of our knowledge, no mathematical study (existence, approximation and convergence of optimal solution) of a TO problem involving heat transfers in anisothermal flows. In addition, in the TO mathematical literature [13, 24, 28], the boundary conditions considered are homogeneous Dirichlet on the whole boundary. This simplify the mathematical analysis of the incompressible Navier-Stokes equation [12, 19, 27, 48]. The first objectives of this paper are then to study a general TO problem involving, as constraint, the Navier-Stokes equation coupled to the heat equation with mixed (homogeneous/inhomogeneous Dirichlet and traction) boundary conditions since the latter are closer to those used in physical situations (see e.g. [1, 45, 46]).

Another topic of interest of this paper is to look for optimized solid with thermal conductivity that are not only constant as it is the case in most TO studies. There already exists some methods to get optimized physical parameter that are piecewise constant [37, 38, 49] but the latter introduce an optimization parameter per constant which may thus lead to large optimization problems. This constraint have been lifted in [50] where an ordered SIMP (Solid Isotropic Material with Penalization) interpolation function for the elastic modulus is introduced. Although this technique could be applied to get optimized piece-wise constant thermal diffusivity, its definition is actually based on *gluing* together power curves which results in a piecewise-defined function with some points where it is not derivable (see [50, Figure 2]). Another goal of this paper is then to introduce a smooth globally defined interpolation function that yields optimized piecewise constant thermal parameters.

Plan of the paper. The paper is now organized as follow: first we end this introduction by clearly defining the optimization problem under study. Next, we study the existence and uniqueness of weak solution to the steady-state incompressible Navier-Stokes equation coupled to a heat equation modelling heat transfer in anisothermal flow. We prove next the convergence of a finite element approximation of the latter

where a discrete optimization parameter is used. After giving some general conditions on the cost function to obtain the existence of, at least, one optimal solution, we then prove the convergence of the discrete optimum toward its continuous counterpart. We introduce next the multi-material interpolation function that will be used to obtain optimized solid with piecewise constant thermal diffusivity. We end this paper with numerical simulations to show the interest of our approach.

Definition of the topology optimization problem. We present now the main ideas leading to the TO problem considered in this paper. First, since the velocity of the fluid vanishes inside the solid, one can use a penalization model as those introduced in [5] in order to write the fluid-solid model as a single system valid on the whole computational domain whose solution converges toward those of the fluid-solid interface problem. Such model involves an indicator function, that is a binary variable, to locate the solid and is thus not suitable for optimization (see e.g. [24]). To bypass this difficulty, some smooth regularization of the indicator function is introduced hence defining another model whose advantage is that the location of the solid now depends on a continuous variable. We emphasize that the approach described above has been used in several works [2, 20] and we describe it below for the case of anisothermal flows.

We assume the fluid occupies a region $\Omega_f \subset \Omega$ and that a solid is embedded in Ω_s such that $\Omega = \Omega_f \cup \Omega_s$. The Boussinesq approximation (see e.g. [46] for the steady case) of the Navier-Stokes equation coupled to convective heat transfer is given by

$$(1.1) \quad \begin{aligned} \nabla \cdot u &= 0 && \text{in } \Omega_f, \\ (u \cdot \nabla) u + \nabla p - \text{Re}^{-1} \Delta u - \text{Ri} \theta e_y &= 0 && \text{in } \Omega_f, \\ \nabla \cdot (u \theta) - \nabla \cdot (\text{Re}^{-1} \text{Pr}^{-1} k(x) \nabla \theta) &= 0 && \text{in } \Omega, \\ u &= 0 && \text{in } \Omega_s, \end{aligned}$$

where Re is the Reynolds number, Ri the Richardson number, Pr is the Prandtl number (see [46, Section 2] for the definition of these constants) and $k(x)$ is the thermal diffusivity which is 1 in the fluid and k_s/k_f in the solid.

As explained above, the solid can be located thanks to a penalization model (see e.g. [5]) for (1.1) that add a term of the form $\eta^{-1} \mathbf{1}_{\Omega_s} u$ in the momentum conservation equation. The latter formally enforces, as $\eta \rightarrow 0$, that $u|_{\Omega_s} = 0$ as well as a no-slip boundary condition on $\partial\Omega_s$. For Problem (1.1), this reads

$$(1.2) \quad \begin{aligned} \nabla \cdot u &= 0 && \text{in } \Omega_f, \\ (u \cdot \nabla) u + \nabla p - \text{Re}^{-1} \Delta u - \text{Ri} \theta \vec{v} + \frac{1}{\eta} \mathbf{1}_{\Omega_s}(x) u &= 0 && \text{in } \Omega, \\ \nabla \cdot (u \theta) - \nabla \cdot (\text{Re}^{-1} \text{Pr}^{-1} k(x) \nabla \theta) &= 0 && \text{in } \Omega, \end{aligned}$$

where $k(x) = \frac{1}{k_f} (k_f \mathbf{1}_{\Omega_s}(x) + k_s(1 - \mathbf{1}_{\Omega_s}(x)))$.

To get the model studied in this paper that acts as constraint in the TO problem, we now consider some smooth regularization $h_\tau(\alpha(x))$ of the indicator function that satisfy $h_\tau(\alpha) \rightarrow 0$ for $\alpha < \alpha_0$ and $h_\tau(\alpha) \rightarrow \alpha_{\max}$ for $\alpha \geq \alpha_0$ and a large enough α_{\max} where the convergence is pointwise. We emphasize that the function $\alpha : x \in \Omega \mapsto \mathbb{R}^+$ is going to be the optimization parameter. In addition, the fluid/solid zones can now be obtained as $\Omega_s := \{x \in \Omega \mid \alpha(x) < \alpha_0\}$, $\Omega_f := \{x \in \Omega \mid \alpha(x) \geq \alpha_0\}$. The dimensionless form of the energy and penalized incompressible Navier-Stokes

equations under Boussinesq assumption are finally written as follows

$$\begin{aligned}
(1.3a) \quad & \nabla \cdot u = 0 && \text{in } \Omega, \\
(1.3b) \quad & (u \cdot \nabla) u = -\nabla p + A\Delta u - h_\tau(\alpha)u + B\theta e_y && \text{in } \Omega, \\
(1.3c) \quad & \nabla \cdot (u\theta) = \nabla \cdot (Ck_\tau(\alpha)\nabla\theta) && \text{in } \Omega,
\end{aligned}$$

where A, B, C are physical constants that are introduced to lighten the overall expressions and reads $A = \frac{1}{\text{Re}}$, $B = \text{Ri}$, $C = \frac{1}{\text{RePr}}$. Regarding the boundary conditions we are going to work with mixed boundary conditions and first assume $\partial\Omega = \Gamma$ can be decomposed as $\Gamma = \Gamma_w \cup \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ with $|\Gamma_w| > 0$, $|\Gamma_{\text{in}}| > 0$, $|\Gamma_{\text{out}}| > 0$ and $\overline{\Gamma_{\text{in}}} \cap \overline{\Gamma_{\text{out}}} = \emptyset$. where Γ_w are the walls, Γ_{in} the inlet/entrance and Γ_{out} is the exit of the computational domain. These boundary conditions can be found in [45, 46] and finally read

$$\begin{aligned}
(1.4a) \quad & u = u_{\text{in}}, & \theta = 0, & \text{in } \Gamma_{\text{in}}, \\
(1.4b) \quad & u = 0, & \partial_n \theta = \phi, & \text{in } \Gamma_w, \\
(1.4c) \quad & A\partial_n u - np = 0, & \partial_n \theta = 0, & \text{in } \Gamma_{\text{out}},
\end{aligned}$$

where ϕ is a given heat flux.

Before deriving a weak formulation of (1.3)-(1.4), we introduce the spaces $X_1^u = \{v \in H^1(\Omega)^d \mid v|_{\Gamma_w} = 0\}$, $X^u = \{v \in H^1(\Omega)^d \mid v|_{(\Gamma_w \cup \Gamma_{\text{in}})} = 0\}$, $X^\theta = \{v \in H^1(\Omega) \mid v|_{\Gamma_{\text{in}}} = 0\}$. A variational formulation of (1.3)-(1.4) then reads:

Find $(u, \theta, p) \in X_1^u \times X^\theta \times L^2(\Omega)$ such that:

$$(1.5) \quad \begin{cases} u|_{\Gamma_{\text{in}}} = u_{\text{in}}, \\ a(\alpha; u, v_1) + b(v_1, p) + c(u, u, v_1) + f(\theta, v_1) = 0, & \forall v_1 \in X^u \\ \tilde{a}(\alpha; \theta, v_2) + \tilde{c}(\theta, u, v_2) = \int_{\Gamma_w} Ck_\tau(\alpha)\phi v_2, & \forall v_2 \in X^\theta \\ b(u, q) = 0, & \forall q \in L^2(\Omega). \end{cases}$$

where

$$\begin{aligned}
a(\alpha; u, v) &= \int_{\Omega} [A\nabla u : \nabla v + h_\tau(\alpha)u \cdot v], \\
b(u, q) &= - \int_{\Omega} q(\nabla \cdot u), \quad c(u, v, w) = \int_{\Omega} (u \cdot \nabla)v \cdot w, \quad f(\theta, v) = - \int_{\Omega} B\theta e_y \cdot v, \\
\tilde{a}(\alpha; \theta, v) &= \int_{\Omega} Ck_\tau(\alpha)\nabla\theta \cdot \nabla v, \quad \tilde{c}(\theta, u, v) = \int_{\Omega} (\nabla\theta \cdot u)v.
\end{aligned}$$

Also, we endow the spaces X^u and X^θ with the following norms: $\|\cdot\|_{X^u}^2 = a(\alpha, \cdot, \cdot)$, $\|\cdot\|_{X^\theta} = \|\cdot\|_{H^1(\Omega)}$. Remark that:

$$(1.6) \quad \|u\|_{L^4(\Omega)} \leq C(\Omega) \|u\|_{H^1(\Omega)} \leq C(\Omega) \|\nabla u\|_{L^2(\Omega)} \leq C(\Omega) \|u\|_{X^u},$$

thanks to the embedding $H^1(\Omega) \subset L^4(\Omega)$ and the Poincaré inequality.

