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# Finite element discretization of an obstacle problem for Naghdi's shell

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**Abstract:** In this work, we present a finite element discretization of a contact model of a Naghdi shell with a rigid body. The local basis-free contact formulation introduced in [2] allows for existence and uniqueness of the discret problem for general shells with discontinuous curvatures and to perform its *a priori* analysis.

## 1 Introduction

We are interested in the discretization of the unilateral contact of a shell with a rigid body. We consider here the Naghdi's shell model in the free-basis formulation introduced by Blouza [5] and Blouza and Le Dret [6]. This formulation relies on the idea of using Cartesian coordinates for the unknowns, which makes easier the treatment of the contact of the shell with an obstacle. A model for the contact of this shell with a rigid body on which it is clamped has first been derived in [2], where its well-posedness is proved under some realistic assumptions. The full problem involves two Lagrange multipliers but we are not really interested by their approximations, even if one of these multipliers is involved in the discrete problem.

The system results into a double mixed problem (i.e., a mixed problem with a double Lagrange multipliers) combining variational equalities and inequalities. The finite element discretization of variational inequalities has already been considered in [8] and more

recently in [15]. So, following their approach, we propose a finite element discretization of our problem relying on the Galerkin method and perform its numerical analysis. *A priori* error estimates can be established through some now standard arguments. As the solution is in  $H^1$ ,  $\mathcal{C}^0$ -Lagrange  $\mathcal{P}_1$  elements should be sufficient. Even if the discretization that we consider is of low order, these estimates prove its convergence.

The article is as follows. We first briefly recall the geometry of the midsurface and Nagdhi shell formulation given in Blouza [5] and Blouza and Le Dret [6]. This formulation involves the displacement vector and the infinitesimal rotation vector, a vector unknown that is tangent to the midsurface. Such tangency cannot be implemented in a conforming way in finite element spaces (a problem that does not occur in the classical covariant formulation). Therefore, in Section 3, we introduce the local basis-free contact model as well as its well-posedness. Section 4 is devoted to finite element discretization of our formulation. We also prove that the discret problem has a unique solution. Finally, in Section 5 we perform the *a priori* analysis of the discretization.

## 2 Notations

Greek indices and exponents take their values in the set  $\{1, 2\}$  and Latin indices and exponents take their values in the set  $\{1, 2, 3\}$ . Unless otherwise specified, the summation convention for indices and exponents is assumed.

Let  $(e_1, e_2, e_3)$  be the canonical orthonormal basis of the Euclidean space  $\mathbb{R}^3$ . We note  $u \cdot v$  the inner product of  $\mathbb{R}^3$ ,  $|u| = \sqrt{u \cdot v}$  the associated Euclidean norm and  $u \wedge v$  the vector product of  $u$  and  $v$ .

Let  $\omega$  be a bounded connected domain in  $\mathbb{R}^2$  with a Lipschitz-continuous boundary  $\partial\omega$ . We consider a shell whose midsurface is given by  $\mathcal{S} = \varphi(\bar{\omega})$  where  $\varphi \in W^{2,\infty}(\omega; \mathbb{R}^3)$  is one-to-one mapping such that the two vectors

$$\mathbf{a}_\alpha(\mathbf{x}) = \partial_\alpha \varphi(\mathbf{x}),$$

are linearly independent at each point  $\mathbf{x}$  of  $\bar{\omega}$ . Thus,

$$\mathbf{a}_3(\mathbf{x}) = \frac{\mathbf{a}_1(\mathbf{x}) \wedge \mathbf{a}_2(\mathbf{x})}{|\mathbf{a}_1(\mathbf{x}) \wedge \mathbf{a}_2(\mathbf{x})|}$$

is the unit normal vector on the midsurface at point  $\varphi(\mathbf{x})$ . The vectors  $\mathbf{a}_i(\mathbf{x})$  define the local covariant basis at point  $\varphi(\mathbf{x})$ . The contravariant basis  $\mathbf{a}^j(\mathbf{x})$  is defined by the relations  $\mathbf{a}_i \cdot \mathbf{a}^j = \delta_i^j$  where  $\delta_i^j$  is the Kronecker symbol. In particular  $\mathbf{a}_3(\mathbf{x})$  coincides with  $\mathbf{a}^3(\mathbf{x})$ . Note that all these vectors belong to  $W^{1,\infty}(\omega; \mathbb{R}^3)$ .

The first and second fundamental forms of the surface are given in covariant components by

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \quad \text{and} \quad b_{\alpha\beta} = \mathbf{a}_3 \cdot \partial_\beta \mathbf{a}_\alpha.$$

We set  $a(\mathbf{x}) = |\mathbf{a}_1(\mathbf{x}) \wedge \mathbf{a}_2(\mathbf{x})|^2$  so that  $\sqrt{a(\mathbf{x})}$  is the area element of the midsurface in the chart  $\varphi$ . Similarly, the length element  $\ell$  on the boundary  $\partial\omega$  is given by  $\sqrt{a^{\alpha\beta}\tau_\alpha\tau_\beta}$ , with the standard summation convention for repeated indices and exponents, the  $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$  being the contravariant components of the first fundamental form and  $(\tau_1, \tau_2)$  being the covariant coordinates of a unit vector tangent to  $\partial\omega$ . The thickness of the shell, denoted by  $e$ , is a positive constant.

In the case of a homogeneous, isotropic material with Young modulus  $E > 0$  and Poisson ratio  $\nu$ ,  $0 \leq \nu < \frac{1}{2}$ , the contravariant components of the elasticity tensor  $a^{\alpha\beta\rho\sigma}$  are given by

$$a^{\alpha\beta\rho\sigma} = \frac{E}{2(1+\nu)}(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}) + \frac{E\nu}{1-\nu^2}a^{\alpha\beta}a^{\rho\sigma}. \quad (2.1)$$

We note that each component of the elasticity tensor belongs to  $L^\infty(\omega)$ . Moreover, this tensor satisfies the usual symmetry properties and is uniformly strictly positive.

