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# A SHORT NOTE ON A PAIR OF MEROMORPHIC FUNCTIONS IN A $p$ -ADIC FIELD, SHARING A FEW SMALL ONES

ALAIN ESCASSUT AND C.C. YANG

ABSTRACT. A new Nevanlinna theorem on  $q$   $p$ -adic small functions is given. Let  $f, g$ , be two meromorphic functions on a complete ultrametric algebraically closed field  $\mathbb{K}$  of characteristic 0, or two meromorphic functions in an open disk of  $\mathbb{K}$ , that are not quotients of bounded analytic functions by polynomials. If  $f$  and  $g$  share 7 small meromorphic functions I.M., then  $f = g$ .

Better results hold when  $f$  and  $g$  satisfy some property of growth. Particularly, if  $f$  and  $g$  have finitely many poles or finitely many zeros and share 3 small meromorphic functions I.M., then  $f = g$ .

## 1. Main results

Let  $\mathbb{K}$  be a complete ultrametric algebraically closed field of characteristic 0. Let us fix  $a \in \mathbb{K}$  and let  $R \in ]0, +\infty[$ . We denote by  $d(a, R^-)$  the disk  $\{x \in \mathbb{K} \mid |x - a| < R\}$ .

We denote by  $\mathcal{A}(\mathbb{K})$  the  $\mathbb{K}$ -algebra of entire functions in  $\mathbb{K}$  and by  $\mathcal{M}(\mathbb{K})$  the field of meromorphic functions which is its field of fractions. We denote by  $\mathcal{A}(d(a, R^-))$  the  $\mathbb{K}$ -algebra of analytic functions in  $d(a, R^-)$  i.e. the set of power series converging in the disk  $d(a, R^-)$  and by  $\mathcal{M}(d(a, R^-))$  the field of meromorphic functions in  $d(a, R^-)$  i.e. the field of fractions of  $\mathcal{A}(d(a, R^-))$ . Moreover, we denote by  $\mathcal{A}_b(d(a, R^-))$  the  $\mathbb{K}$ -algebra of functions  $f \in \mathcal{A}(d(a, R^-))$  that are bounded in  $d(a, R^-)$ , by  $\mathcal{M}_b(d(a, R^-))$  its field of fractions and we put  $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$ .

We define  $N(r, f)$  ([1], chapter 40 or [3], chapter 2) in the same way as for complex meromorphic functions [2]. Let  $f$  be a meromorphic function in all  $\mathbb{K}$  (resp. in  $d(0, R^-)$ ) having no zero and no pole at 0. Let  $(a_n)_{n \in \mathbb{N}}$  be the sequence of poles of  $f$ , of respective order  $s_n$ , with  $|a_n| \leq |a_{n+1}|$  and, given  $r > 0$ , (resp.  $r \in ]0, R[$ ), let  $q(r)$  be such that  $|a_{q(r)}| \leq r, |a_{q(r)+1}| > r$ . We then denote by  $N(r, f)$  the counting function of the zeros of  $f$ , counting multiplicity,

as usual: for all  $r > 0$ , we put  $N(r, f) = \sum_{j=0}^{q(r)} s_j (\log |a_j| - \log(r))$ . Moreover, we

denote by  $\overline{N}(r, f)$  the counting function of the poles of  $f$ , ignoring multiplicity

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as  $\sum_{j=0}^{q(r)} (\log |a_j| - \log(r))$ . Next, we define the counting function of zeros of  $f$  as

$$Z(r, f) = N(r, \frac{1}{f}) \text{ and we put } \bar{Z}(r, f) = \bar{N}(r, \frac{1}{f}).$$

Similarly, considering a function  $f \in \mathcal{M}(d(a, R^-))$ , we denote by  $N(r, f)$  the counting function of the poles of  $f$ , counting multiplicity ( $0 < r < R$ ) counting multiplicity, as  $N(r, f) = \sum_{j=0}^{q(r)} s_j (\log |a_j - a| - \log(r))$ , the counting function of

the poles of  $f$ , ignoring multiplicity as  $\bar{N}(r, f) = \sum_{j=0}^{q(r)} (\log |a_j - a| - \log(r))$  ( $0 <$

$r < R$ ), we define the counting function of zeros of  $f$  as  $Z(r, f) = N(r, \frac{1}{f})$  and

$$\text{we put } \bar{Z}(r, f) = \bar{N}(r, \frac{1}{f}).$$

Finally, in each situation, we put  $T(r, f) = \max(Z(r, f), N(r, f))$ . Then  $T(r, f)$  is strictly increasing and has most properties of the characteristic function of a complex function, concerning operations (see [1], chapter 40 and [3], chapter 2.)

