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# How Reversibility Can Solve Traditional Questions: The Example of Hereditary History-Preserving Bisimulation

Clément Aubert 

School of Computer and Cyber Sciences, Augusta University, USA  
caubert@augusta.edu

Ioana Cristescu

Tarides, Paris  
ioana@tarides.com

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## Abstract

Reversible computation opens up the possibility of overcoming some of the hardware’s current physical limitations. It also offers theoretical insights, as it enriches multiple paradigms and models of computation, and sometimes retrospectively enlightens them. Concurrent reversible computation, for instance, offered interesting extensions to the Calculus of Communicating Systems, but was still lacking a natural and pertinent bisimulation to study processes equivalences. Our paper formulates an equivalence exploiting the two aspects of reversibility: backward moves and memory mechanisms. This bisimulation captures classical equivalences relations for denotational models of concurrency (History- and hereditary history-preserving bisimulation, (H)HPB), that were up to now only partially characterized by process algebras. This result gives an insight on the expressiveness of reversibility, as both backward moves and a memory mechanism—providing ‘backward determinism’—are needed to capture HHPB.

**2012 ACM Subject Classification** Theory of computation → Program semantics; Computing methodologies → Concurrent programming languages

**Keywords and phrases** Formal semantics, Process algebras and calculi, Distributed and reversible computation, Configuration structures, Reversible CCS, Bisimulation

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## 1 Introduction

**The Benefits of Reversible Computation** Future progresses in computing may heavily rely on reversibility [16]. The foreseeable limitations of conventional semiconductor technology, Lauder’s principle [21]—promising low-energy consumption for reversible computers—and quantum computing [23]—intrinsically reversible and now within reach [1]—motivated a colossal push toward a better understanding of reversible computation. Those efforts have given birth to new paradigms [15], richer models of computation (e.g. for automata [19], Petri nets [14, 24], Turing machines [3]) and richer semantics, sometimes for preexisting calculus like the Calculus of Communicating Systems (CCS) and the  $\pi$ -calculus [9, 10]. Those new perspectives sometimes additionally give in retrospect a better understanding of ‘traditional’ (i.e., irreversible) computation, and our contribution illustrates this latter aspect.

**Summary** In brief terms, we offer a solution to an open problem on classes of equivalence of (non-reversible) concurrent processes thanks to reversibility. *Hereditary history-preserving bisimulation* (HHPB) is considered ‘the gold standard’ for establishing equivalence classes on ‘true’ models of concurrency [20, 32]. However, no relation expressed in syntactical terms (e.g. on CCS) was known to capture them, despite intensive efforts: previous results [2, 25]

characterized HHPB on limited classes of processes, that excluded CCS terms as simple as  $a.a$ ,  $a + a$ ,  $a \mid a$  or containing similar patterns. We prove in this paper a somewhat expected result, namely that adding mechanisms to reverse the computation and keep track of the past, thus making CCS ‘backwards deterministic’ [6], enables syntactical characterizations of HHPB. This result uses natural reformulations of canonical CCS bisimulations, provides a ‘meaningful’ bisimulation for reversible calculi, and furthermore validates the mechanism we use to reverse the computation in a concurrent set-up. It also connects with previous semantics of reversible processes [2, 18] and supports categorical treatment.

**Reversing Concurrent Computation** Reversible systems can backtrack and return to a previous state. Implementing reversibility often requires a mechanism to record the history of the execution. Ideally, this history should be *complete*, so that every forward step can be backtracked, and *minimal*, so that only the relevant information is saved. Concurrent programming languages have additional requirements: the history should be *distributed*, to avoid centralization, and should prevent steps that required a synchronization with other parts of the program to backtrack without undoing this synchronization. To fulfill those requirements, Reversible CCS (RCCS) [11, 12] uses *memories* attached to the threads of a process, and CCS with Communication Keys (CCSK) [26] ‘marks’ each occurrence of an action. The two calculi are equi-expressive [22] and are conservative extensions over CCS, and in this paper we will use RCCS, to benefit from its previous operational semantics [2].

**Equivalences for Denotational Models of Concurrency** A theory of concurrent computation, be it reversible or irreversible, relies not only on a syntax and a model, but also on ‘meaningful’ behavioral equivalences. This paper defines new bisimulations on RCCS, and proves their relevance by connecting them to bisimulations on configuration structures [36], a classical denotational model for concurrency. In configuration structures, an *event* represents an execution step, and a *configuration*—a set of events that occurred—represents a state. A forward transition is then represented as moving from a configuration to one of its supersets. Backward transitions have a ‘built-in’ representation: it suffices to move from a configuration to one of its subsets. Among the behavioral equivalences on configuration structures, some of them, like *history-* and *hereditary history-preserving bisimulation* (HPB and HHPB) [4, 6, 17, 27, 28, 31], use that ‘built-in’ notion of reversibility. HPB and HHPB are usually regarded as the ‘canonical’ [25, p. 94] or ‘strongest desirable true’ [28, p. 2] concurrency equivalences because they preserve causality, branching, their interplay, and are the coarser (or finer, for HPB) reasonable equivalences with these properties [33, p. 309]. HHPB also naturally equal path bisimulations [20], an elegant notion of equivalence relying on category theory, and can be captured concisely using event identifier logic [28]. However, no relation on labeled transition systems (LTS), being CCS, RCCS, or CCSK, was known to capture it on all processes.

**Encoding Reversible Processes in Configuration Structures** There have been multiple attempts [2, 26] to transfer equivalences defined on denotational models, by construction adapted for reversibility, into reversible process algebras. Showing that an equivalence on configuration structures corresponds to one on processes starts by defining an encoding of the latter in the former. Encodings, for RCCS [2] and CCSK [18] alike, generally consider only *reachable* reversible processes, and map them to one particular configuration in the encoding of its *origin*, the CCS ‘memory-less’ process to which it can backtrack. We come back to the relation between this encoding and our current work in conclusion.

**Contribution** This paper improves on previous results [2, 25] by defining relations on syntactical terms that correspond to HHPB on *all* processes, and hence are proven to be ‘the right’ bisimulation to study reversible computation. This result relies on encoding the processes’ memories into *identified configuration structures*, an extension to configuration structures that constitutes our second contribution.

RCCS is exploited as a syntactic tool to decide HHPB for CCS processes, by endowing them with **1.** backtracking capabilities and **2.** a memory mechanism. This gives a precious insight on the expressiveness of reversibility, as we show that having only one of those tools is not enough to define relations that capture HHPB. The memories attached to a process are no longer only a syntactic layer to implement reversibility, but become essential to define and capture equivalences, thanks to the ‘backward determinism’ they provide.

Recursion is not treated in this paper: its treatment amounts to unfolding processes and structures up to a certain level, and it strongly suggests that there are not much insight to gain from its development, that we reserve as a technical appendix for future work.

**Related Work** The correspondences between HHPB and back-and-forth bisimulations for restricted classes of processes [2, 25] inspired some of the work presented here. This correspondence has been studied in the denotational world, on structures without auto-concurrency [6].

The insight that ‘backward determinism’ is essential to capture HHPB is also used in the event identifier logic [28] and in one of its sub-system [5], that both characterize HHPB without restricting the structures considered. As in our case, the authors exploits identifiers to eliminate constraints due to label ‘duplication’. Our work is similar, taking place in the operational world instead of the logical one. Note that we do not introduce extra syntax constructions on the LTS, but simply use the ones provided by RCCS, that we use ‘as is’.

In the operational semantics, the previous attempts to characterize HHPB wrongly focused on the backward capabilities of processes and considered the memory mechanisms only as a tool to achieve it. As a result, those bisimulations could not decide that the encoding of  $(a.a) | b$  and  $a | a | b$  were not HHPB. Instead, their characterizations of HHPB were applicable only on ‘non-repeating’ [25] or ‘singly-labeled’ [2] processes. By integrating the memory mechanism, the equivalence relation we introduce and consider can correctly determine that these two processes are not HHPB, as we detail in Example 27.

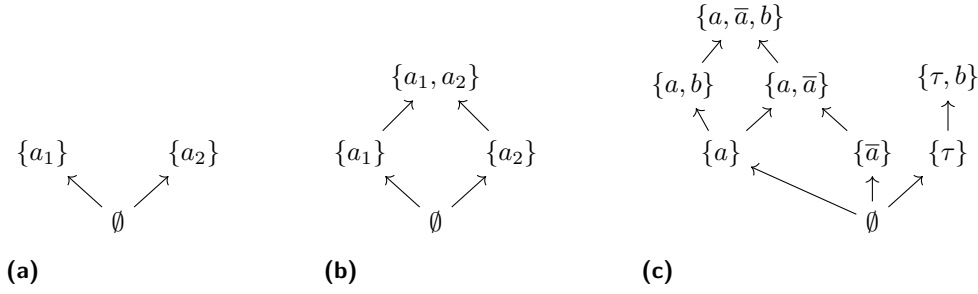
Finally, our approach shares similarity with causal trees—in the sense that only *part of the execution*, its ‘past’, is encoded in a denotational representation—which were used to characterize HPB for CCS terms [13]. Capturing (H)HPB with novel techniques can also impact model-checking and decidability issues [5], but we leave this as future work.

## 2 Denotational and Operational Models for the Reversible CCS

We write  $\subseteq$  the set inclusion,  $\mathcal{P}$  the power set,  $\setminus$  the set difference,  $f : A \rightarrow B$  (resp.  $f : A \dashrightarrow B$ ) for the (resp. partial) function from  $A$  to  $B$ ,  $f|_C$  the restriction of  $f$  to  $C \subseteq A$ , and  $f \cup \{a \mapsto b\}$  the function defined as  $f$  on  $A$  that additionally maps  $a \notin A$  to  $b$ .