In this context, the covariant components of the change of metric tensor read

$$\gamma_{\alpha\beta}(\mathbf{u}) = \frac{1}{2}(\partial_\alpha \mathbf{u} \cdot \mathbf{a}_\beta + \partial_\beta \mathbf{u} \cdot \mathbf{a}_\alpha), \quad (2.2)$$

the covariant components of the change of transverse shear tensor read

$$\delta_{\alpha 3}(\mathbf{u}, \mathbf{r}) = \frac{1}{2}(\partial_\alpha \mathbf{u} \cdot \mathbf{a}_3 + \mathbf{r} \cdot \mathbf{a}_\alpha), \quad (2.3)$$

and the covariant components of the change of curvature tensor read

$$\chi_{\alpha\beta}(\mathbf{u}, \mathbf{r}) = \frac{1}{2}(\partial_\alpha \mathbf{u} \cdot \partial_\beta \mathbf{a}_3 + \partial_\beta \mathbf{u} \cdot \partial_\alpha \mathbf{a}_3 + \partial_\alpha \mathbf{r} \cdot \mathbf{a}_\beta + \partial_\beta \mathbf{r} \cdot \mathbf{a}_\alpha), \quad (2.4)$$

as introduced in [6]. Note that all these quantities make sense for shells with little regularity, and are easily expressed with the Cartesian coordinates of the unknowns and geometrical data.

We assume that the boundary  $\partial\omega$  of the chart domain is divided into two parts:  $\gamma_0$  on which the shell is clamped and the complementary part  $\gamma_1 = \partial\omega \setminus \gamma_0$  on which the shell is subjected to applied tractions and moments. From now on, we suppose that  $\gamma_0$  has a finite number of connected components and a strictly positive 1-dimensional measure. To take into account the boundary conditions, we define the space

$$H_{\gamma_0}^1(\omega; \mathbb{R}^3) = \{\mu \in H^1(\omega; \mathbb{R}^3); \mu = 0 \text{ on } \gamma_0\}. \quad (2.5)$$

Let us now consider the function space introduced in [6], which is appropriate in the context of shells with curvature discontinuities,

$$\mathbb{V}(\omega) = \{V = (\mathbf{v}, \mathbf{s}) \in H_{\gamma_0}^1(\omega; \mathbb{R}^3) \times H_{\gamma_0}^1(\omega; \mathbb{R}^3); \mathbf{s} \cdot \mathbf{a}_3 = 0 \text{ in } \omega\}. \quad (2.6)$$

It is a subspace of the space

$$\mathbb{X}(\omega) = H_{\gamma_0}^1(\omega; \mathbb{R}^3) \times H_{\gamma_0}^1(\omega; \mathbb{R}^3),$$

which is endowed with the natural Hilbert norm

$$\|V\|_{\mathbb{X}(\omega)} = (\|\mathbf{v}\|_{H^1(\omega; \mathbb{R}^3)}^2 + \|\mathbf{s}\|_{H^1(\omega; \mathbb{R}^3)}^2)^{1/2}. \quad (2.7)$$

We recall the variational formulation of the problem corresponding to the linear Naghdi model for shells with little regularity, where the unknowns are the displacement  $\mathbf{u}$  of the shell and the rotation  $\mathbf{r}$  of orthogonal fibers to its midsurface. For data  $(\mathbf{f}, \mathbf{N}, \mathbf{M})$  in  $L^2(\omega; \mathbb{R}^3) \times L^2(\gamma_1; \mathbb{R}^3) \times L^2(\gamma_1; \mathbb{R}^3)$ , it reads : Find  $U = (\mathbf{u}, \mathbf{r})$  in  $\mathbb{V}(\omega)$  such that

$$a(U, V) = \mathcal{L}(V), \quad \forall V \in \mathbb{V}(\omega) \quad (2.8)$$

where the bilinear form  $a(\cdot, \cdot)$  is defined by (with the standard summation convention)

$$\begin{aligned} a(U, V) = & \int_{\omega} \left\{ e a^{\alpha\beta\rho\sigma} \left[ \gamma_{\alpha\beta}(\mathbf{u}) \gamma_{\rho\sigma}(\mathbf{v}) + \frac{e^2}{12} \chi_{\alpha\beta}(U) \chi_{\rho\sigma}(V) \right] \right. \\ & \left. + 2e \frac{E}{1+\nu} a^{\alpha\beta} \delta_{\alpha 3}(U) \delta_{\beta 3}(V) \right\} \sqrt{a} \, dx, \end{aligned} \quad (2.9)$$

and the linear form  $\mathcal{L}(\cdot)$  is given by

$$\mathcal{L}(V) = \int_{\omega} \mathbf{f} \cdot \mathbf{v} \sqrt{a} \, dx + \int_{\gamma_1} (\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \mathbf{s}) \ell \, d\tau. \quad (2.10)$$

The data  $\mathbf{f}$ ,  $\mathbf{N}$  and  $\mathbf{M}$  represent a given resultant force density, an applied traction density and an applied moment density, respectively. The following ellipticity property is proved in [6]: There exists a constant  $\alpha > 0$  such that

$$\forall V \in \mathbb{V}(\omega), \quad a(V, V) \geq \alpha \|V\|_{\mathbb{X}(\omega)}^2. \quad (2.11)$$

It yields that this problem admits a unique solution  $U$  in  $\mathbb{V}(\omega)$ .