A function  $w \in \mathcal{M}(\mathbb{K})$  (resp.  $w \in \mathcal{M}(d(a, R^-))$ ) is called a *small function with respect to  $f$*  if  $\lim_{r \rightarrow +\infty} \frac{T(r, w)}{T(r, f)} = 0$  (resp.  $\lim_{r \rightarrow R^-} \frac{T(r, w)}{T(r, f)} = 0$ ) and we denote by  $\mathcal{M}_f(\mathbb{K})$  (resp.  $\mathcal{M}_f(d(a, R^-))$ ) the set of small functions with respect to  $f$ .

Two functions  $f, g \in \mathcal{M}(\mathbb{K})$  (resp.  $f, g \in \mathcal{M}_u(d(a, R^-))$ ) are said to *share a small function I.M.*  $w \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  (resp.  $w \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ ) if  $f(z) = w(z)$  holds if and only if  $g(z) = w(z)$ .

A function  $F \in \mathcal{M}(\mathbb{K})$  (resp.  $F \in \mathcal{M}(d(a, R^-))$ ) will be called *climbing* if  $\liminf_{r \rightarrow +\infty} \frac{Z(r, F)}{N(r, F)} > 1$  (resp.  $\liminf_{r \rightarrow R^-} \frac{Z(r, F)}{N(r, F)} > 1$ ).

A function  $F \in \mathcal{M}(\mathbb{K})$  (resp.  $F \in \mathcal{M}(d(a, R^-))$ ) will be called *downing* if  $\frac{1}{F}$  is climbing.

A function  $F \in \mathcal{M}(\mathbb{K})$  (resp.  $F \in \mathcal{M}(d(a, R^-))$ ) will be called *strongly climbing* if  $N(r, f) = o(Z(r, f))$ ,  $r \rightarrow +\infty$  (resp.  $r \rightarrow R$ ).

A function  $F \in \mathcal{M}(\mathbb{K})$  (resp.  $F \in \mathcal{M}(d(a, R^-))$ ) will be called *strongly downing* if  $\frac{1}{F}$  is strongly climbing.

**Remark:** If a function  $f \in \mathcal{M}(\mathbb{K})$  (resp.  $f \in \mathcal{M}(d(a, R^-))$ ) has finitely many poles and infinitely many zeros, it is obviously strongly climbing.

Here we aim at studying the problem of two meromorphic functions sharing a few small meromorphic functions I.M. in order to show that these two functions are equal. Indeed thanks to Yamanoi's Nevanlinna Second Main Theorem, a similar result is known in complex analysis when two meromorphic functions share 5 small meromorphic functions. But in p-adic analysis, no Theorem similar

to Yamanoi's Theorem is known. However, here we give Theorem 2 which makes a tool other than Yamanoi's Theorem to derive some similar results as follows.

**Theorem 1:** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental (resp.  $f, g \in \mathcal{M}_u(d(a, R^-))$ ), be distinct and share  $q$  distinct small functions I.M.  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, q$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  ( $j = 1, \dots, q$ )) other than the constant  $\infty$ . Then*

$$\sum_{j=1}^q \overline{Z}(r, f - w_j) \leq \overline{Z}(r, f - g) + o(T(r, f)) + o(T(r, g)).$$