Eventually, the following general TO problem is studied in this paper

$$\begin{aligned}
& \min \mathcal{J}(\alpha, u, \theta, p) \\
& \text{s.t.} \quad \begin{cases} (u, \theta, p) \text{ solution of (1.5) parametrized by } \alpha, \\ \alpha \in \mathcal{U}_{ad}, \end{cases}
\end{aligned}$$

where \mathcal{J} is a given cost function.

2. Study of the PDE system and its discretization. This section will focus on the study of the underlying PDE system in the TO problem, namely on (1.5). The study will first be focused on proving the existence of a unique solution to (1.5) using a fixed point approach. We will afterwards analyze the finite-element discretization of (1.5), proving once again the existence of a unique solution and more importantly, the convergence of the discretized solution toward the continuous one.

2.1. Existence and uniqueness. We begin our analysis with some specification on our model.

ASSUMPTION 2.1. \bullet α is bounded with bounded variation on Ω , has non-negative values and a prescribed bound on its total variation, i.e. for some $\alpha_{max} > 0$ and $\kappa > 0$,

$$\alpha \in \mathcal{U}_{ad} = \{\alpha \in BV(\Omega) : 0 \leq \alpha(x) \leq \alpha_{max} \text{ a.e. on } \Omega, |D\alpha|(\Omega) \leq \kappa\}.$$

Remark that \mathcal{U}_{ad} is a convex, closed, weak-* closed subset of $BV(\Omega)$.

- \bullet We suppose that there exists $k_{min} > 0$ such that, for all $x \in \Omega$, $k_{min} \leq k_\tau(\alpha(x))$ and $0 \leq h_\tau(\alpha(x))$.
- \bullet h_τ and k_τ are bounded and continuous on $[0, \alpha_{max}]$.
- \bullet There exists $V \in H^1(\Omega)^d$ such that $\text{div}(V) = 0$, $V|_{\Gamma_w} = 0$ and $V|_{\Gamma_{in}} = u_{in}$.

Concerning the last assumption, one can prove that for all $u_{in} \in H_{00}^{1/2}(\Gamma_{in})^d$ (this is the set of $\psi \in H^{1/2}(\Gamma_{in})$ such that $\tilde{\psi} \in H^{1/2}(\partial\Omega)$ where $\tilde{\psi}$ denotes the extension by zero), such V exists and verify $\|V\|_{H^1(\Omega)^d} \leq C \|u_{in}\|_{H_{00}^{1/2}(\Gamma_{in})^d}$. We use this assumption in order to deal with the inhomogeneous Dirichlet boundary condition on Γ_{in} . One can therefore write $u = w + V$, where $w \in X^u$ satisfies:

$$(2.1) \quad \begin{aligned} & \text{Find } (w, \theta, p) \in X^u \times X^\theta \times L^2(\Omega) \text{ such that:} \\ & \begin{cases} a(\alpha; w, v_1) + b(v_1, p) = \langle G(\alpha; w) + F(\theta), v_1 \rangle, & \forall v_1 \in X^u, \\ \tilde{a}(\alpha; \theta, v_2) + \tilde{c}(\theta, w + V, v_2) = \langle \tilde{G}(\alpha), v_2 \rangle, & \forall v_2 \in X^\theta, \\ b(w, q) = 0, & \forall q \in L^2(\Omega), \end{cases} \end{aligned}$$

where: $\langle G(\alpha; w) + F(\theta), v \rangle = -c(w + V, w + V, v) - f(\theta, v) - a(\alpha, V, v)$, $\langle \tilde{G}(\alpha), v \rangle = \int_{\Gamma_w} k_\tau(\alpha) \phi v$. Note that (2.1) is a fixed point equation equivalent to (1.5). In order to study the well-posedness of (2.1), we consider first the following linear problem:

$$(2.2) \quad \begin{aligned} & \text{Find } (w, \theta, p) \in X^u \times X^\theta \times L^2(\Omega) \text{ such that:} \\ & \begin{cases} a(\alpha; w, v_1) + b(v_1, p) = \langle G(\alpha; \tilde{w}) + F(\theta), v_1 \rangle, & \forall v_1 \in X^u, \\ \tilde{a}(\alpha; \theta, v_2) + \tilde{c}(\theta, \tilde{w} + V, v_2) = \langle \tilde{G}(\alpha), v_2 \rangle, & \forall v_2 \in X^\theta, \\ b(w, q) = 0, & \forall q \in L^2(\Omega). \end{cases} \end{aligned}$$

for some fixed $\tilde{w} \in X^u$. We start our analysis with estimates on the solution of (2.2).

PROPOSITION 2.2. *Problem (2.2) has a unique solution $(w, \theta, p) \in X^u \times X^\theta \times L^2(\Omega)$ that satisfies*

$$\|\theta\|_{X^\theta} \leq \frac{\tilde{g}_0(\Omega)(1 + \varepsilon)}{\varepsilon C k_{min}} \left\| \tilde{G}(\alpha) \right\| \leq \frac{\tilde{g}_0(\Omega)(1 + \varepsilon)}{\varepsilon C k_{min}} \|k_\tau\|_\infty \|\phi\|_{L^2(\Gamma)},$$

$$\|w\|_{X^u} \leq \|G(\alpha; \tilde{w}) + F(\theta)\|_{(X^u)'},$$

$$\|p\|_{L^2(\Omega)} \leq \frac{2}{\beta} \|G(\alpha; \tilde{w}) + F(\theta)\|_{(X^u)'},$$

where $\|a\|$ (resp. a_0) is the norm (resp. coercivity constant) of $a(\alpha, \cdot, \cdot)$.

Proof. One can easily prove that the application $\tilde{b} : (\theta, v_2) \mapsto \tilde{a}(\alpha; \theta, v_2) + \tilde{c}(\theta, \tilde{w} + V, v_2)$ is bilinear and complies with the estimates:

$$|\tilde{b}(\theta, v_2)| \leq C(\Omega) \max\{\|\tilde{w} + V\|_{X^u}, C\|k_\tau\|_\infty\} \|\theta\|_{X^\theta} \|v_2\|_{X^\theta},$$

$$\tilde{b}(\theta, \theta) \geq C(\Omega) (Ck_{\min} - \|\tilde{w} + V\|_{X^u}) \|\theta\|_{X^\theta}^2.$$

Therefore, if one chooses \tilde{w} such that $\|\tilde{w} + V\|_{X^u} \leq \frac{Ck_{\min}}{1+\varepsilon}$ for some $\varepsilon > 0$, one proves that \tilde{b} is continuous and coercive with constant $C(\Omega)Ck_{\min} \frac{\varepsilon}{1+\varepsilon}$. Therefore, thanks to the Lax-Milgram theorem, one proves that there exists a unique function θ solving the second equation of (2.2) and respecting the following estimate:

$$\|\theta\|_{X^\theta} \leq \frac{1+\varepsilon}{\varepsilon C(\Omega)Ck_{\min}} \left\| \tilde{G}(\alpha) \right\|_{(X^\theta)'}, \leq \frac{\tilde{g}_0(\Omega)(1+\varepsilon)}{\varepsilon Ck_{\min}} \|k_\tau\|_\infty \|\phi\|_{L^2(\Gamma)},$$

where $\tilde{g}_0(\Omega)$ is a positive constant that only depends on Ω .

We are left with an equation satisfied by (w, p) , which is a standard linear saddle-point problem. Since the bilinear form $a(\alpha, \cdot, \cdot)$ is continuous and coercive (with respect to the norm defined on X^u), it only remains to prove that the bilinear form b complies with an inf-sup condition. Adapting the result of [10, p.6, Eq. (2.13)], one proves that there exists a constant $\beta > 0$ such that

$$\inf_{q \in L^2(\Omega) \setminus \{0\}} \sup_{v \in X^u \setminus \{0\}} \frac{b(v, q)}{\|v\|_{X^u} \|q\|_{L^2(\Omega)}} \geq \beta.$$

We eventually conclude using [26, II.1, Proposition 1.3]. \square

We also need some upper bound and Lipschitz condition on $G(\alpha; \cdot)$ in order to state a fixed point result. This is done in the following Lemma.

LEMMA 2.3. *The nonlinear function $G(\alpha; \cdot) + F(\cdot) : X^u \times X^\theta \rightarrow (X^u)'$ satisfies the following estimates:*

$$\|G(\alpha; \tilde{w}) + F(\theta)\|_{(X^u)'} \leq C \left(\|u_{in}\|_{H_{00}^{1/2}(\Gamma_{in})^d}^2 + \|u_{in}\|_{H_{00}^{1/2}(\Gamma_{in})^d} + B \|\theta\|_{X^\theta} + \|\tilde{w}\|_{X^u}^2 \right),$$

$$\begin{aligned} & \|(G(\alpha; w_1) + F(\theta_1)) - (G(\alpha; w_2) + F(\theta_2))\|_{(X^u)'} \leq \\ & C_L \left((\|w_1\|_{X^u} + \|w_2\|_{X^u} + \|u_{in}\|_{H_{00}^{1/2}(\Gamma_{in})^d}) \|w_1 - w_2\|_{X^u} + B \|\theta_1 - \theta_2\|_{X^\theta} \right), \end{aligned}$$

where C, C_L are positive constants that only depend on Ω .