### 3 The contact model

We are interested in studying the contact of this shell with a rigid obstacle contained in the half-space  $\mathbf{z} \cdot \mathbf{e}_3 < 0$  and such that its boundary occupies the whole plane  $\mathbf{z} \cdot \mathbf{e}_3 = 0$ . So, from now on, we assume without restriction that the function  $\varphi$  satisfies  $\varphi(\mathbf{x}) \cdot \mathbf{e}_3 > 0$  for all  $\mathbf{x}$  in  $\bar{\omega}$ . Thus, the contact occurs on part of the lower surface of the shell, namely on the surface  $\{\varphi(\mathbf{x}) - \frac{e}{2} \mathbf{a}_3, \mathbf{x} \in \bar{\omega}\}$ , see ([2], Section 3) for more details. We are thus led to set:

$$\Phi(\mathbf{x}) = \left( \frac{e}{2} \mathbf{a}_3(\mathbf{x}) - \varphi(\mathbf{x}) \right) \cdot \mathbf{e}_3, \quad (3.1)$$

and let us introduce the convex set:

$$\mathbb{K}_\Phi(\omega) = \left\{ V \in \mathbb{V}(\omega); \left( \mathbf{v} - \frac{e}{2} \mathbf{s} \right) \cdot \mathbf{e}_3 \geq \Phi \text{ a.e. in } \omega \right\}. \quad (3.2)$$

We first consider the problem: *Find  $U$  in  $\mathbb{K}_\Phi(\omega)$  such that*

$$a(U, V - U) \geq \mathcal{L}(V - U), \quad \forall V \in \mathbb{K}_\Phi(\omega). \quad (3.3)$$

It is checked in ([2], Section 3) that this problem is the basic one for the contact. Moreover, it follows from the ellipticity property (2.11), combined with the theorem of Lions and Stampacchia [14] that this problem has a unique solution  $U$  in  $\mathbb{K}_\Phi(\omega)$  whenever  $\mathbb{K}_\Phi(\omega)$  is not empty (sufficient conditions for this are given later on).

However, from a numerical point of view, we immediately encounter a problem since the constraint  $\mathbf{r} \cdot \mathbf{a}_3 = 0$  clearly cannot be implemented in a conforming way for a general shell. We thus introduce the convex set in which the unknowns are the displacement  $u$  and  $r$ , elements of the space  $\mathbb{X}(\omega)$  without any constraint:

$$\mathbb{N}_\Phi(\omega) = \left\{ V \in \mathbb{X}(\omega); \left( \mathbf{v} - \frac{e}{2} \mathbf{s} \right) \cdot \mathbf{e}_3 \geq \Phi \text{ a.e. in } \omega \right\}, \quad (3.4)$$

together with the space  $\mathbb{M}(\omega) = H_{\gamma_0}^1(\omega)$ , and consider the problem: *Find  $(U, \psi)$  in  $\mathbb{N}_\Phi(\omega) \times \mathbb{M}(\omega)$  such that*

$$\begin{aligned} a(U, V - U) + b(V - U, \psi) &\geq \mathcal{L}(V - U), \quad \forall V \in \mathbb{N}_\Phi(\omega), \\ b(U, \chi) &= 0, \quad \forall \chi \in \mathbb{M}(\omega), \end{aligned} \quad (3.5)$$

where the form  $b(\cdot, \cdot)$  is now defined by

$$b(V, \chi) = \int_{\omega} \partial_\alpha(\mathbf{s} \cdot \mathbf{a}_3) \partial_\alpha \chi \, d\mathbf{x}. \quad (3.6)$$

It is readily checked that, for any solution  $(U, \psi)$  of problem (3.5), its part  $U$  is a solution of problem (3.3).

Let us check now the well-posedness of this problem. We need a preliminary lemma for that. Let  $\mathbb{N}_{\sharp}(\omega)$  be the following subspace

$$\mathbb{N}_{\sharp}(\omega) = \left\{ V \in \mathbb{X}(\omega); \left( \mathbf{v} - \frac{e}{2} \mathbf{s} \right) \cdot \mathbf{e}_3 = 0 \text{ a.e. in } \omega \right\} \quad (3.7)$$

**Lemma 3.1** *There exists a constant  $\beta > 0$  such that the following inf-sup condition holds*

$$\forall \chi \in \mathbb{M}(\omega), \quad \sup_{V \in \mathbb{N}_{\sharp}(\omega)} \frac{b(V, \chi)}{\|V\|_{\mathbb{X}(\omega)}} \geq \beta \|\chi\|_{\mathbb{M}(\omega)}. \quad (3.8)$$

*Proof.* Let  $\chi$  be any function in  $\mathbb{M}(\omega)$ . Then, it follows from the regularity of  $\mathbf{a}_3$  that both functions  $\mathbf{s} = \chi \mathbf{a}_3$  and  $\mathbf{v} = \frac{e}{2} \mathbf{s}$  belong to  $H_{\gamma_0}^1(\omega; \mathbb{R}^3)$  and even that  $V = (\mathbf{v}, \mathbf{s})$  belongs to  $\mathbb{N}_{\sharp}(\omega)$ . With this choice of  $V$ , we have

$$b(V, \chi) = \|\chi\|_{\mathbb{M}(\omega)}^2 \quad \text{and} \quad \|V\|_{\mathbb{X}(\omega)} \leq c \|\chi\|_{\mathbb{M}(\omega)},$$

whence the desired inf-sup condition.

**Theorem 3.1** *Assume that the function  $\Phi$  satisfies*

$$\Phi(\mathbf{x}) \leq 0 \text{ for a.e. } \mathbf{x} \text{ in } \omega \quad \text{and} \quad \Phi(\mathbf{x}) = 0 \text{ for a.e. } \mathbf{x} \text{ on } \gamma_0. \quad (3.9)$$

*Then, for any data  $(\mathbf{f}, \mathbf{N}, \mathbf{M})$  in  $L^2(\omega; \mathbb{R}^3) \times L^2(\gamma_1; \mathbb{R}^3) \times L^2(\gamma_1; \mathbb{R}^3)$ , problem (2.16) has a unique solution  $(U, \psi)$  in  $\mathbb{X}(\omega) \times \mathbb{M}(\omega)$ .*

*Proof.* We prove successively the existence and the uniqueness.