**Theorem 2:** *Let  $f \in \mathcal{M}(\mathbb{K})$  be transcendental (resp.  $f \in \mathcal{M}_u(d(a, R^-))$ ) and let  $w_j \in \mathcal{M}_f(\mathbb{K})$  ( $j = 1, \dots, q$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-))$ ) be  $q$  distinct small functions other than the constant  $\infty$ . Then*

$$qT(r, f) \leq 3 \sum_{j=1}^q \overline{Z}(r, f - w_j) + o(T(r, f)).$$

Moreover, if  $f$  is strongly climbing, then

$$qT(r, f) \leq 2 \sum_{j=1}^q \overline{Z}(r, f - w_j) + o(T(r, f)).$$

**Corollary 1:** *Let  $f \in \mathcal{A}(\mathbb{K})$  be transcendental (resp. Let  $f \in \mathcal{A}_u((0, R^-))$ ) and let  $w_j \in \mathcal{M}_f(\mathbb{K})$  ( $j = 1, \dots, q$ ) (resp.  $w_j \in \mathcal{M}_f(d(0, R^-))$ , ( $j=1, \dots, q$ )) be  $q$  distinct small functions other than the constant  $\infty$ . Then*

$$qT(r, f) \leq 2 \sum_{j=1}^q \overline{Z}(r, f - w_j) + o(T(r, f)).$$

**Theorem 3:** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental (resp.  $f, g \in \mathcal{M}_u(d(a, R^-))$ ) be distinct and share 7 distinct small functions (other than the constant  $\infty$ ) I.M.  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, 7$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  ( $j = 1, \dots, 7$ )). Then  $f = g$ .*

Moreover, if  $f$  and  $g$  are climbing and share 6 distinct small functions (other than the constant  $\infty$ ) I.M. then  $f = g$ .

**Corollary 2:** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  be downing and share 6 distinct small functions (other than the constant 0) I.M.  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, 6$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  ( $j = 1, \dots, 6$ )). Then  $f = g$ .*

**Theorem 4:** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental, strongly climbing and share 3 distinct small functions (other than the constant  $\infty$ ) I.M.  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, 2, 3$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  ( $j = 1, 2, 3$ )). Then  $f = g$ .*

**Corollary 3:** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental, strongly downing and share 3 distinct small functions (other than the constant 0) I.M.  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, 2, 3$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  ( $j = 1, 2, 3$ )). Then  $f = g$ .*

**Corollary 4:** *Let  $f, g \in \mathcal{A}(\mathbb{K})$  be transcendental (resp.  $f, g \in \mathcal{A}_u(d(a, R^-))$ ) and share 3 distinct small functions (other than the constant  $\infty$ ) I.M.  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, 2, 3$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  ( $j = 1, 2, 3$ )). Then  $f = g$ .*

**Remarks:** The results known in complex analysis suggest that the number 7 obtained in Theorem 2 might be improved, concerning  $p$ -adic meromorphic functions. On the contrary, concerning analytic functions, the number 3 obtained in Theorem 4 seems to be the best possible. In Theorem 41.1 of [1] as in Theorem 3.2 of [3], it is shown that two entire functions  $f, g$  sharing 2 constants are equal and in Theorem 41.2 of [1], it is shown that two unbounded analytic functions  $f, g \in \mathcal{A}_u(d(0, R^-))$  sharing 3 constants are equal. Here we see that this last statement is generalized.

The question whether Corollary 3 is sharp is interesting in  $\mathcal{A}_u(d(a, R^-))$ . If  $\mathbb{K}$  had positive characteristic, it would be easy to make a pair of distinct entire functions  $f$  and  $g$  sharing IM two constants. But then the Nevanlinna Theory used here would not apply. On the other hand, Theorem 41.1 of [1] shows that in  $\mathcal{A}(\mathbb{K})$  it is not sharp for constants.

## 2. The proofs

In all the proofs, we can obviously assume that the disk  $d(a, R^-)$  is  $d(0, R^-)$ . In order to prove the theorems, we need to state the following four lemmas.

**Lemma 1:** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental (resp. let  $f, g \in \mathcal{M}_u(d(0, R^-))$ ) and let  $M(r) = (\max(T(r, f), T(r, g)))$ . Then  $T(r, f+g) \leq T(r, f) + T(r, g) + O(1)$ . Moreover, if  $f$  and  $g$  are climbing, then there exists  $\lambda > 0$  and  $S > 0$  (resp.  $S \in ]0, R[$ ) such that*

$$T(r, f+g) \leq (2 - \lambda) \max(T(r, f), T(r, g)) + O(1), \forall r > S$$

(resp.  $\forall r \in ]S, R[$ ).