Proof. From the inequality (1.6), together with the Hölder inequality, one proves: $|c(u, v, w)| \leq C_{NL} \|u\|_{X^u} \|v\|_{X^u} \|w\|_{X^u}$ where $C_{NL} > 0$ only depends on Ω . Using now

the bound on V , we get

$$\begin{aligned}
 |\langle G(\alpha; w) + F(\theta), v \rangle_{(X^u)', X^u}| &\leq C_{NL} \|v\|_{X^u} \|w + V\|_{X^u}^2 + B \|\theta\|_{X^\theta} \|v\|_{X^u} + \\
 &\quad \|V\|_{X^u} \|v\|_{X^u} \\
 &\leq \|v\|_{X^u} \left(C_{NL} (2M^2 \|u_{in}\|_{H_{00}^{1/2}(\Gamma_{in})^d}^2 + \|w\|_{X^u}^2) \right. \\
 &\quad \left. + B \|\theta\|_{X^\theta} + M \|u_{in}\|_{H_{00}^{1/2}(\Gamma_{in})^d} \right) \\
 &\leq \|v\|_{X^u} C(\Omega) \left(\|u_{in}\|_{H_{00}^{1/2}(\Gamma_{in})^d}^2 + \|w\|_{X^u}^2 + \right. \\
 &\quad \left. B \frac{\tilde{g}_0(\Omega)(1+\varepsilon)}{\varepsilon C k_{\min}} \|\theta\|_{X^\theta} + \|u_{in}\|_{H_{00}^{1/2}(\Gamma_{in})^d} \right)
 \end{aligned}$$

Taking the supremum over $v \in X^u$ with $\|v\|_{X^u} \leq 1$ yields the first estimates. Concerning the second estimate, note that:

$$\begin{aligned}
 &\langle G(\alpha; w_1) + F(\theta_1) \rangle - \langle G(\alpha; w_2) + F(\theta_2), v \rangle_{(X^u)', X^u} \\
 &= c(w_2 + V, w_2 + V, v) - c(w_1 + V, w_1 + V, v) + f(\theta_2, v) - f(\theta_1, v) \\
 &= \int_{\Omega} (((w_1 + V) \cdot \nabla)(w_1 + V) - ((w_2 + V) \cdot \nabla)(w_2 + V)) \cdot v \\
 &\quad - \int_{\Omega} B(\theta_2 - \theta_1) e_y \cdot v
 \end{aligned}$$

For two vector fields a and b , one has the following bound: $|a \cdot \nabla a - b \cdot \nabla b| \leq |a - b| |\nabla a| + |b| |\nabla(a - b)|$. Therefore there exists $C_L > 0$ such that:

$$\begin{aligned}
 &|\langle G(\alpha; w_1) + F(\theta_1) \rangle - \langle G(\alpha; w_2) + F(\theta_2), v \rangle_{(X^u)', X^u}| \\
 &\leq C_L \|v\|_{X^u} \left(\|w_1\|_{X^u} + \|w_2\|_{X^u} + \|u_{in}\|_{H_{00}^{1/2}(\Gamma_{in})^d} \right) \|w_1 - w_2\|_{X^u} \\
 &\quad + B \|\theta_1 - \theta_2\|_{X^\theta}
 \end{aligned}$$

Taking once again the supremum over $v \in X^u$ with $\|v\|_{X^u} \leq 1$ finishes the proof. \square

Let us now move back to the non-linear problem (2.1). As stated before, we use the properties we have proved on (2.2) in order to state some fixed point result to prove the existence of solution to (2.1).

THEOREM 2.4. *Given any $\alpha \in \mathcal{U}_{ad}$ and given B , $\|u_{in}\|_{H_{00}^{1/2}(\Gamma_{in})}$ and $\|\phi\|_{L^2(\Gamma)}$ small enough (see (2.3)-(2.5)), there exists a unique solution $(w, \theta, p) \in X^u \times X^\theta \times L^2(\Omega)$ to the variational problem (2.1).*

Proof. Denote by $\mathcal{S}(\tilde{w}) : (X^u)' \times (X^\theta)' \rightarrow X^u \times X^\theta \times L^2(\Omega)$, $\mathcal{S}(\tilde{w}) : (F, G) \mapsto (w, p, \theta)$, the (linear) solution map of the linear problem:

$$\begin{aligned}
 &\text{Find } (w, \theta, p) \in X^u \times X^\theta \times L^2(\Omega) \text{ such that:} \\
 &\begin{cases} a(\alpha; w, v_1) + b(v_1, p) = \langle F, v_1 \rangle, & \forall v_1 \in X^u, \\ \tilde{a}(\alpha; \theta, v_2) + \tilde{c}(\theta, \tilde{w} + V, v_2) = \langle G, v_2 \rangle, & \forall v_2 \in X^\theta, \\ b(w, q) = 0, & \forall q \in L^2(\Omega), \end{cases}
 \end{aligned}$$

for some $(F, G) \in (X^u)' \times (X^\theta)'$. The operator $\mathcal{S}(\tilde{w})$ is well-defined for $\|\tilde{w} + V\|_{X^u} \leq$

$\frac{Ck_{\min}}{1+\varepsilon}$ It is also continuous and [Proposition 2.2](#) gives the estimate

$$\begin{aligned} \|\mathcal{S}(\tilde{w})[F, G]\|_{X^u \times X^\theta} &\leq C_S \left(\|F\|_{(X^u)'} + \|G\|_{(X^\theta)'} \right), \\ C_S &= \max \left\{ \frac{\tilde{g}_0(\Omega)(1+\varepsilon)}{\varepsilon Ck_{\min}}, 1 + \frac{2}{\beta} \right\}. \end{aligned}$$

Problem (2.1) then becomes the fixed point equation:

$$(w, \theta, p) = \mathcal{T}(w, \theta, p),$$

where $\mathcal{T}(w, \theta, p) = \mathcal{S}(w)[G(\alpha, w) + F(\theta), \tilde{G}(\alpha)]$. Denote, for $R > 0$, the set

$$B_R = \{(w, \theta, p) \in X^u \times X^\theta \times L^2(\Omega) : \|w\|_{X^u} + \|\theta\|_{X^\theta} + \|p\|_{L^2(\Omega)} \leq R\}.$$

Thanks to [Proposition 2.2](#) and [Lemma 2.3](#), we have the estimate

$$\begin{aligned} \|\mathcal{T}(w, \theta, p)\|_{X^u \times X^\theta \times L^2(\Omega)} &\leq C_S \left(\|G(\alpha, w) + F(\theta)\|_{(X^u)'} + \|\tilde{G}(\alpha)\|_{(X^\theta)'} \right) \\ &\leq C_S \left(C_1 + \|w\|_{X^u}^2 + B \|\theta\|_{X^\theta} \right), \end{aligned}$$

where

$$C_1 = C(\Omega) \left(\|\tilde{G}(\alpha)\|_{(X^\theta)'} + \|u_{\text{in}}\|_{H_{00}^{1/2}(\Gamma_{\text{in}})^d}^2 + \|u_{\text{in}}\|_{H_{00}^{1/2}(\Gamma_{\text{in}})^d} \right).$$

We assume $B^2 C_S^2 - 4C_1 C_S^2 - 2BC_S + 1 \geq 0$. This reduces to

$$(2.3) \quad 4C_1 C_S^2 \leq (BC_S - 1)^2,$$

which amount to have the source terms small enough. Assuming now that $(w, \theta, p) \in B_R$ with R such that

$$(2.4) \quad R \leq R_0, \quad R_0 = \frac{1 - BC_S + \sqrt{B^2 C_S^2 - 4C_1 C_S^2 - 2BC_S + 1}}{2C_S},$$

we obtain that \mathcal{T} maps B_R to B_R for any $R \leq R_0$.

We now prove that $\mathcal{T} : B_R \rightarrow B_R$ is a contraction mapping. Let (w_1, θ_1, p_1) and $(w_2, \theta_2, p_2) \in B_R$. We have $\mathcal{T}(w_1, \theta_1, p_1) - \mathcal{T}(w_2, \theta_2, p_2) = \mathcal{M}_1 + \mathcal{M}_2$, where

$$\begin{aligned} \mathcal{M}_1 &= \{\mathcal{S}(w_1) - \mathcal{S}(w_2)\} [G(\alpha, w_1) + F(\theta_1), \tilde{G}(\alpha)], \\ \mathcal{M}_2 &= \mathcal{S}(w_2) [(G(\alpha, w_1) + F(\theta_1)) - (G(\alpha, w_2) + F(\theta_2)), 0]. \end{aligned}$$

[Lemma 2.3](#) gives the next bound for \mathcal{M}_2

$$\begin{aligned} \|\mathcal{M}_2\|_{X^u \times X^\theta \times L^2(\Omega)} &\leq C_S C_L \max \left\{ 2R + \|u_{\text{in}}\|_{H_{00}^{1/2}(\Gamma_{\text{in}})^d}, B \right\} \\ &\quad \times (\|w_1 - w_2\|_{X^u} + \|\theta_1 - \theta_2\|_{X^\theta}). \end{aligned}$$

The bound for \mathcal{M}_1 can be obtained by noting that the operator $\{\mathcal{S}(w_1) - \mathcal{S}(w_2)\} [F, G]$ verifies the identity

$$\{\mathcal{S}(w_1) - \mathcal{S}(w_2)\} [F, G] = \mathcal{S}(w_1)[0, \tilde{C}],$$

where $\tilde{C} \in (X^\theta)'$ is defined as $\langle \tilde{C}, v_2 \rangle = \tilde{c}(\theta(w_2, \theta_2, p_2), w_2 - w_1, v_2)$, with $\theta(w_2, \theta_2, p_2)$ being the temperature defined thanks to the operator $\mathcal{S}(w_2)[F, G]$. We then have the next upper bound

$$\begin{aligned} \|\mathcal{M}_1\|_{X^u \times X^\theta \times L^2(\Omega)} &\leq C_S \left\| \tilde{C} \right\|_{(X^\theta)'} \leq C_S C(\Omega) \|\theta(w_2, \theta_2, p_2)\|_{X^\theta} \|w_1 - w_2\|_{X^u} \\ &\leq C_S^2 C(\Omega) \left(\|G(\alpha, w_2) + F(\theta_2)\|_{(X^u)'} + \|\tilde{G}(\alpha)\|_{(X^\theta)'} \right) \\ &\quad \|w_1 - w_2\|_{X^u} \\ &\leq C_S^2 C(\Omega) (C_1 + R^2 + BR) \|w_1 - w_2\|_{X^u}. \end{aligned}$$