i) It follows from (3.9) that  $\mathbb{K}_{\Phi}(\omega)$  contains at least the pair  $(\mathbf{0}, \mathbf{0})$ , hence is not empty. So, for fixed appropriate data  $(\mathbf{f}, \mathbf{N}, \mathbf{M})$ , we consider the solution  $U$  of problem (3.3). Note that, since it belongs to  $\mathbb{V}(\omega)$ , it satisfies the second line of (3.5). On the other hand, applying (3.3) with  $V$  replaced by  $U \pm W$  for any  $W$  in the space

$$\mathbb{K}_{\sharp}(\omega) = \left\{ V \in \mathbb{V}(\omega); \left( \mathbf{v} - \frac{e}{2} \mathbf{s} \right) \cdot \mathbf{e}_3 = 0 \text{ a.e. in } \omega \right\},$$

we derive

$$\forall W \in \mathbb{K}_{\sharp}(\omega), \quad a(U, W) = \mathcal{L}(W).$$

Thus, it follows from the inf-sup condition stated in Lemma 2.1 that there exists a  $\psi$  in  $\mathbb{M}(\omega)$  such that

$$\forall W \in \mathbb{N}_{\sharp}(\omega), \quad b(W, \psi) = \mathcal{L}(W) - a(U, W).$$

To conclude, we observe by using once more that, for any  $V$  in  $\mathbb{N}_\Phi(\omega)$ , there exists a  $W$  in  $\mathbb{N}_\#(\omega)$  such that  $b(W, \psi) = b(V, \psi)$ , so that  $V - W$  belongs to  $\mathbb{K}_\Phi(\omega)$ , Applying problem (3.3) thus yields

$$a(U, V - W - U) \geq \mathcal{L}(V - W - U),$$

whence, from the definition of  $\psi$  and  $W$ ,

$$a(U, V - U) + b(W, \psi) = a(U, V - U) + b(V, \psi) \geq \mathcal{L}(V - U).$$

Finally, since  $U$  satisfies the second line of (3.5), we obtain

$$a(U, V - U) + b(V - U, \psi) \geq \mathcal{L}(V - U).$$

Thus, the pair  $(U, \psi)$  satisfies the first line of problem (3.5), so it is a solution of this problem.

ii) Let  $(U_1, \psi_1)$  and  $(U_2, \psi_2)$  be two solutions of problem (3.5). Obviously,  $U_1$  and  $U_2$  are solutions of problem (3.3), hence the theorem of Lions and Stampacchia yields that they coincide. As previously, we have

$$\forall W \in \mathbb{N}_\#(\omega), \quad b(W, \psi_i) = \mathcal{L}(W) - a(U_i, W)$$

and, since  $U_1$  and  $U_2$  are equal, the same property for  $\psi_1$  and  $\psi_2$  follows from Lemma 2.1, whence the uniqueness property.

**Remarks 3.1** i) Note that assumption (3.9) is rather realistic from a physical point of view: The first part means that the undeformed shell is above the obstacle and the second part says that the shell is clamped on the obstacle.

ii) In fact, problem (2.14) has two implicitly associated Lagrange multipliers: the quantity  $\psi$  to enforce the tangency character of the rotation and the reaction of the obstacle  $\lambda$ . Indeed, the full problem, introduced in [2], reads *Find*  $(U, \psi, \lambda)$  in  $\mathbb{X}(\omega) \times \mathbb{M}(\omega) \times \Lambda$  such that

$$\begin{aligned} a(U, V) + b(V, \psi) - c(V, \lambda) &= \mathcal{L}(V), & \forall V \in \mathbb{X}(\omega), \\ b(U, \chi) &= 0, & \forall \chi \in \mathbb{M}(\omega), \\ c(U, \mu - \lambda) &\geq \langle \Phi \sqrt{a}, \mu - \lambda \rangle, & \forall \mu \in \Lambda, \end{aligned} \tag{3.10}$$

where the form  $c(\cdot, \cdot)$  is given by

$$c(V, \mu) = \left\langle \left( \mathbf{v} - \frac{e}{2} \mathbf{s} \right) \cdot \mathbf{e}_3 \sqrt{a}, \mu \right\rangle, \tag{3.11}$$

and  $\Lambda$  is the cone of nonnegative distributions in the dual space of  $H_{\gamma_0}^1(\omega)$  (see [2], Section 4, for details). We refer to [2], Thm 4.8, for the well-posedness of this problem, we prefer not to use it for the discretization.



## 4 The discrete problem and its well-posedness

Without restriction, we now assume that  $\omega$  is a polygon. Let  $(\mathcal{T}_h)_h$  be a regular family of triangulations of  $\omega$  (by triangles), in the sense that, for each  $h$ :

- $\bar{\omega}$  is the union of all elements of  $\mathcal{T}_h$ ;
- The intersection of two different elements of  $\mathcal{T}_h$ , if not empty, is a vertex or a whole edge of both of them;
- The ratio of the diameter  $h_K$  of any element  $K$  of  $\mathcal{T}_h$  to the diameter of its inscribed circle is smaller than a constant  $\sigma$  independent of  $h$ .

As usual,  $h$  stands for the maximum of the diameters  $h_K$ ,  $K \in \mathcal{T}_h$ . We make the further and non restrictive assumption that both  $\bar{\gamma}_0$  and  $\bar{\gamma}_1$  are the union of whole edges of elements of  $\mathcal{T}_h$ .

Since we are mostly interested in shells with little regularity – otherwise classical formulations would apply – it is presumably not useful to look for higher order elements in the hope of improving the rate of convergence. Indeed, in the case of such a shell, the underlying system of PDEs has nonsmooth coefficients. It is therefore unclear whether elliptic regularity can be applied to yield even an  $H^2$ -regularity, let alone  $H^{k+1}$ -regularity with  $k \geq 1$ . Note, however, that if the midsurface chart is smooth and we want to use our formulation nonetheless for simplicity as compared to the classical approach, then elliptic regularity will apply.