### 2.1 Identified Configuration Structures

Labeled configuration structures [34, 35] are a classical non-interleaving model of concurrent computation, that we enrich here with identifiers. We then show that they support categorical understanding and the usual operations just as well as their ‘un-identified’ variations.



■ **Figure 1** Examples of  $\mathcal{I}$ -structures

► **Definition 1** ((Identified) Configuration structure). A configuration structure  $\mathcal{C}$  is a tuple  $(E, C, L, \ell)$  where  $E$  is a set of events,  $L$  is a set of labels,  $\ell : E \rightarrow L$  is a labeling function and  $C \subseteq \mathcal{P}(E)$  is a set of subsets satisfying:

$$\begin{aligned} \forall x \in C, \forall e \in x, \exists z \in C \text{ finite}, e \in z, z \subseteq x & \quad (\text{Finiteness}) \\ \forall x \in C, \forall d, e \in x, d \neq e \Rightarrow \exists z \in C, z \subseteq x, d \in z \iff e \notin z & \quad (\text{Coincidence Freeness}) \\ \forall X \subseteq C, \exists y \in C \text{ finite}, \forall x \in X, x \subseteq y \Rightarrow \bigcup X \in C & \quad (\text{Finite Completeness}) \\ \forall x, y \in C, x \cup y \in C \Rightarrow x \cap y \in C & \quad (\text{Stability}) \end{aligned}$$

If  $\mathcal{C}$  also has a set of identifiers  $I$  and an identifying function  $m : E \rightarrow I$  satisfying:

$$\forall x \in C, \forall e_1, e_2 \in x, m(e_1) \neq m(e_2) \quad (\text{Collision Freeness})$$

then we write  $\mathcal{C} \oplus m$ , and say that  $\mathcal{I} = (E, C, L, \ell, I, m)$  is an identified configuration structure, or  $\mathcal{I}$ -structure, and call  $\mathcal{C}$  the underlying configuration structure of  $\mathcal{I}$ .

We denote with  $\mathbf{0}$  both the  $\mathcal{I}$ -structure and its underlying configuration structure with  $C = \{\emptyset\}$ , and, for  $x, y \in C$ , we write  $x \xrightarrow{e} y$  and  $y \xrightarrow{\ell} x$  if  $x = y \cup \{e\}$ .

We omit the identifiers when representing  $\mathcal{I}$ -structures and write the label for the event (with a subscript if multiple events shares a label).

► **Definition 2** (Causality, concurrency, and maximality). For all  $\mathcal{I}$ ,  $x \in C$  and  $d, e \in x$ , the causality relation on  $x$  is given by  $d <_x e$  iff  $d \leq_x e$  and  $d \neq e$ , where  $d \leq_x e$  iff for all  $y \in C$  with  $y \subseteq x$ , we have  $e \in y \Rightarrow d \in y$ . The concurrency relation on  $x$  is given by  $d \text{ co}_x e$  iff  $\neg(d <_x e \vee e <_x d)$ . Finally,  $x$  is a maximal configuration in  $\mathcal{I}$  if  $\forall y \in C, x = y$  or  $x \not\subseteq y$ .

► **Example 3.** Consider Fig. 1, where we let the events have distinct arbitrary identifiers: two events with complement names as labels can happen at the same time (Fig. 1c), in which case they are labeled with  $\tau$ , as is usual in CCS (Sect. 2.2). In Fig. 1c,  $a <_{\{a,b\}} b$ ,  $a <_{\{a,\bar{a}\}} \bar{a}$  and  $\tau <_{\{\tau,b\}} b$ ; and in Fig. 1b,  $a_1 \text{ co}_{\{a_1,a_2\}} a_2$ . An  $\mathcal{I}$ -structure can have one (Fig. 1b) or multiple (Fig. 1a and 1c) maximal configurations.

**Categorical point of view** We remind in Appendix A that configuration structures and ‘structure-preserving’ functions form a category. We also prove that a similar category can be defined with  $\mathcal{I}$ -structures as objects and define a forgetful functor that returns the underlying configuration structure. This development supports the interest and validity of studying  $\mathcal{I}$ -structures, but can be omitted, except for the notion of morphisms:

► **Definition 4** (Morphism of  $\mathcal{I}$ -structure). A morphism  $f = (f_E, f_L, f_C, f_m)$  between  $\mathcal{I}_1$  and  $\mathcal{I}_2$  is given by  $f_E : E_1 \rightarrow E_2$  such that  $\ell_2(f_E(e)) = f_L(\ell_1(e))$ , for  $f_L : L_1 \rightarrow L_2$ ;  $f_C : C_1 \rightarrow C_2$

defined as  $f_C(x) = \{f_E(e) \mid e \in x\}$ , and  $f_m : I_1 \rightarrow I_2$  such that  $f_m(m_1(e)) = m_2(f_E(e))$ . We often write  $f$  for all the components, and write  $\mathcal{I}_1 \cong \mathcal{I}_2$  if  $f$  is an isomorphism.

**Operations on  $\mathcal{I}$ -configurations** Operations on  $\mathcal{I}$ -structures are conservative extensions over their counterparts on configuration structures—forgetting about event identifiers gives back the ‘un-identified’ definition [35]—, except for the parallel composition. The intuition here is that configuration structures encode CCS processes, and  $\mathcal{I}$ -structures encode memories of RCCS processes, where parallel composition has a different meaning. Examples of those operations will be given in Sect. 2.4, after introducing the calculi in Sect. 2.2 and the encodings in Sect. 2.3. Sect. C.1 is devoted to proving the correctness of those operations.

Given two sets  $A, B$ , and a symbol  $\star \notin A \cup B$  denoting *undefined*, we write  $C^\star = C \cup \{\star\}$  if  $\star \notin C$  and define the partial product [35, Appendix A] of  $A$  and  $B$  to be

$$A \times_\star B = \{(a, \star) \mid a \in A\} \cup \{(\star, b) \mid b \in B\} \cup \{(a, b) \mid a \in A, b \in B\}$$

and the two projections to be  $\pi_1 : A \times_\star B \rightarrow A^\star$  and  $\pi_2 : A \times_\star B \rightarrow B^\star$ .

► **Definition 5** (Operations on  $\mathcal{I}$ -structures). Given  $\mathcal{I}_i = (E_i, C_i, L_i, \ell_i, I_i, m_i)$ , for  $i = 1, 2$ ,

**The relabeling** of  $\mathcal{I}_1$  along  $\ell' : E_1 \rightarrow L$  is  $\mathcal{I}_1[\ell'/\ell_1] = (E_1, C_1, L, \ell', I_1, m_1)$ .

**The reidentifying** of  $\mathcal{I}_1$  along  $m : E_1 \rightarrow I$  is  $\mathcal{I}_1[m/m_1] = (E_1, C_1, L_1, \ell_1, I, m)$ , provided  $\mathcal{I}_1[m/m_1]$  respects the Collision Freeness condition.

**The restriction of a set of events**  $A \subseteq E_1$  in  $\mathcal{I}_1$  is  $\mathcal{I}_1 \upharpoonright_A = (E, C, L, \ell_1 \upharpoonright_E, I, m_1 \upharpoonright_E)$ , where  $E = E_1 \setminus A$ ,  $C = \{x \mid x \in C_1 \text{ and } x \cap A = \emptyset\}$ ,  $L = \{a \mid \exists e \in E_1 \setminus A, \ell_1(e) = a\}$  and  $I = \{i \mid \exists e \in E_1 \setminus A, m_1(e) = i\}$ .

**The restriction of a set of labels**  $L \subseteq L_1$  in  $\mathcal{I}_1$  is  $\mathcal{I}_1 \upharpoonright_L = \mathcal{I}_1 \upharpoonright_{E_1^L}$  where  $E_1^L = \{e \in E_1 \mid \ell_1(e) \in L\}$ . We write  $\mathcal{I}_1 \upharpoonright_a$ , when the restricting set of labels  $L$  is the singleton  $\{a\}$ .

**The prefixing** of  $\mathcal{I}_1$  by the label  $a$  is  $a.\mathcal{I}_1 = (E, C, L, \ell, I, m)$  where

- $E = E_1 \cup \{e\}$ , for  $e \notin E_1$ ,
- $C = \{x \mid x = \emptyset \text{ or } \exists x' \in C_1, x = x' \cup e\}$ ,
- $L = L_1 \cup \{a\}$ ,
- $\ell = \ell_1 \cup \{e \mapsto a\}$ ,
- $I = I_1 \cup \{i\}$ , for  $i \notin I_1$
- $m = m_1 \cup \{e \mapsto i\}$ .

**The postfixing** of  $(a, i)$  to  $\mathcal{I}_1$  is defined if  $i \notin I_1$  as  $\mathcal{I}_1 : (a, i) = (E, C, L, \ell, I, m)$  where everything is as in  $a.\mathcal{I}_1$ , except that  $C = C_1 \cup \{x \cup \{e\} \mid x \in C_1 \text{ is maximal and finite}\}$ .

**The nondeterministic choice** of  $\mathcal{I}_1$  and  $\mathcal{I}_2$  is  $\mathcal{I}_1 + \mathcal{I}_2 = (E, C, L, \ell, I, m)$ , where

- $E = \{\{1\} \times E_1\} \cup \{\{2\} \times E_2\}$  with  $\pi_1 : E \rightarrow \{1, 2\}$  and  $\pi_2 : E \rightarrow E_1 \cup E_2$ ,
- $C = \{\{i\} \times x \mid x \in C_i\}$ ,
- $L = L_1 \cup L_2$ ,
- $\ell(e) = \ell_i(\pi_2(e))$  for  $\pi_1(e) = i$ ,
- $I = I_1 \cup I_2$ ,
- $m(e) = m_i(\pi_2(e))$  for  $\pi_1(e) = i$ .