Gathering the previous estimates, we obtain

$$\|\mathcal{T}(w_1, \theta_1, p_1) - \mathcal{T}(w_2, \theta_2, p_2)\|_{X^u \times X^\theta \times L^2(\Omega)} \leq C_{\text{Lip}} (\|w_1 - w_2\|_{X^u} + \|\theta_1 - \theta_2\|_{X^\theta}),$$

with

$$C_{\text{Lip}} := \max \left\{ C_S C_L B, C_S C_L \left(2R + \|u_{\text{in}}\|_{H_{00}^{1/2}(\Gamma_{\text{in}})^d} \right), C_S^2 C(\Omega) (C_1 + R^2 + BR) \right\}.$$

Assuming now the source terms are small enough so that

$$(2.5) \quad \|u_{\text{in}}\|_{H_{00}^{1/2}(\Gamma_{\text{in}})^d} < \frac{1}{C_S C_L}, \quad 4C_1 C_S^2 C(\Omega)^2 < B^2 C_S^2 C(\Omega)^2 + 4C(\Omega),$$

and that

$$(2.6) \quad \begin{aligned} C_S C_L B &< 1, \quad R \leq \min\{R_1, R_2\}, \\ R_1 &= \frac{1}{2} \left(\frac{1}{C_S C_L} - \|u_{\text{in}}\|_{H_{00}^{1/2}(\Gamma_{\text{in}})^d} \right), \\ R_2 &= \frac{1}{2C_S C(\Omega)} \left(\sqrt{B^2 C_S^2 C(\Omega)^2 - 4C_1 C_S^2 C(\Omega)^2 + 4C(\Omega)} - C(\Omega) B C_S \right). \end{aligned}$$

One finally proves that for any $R \leq \min(R_0, R_1, R_2)$, $\mathcal{T} : B_R \rightarrow B_R$ is a contraction if one takes B , $\|u_{\text{in}}\|_{H_{00}^{1/2}(\Gamma_{\text{in}})^d}$ and $\|\phi\|_{L^2(\Gamma)}$ small enough (see (2.3)-(2.5)). Banach fixed point Theorem then proves the Theorem. \square

REMARK 2.5. Let us consider $h_\tau = \frac{1}{\tau} 1_{\Omega_s}$ where the solid is located in $\Omega_s \subset \Omega$. *Theorem 2.4* ensures the existence and uniqueness of a solution to Problem (2.1) that satisfy the bound $\|w\|_{X^u} + \|\theta\|_{X^\theta} + \|p\|_{L^2(\Omega)} \leq R$, where R does not depend on τ (see (2.3)-(2.5)). Since $\|\cdot\|_{X^u} = \sqrt{a(\alpha; \cdot, \cdot)}$, we get $\|\sqrt{h_\tau} w\|_{L^2(\Omega)} = \frac{1}{\sqrt{\tau}} \|w\|_{L^2(\Omega_s)} \leq R$, from which we infer $\|w\|_{L^2(\Omega_s)} \leq R\sqrt{\tau}$. In addition, from the the multiplicative trace inequality, we have

$$\|w\|_{L^2(\partial\Omega_s)} \leq C \sqrt{\|w\|_{L^2(\Omega_s)} \|w\|_{H^1(\Omega)}} \leq C \sqrt{R} \tau^{1/4} \|w\|_{X^u} \leq CR \tau^{1/4},$$

where $C > 0$ is a generic constant. Therefore the velocity of the fluid vanishes in the solid as τ goes to 0 and it satisfies the no-slip boundary condition on $\partial\Omega_s$. It is finally worth noting that we obtain similar convergence rates as those proved in [5, Corollary 4.1, Lemma 4.4] where incompressible unsteady Navier-Stokes equations with homogeneous Dirichlet boundary on $\partial\Omega$ were considered.

2.2. Convergence of a finite element approximation. We now move on to the analysis of the discretization of (2.1). We consider a quasi-uniform family of triangulations (see [22, Definition 1.140]) $\{\mathcal{T}_h\}_{h>0}$ of Ω whose elements are triangles ($d = 2$) or tetrahedrons ($d = 3$). We emphasize that $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$. The parameter h_K is the diameter of the circle or sphere inscribed in the cell $K \in \mathcal{T}_h$ and we set $h = \sup_{K \in \mathcal{T}_h} h_K$. We consider the Taylor-Hood finite element [47] that uses piecewise polynomial approximations $(w_h, \theta_h, p_h) \in X_h^u \times X_h^\theta \times M_h$ of $(w, \theta, p) \in X^u \times X^\theta \times L^2(\Omega)$ with

$$\begin{aligned} X_h^u &= \{v_h^u \in X^u \mid \forall K \in \mathcal{T}_h, v_h^u|_K \in \mathbb{P}_2(K)\}, \\ X_h^\theta &= \{v_h^\theta \in X^\theta \mid \forall K \in \mathcal{T}_h, v_h^\theta|_K \in \mathbb{P}_2(K)\}, \\ M_h &= \{q_h \in \mathcal{C}^0(\bar{\Omega}) \mid \forall K \in \mathcal{T}_h, q_h|_K \in \mathbb{P}_1(K)\}, \end{aligned}$$

and we denote by $\mathcal{I}_{X_h^u} : X^u \rightarrow X_h^u$ the finite element interpolate on X_h^u . Regarding the discretization of α by some α_h , we use:

$$\mathcal{K}_h = \{\alpha_h \in L^\infty(\Omega) : \alpha_h|_K \in \mathbb{P}_0(K), \forall K \in \mathcal{T}_h\}.$$

Let us now consider the following discretized variational problem: given $\alpha_h \in \mathcal{K}_h$:

$$(2.7) \quad \begin{aligned} &\text{Find } (w_h, \theta_h, p_h) \in X_h^u \times X_h^\theta \times M_h \text{ such that:} \\ &\begin{cases} a(\alpha_h; w_h, v_h^u) + b(v_h^u, p_h) = \langle G(\alpha_h; w_h) + F(\theta_h), v_h^u \rangle, & \forall v_h^u \in X_h^u, \\ \tilde{a}(\alpha_h; \theta_h, v_h^\theta) + \tilde{c}(\theta_h, w_h + \mathcal{I}_{X_h} V, v_h^\theta) = \langle \tilde{G}(\alpha_h), v_h^\theta \rangle, & \forall v_h^\theta \in X_h^\theta, \\ b(w_h, q_h) = 0, & \forall q_h \in M_h. \end{cases} \end{aligned}$$

Throughout this section, we make the following assumption on \mathcal{T}_h :

ASSUMPTION 2.6. *At least an edge ($d=2$) or a face ($d=3$) of an element of \mathcal{T}_h is contained in Γ_{out} .*

As noted in [11], this assumption is fulfilled for h small enough. Assuming **Assumption 2.6**, [11, Lemma 3.2] proves that there exists $\beta^* > 0$ such that:

$$\inf_{q_h \in M_h \setminus \{0\}} \sup_{v_h \in X_h^u} \frac{b(v_h, q_h)}{\|v_h\|_{X^u} \|q_h\|_{L^2(\Omega)}} \geq \beta^*.$$

Therefore, [29, Theorem 4.1] and a similar proof as for **Theorem 2.4** (using a fixed point approach) prove that Problem (2.7) admits a unique solution for h small enough which complies with:

$$(2.8) \quad \|w_h\|_{X^u} + \|\theta_h\|_{X^\theta} + \|p_h\|_{L^2(\Omega)} \leq R,$$

for some R that does not depend on h (see (2.3)-(2.5)).

We now prove convergence of the discretized solutions to the continuous ones. Consider a sequence of controls α_h which converges to a control α in the weak-* topology, which means $\alpha_h \rightarrow \alpha$ strongly in L^1 and $D\alpha_h \xrightarrow{*} D\alpha$ weakly-* in $\mathcal{M}_b(\Omega)$, the space of bounded Radon measure.

Given this definition, we prove the convergence of the finite element approximation toward the continuous solution of (2.1).

THEOREM 2.7. *Suppose B , $\|u_{in}\|_{H_{00}^{1/2}(\Gamma_{in})}$ and $\|\phi\|_{L^2(\Gamma)}$ are small enough for the solutions of (2.1) and (2.7) to exist for $h > 0$ small enough, and that $\alpha_h \xrightarrow{*} \alpha$ in $BV(\Omega)$. Denote by (w_h, θ_h, p_h) the solution of (2.7) using α_h , and by (w, θ, p) the solution of (2.1) using α . We then have the following convergence:*

$$\lim_{h \rightarrow 0} (\|w_h - w\|_{X^u} + \|\theta_h - \theta\|_{X^\theta} + \|p_h - p\|_{L^2(\Omega)}) = 0.$$

Proof. Since the presence of $\mathcal{I}_{X_h^u} V$ in (2.7) does not add any trouble in the following analysis, it is omitted for this proof.