Therefore, let us introduce the basic approximation spaces:

$$\begin{aligned} \mathbb{M}_h &= \{ \mu_h \in H^1(\omega); \forall K \in \mathcal{T}_h, \mu_h|_K \in \mathcal{P}_1(K) \}, \\ \mathbb{X}_h^{\gamma_0} &= \{ v_h \in H_{\gamma_0}^1(\omega; \mathbb{R}^3); \forall K \in \mathcal{T}_h, v_h|_K \in \mathcal{P}_1(K) \}, \end{aligned} \quad (4.1)$$

where  $\mathcal{P}_1(K)$  stands for the space of restrictions to  $K$  of affine functions on  $\mathbb{R}^2$ . The spaces that are involved in the discrete problem are then

$$\mathbb{M}_h^{\gamma_0} = \mathbb{M}_h \cap H_{\gamma_0}^1(\omega), \quad \mathbb{X}_h = \mathbb{X}_h^{\gamma_0} \times \mathbb{X}_h^{\gamma_1}. \quad (4.2)$$

Let  $\mathcal{I}_h$  denote the Lagrange interpolation operator with values in  $(\mathbb{M}_h)^3$ . Since  $\varphi$  is in  $W^{2,\infty}(\omega; \mathbb{R}^3)$  then the function  $\Phi$  is continuous on  $\bar{\omega}$ . Therefore, we define

$$\Phi_h = \mathcal{I}_h \Phi.$$

and introduce the discrete convex set:

$$\mathbb{N}_{h,\Phi} = \left\{ V_h = (\mathbf{v}_h, \mathbf{s}_h) \in \mathbb{X}_h; \mathcal{I}_h \left( (\mathbf{v}_h - \frac{e}{2} \mathbf{s}_h) \cdot \mathbf{e}_3 \right) \geq \Phi_h \text{ a.e. in } \omega \right\} \quad (4.3)$$

Note that this set is not very difficult to build since the condition in it turns out to be equivalent to the pointwise one: For all vertices  $\mathbf{a}$  of elements of  $\mathcal{T}_h$ ,

$$\left(\mathbf{v}_h(\mathbf{a}) - \frac{e}{2} \mathbf{s}_h(\mathbf{a})\right) \cdot \mathbf{e}_3 \geq \Phi_h(\mathbf{a}). \quad (4.4)$$

**Remark 4.1** The choice of  $\mathbb{N}_{h,\Phi}$  is of course the key point for the construction of our discrete problem. Note however in view of (4.4) and due to the possible high variations of  $\Phi$  that it seems impossible to choose it as included in  $\mathbb{N}_\Phi$ .

We are thus in a position to construct the discrete problem by the Galerkin method applied to problem (3.5) with a small modification. It reads: Find  $(U_h, \psi_h)$  in  $\mathbb{N}_{h,\Phi} \times \mathbb{M}_h^{\gamma_0}$  such that

$$\begin{aligned} a(U_h, V_h - U_h) + b_h(V_h - U_h, \psi_h) &\geq \mathcal{L}(V_h - U_h), & \forall V_h \in \mathbb{N}_{h,\Phi}, \\ b_h(U_h, \chi_h) &= 0, & \forall \chi_h \in \mathbb{M}_h^{\gamma_0}, \end{aligned} \quad (4.5)$$

where the form  $b_h(\cdot, \cdot)$  is defined on smooth enough functions  $V$  and  $\chi$  by

$$b_h(V, \chi) = \int_{\omega} \partial_\alpha \mathcal{I}_h(\mathbf{s} \cdot \mathbf{a}_3) \partial_\alpha \chi \, d\mathbf{x}. \quad (4.6)$$

**Remark 4.2** In the implementation of this problem, approximations of the scalar coefficients  $a^{\alpha\beta}$ ,  $a^{\alpha\beta\rho\sigma}$ ,  $\sqrt{a}$  and  $\ell$  in the space  $\mathbb{M}_h$  are introduced together with approximations of the vectors  $\mathbf{a}_k$  and  $\partial_\alpha \mathbf{a}_3$  in the space  $(\mathbb{M}_h)^3$ . This leads to a modified discrete problem where the forms  $a(\cdot, \cdot)$ ,  $b(\cdot, \cdot)$  and  $\mathcal{L}$  are replaced by  $a_h(\cdot, \cdot)$ ,  $b_h(\cdot, \cdot)$  and  $\mathcal{L}_h$ , respectively. However it has been checked in [7] for the *a priori* analysis and in [4] for the *a posteriori* analysis that this modification only adds technical difficulties in the proofs, without any major change in the final estimates. So we prefer to skip it in what follows.

Proving the well-posedness of problem (4.5) requires several steps. We first introduce the kernel

$$\mathbb{V}_h = \{V_h \in \mathbb{X}_h; \forall \chi_h \in \mathbb{M}_h^{\gamma_0}, b_h(V_h, \chi_h) = 0\}. \quad (4.7)$$

It is readily checked that, for any solution  $(U_h, \psi_h)$  of problem (4.5), its part  $U_h$  is a solution of the problem: Find  $U_h$  in  $\mathbb{N}_{h,\Phi} \cap \mathbb{V}_h$  such that:

$$a(U_h, V_h - U_h) \geq \mathcal{L}(V_h - U_h), \quad \forall V_h \in \mathbb{N}_{h,\Phi} \cap \mathbb{V}_h. \quad (4.8)$$

The next lemma provides a characterization of  $\mathbb{V}_h$ .

**Lemma 4.1** *The space  $\mathbb{V}_h$  coincides with the space of functions  $V_h = (\mathbf{v}_h, \mathbf{s}_h)$  in  $\mathbb{X}_h$  such that:*

$$\mathcal{I}_h(\mathbf{s}_h \cdot \mathbf{a}_3) = 0 \quad \text{in } \bar{\omega}. \quad (4.9)$$

*Proof.* By taking  $\chi_h$  equal to  $\mathcal{I}_h(\mathbf{s}_h \cdot \mathbf{a}_3)$  in the definition of  $\mathbb{V}_h$ , we obtain that the gradient of  $\mathcal{I}_h(\mathbf{s}_h \cdot \mathbf{a}_3)$  is zero. Thus, the desired property follows from its nullity on  $\gamma_0$ .

This lack of conformity of the discretization makes the proof of the well-posedness of problem (4.8) a little more complex. We begin with a technical lemma and, from now on, we assume that  $\mathbf{a}_3$  belongs to  $W^{2,\infty}(\omega; \mathbb{R}^3)$ .