Furthermore, if  $f$  and  $g$  are strongly climbing then

$$T(r, f+g) \leq M(r) + o(M(r)), \forall r > S$$

(resp.  $\forall r \in ]S, R[$ ).

**Proof:** The general statement concerning meromorphic functions  $f, g$  that are not supposed to be climbing is well known and comes, for instance from Theorem 40.8 in [1]. In all the sproof, we set  $M(r) = \max(T(r, f), T(r, g))$  and for simplicity, we suppose first that  $f$  and  $g$  belong to  $\mathcal{M}(\mathbb{K})$ .

Suppose now that  $f$  and  $g$  are just climbing. By Lemma 32.3 in [1], we can write  $f$  and  $g$  in the form  $f = \frac{h_1}{\ell_1}$  and  $g = \frac{h_2}{\ell_2}$ , with  $h_1, h_2, \ell_1, \ell_2 \in \mathcal{A}(\mathbb{K})$  where  $h_i$  and  $\ell_i$  have no common zeros [1]. Then, applying hypotheses on  $f$  and  $g$ , there

exists  $S > 0$  and  $\lambda > 0$ , such that  $\frac{T(r, \ell_1)}{M(r)} \leq 1 - \lambda$  and  $\frac{T(r, \ell_2)}{M(r)} \leq 1 - \lambda \forall r > S$ .

Consequently,  $T(r, h_1 \ell_2) \leq (2 - \lambda)M(r)$ ,  $T(r, h_2 \ell_1) \leq (2 - \lambda)M(r) \forall r > S$ , hence  $T(r, h_1 \ell_2 - h_1 \ell_2) \leq (2 - \lambda)M(r) + O(1) \forall r > S$  while  $T(r, \ell_1 \ell_2) \leq (2 - 2\lambda)M(r)$ . Consequently

$$T(r, f + g) \leq (2 - \lambda)M(r) + O(1) \forall r > S.$$

Let us now suppose that  $f$  and  $g$  strongly climbing. Let us write again  $f$  and  $g$  in the form  $\frac{h_1}{\ell_1}$  and  $g = \frac{h_2}{\ell_2}$ , with  $h_1, h_2, \ell_1, \ell_2 \in \mathcal{A}(\mathbb{K})$  where  $h_i$  and  $\ell_i$  have no common zeros. For every  $r$ , set  $M(r) = \max(T(r, f), T(r, g))$ . Since  $h_1, h_2, \ell_1, \ell_2$  belong to  $\mathcal{A}(\mathbb{K})$ , by Theorem 40.8 of [1], we have:

$$T(r, h_1 \ell_2 - h_2 \ell_1) \leq \max(T(r, h_1 \ell_2), T(r, h_2 \ell_1)) + o(M(r))$$

Consequently, we can write  $T(r, \ell_1 \ell_2) \leq T(r, \ell_1) + T(r, \ell_2) \leq M(r) + o(M(r)) \forall r > S$  therefore  $T(r, f + g) \leq M(r) + o(M(r))$ .

Suppose now that  $f$  and  $g$  belong to  $\mathcal{M}_u(d(0, R^-))$ . Without loss of generality, we can assume that the field  $\mathbb{K}$  is spherically complete because the Nevanlinna functions  $Z(r, f)$ ,  $N(r, f)$ ,  $T(r, f)$  are the same in a spherically algebraically closed extension of  $\mathbb{K}$ . In such a field, we can write  $f$  in the form  $\frac{h_1}{\ell_1}$  and  $g$  in the form  $\frac{h_2}{\ell_2}$  where  $h_i$  and  $\ell_i$  have no common zero and then we can make the same reasonings as in  $\mathcal{M}(\mathbb{K})$ . That ends the proof of Lemma 1.

The following Lemma 2 is Lemma 40.10 in [1] (see also Proposition 2.5 in [3]):

**Lemma 2:** *Let  $f \in \mathcal{M}(\mathbb{K})$ . Then  $T(r, f) \leq O(\log(r))$  in  $]0, +\infty[$  if and only if  $f$  belongs to  $\mathbb{K}(x)$ . Let  $f \in \mathcal{M}(d(0, R^-))$ . Either  $f \in \mathcal{M}_b(d(0, R^-))$  and then  $T(r, f)$  is bounded in  $]0, R[$  or  $f \in \mathcal{M}_u(d(0, R^-))$  and then  $\lim_{r \rightarrow R^-} T(r, f) = +\infty$ .*

The following Lemma 3 comes from Theorems 43.10 and 43.11 in [1] (see also Theorem 2.21 in [3]).