**The product** of  $\mathcal{I}_1$  and  $\mathcal{I}_2$  is  $\mathcal{I}_1 \times \mathcal{I}_2 = (E, C, L, \ell, I, m)$ , where:

- $E = E_1 \times_\star E_2$ , with  $\pi_i : E \rightarrow E_i^\star$  the projections of the partial product,
- For  $i \in \{1, 2\}$ , define the projections  $\gamma_i : \mathcal{I}_1 \times \mathcal{I}_2 \rightarrow \mathcal{I}_i$  and the configurations  $x \in C$ :

$$\forall e \in E, \gamma_i(e) = \pi_i(e), \gamma_i(\ell_i(e)) = \ell_i(\pi_i(e)), \gamma_i(m_i(e)) = m_i(\pi_i(e))$$

$$\gamma_i(x) \in C_i, \text{ with } \gamma_i(x) = \{e_i \mid \pi_i(e) = e_i \neq \star \text{ and } e \in x\}$$

$$\forall e, e' \in x, \pi_1(e) = \pi_1(e') \neq \star \text{ or } \pi_2(e) = \pi_2(e') \neq \star \Rightarrow e = e'$$

$$\forall e \in x, \exists z \subseteq x \text{ finite, } \gamma_i(z) \in C_i, e \in z$$

$$\forall e, e' \in x, e \neq e' \Rightarrow \exists z \subseteq x, \gamma_i(z) \in C_i, e \in z \iff e' \notin z$$

$$\begin{aligned} \blacksquare \ell : E \rightarrow L = L_1 \times_{\star} L_2 \text{ is } \ell(e) &= \begin{cases} (\ell_1(e_1), \star) & \text{if } \pi_2(e) = \star \\ (\star, \ell_2(e_2)) & \text{if } \pi_1(e) = \star \\ (\ell_1(e_1), \ell_2(e_2)) & \text{otherwise} \end{cases} \\ \blacksquare m : E \rightarrow I = I_1 \times_{\star} I_2 \text{ is } m(e) &= \begin{cases} (m_1(\pi_1(e)), \star) & \text{if } \pi_2(e) = \star \\ (\star, m_2(\pi_2(e))) & \text{if } \pi_1(e) = \star \\ (m_1(\pi_1(e)), m_2(\pi_2(e))) & \text{otherwise} \end{cases} \end{aligned}$$

We now recall the definition of parallel composition for configuration structure, and detail the definition for  $\mathcal{I}$ -structures. Parallel composition consists of a combination of product, relabelling, reidentifying for the  $\mathcal{I}$ -structures, and restriction. For the relabelling operation, we use a *synchronization algebra* [37]  $(S, \star, \perp)$  consisting of a commutative and associative operation  $\bullet$  on a set of labels  $S \cup \{\star, \perp\}$ , where  $\{\star, \perp\} \not\subseteq S$  and such that  $a \bullet \star = a$  (e.g.  $\star$  is the identity element) and  $a \bullet \perp = \perp$  (e.g.  $\perp$  is the zero element), for all  $a \in S$ . To avoid repetition, below we assume given  $(S, \star, \perp)$ , such that  $S \subseteq L_1 \cup L_2$ . We give examples of synchronization algebras in Sect. 2.3.

► **Definition 6** (Parallel composition of configuration structures). The parallel composition of  $\mathcal{C}_1$  and  $\mathcal{C}_2$  is  $\mathcal{C}_1 \mid_S \mathcal{C}_2 = ((\mathcal{C}_1 \times \mathcal{C}_2)[\ell'/\ell])\upharpoonright_{\perp}$  where  $\ell : E_1 \times_{\star} E_2 \rightarrow L_1 \cup L_2$  is the labeling function from the product, and  $\ell' : E_1 \times_{\star} E_2 \rightarrow L_1 \cup L_2 \cup \{\perp\}$  is defined as  $\ell'(e) = \ell_1(e_1) \bullet \ell_2(e_2)$ .

► **Definition 7** (Parallel composition of  $\mathcal{I}$ -structures). The parallel composition of  $\mathcal{I}_1$  and  $\mathcal{I}_2$ , is  $\mathcal{I}_1 \mid_S \mathcal{I}_2 = (\mathcal{I}_3[m'/m_3][\ell'/\ell_3])\upharpoonright_{\perp}$  where  $\mathcal{I}_3 = (E_3, C_3, L_3, \ell_3, I_3, m_3)$  is  $\mathcal{I}_1 \times \mathcal{I}_2$ , and

►  $m' : E_3 \rightarrow I_1 \cup I_2 \cup \{\perp_k \mid k \in I_1 \times_{\star} I_2\}$  is defined as follows, for  $i \neq \star$ :

$$m'(e) = \begin{cases} i & \text{if } m_3(e) = (i, i) & \text{(Sync. or Fork)} \\ i & \text{if } m_3(e) = (i, \star) \wedge \forall e_2 \in E_2, m_2(e_2) \neq i & \text{(Extra } \star.1) \\ i & \text{if } m_3(e) = (\star, i) \wedge \forall e_1 \in E_1, m_1(e_1) \neq i & \text{(Extra } \star.2) \\ \perp_k & \text{otherwise, with } m_3(e) = k & \text{(Error)} \end{cases}$$

►  $\ell' : E_3 \rightarrow L_1 \cup L_2 \cup \{\perp\}$  maps  $e$  to  $\perp$  if  $m'(e) = \perp_k$ , and to  $\ell_1(e_1) \bullet \ell_2(e_2)$  otherwise.

Parallel composition removes from the product the pairs of events that represent ‘non-realizable’ interactions: for configuration structures, pairs of events that do not represent *possible and future synchronizations*; for  $\mathcal{I}$ -structures, pairs of events that do not represent *past synchronizations or forks*. Definitions 11 and 12 will detail how those operations are used to encode a process or a memory, and both types of parallel compositions will be illustrated in Examples 14 and 15.

## 2.2 Concurrent Communicating Calculi

Let  $I = \mathbb{N}$  be a set of *identifiers* and  $i, j$  range over it. Let  $N = \{a, b, c, \dots\}$  be a set of *names* and  $\bar{N} = \{\bar{a}, \bar{b}, \bar{c}, \dots\}$  its set of *co-names*. We define the set of labels  $L = N \cup \bar{N} \cup \{\tau\}$ , and use  $\alpha$  (resp.  $\lambda, \mu, \nu$ ) to range over  $L$  (resp.  $L \setminus \{\tau\}$ ). The *complement* of a name is given by a bijection  $\bar{\cdot} : N \rightarrow \bar{N}$ , whose inverse is also written  $\bar{\cdot}$ , and that we extend to  $\tau$ , i.e.  $\bar{\tau} = \tau$ .

► **Definition 8** (RCCS Processes). The set of reversible processes  $R$  is built on top of the set

of CCS processes by adding memory stacks to the threads:

$$\begin{aligned}
P, Q &:= P \mid Q \parallel \sum_{i \geq 0} \lambda_i.P_i \parallel P \setminus a && \text{(CCS Processes)} \\
e &:= \langle i, \lambda, P \rangle && \text{(Memory Events)} \\
m &:= \emptyset \parallel \gamma.m \parallel e.m && \text{(Memory Stacks)} \\
T &:= m \triangleright P && \text{(Reversible Threads)} \\
R, S &:= T \parallel R \mid S \parallel R \setminus a && \text{(RCCS Processes)}
\end{aligned}$$

We denote  $I(e)$  (resp  $I(m)$ ,  $I(R)$ ) the set of identifiers occurring in  $e$  (resp.  $m$ ,  $R$ ), and let  $\text{nm}(m) = \{\alpha \mid \alpha \in \mathbb{N} \text{ or } \bar{\alpha} \in \bar{\mathbb{N}} \text{ occurs in } m\}$  be the set of (co-)names occurring in  $m$ .

Note that the nullary case of the sum<sup>1</sup> gives the inactive process, denoted  $0$ , and that the unary case gives the ‘usual’ prefixing of a process by a label, and we write e.g.  $a.P$  for  $\sum_1 \lambda_i.P_i$  with  $\lambda_1 = a$  and  $P_1 = P$ . We assume sum to be associative and often consider only its binary case, that we denote with the  $+$  sign. We often forget about the trailing  $\emptyset$  in the memory stack, or the inactive process  $0$  and write e.g.  $e \triangleright a \mid (b + \bar{c})$  for  $e.\emptyset \triangleright (a.0 \mid (b.0 + \bar{c}.0))$ . We work up to the structural congruence  $\equiv$  of CCS—that we suppose familiar to the reader—and write e.g.  $\emptyset \triangleright (P_1 \mid P_2 \mid P_3)$  without parenthesis since  $(P_1 \mid P_2) \mid P_3 \equiv P_1 \mid (P_2 \mid P_3)$ . Finally, alpha-equivalence is written  $=_\alpha$  and supposed familiar as well.

In a memory event  $\langle i, \lambda, P \rangle$ , the  $P$  component represents the process that was not chosen in a non-deterministic transition, but that can be restored if the process wants to go back. The ‘fork’ symbol  $\gamma$  tracks when a memory stack is split between two threads.