**Lemma 4.2** *The following estimate holds for any function  $\mathbf{s}_h$  in  $\mathbb{M}_h^3$ :*

$$\|\mathbf{s}_h \cdot \mathbf{a}_3 - \mathcal{I}_h(\mathbf{s}_h \cdot \mathbf{a}_3)\|_{H^1(\omega)} \leq c h \|\mathbf{s}_h\|_{H^1(\omega; \mathbb{R}^3)}. \quad (4.10)$$

*Proof.* The standard properties of the Lagrange interpolation operator with values in  $\mathbb{M}_h^{\gamma_0}$  yield for all  $K$  in  $\mathcal{T}_h$  that

$$\|\mathbf{s}_h \cdot \mathbf{a}_3 - \mathcal{I}_h(\mathbf{s}_h \cdot \mathbf{a}_3)\|_{H^1(K)} \leq c h |\mathbf{s}_h \cdot \mathbf{a}_3|_{H^2(K)}.$$

To evaluate  $|\mathbf{s}_h \cdot \mathbf{a}_3|_{H^2(K)}$ , we observe that, since  $\mathbf{s}_h|_K$  belongs to  $\mathcal{P}_1(K)^3$ , each partial derivative  $\partial_{\alpha\beta}(\mathbf{s}_h \cdot \mathbf{a}_3)$  can be written as

$$\partial_{\alpha\beta}(\mathbf{s}_h \cdot \mathbf{a}_3) = \partial_\alpha \mathbf{s}_h \cdot \partial_\beta \mathbf{a}_3 + \partial_\beta \mathbf{s}_h \cdot \partial_\alpha \mathbf{a}_3 + \mathbf{s}_h \cdot \partial_{\alpha\beta} \mathbf{a}_3,$$

whence the desired result.

**Remark 4.3** By switching to the reference triangle  $\hat{K}$  and using the imbedding of  $H^1(\hat{K})$  into all  $L^p(\hat{K})$ ,  $1 \leq p < +\infty$ , we can prove an analogue of (4.10) with a weaker assumption on  $\mathbf{a}_3$ . We did not use this modification for simplicity.

We are thus in a position to derive the following ellipticity property.

**Lemma 4.3** *There exists a real number  $h_0 > 0$  and a constant  $\alpha_* > 0$  such that, for all  $h \leq h_0$ ,*

$$a(V_h, V_h) \geq \alpha_* \|V_h\|_{\mathbb{X}(\omega)}^2, \quad \forall V_h \in \mathbb{V}_h. \quad (4.11)$$

*Proof.* For any  $V_h = (\mathbf{v}_h, \mathbf{s}_h)$  in  $\mathbb{V}_h$ , the function  $V_h - W$ , with  $W = (0, (\mathbf{s}_h \cdot \mathbf{a}_3)\mathbf{a}_3)$ , belongs to  $\mathbb{V}(\omega)$ . So, applying the ellipticity property (2.11) yields

$$a(V_h - W, V_h - W) \geq \alpha \|V_h - W\|_{\mathbb{X}(\omega)}^2.$$

Denoting by  $c$  the norm of  $a(\cdot, \cdot)$  on  $\mathbb{X}(\omega)$ , we thus derive

$$a(V_h, V_h) \geq \alpha \|V_h\|_{\mathbb{X}(\omega)}^2 - 2\alpha \|V_h\|_{\mathbb{X}(\omega)} \|W\|_{\mathbb{X}(\omega)} - 2c \|V_h\|_{\mathbb{X}(\omega)} \|W\|_{\mathbb{X}(\omega)} - c \|W\|_{\mathbb{X}(\omega)}^2.$$

Moreover, since  $V_h$  belongs to  $\mathbb{V}_h$ , it follows from Lemma 4.1 that the quantity  $W$  is equal to  $(0, (\mathbf{s}_h \cdot \mathbf{a}_3 - \mathcal{I}_h(\mathbf{s}_h \cdot \mathbf{a}_3))\mathbf{a}_3)$ , so that the desired ellipticity property is a direct consequence of Lemma 4.2.

This property yields that, for  $h \leq h_0$ , problem (4.8) is well-posed. The arguments for proving that problem (4.5) has a unique solution are exactly the same as in Section 2, we only give an abridged version of the proofs where necessary. First note that the formula

$$b_h(V, \chi) = b(V, \chi) - \int_{\Omega} \partial_{\alpha} (s \cdot \mathbf{a}_3 - \mathcal{I}_h(s \cdot \mathbf{a}_3)) \partial_{\alpha} \chi \, d\mathbf{x},$$

and Lemma 4.2 yield the continuity of  $b_h(\cdot, \cdot)$  on  $\mathbb{X}_h \times \mathbb{M}_h$ ; moreover its norm is bounded independently of  $h$ . We also define the set

$$\mathbb{N}_{h,\sharp} = \left\{ V_h \in \mathbb{X}_h; \mathcal{I}_h((\mathbf{v}_h - \frac{\epsilon}{2} \mathbf{s}_h) \cdot \mathbf{e}_3) = 0 \text{ a.e. in } \omega \right\}. \quad (4.12)$$

**Lemma 4.4** *There exists a real number  $h_* > 0$  and a constant  $\beta_* > 0$  such that the following inf-sup condition holds, for all  $h \leq h_*$ ,*

$$\forall \chi_h \in \mathbb{M}_h, \quad \sup_{V_h \in \mathbb{N}_{h,\sharp}} \frac{b_h(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}(\omega)}} \geq \beta_* \|\chi_h\|_{\mathbb{M}(\omega)}. \quad (4.13)$$