**Lemma 3:** *Let  $f \in \mathcal{M}(\mathbb{K})$  be transcendental (resp. let  $f \in \mathcal{M}_u(d(0, R^-))$ ) and let  $w_1, w_2, w_3 \in \mathcal{M}_f(\mathbb{K})$  (resp. let  $w_1, w_2, w_3 \in \mathcal{M}_f(d(0, R^-))$ ). Then*

$$T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + \bar{Z}(r, f - w_3) + o(T(r, f))$$

Moreover, if  $f$  is strongly climbing, then

$$T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + o(T(r, f)).$$

**General remark:** By Lemma 2, all functions  $h$  we will consider in  $\mathcal{M}(\mathbb{K})$  (resp. in  $\mathcal{M}_u(d(0, R^-))$ ) satisfy  $\lim_{r \rightarrow +\infty} T(r, h) = +\infty$ , (resp.  $\lim_{r \rightarrow R} T(r, h) = +\infty$ ).

**Proof of Theorem 1:** Suppose that  $f$  and  $g$  belong to  $\mathcal{M}(\mathbb{K})$ , are distinct and share  $q$  distinct small functions I.M.  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, q$ ) (resp.  $w_j \in \mathcal{M}_f(d(0, R^-)) \cap \mathcal{M}_g(d(0, R^-))$ ) ( $j = 1, \dots, q$ ).

Let  $b$  be a zero of  $f - w_i$  for a certain index  $i$ . Then it is also a zero of  $g - w_i$ . Suppose that  $b$  is counted several times in the sum  $\sum_{j=1}^q \bar{Z}(r, f - w_j)$ , which means that it is a zero of another function  $f - w_k$  for a certain index  $k \neq i$ . Then we have  $w_i(b) = w_k(b)$  and hence  $b$  is a zero of the function  $w_i - w_k$  which belongs to  $\mathcal{M}_f(\mathbb{K})$ . Now, put  $\tilde{Z}(r, f - w_1) = \bar{Z}(r, f - w_1)$  and for each  $j > 1$ , let  $\tilde{Z}(r, f - w_j)$  be the counting function of zeros of  $f - w_j$  in the disk  $d(0, r^-)$  ignoring multiplicity and avoiding the zeros already counted as zeros of  $f - w_k$  for some  $k < j$ . Consider now the sum  $\sum_{j=1}^q \tilde{Z}(r, f - w_j)$ . Since the functions  $w_i - w_j$  belong to  $\mathcal{M}_f(\mathbb{K})$ , clearly, we have

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) = \sum_{j=1}^q \tilde{Z}(r, f - w_j) = o(T(r, f))$$

Now, it is clear, from the assumption, that  $f(x) - w_j(x) = 0$  implies  $g(x) - w_j(x) = 0$  and hence  $f(x) - g(x) = 0$ . Since  $f - g$  is not the identically zero function, it follows that

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g).$$

Consequently,

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g) + o(T(r, f)) + o(T(r, g)).$$

Now, if  $f$  and  $g$  belong to  $\mathcal{M}(d(0, R^-))$ , the proof is exactly the same.

**Proof of Theorems 2:** By Lemma 3, for every triplet  $(i, j, k)$  such that  $1 \leq i \leq j \leq k \leq q$ , we can write

$$T(r, f) \leq \bar{Z}(r, f - w_i) + \bar{Z}(r, f - w_j) + \bar{Z}(r, f - w_k) + o(T(r, f)).$$

The number of such inequalities is  $C_q^3$ . Summing up, we obtain

$$(1) \quad C_q^3 T(r, f) \leq \sum_{(i,j,k), 1 \leq i \leq j \leq k \leq q} \bar{Z}(r, f - w_i) + \bar{Z}(r, f - w_j) + \bar{Z}(r, f - w_k) + o(T(r, f)).$$