► **Definition 9** (Structural equivalence [2, 11]). *Structural equivalence on  $R$  is the smallest equivalence relation generated by the following rules:*

$$\begin{aligned}
\frac{P =_\alpha Q}{m \triangleright P \equiv m \triangleright Q} &&& (\alpha\text{-Conversion}) \\
m \triangleright (P \mid Q) \equiv (\gamma.m \triangleright P \mid \gamma.m \triangleright Q) &&& (\text{Distribution of Memory}) \\
m \triangleright P \setminus a \equiv (m \triangleright P) \setminus a \text{ with } a \notin \text{nm}(m) &&& (\text{Scope of Restriction})
\end{aligned}$$

The labeled transition system for RCCS is given by the rules of Fig. 2. We use  $\xrightarrow{i:\alpha}$  for the union of  $\xrightarrow{i:\alpha}$  (forward) and  $\xrightarrow{i:\alpha}$  (backward transition), and if there are indices  $i_1, \dots, i_n$  and labels  $\alpha_1, \dots, \alpha_n$  such that  $R_1 \xrightarrow{i_1:\alpha_1} \dots \xrightarrow{i_n:\alpha_n} R_n$ , then we write  $R_1 \Longrightarrow R_n$ . Sect. 2.4 will provide examples of executions, but it should be noted that a process  $m \triangleright a.P$  is allowed to make a transition with label  $a$  and identifier  $i \notin I(m)$  using  $\text{act.}$  and add the event  $\langle i, a, 0 \rangle$  to the memory stack  $m$  if  $i \notin I(m)$ . Conversely, a process  $\langle i, a, 0 \rangle.m \triangleright P$  can do a backward transition using  $\text{act.}_*$  with label  $a$  and identifier  $i$  and become  $m \triangleright a.P$ . This system is a conservative extension over the LTS of CCS with prefixed sum, simply adding indices and backward transitions.

► **Definition 10** (Reachable [2, Lemma 1]). *For all  $R$ , if there is a CCS process  $P$  such that  $\emptyset \triangleright P \Longrightarrow R$ , we say that  $R$  is reachable, that  $P$  is the unique origin of  $R$  and write it  $O_R$ .*

An important result [11, Lemma 10] furthermore states that the trace  $\emptyset \triangleright P \Longrightarrow R$  is forward-only<sup>2</sup>. Also, note that multiple RCCS processes can have the same origin, but that

<sup>1</sup> This version of sum is used for simplifying the presentation of the LTS in Fig. 2.

<sup>2</sup> Traces and trace equivalences for RCCS are reminded in Appendix B: they are needed for some proofs and they are similar to their CCS’s counterpart [8], but are not required to understand our results.



$$\begin{array}{c}
i \notin l(m) \frac{}{(m \triangleright \lambda.P + Q) \xrightarrow{i:\lambda} \langle i, \lambda, Q \rangle . m \triangleright P} \text{act.} \\
i \notin l(m) \frac{}{\langle i, \lambda, Q \rangle . m \triangleright P \xrightarrow{i:\lambda} m \triangleright (\lambda.P + Q)} \text{act.*} \\
i \notin l(S) \frac{R \xrightarrow{i:\alpha} R'}{R \mid S \xrightarrow{i:\alpha} R' \mid S} \text{par.L} \\
\frac{R \xrightarrow{i:\alpha} R' \quad a \notin \alpha}{R \setminus a \xrightarrow{i:\alpha} R' \setminus a} \text{res.} \\
\frac{R \xrightarrow{i:\lambda} R' \quad S \xrightarrow{i:\bar{\lambda}} S'}{R \mid S \xrightarrow{i:\tau} R' \mid S'} \text{syn.} \\
\frac{R \xrightarrow{i:\bar{\lambda}} R' \quad S \xrightarrow{i:\lambda} S'}{R \mid S \xrightarrow{i:\tau} R' \mid S'} \text{syn.*} \\
i \notin l(S) \frac{S \xrightarrow{i:\alpha} S'}{R \mid S \xrightarrow{i:\alpha} R \mid S'} \text{par.R} \\
\frac{R_1 \equiv R \quad R \xrightarrow{i:\alpha} R' \quad R' \equiv R'_1}{R_1 \xrightarrow{i:\alpha} R'_1} \equiv
\end{array}$$

■ **Figure 2** Rules of the labeled transition system (LTS)

reachable RCCS processes have one unique origin (up to structural equivalence). We consider only reachable terms: unreachable terms are ‘dysfunctional’ and their memory is considered *not coherent* [12], as they can not ‘rewind’ back to an origin process.

### 2.3 Processes and Memories as (Identified) Configuration Structures

In the definitions below, we write  $S$  for a synchronization algebra  $(S, \star, \perp)$  with  $S = \text{NU}\bar{\text{N}} \cup \{\tau\}$ .

► **Definition 11** (Encoding CCS processes [37]). *Given a CCS process  $P$ , its encoding  $\llbracket P \rrbracket$  as a configuration structure is built inductively:*

$$\begin{array}{lll}
\llbracket \lambda.P + Q \rrbracket = \llbracket \lambda.P \rrbracket + \llbracket Q \rrbracket & \llbracket P \mid Q \rrbracket = \llbracket P \rrbracket \mid_S \llbracket Q \rrbracket & \llbracket P \setminus a \rrbracket = \llbracket P \rrbracket \upharpoonright_{\{a, \bar{a}\}} \\
\llbracket \lambda.P \rrbracket = \lambda.\llbracket P \rrbracket & \llbracket 0 \rrbracket = \mathbf{0} &
\end{array}$$

where  $S$  includes  $\alpha \bullet \bar{\alpha} = \tau$  and  $\alpha \bullet \beta = \perp$ , if  $\beta \neq \bar{\alpha}$ .

► **Definition 12** (Encoding RCCS memories). *Given a RCCS process  $R$ , the encoding  $\llbracket R \rrbracket$  of its memory as an  $\mathcal{I}$ -structure is built by induction on the process and on the memory:*

$$\begin{array}{lll}
\llbracket m \triangleright P \rrbracket = \llbracket m \rrbracket & \llbracket R_1 \mid R_2 \rrbracket = \llbracket R_1 \rrbracket \mid_S \llbracket R_2 \rrbracket & \llbracket R \setminus a \rrbracket = \llbracket R \rrbracket \\
\llbracket \langle i, \lambda, P \rangle . m \rrbracket = \llbracket m \rrbracket :: (\lambda, i) & \llbracket \emptyset \rrbracket = \mathbf{0} & \llbracket \gamma . m \rrbracket = \llbracket m \rrbracket
\end{array}$$

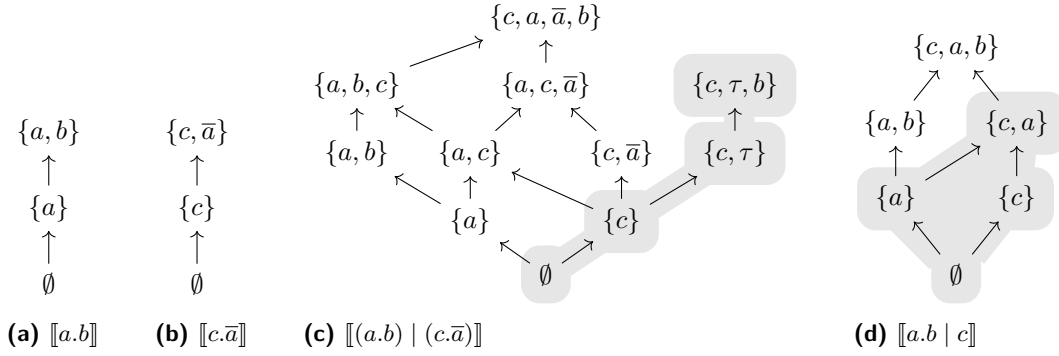
where  $S$  includes  $\alpha \bullet \bar{\alpha} = \tau$ ;  $\alpha \bullet \alpha = \alpha$  and  $\alpha \bullet \beta = \perp$  if  $\beta \notin \{\bar{\alpha}, \alpha\}$ .

### 2.4 Examples

We now illustrate the execution of RCCS processes, the encoding of CCS processes and of RCCS memories, and how they relate.

► **Example 13** (Executing a RCCS process). An example of forward-only trace is:

$$\begin{array}{lll}
\emptyset \triangleright (a.b \mid c.\bar{a}) \equiv & (\gamma.\emptyset \triangleright a.b) \mid (\gamma.\emptyset \triangleright c.\bar{a}) & \text{(Distribution of Memory)} \\
\frac{1:c}{\rightarrow} & (\gamma.\emptyset \triangleright a.b) \mid (\langle 1, c, 0 \rangle . \gamma.\emptyset \triangleright \bar{a}) & \text{(act.)} \\
\frac{2:\tau}{\rightarrow} & (\langle 2, a, 0 \rangle . \gamma.\emptyset \triangleright b) \mid (\langle 2, \bar{a}, 0 \rangle . \langle 1, c, 0 \rangle . \gamma.\emptyset \triangleright 0) & \text{(syn.)} \\
\frac{3:b}{\rightarrow} & (\langle 3, b, 0 \rangle . \langle 2, a, 0 \rangle . \gamma.\emptyset \triangleright 0) \mid (\langle 2, \bar{a}, 0 \rangle . \langle 1, c, 0 \rangle . \gamma.\emptyset \triangleright 0) & \text{(act.)}
\end{array}$$



■ **Figure 3**  $\mathcal{I}$ -structures for Examples 14, 15, 16 and 20, with the CCS term their underlying configuration encode in caption.