*Proof.* For any function  $\chi_h$  in  $\mathbb{M}_h$ , the pair  $V_h = (\mathbf{v}_h, \mathbf{s}_h)$  with  $\mathbf{s}_h = \mathcal{I}_h(\chi_h \mathbf{a}_3)$  and  $\mathbf{v}_h = \frac{\epsilon}{2} \mathbf{s}_h$  belongs to  $\mathbb{N}_{h,\sharp}$ . We have

$$\begin{aligned} b_h(V_h, \chi_h) &\geq \|\chi_h\|_{\mathbb{M}(\omega)}^2 - \|\mathbf{s}_h \cdot \mathbf{a}_3 - \mathcal{I}_h(\mathbf{s}_h \cdot \mathbf{a}_3)\|_{H^1(\omega)} \|\chi_h\|_{\mathbb{M}(\omega)} \\ &\quad - \|\chi_h \mathbf{a}_3 - \mathcal{I}_h(\chi_h \mathbf{a}_3)\|_{H^1(\omega; \mathbb{R}^3)} \|\chi_h\|_{\mathbb{M}(\omega)}, \end{aligned}$$

and from Lemma 4.2 and an extension of it this quantity is larger than  $c \|\chi_h\|_{\mathbb{M}(\omega)}^2$  for  $h$  small enough. Lemma 4.2 also implies that

$$\|V_h\|_{\mathbb{X}(\omega)} \leq c \|\chi_h\|_{\mathbb{M}(\omega)}.$$

This gives the desired inf-sup condition.

**Theorem 4.1** *Assume that the function  $\Phi$  satisfies (3.9). Then, for any data  $(\mathbf{f}, \mathbf{N}, \mathbf{M})$  in  $L^2(\omega)^3 \times L^2(\gamma_1)^3 \times L^2(\gamma_1)^3$ , problem (4.5) has a unique solution  $(U_h, \psi_h)$  in  $\mathbb{X}_h \times \mathbb{M}_h$ .*

*Proof.* It relies on exactly the same arguments as for Theorem 3.1, now combined with the new ellipticity and inf-sup properties in Lemmas 4.3 and 4.4.

## 5 A priori analysis of the error

Our problem is formulated as variational inequalities in  $H^1$ . In order to prove the convergence of non-conforming finite element approximations, we follow the standard approach for variational inequalities in mixed problems, see [8] and [15] for instance. Therefore, we first state the following version of the Strang's lemma, which is not fully obvious and involves the full problem (3.10).

**Lemma 5.1** *Assume that the part  $\lambda$  of the solution of problem (3.10) belongs to  $L^2(\omega)$ . Then, the following estimate holds between the part  $U$  of the solution of this problem and the part  $U_h$  of the solution of problem (4.5)*

$$\begin{aligned} \|U - U_h\|_{\mathbb{X}(\omega)} &\leq c \left( \inf_{V_h \in \mathbb{N}_{h,\Phi} \cap \mathbb{V}_h} (\|U - V_h\|_{\mathbb{X}(\omega)} + \|U - V_h\|_{L^2(\omega; \mathbb{R}^3)}^{\frac{1}{2}}) \right. \\ &\quad \left. + \inf_{\chi_h \in \mathbb{M}_h} \|\psi - \chi_h\|_{\mathbb{M}(\omega)} + \|\Phi - \Phi_h\|_{L^2(\omega; \mathbb{R}^3)}^{\frac{1}{2}} \right), \end{aligned} \quad (5.1)$$

for a constant  $c$  depending on  $\|\lambda\|_{L^2(\omega)}$ .

*Proof.* Let  $U_h$  be an approximation of  $U$  in  $\mathbb{N}_{h,\Phi} \cap \mathbb{V}_h$ . It follows from problem (4.8) that

$$a(U_h - V_h, U_h - V_h) \leq \mathcal{L}(U_h - V_h) - a(V_h, U_h - V_h).$$

So applying problem (3.10) leads to

$$a(U_h - V_h, U_h - V_h) \leq a(U - V_h, U_h - V_h) + b(U_h - V_h, \psi) - c(U_h - V_h, \lambda). \quad (5.2)$$

Owing to Lemma 4.3, the difficulty is now to evaluate the quantities  $b(U_h - V_h, \psi)$  and  $c(U_h - V_h, \lambda)$ . First, since both  $U_h$  and  $V_h$  belong to  $\mathbb{N}_{h,\Phi} \cap \mathbb{V}_h$ , it is readily checked that, for any  $\chi_h$  in  $\mathbb{M}_h$ ,

$$b(U_h - V_h, \psi) = b(U_h - V_h, \psi - \chi_h). \quad (5.3)$$

On the other hand, since  $\lambda$  is nonnegative, we deduce from the definition of  $\mathbb{N}_{h,\Phi}$  that

$$-c(U_h, \lambda) \leq -\langle \Phi_h \sqrt{a}, \lambda \rangle.$$

We also observe that  $\Lambda$  contains the zero function, whence, by applying the third line of problem (3.10) with  $\mu = 0$ ,

$$-c(U_h - V_h, \lambda) \leq -c(U - V_h, \lambda) + \langle (\Phi - \Phi_h) \sqrt{a}, \lambda \rangle. \quad (5.4)$$

By combining the ellipticity property of the form  $a(\cdot, \cdot)$ , see Lemma 4.3, and the continuity of all forms involved in the previous lines, since the function  $\lambda$  belongs to  $L^2(\omega)$ . we derive

$$\begin{aligned} \alpha_* \|U_h - V_h\|_{\mathbb{X}(\omega)}^2 &\leq c \left( \|U - V_h\|_{\mathbb{X}(\omega)} + \|\psi - \chi_h\|_{\mathbb{M}(\omega)} \right) \|U_h - V_h\|_{\mathbb{X}(\omega)} \\ &\quad + c' \left( \|U - V_h\|_{L^2(\omega; \mathbb{R}^3)^2} + \|\Phi - \Phi_h\|_{L^2(\omega; \mathbb{R}^3)} \right). \end{aligned}$$

Then, the desired estimate follows from a triangle inequality.

Now, it remains to construct approximations  $V_h$  of  $U$  and  $\chi_h$  of  $\psi$  satisfying the previous conditions. Assuming that  $\psi$  is continuous, we take  $\chi_h$  equal to  $\mathcal{I}_h \psi$ . Then, bounding the term  $\|\psi - \chi_h\|_{\mathbb{M}(\omega)}$  is fully standard. We need a lemma for the approximation of  $U$ .