As proved in inequality (2.8), the sequence (w_h, θ_h, p_h) is uniformly bounded in $X^u \times X^\theta \times L^2(\Omega)$ with respect to h . Therefore, there exist $(w, \theta, p) \in X^u \times X^\theta \times L^2(\Omega)$ and subsequences such that:

$$(w_{h_k}, \theta_{h_k}, p_{h_k}) \rightharpoonup (w, \theta, p) \text{ weakly in } H^1(\Omega)^{d+1} \times L^2(\Omega),$$

$$(w_{h_k}, \theta_{h_k}) \rightarrow (w, \theta) \text{ strongly in } L^4(\Omega)^{d+1}.$$

Part 1 Let us prove that the sequence weakly converges. Let $(v^u, v^\theta, q) \in \mathcal{C}^\infty(\bar{\Omega})^{d+2}$. There exists a sequence $(v_h^u, v_h^\theta, q_h) \in X_h^u \times X_h^\theta \times M_h$ that strongly converges to (v^u, v^θ, q) in $X^u \times X^\theta \times L^2(\Omega)$. Using the convergence of the subsequences, one proves easily: $b(v_{h_k}^u, p_{h_k}) \rightarrow b(v^u, p)$, $b(w_{h_k}, q_{h_k}) \rightarrow b(w, q)$, $c(w_{h_k}, w_{h_k}, w_{h_k}^u) \rightarrow c(w, w, v^u)$, $f(\theta_{h_k}, v_{h_k}^u) \rightarrow f(\theta, v^u)$, $\tilde{c}(\theta_{h_k}, w_{h_k}, v_{h_k}^\theta) \rightarrow \tilde{c}(\theta, w, v^\theta)$.

Also, one proves:

$$\begin{aligned} |a(\alpha_{h_k}, w_{h_k}, v_{h_k}^u) - a(\alpha, w, v^u)| &\leq |A| \left| \int_{\Omega} \nabla w_{h_k} : \nabla v_{h_k} - \nabla w : \nabla v \right| \\ &\quad + \|h_\tau\|_\infty \left| \int_{\Omega} w_{h_k} \cdot v_{h_k}^u - w \cdot v^u \right| \\ &\xrightarrow{k \rightarrow +\infty} 0, \end{aligned}$$

$$\begin{aligned} |\tilde{a}(\alpha_{h_k}, \theta_{h_k}, v_{h_k}^\theta) - \tilde{a}(\alpha, \theta, v^\theta)| &\leq C \|k_\tau\|_\infty \left| \int_{\Omega} \nabla \theta_{h_k} \cdot \nabla v_{h_k} - \nabla \theta \cdot \nabla v \right| \\ &\xrightarrow{k \rightarrow +\infty} 0, \end{aligned}$$

$$\left| \int_{\Gamma_w} k_\tau(\alpha_{h_k}) \phi v_{h_k}^\theta - \int_{\Gamma_w} k_\tau(\alpha) \phi v^\theta \right| \leq \|k_\tau\|_\infty \left| \int_{\Gamma_w} \phi (v_{h_k}^\theta - v^\theta) \right| \rightarrow 0, .$$

and $a(\alpha_{h_k}, V, v_{h_k}^u) \rightarrow a(\alpha, V, v^u)$. It finally proves that the limit (w, θ, p) satisfies (2.1) for all $(v^u, v^\theta, q) \in \mathcal{C}^\infty(\bar{\Omega})^{d+2}$. The density of smooth functions in $X^u \times X^\theta \times L^2(\Omega)$ ensures that $(w, \theta, p) \in X^u \times X^\theta \times L^2(\Omega)$ satisfies (2.1) for all functions $(v^u, v^\theta, q) \in X^u \times X^\theta \times L^2(\Omega)$. Thus, $(w_{h_k}, \theta_{h_k}, p_{h_k})$ weakly converges toward (w, θ, p) solution of (2.1).

Part 2 Let us now show that w_h and θ_h strongly converge in $X^u \times X^\theta$. Taking $v_{h_k}^u = w_{h_k}$ and $v_{h_k}^\theta = \theta_{h_k}$ in (2.7), one gets:

$$\begin{aligned} (2.9) \quad &\int_{\Omega} (A|\nabla w_{n_k}|^2 + h_\tau(\alpha_{n_k})|w_{n_k}|^2 + (w_{n_k} \cdot \nabla)w_{n_k} \cdot w_{n_k} - B\theta_{n_k} e_y \cdot w_{n_k}) \\ &= \int_{\Omega} (A\nabla V : \nabla w_{n_k} + h_\tau(\alpha_{n_k})V \cdot w_{n_k}), \end{aligned}$$

$$(2.10) \quad \int_{\Omega} (Ck_{\tau}(\alpha_{n_k})|\nabla\theta_{n_k}|^2 + \theta_{n_k}(\nabla\theta_{n_k} \cdot w_{n_k})) = \int_{\Gamma_w} k_{\tau}(\alpha_{n_k})\phi\theta_{n_k}.$$

From (2.9), one gets:

$$\begin{aligned} A\|\nabla w_{h_k}\|_{L^2(\Omega)}^2 &= \int_{\Omega} (A\nabla V : \nabla w_{h_k} + h_{\tau}(\alpha_{h_k})V \cdot w_{h_k}) \\ &\quad + \int_{\Omega} (B\theta_{h_k}e_y \cdot w_{h_k} - (w_{h_k} \cdot \nabla)w_{h_k} \cdot w_{h_k}) \\ &\quad - \int_{\Omega} h_{\tau}(\alpha_{h_k})|w_{h_k}|^2 \\ &\rightarrow A\|\nabla w\|_{L^2(\Omega)}^2. \end{aligned}$$

Therefore, we have proved that $w_{h_k} \rightharpoonup w$ and $\|\nabla w_{h_k}\|_{L^2(\Omega)} \rightarrow \|\nabla w\|_{L^2(\Omega)}$. It eventually proves that $w_{h_k} \rightarrow w$ strongly in X^u . Also, using the same technique, one proves that $C\left\|\sqrt{k_{\tau}(\alpha_{h_k})}\nabla\theta_{h_k}\right\|_{L^2(\Omega)}^2 \rightarrow C\left\|\sqrt{k_{\tau}(\alpha)}\nabla\theta\right\|_{L^2(\Omega)}^2$, and since $\theta_{h_k} \rightharpoonup \theta$, it proves that $\theta_{h_k} \rightarrow \theta$ strongly in X^{θ} .

Part 3 Using the discrete inf-sup condition, we have

$$\|p - p_{h_k}\|_{L^2(\Omega)} \leq \frac{1}{\beta^*} \sup_{v_{h_k} \in B_h} (b(v_{h_k}, p - p_{h_k})),$$

where $B_h = \{u_h \in X_h^u : \|u_h\|_{X^u} = 1\}$. Since $(w_{h_k}, \theta_{h_k}, p_{h_k})$ satisfies (2.7) and (w, θ, p) satisfies (2.1), one proves:

$$\begin{aligned} |b(v_{h_k}, p) - b(v_{h_k}, p_{h_k})| &\leq |a(\alpha_{h_k}; w_{h_k}, v_{h_k}) - a(\alpha; w, v_{h_k})| \\ &\quad + |f(\theta_{h_k}, v_{h_k}) - f(\theta, v_{h_k})| \\ &\quad + |c(w_{h_k}, w_{h_k}, v_{h_k}) - c(w, w, v_{h_k})| \\ &\rightarrow 0, \end{aligned}$$

where the last limit comes from the strong limit of (w_{h_k}, θ_{h_k}) proved before. It therefore proves that $\lim_{k \rightarrow +\infty} \sup_{v_{h_k} \in B_h} (b(v_{h_k}, p) - b(v_{h_k}, p_{h_k})) = 0$. Eventually, it proves that $p_{h_k} \rightarrow p$ strongly in $L^2(\Omega)$.

To summarise, we have proved that there exists a subsequence $(w_{h_k}, \theta_{h_k}, p_{h_k})$ which converges strongly to a solution (w, θ, p) of (2.1) when $\alpha_h \xrightarrow{*} \alpha$.

Part 4 Let us eventually prove that the whole sequence actually converges. Denote by $S_h = (w_h, \theta_h, p_h)$ a sequence of solutions and $S = (w, \theta, p)$. Since S_h is bounded, so is every subsequence S_{h_k} of S_h . Therefore, we can extract another subsequence of S_{h_k} which will also converge to S using the same arguments as in Part 1-3 and by uniqueness of the solution to (2.1). Therefore, every subsequence of (S_h) has a further subsequence that strongly converges to S , and using Urysohn's subsequence principle, one proves that the whole sequence (S_h) strongly converges to S . \square

REMARK 2.8. *Theorem 2.7 gives the convergence, as $h \rightarrow 0$, of the finite element approximation of (2.1). There is however no additional information on the rate of convergence. Optimal error estimates can actually be obtained using results from [14] (see also [30]). Nevertheless, these require the solution to Problem (2.1) to be more regular (like e.g. $(w, \theta, p) \in H^2(\Omega)^d \times H^2(\Omega) \times H^1(\Omega)$).*

3. Optimization problem. Now that we have proved that the system (1.5) is well-posed, we now tackle the optimal control problem involving it, namely:

$$(3.1) \quad \begin{aligned} & \min \mathcal{J}(\alpha, w, \theta, p) \\ & \text{s.t.} \quad \begin{cases} (w, \theta, p) \text{ solution of (2.1) parametrized by } \alpha, \\ \alpha \in \mathcal{U}_{ad}, \end{cases} \end{aligned}$$

where \mathcal{J} is a real-valued cost functional. We will follow the approach used in section 2, namely we first study the well-posedness of (3.2) and then the discretization of it. We end up with a proof of a necessary condition of optimality for (3.1).

3.1. Continuous problem. We start this study with the existence of a solution to (3.1).

THEOREM 3.1. *Suppose:*

- (A1) $\forall (\alpha, w, \theta, p) \in \mathcal{U}_{ad} \times X^u \times X^\theta \times L^2(\Omega)$, $\inf_{\mathcal{U}_{ad} \times X^u \times X^\theta} \mathcal{J} > -\infty$.
- (A2) \mathcal{J} is lower semi-continuous w.r.t. the (weak-*, weak, weak, weak) topology of $\mathcal{U}_{ad} \times X^u \times X^\theta$.

Then the optimization problem (3.1) has at least one solution in $\mathcal{U}_{ad} \times X^u \times X^\theta \times L^2(\Omega)$.