In this sum, for each index  $i$ , the number of terms  $\bar{Z}(r, f - w_i)$  is clearly  $C_{q-1}^2$ . Consequently, by (1) we obtain

$$C_q^3 T(r, f) \leq C_{q-1}^2 \sum_{i=1}^q \bar{Z}(r, f - w_i) + o(T(r, f))$$

and hence

$$\frac{q}{3}T(r, f) \leq \sum_{i=1}^q \bar{Z}(r, f - w_i) + o(T(r, f)).$$

Suppose now that  $f$  is strongly climbing. By Lemma 3, for every pair  $(i, j)$  such that  $1 \leq i \leq j \leq q$ , we have

$$T(r, f) \leq \bar{Z}(r, f - w_i) + \bar{Z}(r, f - w_j) + o(T(r, f)).$$

The number of such inequalities is then  $C_q^2$ . Summing up we now obtain

$$(2) \quad C_q^2 T(r, f) \leq \sum_{(i,j), 1 \leq i \leq j \leq q} \bar{Z}(r, f - w_i) + \bar{Z}(r, f - w_j) + o(T(r, f)).$$

In this sum, for each index  $i$ , the number of terms  $\bar{Z}(r, f - w_i)$  is clearly  $C_{q-1}^1 = q - 1$ . Consequently, by (1) we obtain

$$C_q^2 T(r, f) \leq (q - 1) \sum_{i=1}^q \bar{Z}(r, f - w_i) + o(T(r, f))$$

and hence

$$\frac{q}{2}T(r, f) \leq \sum_{i=1}^q \bar{Z}(r, f - w_i) + o(T(r, f)).$$

**Proof of Theorems 3 and 4:** In all the proofs of Theorems 3 and 4, we put  $M(r) = \max(T(r, f), T(r, g))$ . Suppose that  $f$  and  $g$  are distinct and share  $q$  small functions I.M.  $w_j$ , ( $1 \leq j \leq q$ ). By Theorem 2, we have

$$qT(r, f) \leq 3 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(T(r, f)).$$

But thanks to Theorem 1, we can derive

$$qT(r, f) \leq 3T(r, f - g) + o(T(r, f))$$

and similiary

$$qT(r, g) \leq 3T(r, f - g) + o(T(r, g))$$

hence

$$(1) \quad qM(r) \leq 3T(r, f - g) + o(M(r)).$$

By Lemma 1, we can derive that

$$qM(r) \leq 3(T(r, f) + T(r, g)) + o(M(r))$$

and hence  $qM(r) \leq 6M(r) + o(M(r))$ . That applies to the situation when  $f$  and  $g$  belong to  $\mathcal{M}(\mathbb{K})$  as well as when  $f$  and  $g$  belong to  $\mathcal{M}_u(d(0, R^-))$ . In the hypotheses of Theorem 3, this is impossible if  $q \geq 7$  and hence the first statement of Theorem 3 is proved.

Suppose now that  $f$  and  $g$  are climbing. Then by Lemma 1 there exists  $\lambda > 0$  and  $S > 0$  (resp.  $S \in ]0, R[$ ) such that  $T(r, f - g) \leq (2 - \lambda)M(r) + O(1) \forall r > S$ , (resp.  $\forall r \in ]S, R[$ ). Consequently, by (1) we obtain  $qM(r) \leq 3(2 - \lambda)M(r) + o(M(r)) \forall r > S$ , (resp.  $\forall r \in ]S, R[$ ). Thus, this inequality is impossible if  $q \geq 6$ .



That finishes proving Theorem 3 when  $f$  and  $g$  are meromorphic transcendental climbing functions.

Similarly, if  $f$  and  $g$  are meromorphic climbing functions in  $d(0, R^-)$ , we can clearly make the same reasoning with  $S \in ]0, R[$  and  $r \in ]S, R[$ . That ends the proof of Theorem 3.

Consider now the hypotheses of Theorem 4. By Lemma 1, Relation (2) gives us

$$qM(r) \leq 2M(r) + o(M(r))$$

which is obviously absurd whenever  $q \geq 3$  and proves that  $f = g$  when  $f$  and  $g$  belong to  $\mathcal{M}(\mathbb{K})$  as well as when  $f$  and  $g$  belong to  $\mathcal{M}_u(d(0, R^-))$ .

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