Reading it from end to beginning and replacing  $\xrightarrow{i}$  with  $\xleftarrow{i}$  gives a *backward-only* trace, that would rewind the process back to its origin. Of course, a trace can mix forward and backward transitions, as we illustrate in Example 20. The memory of this process is encoded in Example 15.

► **Example 14** (Encoding CCS processes). We can see the  $\mathcal{I}$ -structures from Fig. 1 as configuration structures obtained by encoding the CCS processes  $a + a$ ,  $a \mid a$ , and  $(a.b) \mid \bar{a}$ . Similarly, we can consider the  $\mathcal{I}$ -structures from Fig. 3—ignoring the grayed out parts for now—as configuration structures. The interested reader can check that the encoding of  $(a.b) \mid (c.\bar{a})$  in Fig. 3c is indeed the result of applying the parallel composition of configurations structures (Definition 6) to the encoding of  $a.b$  in Fig. 3a and  $c.\bar{a}$  in Fig. 3b. Lastly, Fig. 3d shows the encoding of  $(a.b) \mid c$ .

The parallel composition of  $\mathcal{I}$ -structures (Definition 7) differs slightly and is new, and hence deserves a detailed example.

► **Example 15** (Encoding RCCS memories). The process obtained at the end of Example 13 has its memory encoded as follows:

$$\begin{aligned} & [(\langle 3, b, 0 \rangle . \langle 2, a, 0 \rangle . \gamma . \emptyset \triangleright 0) \mid (\langle 2, \bar{a}, 0 \rangle . \langle 1, c, 0 \rangle . \gamma . \emptyset \triangleright 0)] \\ &= [\langle 3, b, 0 \rangle . \langle 2, a, 0 \rangle . \gamma . \emptyset \triangleright 0] \mid [\langle 2, \bar{a}, 0 \rangle . \langle 1, c, 0 \rangle . \gamma . \emptyset \triangleright 0] \\ &= [\langle 3, b, 0 \rangle . \langle 2, a, 0 \rangle . \gamma . \emptyset] \mid [\langle 2, \bar{a}, 0 \rangle . \langle 1, c, 0 \rangle . \gamma . \emptyset] \end{aligned}$$

Letting  $E = L = \{a, \bar{a}, b, c\}$ ,  $\ell = \text{id}$ ,  $I = \{1, 2, 3\}$ , using  $[\gamma . \emptyset] = [\emptyset] = \mathbf{0}$  and the postfixing:

$$\begin{aligned} [\langle 3, b, 0 \rangle . \langle 2, a, 0 \rangle . \gamma . \emptyset] &= (\{a, b\}, \{\emptyset, \{a\}, \{a, b\}\}, L, \ell, I, \{a \mapsto 2, b \mapsto 3\}) \\ [\langle 2, \bar{a}, 0 \rangle . \langle 1, c, 0 \rangle . \gamma . \emptyset] &= (\{c, \bar{a}\}, \{\emptyset, \{c\}, \{c, \bar{a}\}\}, L, \ell, I, \{c \mapsto 1, \bar{a} \mapsto 2\}) \end{aligned}$$

Those are displayed in Fig. 3a and 3b, and their product (which is the first step to compute their parallel composition) gives the following sets of events and identifiers:

Event	$(a, \star)$	$(b, \star)$	$(\star, c)$	$(\star, \bar{a})$	$(a, c)$	$(a, \bar{a})$	$(b, c)$	$(b, \bar{a})$
Identifier	$(2, \star)$	$(3, \star)$	$(\star, 1)$	$(\star, 2)$	$(2, 1)$	$(2, 2)$	$(3, 1)$	$(3, 2)$

Re-identifying and re-labeling according to the definition of parallel composition gives:

Event	$(a, \star)$	$(b, \star)$	$(\star, c)$	$(\star, \bar{a})$	$(a, c)$	$(a, \bar{a})$	$(b, c)$	$(b, \bar{a})$
Re-identified	$\perp_{(2, \star)}$	3	1	$\perp_{(\star, 2)}$	$\perp_{(2, 1)}$	2	$\perp_{(3, 1)}$	$\perp_{(3, 2)}$
Re-labeled	$\perp$	$b$	$c$	$\perp$	$\perp$	$\tau$	$\perp$	$\perp$

Indeed, if two events occur at the same time with the same identifier (Sync. or Fork), then their identifier is simply picked. Hence,  $m'(a, \bar{a}) = 2$ . If only one event is present in the pair, and no event on the other component have the same identifier (Extra  $\star.1$ , Extra  $\star.2$ ), then this event's identifier is picked. Hence,  $m'(b, \star) = 3$  and  $m'(\star, c) = 1$ . The remaining cases get re-identified with  $\perp_k$  (Error). Finally,  $(b, \star)$ ,  $(\star, c)$  and  $(a, \bar{a})$  gets relabeled with  $b$ ,  $c$  and  $\tau$  respectively, and after restricting to the label  $\perp$  we obtain the grayed out part of Fig. 3c.

Observe that in this last example, the structure underlying the encoding of the memory is just a particular 'path' in the encoding of the origin. We can observe this intuition again with the following example:

► **Example 16** (Memory and origin). The encoding of the memory resulting from the execution  $\emptyset \triangleright ((a.b) \mid c) \xrightarrow{1:c} \xrightarrow{2:a} (\langle 2, a, 0 \rangle. \gamma. \emptyset \triangleright b) \mid (\langle 1, c, 0 \rangle. \gamma. \emptyset \triangleright 0)$  is the grayed out part in Fig. 3d, with  $m(c) = 1$  and  $m(a) = 2$ . We name this process  $R_1$  and come back to it in Example 20.

## 2.5 Operational Correspondence

Before studying bisimulations on configuration structures and processes, we prove the operational correspondence<sup>3</sup> between RCCS processes and the encodings of their memories in  $\mathcal{I}$ -structures (Lemma 19, cf. also Sect. C.2). Events in  $\mathcal{I}$ -structures resulting from the encoding of a process have different identifiers, and they are either causally linked or concurrent.

► **Lemma 17** (Memories give posets). *For all  $R$ , letting  $x$  be the maximal configuration in  $[R]$  (Definition 2),  $([R], \subseteq)$  is a partially ordered set (poset) with maximal element  $x$ .*

This is proved by induction on  $R$  and illustrated by Examples 15 and 16. However, having at most one maximal configuration does not imply that one particular event has to be 'the last one' introduced. We use the following definition to make it formal.

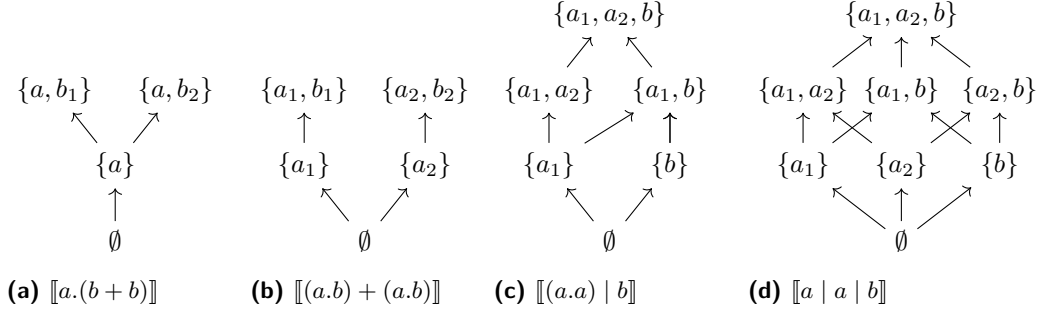
► **Definition 18** (Maximal event). *An event  $e$  is maximal in  $\mathcal{I}$  if there is no event  $e'$  such that  $e <_x e'$ , for  $x$  a maximal configuration of  $\mathcal{I}$ .*

For instance, the encoding of the memory of Example 16, pictured in Fig. 3d, has two maximal events, labeled  $a$  and  $c$ . We can now state the main result of this section:

► **Lemma 19** (Operational Correspondence). *For all  $R$  and  $S$ , writing  $(E_R, C_R, \ell_R, I_R, m_R)$  for  $[R]$  and similarly for  $S$ , if  $R \xrightarrow{i:\alpha} S$  or  $S \xrightarrow{j:\alpha} R$ , then there exists  $e \in E_S$  maximal in  $[S]$  with  $m_S(e) = i$  s.t.  $[R] \cong [S] \upharpoonright_{\{e\}}$ . For all  $R$  and  $e$  a maximal event in  $[R]$ , there is a transition  $R \xrightarrow{m_R(e):\ell_R(e)} S$  with  $[S] \cong [R] \upharpoonright_{\{e\}}$ .*

For the first part, it suffices to show that the forward transition triggers the creation of a maximal event with the same identifier, and nothing else, and that this event can be 'traced' in  $[S]$ . It uses intermediate lemma (Lemmas 43–45) showing how maximal events are preserved by certain operations on  $\mathcal{I}$ -structures. The result follows easily for backward transition, but the last part is more involved: it requires to show that  $e$  can be mapped to a particular transition in the trace from  $O_R$  to  $R$ , and, using a notion of trace equivalence, that this particular transition can be 'postponed' and done last, so that  $R$  can backtrack on it.

<sup>3</sup> Similar to the operational correspondance between configuration structures and CCS processes [7, Section 3] i.e., if  $P \xrightarrow{\alpha} Q$ , then  $\llbracket P \rrbracket = \llbracket Q \rrbracket \upharpoonright_{\{e\}}$ , where  $e$  is an event in  $\llbracket P \rrbracket$  such that  $\ell(e) = \alpha$ .