Proof. We recall $U_{ad} \subset BV(\Omega)$ is a weak-* closed subset of $BV(\Omega)$. Let $(\alpha_n) \subset U_{ad}$ be a sequence uniformly bounded in U_{ad} and converging to $\alpha \in BV(\Omega)$. One can therefore prove that $\alpha_n \xrightarrow{*} \alpha$ in U_{ad} . Let (w_n, θ_n, p_n) be the solutions of (the continuous) problem (2.1) parametrized by α_n , and (w, θ, p) the solution of (2.1) parametrized by α . Using the same technique as in the proof of Theorem 2.7, one can get that $(w_n, \theta_n, p_n) \rightarrow (w, \theta, p)$ strongly in $X^u \times X^\theta \times L^2(\Omega)$. In other words, one can similarly prove that the mapping

$$\alpha \in (U, \text{weak-}^*) \mapsto (w, \theta, p) \in (X^u \times X^\theta \times L^2(\Omega), \text{strong})$$

is continuous. The proof is now based on minimizing sequence and can be adapted for instance from [31, Theorem 2.1]. \square

3.2. Discretized problem. We now turn our attention to the discretization of Problem (3.1), which reads:

$$(3.2) \quad \begin{aligned} & \min \mathcal{J}(\alpha_h, w_h, \theta_h, p_h) \\ & \text{s.t.} \quad \begin{cases} (w_h, \theta_h, p_h) \text{ solution of (2.7) parametrized by } \alpha_h, \\ \alpha_h \in \mathcal{U}_h = \mathcal{U}_{ad} \cap \mathcal{K}_h. \end{cases} \end{aligned}$$

where we remind that $\mathcal{K}_h = \{\alpha_h \in L^\infty(\Omega) : \alpha_h|_T \in \mathbb{P}_0(T), \forall T \in \mathcal{T}_h\}$.

As in subsection 2.2, we will focus on proving some convergence result of the finite element discretization (3.2) toward a solution of the continuous problem (3.1).

THEOREM 3.2. *Assume Assumptions (A1)-(A2) from Theorem 3.1 are verified and that the cost function \mathcal{J} is continuous with respect to the (weak-*, strong, strong, strong) topology of $BV(\Omega) \times X^u \times X^\theta \times L^2(\Omega)$. Let $(\alpha_h^*, w_h, \theta_h, p_h) \in \mathcal{U}_h \times X_h^u \times X_h^\theta \times M_h$ be an optimal solution of (3.2). Then $(\alpha_h) \subset \mathcal{U}_h$ is a bounded sequence. Furthermore, there exists $(\alpha^*, w^*, \theta^*, p^*) \in \mathcal{U}_{ad} \times X^u \times X^\theta \times L^2(\Omega)$ such that a subsequence of $(\alpha_h, w_h, \theta_h, p_h)$ converges (weak-*, strong, strong, strong) to $(\alpha^*, w^*, \theta^*, p^*)$ and*

$$\mathcal{J}(\alpha^*, w^*, \theta^*, p^*) \leq \mathcal{J}(\alpha, w, \theta, p), \quad \forall (\alpha, w, \theta, p) \in \mathcal{U}_{ad} \times X^u \times X^\theta \times L^2(\Omega).$$

Hence, any accumulation point of $(\alpha_h, w_h, \theta_h, p_h)$ is an optimal solution of (3.1).

Proof. The proof can be adapted from [18, Theorem 15] (see also [32, Theorem 3]) but we give it for the sake of completeness.

Remark that $(\alpha_h^*) \subset \mathcal{U}_h \subset \mathcal{U}_{ad}$ is bounded uniformly in $BV(\Omega)$; thus, it admits a weak-* accumulation point α^* , and by [Theorem 2.7](#), (w_h, θ_h, p_h) converges to (w^*, θ^*, p^*) . The lower semi-continuity of \mathcal{J} ensures that $\mathcal{J}(\alpha^*, w^*, \theta^*, p^*) \leq \liminf_{h \rightarrow 0} \mathcal{J}(\alpha_h^*, w_h^*, \theta_h^*, p_h^*)$, where the subsequence is still denoted like the sequence. Let $\alpha \in \mathcal{U}_{ad}$. As stated in [8, Introduction], using the density of smooth functions in BV , there exists a sequence $\alpha_h \in \mathcal{U}_h$ such that $\alpha_h \xrightarrow{*} \alpha$ in BV . From [Theorem 2.7](#), one gets that the solutions (w_h, θ_h, p_h) of Problem (2.7) associated to α_h converge strongly in $X^u \times X^\theta \times L^2(\Omega)$ to the solution (w, θ, p) of Problem (2.1) associated with α . The continuity of \mathcal{J} ensures that $\mathcal{J}(\alpha_h, w_h, \theta_h, p_h) \rightarrow \mathcal{J}(\alpha, w, \theta, p)$. But since $\mathcal{J}(\alpha_h^*, w_h^*, \theta_h^*, p_h^*) \leq \mathcal{J}(\alpha_h, w_h, \theta_h, p_h)$, for all $\alpha_h \in \mathcal{U}_h$, one proves:

$$\begin{aligned} \mathcal{J}(\alpha^*, w^*, \theta^*, p^*) &\leq \liminf_{h \rightarrow 0} \mathcal{J}(\alpha_h^*, w_h^*, \theta_h^*, p_h^*) \\ &\leq \liminf_{h \rightarrow 0} \mathcal{J}(\alpha_h, w_h, \theta_h, p_h) \\ &= \mathcal{J}(\alpha, w, \theta, p) \quad \forall \alpha \in \mathcal{U}_{ad}. \quad \square \end{aligned}$$

3.3. First order necessary conditions. As it can be found in [35, Section 1.7.2], one can define for (3.1) some optimality conditions. Denote by $e(w, \theta, p, \alpha) = 0$ the set of equations defining (2.1) and suppose that $\partial_{(w, \theta, p)} e(w(\alpha), \theta(\alpha), p(\alpha), \alpha)$ has a bounded inverse for all $\alpha \in U_{ad}$. Assuming that $\mathcal{J}_1 : \alpha \mapsto \mathcal{J}(\alpha, w(\alpha), \theta(\alpha), p(\alpha))$ is differentiable, one gets that an optimum α^* satisfies:

$$\langle \nabla_\alpha \mathcal{J}_1(\alpha^*), \alpha - \alpha^* \rangle_{(L^\infty(\Omega))', L^\infty(\Omega)} \geq 0, \quad \forall \alpha \in U_{ad}.$$

We only need to prove that $\partial_{(w, \theta, p)} e(w(\alpha), \theta(\alpha), p(\alpha), \alpha)$ has a bounded inverse, which implies the study of the linearization of (2.1). Computing the Fréchet derivative of the operator involved in (2.1) at $(u, \theta, p) \in X_1^u \times X^\theta \times L^2(\Omega)$, we have to prove that, for any source term $(f_1, f_2, f_3) \in (X_1^u)' \times (X^\theta)' \times L^2(\Omega)$, (ν, ζ, ρ) is the unique solution to the linearization of (2.1), which reads:

$$(3.3) \quad \begin{cases} \text{Find } (\nu, \zeta, \rho) \in X_1^u \times X^\theta \times L^2(\Omega) \text{ such that} \\ \left\{ \begin{array}{ll} a(\alpha; \nu, v_1) + b(v_1, \rho) + c'(u, \nu, v_1) + f(\zeta, v_1) = \langle f_1, v_1 \rangle, & \forall v_1 \in X^u \\ \tilde{a}(\alpha, \zeta, v_2) + \tilde{c}(\zeta, u, v_2) + \tilde{c}(\theta, \nu, v_2) = \langle f_2, v_2 \rangle & \forall v_2 \in X^\theta \\ b(\nu, q) = \langle f_3, q \rangle, & \forall q \in L^2(\Omega). \end{array} \right. \end{cases}$$

where

$$c'(u, \nu, v) = \int_{\Omega} ((u \cdot \nabla) \nu \cdot v + (\nu \cdot \nabla) u \cdot v).$$

THEOREM 3.3. *There exists a unique solution to (3.3) which is continuous with respect to the source term (f_1, f_2, f_3) .*

Proof. Define the linear operators $A, B, F, \tilde{C}, \tilde{A}$ as

$$\begin{aligned} \langle A\nu, v_1 \rangle &= a(\alpha; \nu, v_1) + c'(u, \nu, v_1), \quad \langle \tilde{C}\nu, v_2 \rangle = \tilde{c}(\theta, \nu, v_2), \\ \langle F\zeta, v_1 \rangle &= f(\zeta, v_1), \quad \langle \tilde{A}\zeta, v_2 \rangle = \tilde{a}(\alpha; \zeta, v_2) + \tilde{c}(\zeta, u, v_2) \\ \langle B\rho, v_1 \rangle &= b(v_1, \rho) \end{aligned}$$

Then, one can understand (3.3) as the following equation in $(X_1^u)' \times (X^\theta)' \times L^2(\Omega)$:

$$\text{Find } (\nu, \zeta, \rho) \in X_1^u \times X^\theta \times L^2(\Omega) \text{ such that}$$

$$\begin{cases} A\nu + B\rho + F\zeta = f_1 \\ \tilde{C}\nu + \tilde{A}\zeta = f_2 \\ B^*\nu = f_3. \end{cases}$$