**Lemma 5.2** *For any function  $U$  in  $\mathbb{K}_\Phi(\omega) \cap H^{s+1}(\omega; \mathbb{R}^3)^2$ ,  $0 < s \leq 1$ , there exists a function  $V_h$  in  $\mathbb{N}_{h, \Phi} \cap \mathbb{V}_h$  such that:*

$$\|U - V_h\|_{\mathbb{X}(\omega)} \leq c h^s \|U\|_{H^{s+1}(\omega; \mathbb{R}^3)^2}, \quad (5.5)$$

and

$$\|U - V_h\|_{L^2(\omega)^6} \leq c h^{s+1} \|U\|_{H^{s+1}(\omega; \mathbb{R}^3)^2}. \quad (5.6)$$

*Proof.* The function  $V_h = \mathcal{I}_h U$  satisfies the desired approximation properties. Moreover, it follows from the definition of  $\mathbb{N}_{h, \Phi}$ , see (4.4), that it belongs to this set and since  $\mathbf{r} \cdot \mathbf{a}_3$  vanishes everywhere, when setting  $V_h = (\mathbf{v}_h, \mathbf{s}_h)$ , we observe that  $\mathbf{s}_h \cdot \mathbf{a}_3$  vanishes at all vertices of elements of  $\mathcal{T}_h$ , so that  $\mathcal{I}_h(\mathbf{s}_h \cdot \mathbf{a}_3)$  is zero. Since only this operator appears in the definition (4.6) of the form  $b_h(\cdot, \cdot)$ , this yields that  $V_h$  also belongs to  $\mathbb{V}_h$ . All those arguments yields the desired estimates.

**Theorem 5.1** *Assume that the solution  $(U, \psi, \lambda)$  of problem (3.10) belongs to  $H^2(\omega; \mathbb{R}^3)^2 \times H^2(\omega) \times L^2(\omega)$  and that the vector  $\mathbf{a}_3$  belongs to  $W^{2,p}(\omega; \mathbb{R}^3)$ ,  $p > 2$ . Then, the following a priori error estimate holds between the solution  $(U, \psi)$  of problem (3.5) and the solution  $(U_h, \psi_h)$  of problem (4.5)*

$$\|U - U_h\|_{\mathbb{X}(\omega)} + \|\psi - \psi_h\|_{\mathbb{M}(\omega)} \leq c h, \quad (5.7)$$

for a constant  $c$  depending on  $\Phi$  and  $(U, \psi, \lambda)$ .

*Proof.* The estimate for  $\|U - U_h\|_{\mathbb{X}(\omega)}$  is easily derived from Lemmas 5.1 and 5.2, since  $\Phi_h$  is equal to  $\mathcal{I}_h \Phi$ . To evaluate  $\|\psi - \psi_h\|_{\mathbb{M}(\omega)}$ , we deduce from the inf-sup condition (4.13) that, for all  $\chi_h$  in  $\mathbb{M}_h$ ,

$$\|\psi_h - \chi_h\|_{\mathbb{M}(\omega)} \leq \beta^{-1} \sup_{W_h \in \mathbb{N}_{h, \Phi}} \frac{b_h(W_h, \psi_h - \chi_h)}{\|W_h\|_{\mathbb{X}(\omega)}}.$$

Applying problem (4.5) with  $V_h = U_h \pm W_h$  (which yields an equality) gives

$$b_h(W_h, \psi_h - \chi_h) = \mathcal{L}(W_h) - a(U_h, W_h) - b_h(W_h, \chi_h).$$

It thus follows from problem (3.10)

$$b_h(W_h, \psi_h - \chi_h) = a(U - U_h, W_h) + b(W_h, \psi) - b_h(W_h, \chi_h) - c(W_h, \lambda)$$

and the definition (4.12) of  $\mathbb{N}_{h,\sharp}$  yields that the last term is zero. We thus obtain, with the notation  $W_h = (\mathbf{w}_h, \mathbf{t}_h)$

$$\|\psi_h - \chi_h\|_{\mathbb{M}(\omega)} \leq c \left( \|U - U_h\|_{\mathbb{X}(\omega)} + \|\psi - \chi_h\|_{\mathbb{M}(\omega)} + \sup_{\mathbf{t}_h \in \mathbb{M}_h^3} \frac{\|\mathbf{t}_h \cdot \mathbf{a}_3 - \mathcal{I}_h(\mathbf{t}_h \cdot \mathbf{a}_3)\|_{\mathbb{M}(\omega)}}{\|\mathbf{t}_h\|_{H^1(\omega; \mathbb{R}^3)}} \right).$$

We take  $\chi_h$  equal to  $\mathcal{I}_h\psi$ . We also observe that, for  $K$  running through  $\mathcal{T}_h$ ,

$$\|\mathbf{t}_h \cdot \mathbf{a}_3 - \mathcal{I}_h(\mathbf{t}_h \cdot \mathbf{a}_3)\|_{H^1(K)} \leq c h |\mathbf{t}_h \cdot \mathbf{a}_3|_{H^2(K)},$$

and, since  $\mathbf{t}_h|_K$  belongs to  $\mathcal{P}_1(K)$ , we conclude thanks to the regularity of  $\mathbf{a}_3$ .

**Remark 5.1** Estimate (5.7) is fully optimal, both for the pair  $U_h = (\mathbf{u}_h, \mathbf{r}_h)$  we are interested in and the Lagrange multiplier  $\psi_h$ . This proves the efficiency of our discretization. Moreover, its convergence can be proved without any regularity assumption on the exact solution.

**Remark 5.2** After the pioneering works [13] by Hlaváček, Haslinger, Nečas, and Lovíšek and [1] by Ainsworth, Oden, and Lee, a huge amount of work has been performed on the *a posteriori* analysis of variational inequalities, see, e.g., [3], [9], [12], [16], and the references therein. However, even for the simpler Naghdi's equations, the residual error indicators first introduced in [4] are very complex. The *a posteriori* analysis of this problem will be the object of a forthcoming paper.

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