■ **Figure 4** Configuration structures for Examples 23 and 27

► **Example 20** (Forward and backward transitions). Looking back at the process of Example 16, we could further have  $R_1 \xrightarrow{1:c} R_2 \xrightarrow{3:b} R_3$ , i.e.

$$\begin{aligned} (\langle 2, a, 0 \rangle. \gamma. \emptyset \triangleright b) | (\langle 1, c, 0 \rangle. \gamma. \emptyset \triangleright 0) &\xrightarrow{1:c} (\langle 2, a, 0 \rangle. \gamma. \emptyset \triangleright b) | \gamma. \emptyset \triangleright c) && (\text{act.}_*) \\ &\xrightarrow{3:b} (\langle 3, b, 0 \rangle. \langle 2, a, 0 \rangle. \gamma. \emptyset \triangleright 0) | \gamma. \emptyset \triangleright c) && (\text{act.}) \end{aligned}$$

We can see using Fig. 3d that  $\llbracket R_1 \rrbracket|_c = \llbracket R_2 \rrbracket$  and that  $\llbracket R_2 \rrbracket = \llbracket R_3 \rrbracket|_b$ .

### 3 Reversible and Truly Concurrent Bisimulations Are the Same

#### 3.1 History-Preserving Bisimulations in Configuration Structures

History-preserving bisimulation (HPB) [4, 28, 29] and hereditary history-preserving bisimulation (HHPB) [4, 6] are equivalences on configuration structures that use label- and order-preserving bijections on events. Below, assume given  $\mathcal{C}_i = (E_i, C_i, L_i, \ell_i)$  for  $i = 1, 2$ .

► **Definition 21** (Label- and order-preserving functions (l&o-p)). *A function  $f : x_1 \rightarrow x_2$ , for  $x_i \in \mathcal{C}_i$ ,  $i \in \{1, 2\}$  is label-preserving if  $\ell_1(e) = \ell_2(f(e))$  for all  $e \in x_1$ . It is order-preserving if  $e_1 \leq_{x_1} e_2 \Rightarrow f(e_1) \leq_{x_2} f(e_2)$ , for all  $e_1, e_2 \in x_1$ . We write that  $f$  is l&o-p if it is both.*

► **Definition 22** (HPB and HHPB). *A relation  $\mathcal{R} \subseteq \mathcal{C}_1 \times \mathcal{C}_2 \times (E_1 \rightarrow E_2)$  such that  $(\emptyset, \emptyset, \emptyset) \in \mathcal{R}$ , and if  $(x_1, x_2, f) \in \mathcal{R}$ , then  $f$  is a l&o-p bijection between  $x_1$  and  $x_2$  and (1) and (2) (resp. (1-4)) hold is called a history- (resp. hereditary history-) preserving bisimulation (HPB, resp. HHPB) between  $\mathcal{C}_1$  and  $\mathcal{C}_2$ .*

$$\forall y_1, x_1 \xrightarrow{e_1} y_1 \Rightarrow \exists y_2, g, x_2 \xrightarrow{e_2} y_2, g \upharpoonright_{x_1} = f, (y_1, y_2, g) \in \mathcal{R} \quad (1)$$

$$\forall y_2, x_2 \xrightarrow{e_2} y_2 \Rightarrow \exists y_1, g, x_1 \xrightarrow{e_1} y_1, g \upharpoonright_{x_1} = f, (y_1, y_2, g) \in \mathcal{R} \quad (2)$$

$$\forall y_1, x_1 \xrightarrow{\tilde{e}_1} y_1 \Rightarrow \exists y_2, g, x_2 \xrightarrow{\tilde{e}_2} y_2, g = f \upharpoonright_{y_1}, (y_1, y_2, g) \in \mathcal{R} \quad (3)$$

$$\forall y_2, x_2 \xrightarrow{\tilde{e}_2} y_2 \Rightarrow \exists y_1, g, x_1 \xrightarrow{\tilde{e}_1} y_1, g = f \upharpoonright_{y_1}, (y_1, y_2, g) \in \mathcal{R} \quad (4)$$

We write that  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are (H)HPB if there exists a (H)HPB relation between them.

Note that HPB and HHPB are two different relations, as e.g.  $(a | (b+c)) + (a | b) + ((a+c) | b)$  and  $(a | (b+c)) + ((a+c) | b)$  have HPB but not HHPB encodings [33].

► **Example 23.** The encoding of the two processes  $a.(b+b)$  and  $(a.b) + (a.b)$ , in Fig. 4a and 4b, are HHPB: the relation can start by mapping  $a$  to  $a_1$  or  $a_2$ , and then maps  $b_1$  or  $b_2$  (depending on the superset reached) to  $b_1$  or  $b_2$ , according to the first choice made. This relation can ‘follow’ the forward and backward movements in both structures, giving a l&o-p bijection. This example also proves that HHPB is not CCS’s structural congruence.

### 3.2 Back-and-forth Bisimulations in Reversible CCS

This section presents the relations we will be using, explain the restrictions on previous attempts to capture HHPB as a relation on process algebra terms [2, 25], and shows why both backward transitions *and* identifiers are needed to capture HHPB.

Below, assume given two reachable processes  $R_1$  and  $R_2$ , and if  $f : A \rightarrow B$  is such that  $f(a) = b$ , we write  $f \setminus \{a \mapsto b\}$  for  $f \upharpoonright_{A \setminus \{a\}}$ .

► **Definition 24** (B&F and SB&F bisimulations). *A relation  $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R} \times (I \rightarrow I)$  such that  $(\emptyset \triangleright O_{R_1}, \emptyset \triangleright O_{R_2}, \emptyset) \in \mathcal{R}$  and if  $(R_1, R_2, f) \in \mathcal{R}$ , then  $f$  is a bijection between  $I(R_1)$  and  $I(R_2)$  and (5–8) hold is called a back-and-forth-bisimulation (B&F) between  $R_1$  and  $R_2$ .*

$$\forall S_1, R_1 \xrightarrow{i:\alpha} S_1 \Rightarrow \exists S_2, g, R_2 \xrightarrow{j:\alpha} S_2, g = f \cup \{i \mapsto j\}, (S_1, S_2, g) \in \mathcal{R} \quad (5)$$

$$\forall S_2, R_2 \xrightarrow{i:\alpha} S_2 \Rightarrow \exists S_1, g, R_1 \xrightarrow{j:\alpha} S_1, g = f \cup \{i \mapsto j\}, (S_1, S_2, g) \in \mathcal{R} \quad (6)$$

$$\forall S_1, R_1 \overset{i:\alpha}{\rightsquigarrow} S_1 \Rightarrow \exists S_2, f, R_2 \overset{j:\alpha}{\rightsquigarrow} S_2, g = f \setminus \{i \mapsto j\}, (S_1, S_2, g) \in \mathcal{R} \quad (7)$$

$$\forall S_2, R_2 \overset{i:\alpha}{\rightsquigarrow} S_2 \Rightarrow \exists S_1, g, R_1 \overset{j:\alpha}{\rightsquigarrow} S_1, g = f \setminus \{i \mapsto j\}, (S_1, S_2, g) \in \mathcal{R} \quad (8)$$

If we remove the requirements on  $f$  and  $g$  in the second part of (5–8), we call  $\mathcal{R}$  a simple back-and-forth bisimulation (SB&F). We write that  $R_1$  and  $R_2$  are B&F (resp. SB&F) if there exists a B&F (resp. SB&F) relation between them.

#### Restrictions and Previous Results

► **Definition 25** (Constraints). *Given  $\mathcal{C}$ , if  $\forall x \in \mathcal{C}, \forall e_1, e_2 \in x, \ell(e_1) = \ell(e_2)$  implies  $e_1 = e_2$ , then  $\mathcal{C}$  is non-repeating [25, Definition 3.5]. If,  $\forall x \in \mathcal{C}, \forall e_1, e_2 \in x, e_1 \text{ co}_x e_2$  and  $\ell(e_1) = \ell(e_2)$  implies  $e_1 = e_2$ , then  $\mathcal{C}$  is without auto-concurrency [33, Definition 9.5]. A process  $R$  is non-repeating (resp. without auto-concurrency) if  $\llbracket O_R \rrbracket$  is.*

Those are the constraints used in showing equivalences between process algebra and configuration structures. We omitted the definition of *singly labeled* [2, Definition 26], as it does not contribute to the understanding of our results. Every non-repeating process is without auto-concurrency, but being non-repeating events and singly labeled are incomparable features. Simple processes can be repeating (e.g.  $a.a$ ), with auto-concurrency (e.g.  $(a.b) \mid a$ ), not singly labeled (e.g.  $a + a$ ), as well as complicated processes using these patterns.

The first syntactical characterization of HHPB was obtained on non-repeating processes, using the ‘forward-reverse bisimulation’ (FR) [26, Definition 6.5], which is essentially defined as B&F, with the additional requirement that  $f = \text{id}$ . The theorem states that non-repeating CCSK processes are FR iff their encoding are HHPB [25, Theorem 5.4]. We argue that FR gives too much importance to the technical apparatus implementing reversibility: processes should be able to pick their identifiers freely, and comparing them when establishing bisimulations should not require identities, but only bijections.

A second attempt [2] to capture HHPB used a *back-and-forth barbed congruence* on RCCS processes which was proven to correspond to HHPB on their encodings for a restricted class of processes as well, the class of singly-labeled processes.

**Pinpointing the Right Reversible Bisimulation** We lift both restrictions in Corollary 31, by proving that B&F captures HHPB on *all* processes. Before doing so, let us note that even though B&F is the right notion to capture HHPB, when restricted to non-repeating processes, which are also without auto-concurrency, it does not use in a meaningful way the identifiers.