Using the same analysis as in the proof of [Proposition 2.2](#), ζ is uniquely determined and continuous with respect to the source terms as soon as ν is. Therefore, $\zeta = \tilde{A}^{-1}(f_2 - \tilde{C}\nu)$. We reintroduce this in the first equation, which now reads: $(A - F\tilde{A}^{-1}\tilde{C})\nu + B\rho = f_1 - \tilde{A}^{-1}f_2$. As we have proved in [Lemma 2.3](#), one gets that:

$$\begin{aligned} \langle A\nu, \nu \rangle &= a(\alpha, \nu, \nu) + c(u, \nu, \nu) + c(\nu, u, \nu), \\ &\geq \|\nu\|_{X^u}^2 - 2C_{NL}\|u\|_{X^u}\|\nu\|_{X^u} \geq (1 - 2C_{NL}R)\|\nu\|_{X^u}^2, \end{aligned}$$

where R was defined in [Theorem 2.4](#). Also, from the proof of [Proposition 2.2](#), one has $\langle \tilde{A}\zeta, \zeta \rangle \geq C(\Omega)(Ck_{min} - \|u\|)\|\zeta\|_{X^\theta}^2 \geq C(\Omega)(Ck_{min} - R)\|\zeta\|_{X^\theta}^2$. Denote $C_{\tilde{A}} = C(\Omega)(Ck_{min} - R)$. Since \tilde{A} is invertible, for all $f \in (X^\theta)'$, there exists $\zeta \in X^\theta$ such that $\tilde{A}\zeta = f$ and:

$$\|\tilde{A}^{-1}f\|_{X^\theta}^2 \leq \frac{1}{C_{\tilde{A}}}\langle f, \tilde{A}^{-1}f \rangle_{(X^\theta)', X^\theta} \leq \frac{1}{C_{\tilde{A}}}\|\tilde{A}^{-1}f\|_{X^\theta}\|f\|_{(X^\theta)'}$$

which proves $\|\tilde{A}^{-1}\| = \sup_{\|f\|_{(X^\theta)', 1}} \|\tilde{A}^{-1}f\|_{X^\theta} \leq \frac{1}{C_{\tilde{A}}}$. Furthermore, one can easily show that: $\|F\| \leq B$, $\|\tilde{C}\| \leq \|\theta\|_{X^\theta} \leq R$. Eventually, all these results leads to:

$$\langle (A - F\tilde{A}^{-1}\tilde{C})\nu, \nu \rangle \geq \left(1 - 2C_{NL}R - \frac{BR}{C(\Omega)(Ck_{min} - R)}\right)\|\nu\|^2.$$

Therefore, if one chooses R small enough (which means that the source terms have to be small enough, as in [Theorem 2.4](#)), one has that, for all $f \in (X_1^u)'$, there exists a unique $\nu \in X_1^u$ such that $(A - F\tilde{A}^{-1}\tilde{C})\nu = f$, thanks to Lax-Milgram theorem. Eventually, noting that (ν, ρ) satisfy a standard saddle-point problem that verify the assumptions of [[30](#), Theorem 4.1, p.59], one proves the existence, uniqueness and continuity with respect to the data of (ν, ζ, ρ) . \square

4. Numerical treatment. We focus in this section on a small presentation of the developments handled to numerically treat Problem (3.1), which will be more deeply presented in a companion paper. There exists a huge literature on the numerical methods for solving programs with PDE-constraints ; see for instance [[6](#), [34](#)]. Most methods rely on the gradient of the cost functional with respect to the design variable (in our case, with respect to α) in order to compute a descent direction ; in turn, one needs to compute the derivative of the state variables with respect to the design variable. Since we use a *differentiate then discretize* approach, this gradient will be computed via the so-called *adjoint system* associated to (3.1), which will be afterward discretized.

4.1. Gradient computation with the adjoint system. From now on, we suppose that $\mathcal{J}(\alpha, u, p, \theta) = \int_\Omega J_\Omega(\alpha, u, \theta, p) + \int_\Gamma J_\Gamma(\alpha, u, \theta, p)$, where $J_\Omega : U_{ad} \times X^u \times X^\theta \times L^2(\Omega) \rightarrow L^1(\Omega)$, $J_\Gamma : U_{ad} \times X^u \times X^\theta \times L^2(\Omega) \rightarrow L^1(\Gamma)$ are Fréchet differentiable

mappings. Slightly adapting the result of [9], one can prove that the adjoint system associated to (3.1) reads:

$$(4.1) \quad \begin{cases} \nabla \lambda^p - h_\tau(\alpha) \lambda^u + \theta \nabla \lambda^\theta + A \Delta \lambda^u + \nabla \lambda^u \cdot u - (\lambda^u \cdot \nabla) u = -\frac{\partial J_\Omega}{\partial u}, \\ B \lambda^u \cdot e_y + u \cdot \nabla \lambda^\theta + \nabla \cdot (C k_\tau(\alpha) \nabla \lambda^\theta) = -\frac{\partial J_\Omega}{\partial \theta}, \\ \nabla \cdot \lambda^u = -\frac{\partial J_\Omega}{\partial p}. \end{cases}$$

$$(4.2) \quad \begin{aligned} \underline{\text{On } \Gamma_w} : \quad & \lambda^u \cdot t = 0, \quad \lambda^u \cdot n = \frac{\partial J_\Gamma}{\partial p}, \quad \lambda^\theta (u \cdot n) + C k_\tau(\alpha) \partial_n \lambda^\theta = \frac{\partial J_\Gamma}{\partial \theta}, \\ \underline{\text{On } \Gamma_{in}} : \quad & \lambda^u \cdot t = 0, \quad \lambda^u \cdot n = \frac{\partial J_\Gamma}{\partial p}, \quad \lambda^\theta = 0, \quad \partial_n \lambda^p = 0, \\ \underline{\text{On } \Gamma_{out}} : \quad & \lambda^\theta (u \cdot n) + C k_\tau(\alpha) \partial_n \lambda^\theta = \frac{\partial J_\Gamma}{\partial \theta}, \\ & \lambda^p n + \lambda^\theta \theta n + A \partial_n \lambda^u + (u \cdot n) \lambda^u = \frac{\partial J_\Gamma}{\partial u}, \end{aligned}$$

where t is a unit tangent vector.

The existence of adjoint solutions to (4.1)-(4.2) has been proved in [Theorem 3.3](#). The gradient of the cost functional then reads:

$$(4.3) \quad \begin{aligned} \nabla_\alpha \mathcal{J} &= \frac{\partial J_\Omega}{\partial \alpha} - \frac{\partial h_\tau}{\partial \alpha}(\alpha) u \cdot \lambda^u - C \frac{\partial k_\tau}{\partial \alpha}(\alpha) \nabla \theta \cdot \nabla \lambda^\theta && \text{on } \Omega, \\ \nabla_\alpha \mathcal{J} &= \frac{\partial J_\Gamma}{\partial \alpha} + C \frac{\partial k_\tau}{\partial \alpha}(\alpha) \phi \lambda^\theta && \text{on } \Gamma_w, \end{aligned}$$

and one has the variational optimality condition:

$$(4.4) \quad \langle \nabla_\alpha \mathcal{J}, \beta - \alpha \rangle_{(L^\infty(\Omega))', L^\infty(\Omega)} \geq 0, \quad \forall \beta \in \mathcal{U}_{ad}.$$

Note that in our case, $\nabla_\alpha \mathcal{J} \in L^1(\Omega) \subset (L^\infty(\Omega))'$, the duality bracket can be written as an integral over Ω . However, this inequality is hardly used in an algorithm, since it relies on a duality product. The following Lemma let us recast inequality (4.4) as a projection on an interval, and will be later used in the projected gradient algorithm.

LEMMA 4.1. *Inequality (4.4) is satisfied if and only if, for almost every $x \in \Omega$:*

$$(4.5) \quad \begin{aligned} \nabla_\alpha \mathcal{J}(\alpha(x), u(x), \theta(x), p(x)) \cdot (\beta - \bar{\alpha}(x)) &\geq 0, \\ \forall \beta &\in [0, \alpha_{max}], \end{aligned}$$

where $\bar{\alpha} = \frac{1}{2}(\check{\alpha} + \hat{\alpha})$ and $\check{\alpha}$ (resp. $\hat{\alpha}$) is the lower (resp. upper) semi-continuous envelope of α . Hence, the optimal α is defined pointwise as any representative of:

$$\bar{\alpha}(x) = \Pi_{[0, \alpha_{max}]}(\bar{\alpha}(x) + \kappa \nabla_\alpha \mathcal{J}(\alpha(x), u(x), \theta(x), p(x))),$$

where $\Pi_{[0, \alpha_{max}]}(x) = \max\{0, \min\{\alpha_{max}, x\}\}$ and any $\kappa > 0$.

Proof. The proof is essentially adapted from [21, Proposition 5.1] and [23, Corollary 1, p. 216]. \square

4.2. Topology optimization for materials with piecewise constant thermal diffusivity.

4.2.1. Choice of the interpolating function. Note first that the existence of optimal solution to (3.1) proved in Theorem 3.1 as well as the convergence of the discrete optimum toward the continuous one (see Theorem 3.2) hold for any function h_τ , k_τ that are continuous and bounded. In this subsection, we give the explicit formula for these functions and also present our approach to find optimized solid with piecewise constant thermal diffusivity, also termed as multi-materials.

We first introduce the next smooth regularization of the Heaviside step function (see e.g. [45])

$$(4.6) \quad \tilde{h}_\tau(y, y_0, a, b) = a + (b - a) \left(\frac{1}{1 + \exp(-\tau(y - y_0))} - \frac{1}{1 + \exp(\tau y_0)} \right),$$

where $y \in [0, y_{\max}]$. It is easy to check the following point-wise convergence

$$(4.7) \quad \lim_{\tau \rightarrow +\infty} \tilde{h}_\tau(y, y_0, a, b) = \begin{cases} a & \text{if } y < y_0, \\ (a + b)/2 & \text{if } y = y_0, \\ b & \text{if } y > y_0. \end{cases}$$

Hence, we set $h_\tau(\alpha) = \tilde{h}_\tau(\alpha, \alpha_0, 0, \alpha_{\max})$, which has the properties wanted for an approximation of the indicator of the solid/fluid region.

We now present the multi-material interpolation function used to look for optimized solid with piecewise constant thermal diffusivity. Our idea to get piece-wise constant thermal conductivity k_j for $j = 1, \dots, N$ is to introduce another design variable φ that interpolates the multiple values of the thermal conductivity. Therefore, k_τ is replaced with

$$(4.8) \quad k_\tau(\alpha, \varphi) = k_f + \frac{\tilde{h}_\tau(\alpha, \alpha_0, 0, \alpha_{\max})}{\alpha_{\max}} (\chi_\tau(\varphi) - k_f),$$

where k_f is the thermal conductivity of the solid and χ_τ is going to interpolate the several constants defining the thermal conductivity. Note that, in the fluid part of the domain, one has $\alpha(x) < \alpha_0$ and then (4.7) ensures that $k_\tau(\alpha, \varphi) \rightarrow k_f$ as $\tau \rightarrow \infty$ for any φ . In the solid part of the domain, $\alpha(x) \geq \alpha_0$ and (4.7) shows that $\tilde{h}_\tau(\alpha, \alpha_0, 0, \alpha_{\max})/\alpha_{\max} \rightarrow 1$ and thus $k_\tau(\alpha, \varphi) \rightarrow k_f + (\chi_\tau(\varphi) - k_f) = \chi_\tau(\varphi)$ as $\tau \rightarrow +\infty$. The definition of $\chi_\tau(\varphi)$ is achieved thanks to a superposition of \tilde{h}_τ as follow

$$(4.9) \quad \chi_\tau(\varphi) = \sum_{j=1}^{N-1} \tilde{h}_\tau(\varphi, \varphi_j, a_j, b_j),$$

where $\varphi_i \in [0, \varphi_{\max}]$ and $\varphi_i < \varphi_j$ for any $i \neq j \in \{1, \dots, N\}$. The constants a_j, b_j are then determined thanks to the following requirements

$$(4.10) \quad \begin{aligned} & \lim_{\tau \rightarrow +\infty} \chi_\tau(\varphi) = k_j \text{ for } \varphi_{j-1} < \varphi < \varphi_j, \quad 2 \leq j \leq N-1, \\ & \lim_{\tau \rightarrow +\infty} \chi_\tau(\varphi) = k_1 \text{ for } \varphi < \varphi_1 \text{ and } \lim_{\tau \rightarrow +\infty} k_\tau(\varphi) = k_N \text{ for } \varphi_{N-1} < \varphi \\ & \lim_{\tau \rightarrow +\infty} \chi_\tau(\varphi_j) = \frac{k_j + k_{j+1}}{2} \text{ for } 1 \leq j \leq N-2. \end{aligned}$$

Note that (4.10) gives $(N - 2) + 2 + (N - 2) = 2(N - 1)$ linear equations which determine the constants a_j, b_j . Using (4.7), this linear system reads

$$(4.11) \quad \begin{aligned} \sum_{l=1}^{N-1} a_l = k_1, \quad \sum_{l=1}^{N-1} b_l = k_N, \quad \sum_{l=1}^{j-1} b_l + \sum_{l=j}^{N-1} a_l = k_j \quad \text{for } 2 \leq j \leq N-1, \\ \frac{a_j + b_j}{2} + \sum_{l=1}^{j-1} b_l + \sum_{l=j+1}^{N-1} a_l = \frac{k_j + k_{j+1}}{2} \quad \text{for } 1 \leq j \leq N-2. \end{aligned}$$

Mutatis mutandis, all the results given in sections 2 and 3 are still valid by replacing α by (α, φ) (and therefore, we also suppose that $\varphi \in \mathcal{U}_{ad}$ with $0 \leq \varphi(x) \leq \varphi_{\max}$ for almost every $x \in \Omega$).

4.2.2. A projected gradient method for PDE constrained programs. Inspired by [34], we use a projected gradient method with alternating direction combined with a continuation method in order to numerically solve the optimal control problem (3.1). The process is presented in Algorithm 4.1, but a more thorough presentation will be made in a companion paper.

Algorithm 4.1 A descent algorithm for (3.1)

- 1: Input : $\tau > 0$ big enough. Initial guess $(\alpha_0, \varphi_0) \in \mathcal{U}_{ad}$, thermal conductivity constants $\{k_j\}$
 - 2: Sort $\{k_j\}$ in ascending order;
 - 3: $k:=0$;
 - 4: **while** stopping criteria is not met **do**
 - 5: Solve (1.3)-(1.4) using fixed point for (u, θ, p) ;
 - 6: Solve (4.1)-(4.2) using (u, θ, p) to get $(\lambda^u, \lambda^p, \lambda^\theta)$;
 - 7: Compute $\nabla_{\alpha_k, \varphi_k} \mathcal{J}$ thanks to (4.3);
 - 8: Use backtracking line search on α_k to get the step length t_k^α ;
 - 9: Set $\alpha_{k+1} = \Pi_{[0, \alpha_{\max}]}(\alpha_k - t_k^\alpha \nabla_{\alpha} \mathcal{J})$ pointwise;
 - 10: Use backtracking line search and a continuation method on φ_k to get φ_{k+1} ;
 - 11: Increase k .
 - 12: **end while**
-

Some details are needed to understand the algorithm:

- The first step consists in solving (1.3)-(1.4) using a fixed point approach, in the same spirit used for the proof of Theorem 2.4. Note that convergence is assured by Theorem 2.4 for small enough source terms.
- In order to find φ_{k+1} , one has to do a continuation method. The reason is that the derivative of k_τ with respect to φ is often too small for computing an acceptable descent step.

4.3. Numerical example. We illustrate our method using the cost function giving the pressure losses which can be written as [46, p. 7, Eq. (10)]:

$$(4.12) \quad \begin{aligned} \min \quad & - \int_{\Gamma} \left(p + \frac{1}{2} |u|^2 \right) u \cdot \vec{n} \, d\sigma + \text{Ri} \int_{\Omega} (u \cdot e_y) \theta \, d\Omega \\ \text{s.t.} \quad & \begin{cases} (u, \theta, p) \text{ solve (1.3) - (1.4) with} \\ \text{Re} = 200, \text{ Ri} = 0.6, \text{ Pr} = 0.71, \\ \alpha \in \mathcal{U}_{ad}, \end{cases} \end{aligned}$$

where \vec{n} denotes the normal to Γ . Ω designs a single horizontal pipe of height H and length L , as sketched in [Figure 1](#).

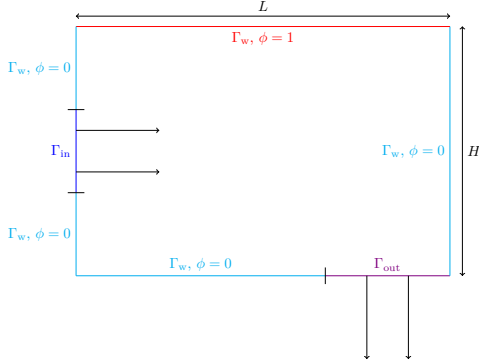


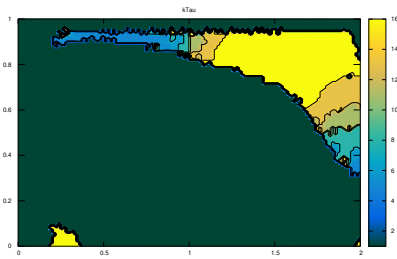
Fig. 1: Sketch of Ω

We suppose that the pipe is heated on a part of the upper side, where $\phi = 1$. We also set $u_{in}(x, y) = c_{in}y(H - y)$ on Γ_{in} , where $c_{in} = 1.8$. [Algorithm 4.1](#) has been implemented using FreeFem++ [[33](#)]. An example is shown in [Figure 2](#), where we plotted the solid distribution in the domain with the thermal diffusivity constants found. We make the following remarks:

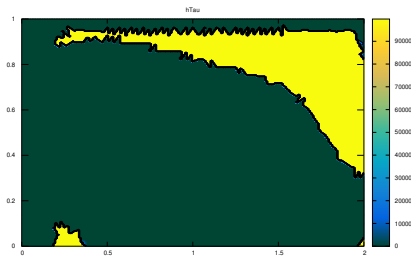
- The cost is well reduced along the different iterations by roughly 4% (going approximately from 1.46×10^{-2} at initialization to approximately

1.38×10^{-2} when the process ends). Remark that in this situation, the fluid doesn't need a lot of energy to move explaining the slight minimization. We noticed that the cost is first sharply reduced by α on the first iteration, and the following iterations were just adjustments made with φ .

- We have a clear border between the solid and the liquid domain with small region of porous media.
- The thermal conductivity constants are taken by big chunks. Note also that the conductivity of the fluid part is well respected.
- Several runs were made with different grid sizes ; the solutions were qualitatively the same, enforcing the result of [Theorem 3.2](#).
- The assumption $|D\alpha_h|(\Omega) \leq \kappa$ is not properly imposed, but the discretization makes it *de facto* verified for some κ big enough.



(a) k_τ .



(b) h_τ .

Fig. 2: Solutions for thermal conductivity in possible values [[5,6,8,11,13,16](#)]

5. Conclusion. We proved the well-posedness of the energy and penalized incompressible Navier-Stokes equations under Boussinesq assumption along with the

convergence of the Taylor-Hood finite element discretization of the model. We also proved some properties of a TO problem, namely the existence of a solution and the convergence of the discretization of this problem. All these results were applied to a TO problem with materials with piecewise constant thermal diffusivity and let us design a numerical method giving interesting results, changing the design of the channel along with the thermal conductivity. However, all these results present some limits: the source terms have to be small enough, and the penalization still contains some hard-coded parameters that need to be hand-tuned. Also, the numerical optimization method presented in this paper is rather simple, and enhancing this method could be an interesting study. On a final note, an other interesting study could focus on keeping the optimization problem without the penalization but rather with mixed state-control constraints that could be non-smooth, but more robust to the a priori chosen parameters.

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