► **Theorem 26** (Collapsing B&F and SB&F). *If  $R_1$  and  $R_2$  are without auto-concurrency, then they are B&F iff they are SB&F.*

The intuition—made formal in Sect. C.3—is that since two concurrent transitions sharing the same label can not be fired at the same time, the identifiers do not add any information. The proof is easy for the forward transitions, and uses an order on the transitions enforced by causality for the backward traces. In the presence of auto-concurrency, the relations differ, e.g. the process with auto-concurrency  $a \mid a$  and  $a.a$  are SB&F but not B&F.

► **Example 27** (Reversibility is *not* ‘just back and forth’). Observe that the bisimulation relation obtained by only considering (5–6) and ignoring the identifiers in Definition 24 is the ‘standard’ CCS bisimulation. Hence, it could seem natural to assume that ‘simply adding the backwards transitions’, i.e. taking (5–8) without the identifiers, giving SB&F, would be ‘the right’ bisimulation for RCCS. Processes like  $(a.a) \mid b$  and  $a \mid a \mid b$  are SB&F, but their encodings, presented in Fig. 4c and 4d, are not HPB and hence not HHPB: SB&F does not account for reversibility in a satisfactory manner.

Both the bijection on identifiers *and* backward transitions are necessary to capture HHPB. Indeed, as suggested by Example 27, ‘simply’ considering forward and backward transitions is not enough. Let us now consider the role of the bijection on identifiers a bit further. A first remark is that Theorem 26 shows that it is easy to overlook the role of identifiers when restricting the class of processes considered. Secondly, we can prove, as an immediate corollary of Theorems 29 and 30, that considering only (5–6) (*with* the identifiers) in Definition 24 gives a characterization of HPB (Corollary 49): if anything, having a bijection between identifiers—thanks to the order on events that can be deduced from it—helps getting closer to ‘truly concurrent’ bisimulation than adding backward transitions does. However, as HPB and HHPB do not coincide, the identifiers are not enough either.

Of course, similar mechanisms could achieve similar results, but it is our hope that reversibility is fully understood as not ‘just’ being about adding backward transitions or memories, but to use both to obtain ‘backward determinism’.

### 3.3 History-Preserving Bisimulations in (R)CCS

Proving our main result (Corollary 31) will use intermediate relations on processes—called HPB and HHPB as well—that use the encoding of the memories into  $\mathcal{I}$ -structures. Those relations are proven to correspond to (H)HPB on the encoding of the processes on one hand (Theorem 29), and the one that characterizes HHPB is proven to coincide with B&F (Theorem 30) on the other hand. The proofs and connections between formalisms are gathered in Sect. C.4.

► **Definition 28** (HPB and HHPB on RCCS). *A relation  $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R} \times (E_1 \rightarrow E_2)$  such that  $(\emptyset \triangleright O_{R_1}, \emptyset \triangleright O_{R_2}, \emptyset) \in \mathcal{R}$  and if  $(R_1, R_2, f) \in \mathcal{R}$  then  $f$  is an isomorphism between  $\lceil R_1 \rceil$  and  $\lceil R_2 \rceil$  and (9) and (10) (resp. (9–12)) hold is called a history- (resp. hereditary history-) preserving bisimulation between  $R_1$  and  $R_2$ .*

$$\forall S_1, R_1 \xrightarrow{i:\alpha} S_1 \Rightarrow \exists S_2, g, R_2 \xrightarrow{j:\alpha} S_2, g \upharpoonright_{\lceil R_1 \rceil} = f, (S_1, S_2, g) \in \mathcal{R} \quad (9)$$

$$\forall S_2, R_2 \xrightarrow{i:\alpha} S_2 \Rightarrow \exists S_1, g, R_1 \xrightarrow{j:\alpha} S_1, g \upharpoonright_{\lceil R_1 \rceil} = f, (S_1, S_2, g) \in \mathcal{R} \quad (10)$$

$$\forall S_1, R_1 \xrightarrow{\tilde{i}:\alpha} S_1 \Rightarrow \exists S_2, f, R_2 \xrightarrow{\tilde{j}:\alpha} S_2, g = f \upharpoonright_{\lceil S_1 \rceil}, (S_1, S_2, g) \in \mathcal{R} \quad (11)$$

$$\forall S_2, R_2 \xrightarrow{\tilde{i}:\alpha} S_2 \Rightarrow \exists S_1, g, R_1 \xrightarrow{\tilde{j}:\alpha} S_1, g = f \upharpoonright_{\lceil S_1 \rceil}, (S_1, S_2, g) \in \mathcal{R} \quad (12)$$

We write that  $R_1$  and  $R_2$  are (H)HPB if there exists a (H)HPB relation between them.

Above, we write  $g|_{\lceil R \rceil}$  for the restriction of each component of  $g$  to  $\lceil R \rceil$ . Note that the definitions above reflect the definition of (H)HPB (Definition 22): the condition  $(\emptyset \triangleright O_{R_1}, \emptyset \triangleright O_{R_2}, \emptyset) \in \mathcal{R}$  is intuitively the counterpart to the condition that  $(\emptyset, \emptyset, \emptyset)$  has to be included in the relation on configuration structures. Also,  $f$  shares similarity with the l&o-p bijection, in the sense that it exists, with  $\text{id}$  as the component on the labels, iff there exists a l&o-p bijection between the unique maximal configurations in  $\lceil R_1 \rceil$  and  $\lceil R_2 \rceil$  (Lemma 46).

Relations defined on RCCS (B&F, SB&F, HPB and HHPB) straightforwardly extend to CCS, by simply stating that  $P_1$  and  $P_2$  are in it if  $\emptyset \triangleright P_1$  and  $\emptyset \triangleright P_2$  are too. Therefore we can state below our results in terms of CCS processes.

► **Theorem 29** (Equivalences).  $P_1$  and  $P_2$  are HHPB (resp. HPB) iff  $\llbracket P_1 \rrbracket$  and  $\llbracket P_2 \rrbracket$  are.

This result can easily be extended to *weak*-HPB and *weak*-HHPB [6, 28], which are defined by removing from (H)HPB on configuration structures (Definition 22) and on RCCS (Definition 28) the condition that  $f$  must be preserved from one step to the next one.

► **Theorem 30** (Equivalence (contd)).  $P_1$  and  $P_2$  are B&F iff they are HHPB.

Theorem 29 (resp. Theorem 30) uses our operational correspondence between RCCS processes (resp. RCCS memories) and their encodings as configuration structures [2, Lemma 6] (resp. as  $\mathcal{I}$ -structure (Lemma 19)) to transition between the semantic and syntactic worlds.

Our main result will come as an immediate corollary of Theorems 29 and 30.

► **Corollary 31** (Main result).  $P_1$  and  $P_2$  are B&F iff  $\llbracket P_1 \rrbracket$  and  $\llbracket P_2 \rrbracket$  are HHPB.

## 4 Concluding Remarks

This work offers a ‘definitive’ answer to the question of finding a meaningful bisimulation for reversible LTS by providing relations that correspond to (H)HPB on their encodings on *all* processes. We believe this contribution is of value because: **1.** This result solves a problem that was open since HHPB was defined [6], nearly 30 years ago, for which despite the use of multiple techniques, only partial results were obtained, **2.** This idea in appearance simple still requires a lot of technical work, as detailed in the Appendix, **3.** The use of reversibility (both the backtracking capability *and* the memory mechanism) is critical to characterize HHPB on syntactical terms. This result also enforces the importance of identifiers in general and not just as part of a backtracking mechanism. Indeed, they are generally already present when concurrency is implemented, e.g. when two `Unix` threads terminate with the same signal, the parent process have the capacity of determining which process sent which signal.

As a byproduct of our result, we also proposed an encoding of RCCS memories into an ‘enriched’ configuration structure, called identified configuration structure. This observation echoes our previous formalism [2] and similar encoding [18] in an interesting way: as mentioned in the Introduction, a reversible process  $R$  was encoded as a pair  $(\llbracket O_R \rrbracket, x_R)$  made of the configuration structure encoding the origin of  $R$ , and a configuration  $x_R$  in it, called the address of  $R$ . The intuition was that we could ‘match’ a partially executed process with a configuration. We can now go further by observing that  $\lceil R \rceil$  is isomorphic to the  $\mathcal{I}$ -structure generated by  $x_R$ , which is everything ‘below’ it. This result (Lemma 48) is used in our proof, and exemplified by Example 16: the encoding of the memory of  $(\langle 2, a, 0 \rangle. \gamma. \emptyset \triangleright b) \mid (\langle 1, c, 0 \rangle. \gamma. \emptyset \triangleright 0)$  corresponds the ‘past’ of the process, whose underlying structure is grayed out in Fig. 3d, and what is left to execute— $b \mid 0$ —corresponds to the ‘future’ of that process, and is represented by the configuration  $\{c, a, b\}$  in Fig. 3d.

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## A Event Structures as Categories

Configuration structures often use the insights provided by the categorical framework [2, 30, 35]. This appendix regroups the categorical treatment of (identified) configuration structures (Definition 1).

► **Definition 32** (Category of configuration structures). *We define  $\mathbb{C}$  the category of configuration structures, where an object is a configuration structure, and a morphism  $f : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  is a triple  $(f_E, f_L, f_C)$  such that*

- $f_L : L_1 \rightarrow L_2$ ;
- $f_E : E_1 \rightarrow E_2$  preserves labels:  $\ell_2(f_E(e)) = f_L(\ell_1(e))$ ;
- $f_C : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  is defined as  $f_C(x) = \{f_E(e) \mid e \in x\}$ .

We write  $\mathcal{C}_1 \cong \mathcal{C}_2$  if there exists an isomorphism between  $\mathcal{C}_1$  and  $\mathcal{C}_2$ .

For simplicity, we often assume that  $L_1 = L_2$ , i.e., that all the configuration structures use the same set of labels, take  $f_L$  to be the identity and remove it from the notation.

► **Definition 33** (Category of  $\mathcal{I}$ -structures). *We define  $\mathbb{D}$  the category of identified configuration structures, where objects are  $\mathcal{I}$ -structures, and a morphism  $f : \mathcal{I}_1 \rightarrow \mathcal{I}_2$  is a tuple  $q = (f, f_m)$  such that*

- $f = (f_E, f_C)$  is a morphism in  $\mathbb{C}$  between the underlying structures of  $\mathcal{I}_1$  and  $\mathcal{I}_2$ ,
- $f_m : I_1 \rightarrow I_2$  preserves identifiers:  $f_m(m_1(e)) = m_2(f_E(e))$ .

We write  $\mathcal{I}_1 \cong \mathcal{I}_2$  if there exists an isomorphism between  $\mathcal{I}_1$  and  $\mathcal{I}_2$ .

Observe that  $\mathbb{C}$  is a subcategory of  $\mathbb{D}$ . In both  $\mathbb{C}$  and  $\mathbb{D}$ , composition is written  $\circ$  and defined componentwise.

► **Lemma 34.** *Identified configuration structures and their morphisms form a category.*

**Proof. Identity** For every  $\mathcal{I}$ -structure  $\mathcal{I} = (E, C, \ell, I, m)$ ,  $\text{id}_{\mathcal{I}} : \mathcal{I} \rightarrow \mathcal{I}$  is defined to be the identity on the underlying configuration structure  $\text{id} : (E, C, \ell) \rightarrow (E, C, \ell)$  from  $\mathbb{C}$ , that trivially preserves identifiers. For any morphism  $f : \mathcal{I}_1 \rightarrow \mathcal{I}_2$ ,  $f \circ \text{id}_{\mathcal{I}_1} = f = \text{id}_{\mathcal{I}_2} \circ f$  is trivial.

**Associativity** for  $f : \mathcal{I}_1 \rightarrow \mathcal{I}_2$ ,  $g : \mathcal{I}_2 \rightarrow \mathcal{I}_3$  and  $h : \mathcal{I}_3 \rightarrow \mathcal{I}_4$ ,  $h \circ (g \circ f) = (h \circ g) \circ f$  is inherited from the associativity in  $\mathbb{C}$ , and since  $f$ ,  $g$  and  $h$  all preserves identifiers.

Hence  $\mathbb{D}$  is a category. ◀

Unsurprisingly, a forgetful functor and an enrichment functor can be defined between those two categories. The only assumption is that we need to suppose that every configuration structure can be endowed with a total ordering  $\preceq$  on its events.

► **Lemma 35.** *The forgetful functor  $\mathcal{F} : \mathbb{D} \rightarrow \mathbb{C}$ , defined by*

- $\mathcal{F}(E, C, \ell, I, m) = (E, C, \ell)$
- $\mathcal{F}(f_E, f_C, f_m) = (f_E, f_C)$

and the enrichment functor  $\mathcal{S} : \mathbb{C} \rightarrow \mathbb{D}$ , defined by

- $\mathcal{S}(E, C, \ell) = (E, C, \ell, I, m)$ , where  $I = \{1, \dots, |E|\}$  for  $|E|$  the cardinality of  $E$ , and

$$m(e) = \begin{cases} 1 & \text{if } \forall e', e \preceq e' \\ i + 1 & \text{if } \exists e', e' \preceq e, m(e') = i \text{ and there is no } e'' \text{ s.t. } e' \preceq e'' \preceq e \end{cases}$$

- For  $(f_E, f_C) : (E_1, C_1, \ell_1) \rightarrow (E_2, C_2, \ell_2)$ ,  $\mathcal{S}(f_E, f_C) = (f_E, f_C, f_m)$ , where we let  $f_m(m_1(e)) = m_2(f_E(e_2))$ .

are functors.

**Proof.** Proving that  $\mathcal{F}$  is a functor is immediate.

Proving that  $\mathcal{S}(\mathcal{C})$  is a  $\mathcal{I}$ -structure is immediate, since our construction of  $m$  trivially insures Collision Freeness. For  $(f_E, f_C) : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ , proving that  $\mathcal{S}(f_E, f_C)$  is a morphism between  $\mathcal{S}(\mathcal{C}_1)$  and  $\mathcal{S}(\mathcal{C}_2)$  is also immediate. For the preservation of the identity, we compute:

$$\begin{aligned} \mathcal{S}(\text{id}_{\mathcal{C}}) &= \mathcal{S}(\text{id}_E, \text{id}_C) \\ &= (\text{id}_E, \text{id}_C, f_m) \end{aligned}$$

where  $f_m(m(e)) = m(\text{id}_E(e)) = m(e)$ , hence  $f_m = \text{id}_I : I \rightarrow I$ ,

$$\begin{aligned} &= (\text{id}_E, \text{id}_C, \text{id}_I) \\ &= \text{id}_{\mathcal{S}(\mathcal{C})} \end{aligned}$$

For the composition of morphisms, given  $f = (f_C, f_E) : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  and  $g = (g_C, g_E) : \mathcal{C}_2 \rightarrow \mathcal{C}_3$ , we write  $\mathcal{S}(\mathcal{C}_i) = (E_i, C_i, \ell_i, I_i, m_i)$  and we compute:

$$\begin{aligned} \mathcal{S}(g) \circ \mathcal{S}(f) &= (g_C, g_E, g_m) \circ (f_C, f_E, f_m) \\ &= (g_C \circ f_C, g_E \circ f_E, g_m \circ f_m) \end{aligned}$$

where, for all  $e \in E_1$ , we compute:

$$\begin{aligned} (g_m \circ f_m)(m_1(e)) &= g_m(f_m(m_1(e))) \\ &= g_m(m_2(f_E(e))) && \text{(Since } f_m \text{ preserves identifiers)} \\ &= m_3(g_E(f_E(e))) && \text{(Since } g_m \text{ preserves identifiers)} \end{aligned}$$

Hence we can conclude:

$$\mathcal{S}(g) \circ \mathcal{S}(f) = \mathcal{S}(g \circ f) \quad \blacktriangleleft$$

► **Remark 36.** In  $\mathbb{D}$ , every morphism  $f = (f_E, f_L, f_C, f_m)$  from  $\mathcal{I}_1$  to  $\mathcal{I}_2$  is actually fully determined by  $f_E$  whenever  $f_L = \text{id}$ . Indeed, given  $f_E : E_1 \rightarrow E_2$ , then we can define for all  $x \in C_1$ ,  $f_C(x) = \{f_E(e) \mid e \in x\}$  and  $f_m$  as  $f_m(m_1(e)) = m_2(f_E(e))$ . We will often make the abuse of notation of writing  $f_E$  for  $f$  and reciprocally.

## B Concurrency in a Trace and Trace Equivalence

We give here a quick reminder on concurrency and causality in CCS [8] and RCCS [11] traces. Aside from the convenient notation  $m_{R/S}$  that represents the memory stack(s) modified by a forward transition from  $R$  to  $S$ , and of the notation  $\vec{a}$  for a list of names  $a_1, \dots, a_n$ , nothing new is introduced in this Section. However, the results reminded below are used in the proofs of Lemma 19 and Theorem 26.

Concurrency on events corresponds to a notion of concurrency on transitions in RCCS traces [11, Definition 7 and Lemma 8]. For this reminder we consider only concurrency and causality for forward transitions, so that CCS intuitions work equally well. We make a remark at the end about extending the concurrency to backward transitions, but it should be noted that forward and backward transitions are not mixed.

Two transitions  $t_1 = R \xrightarrow{i_1:\alpha_1} R_1$  and  $t_2 = R' \xrightarrow{i_2:\alpha_2} R_2$  are *composable* if  $R_1 = R'$ , and in this case, doing  $t_1$  then  $t_2$  is written as the composition  $t_1; t_2$ . Given  $n$  composable

transitions  $t_i : R_i \xrightarrow{i:\alpha_i} R_{i+1}$  and their composition  $t_1; \dots; t_n$ , we say that  $t_i$  is a *direct cause* of  $t_k$  for  $1 \leq i < k \leq n$  and write  $t_i < t_k$  (or, for short,  $i < k$ ) if there is a memory stack  $m$  in  $R_{i+1}$  and a memory stack  $m'$  in  $R_{k+1}$  such that  $m < m'$ , where the order on memory stacks is given by prefix ordering. Note that, if they exist,  $m$  and  $m'$  are unique, as memory events in reachable processes all have a different pairs (identifier, label).

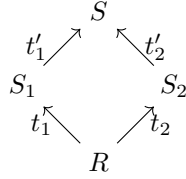
Let  $R \xrightarrow{i:\alpha} S$  be a transition. If  $\alpha \neq \tau$ , we write  $m_{R/S} = \{m\}$  where

$$\begin{aligned} R &= (\dots ((R_3 \mid ((R_1 \mid (m \triangleright P)) \mid R_2) \setminus \vec{b}^1) \mid R_4) \setminus \vec{b}^2 \dots \mid R_n) \setminus \vec{b}^m \\ S &= (\dots ((R_3 \mid ((R_1 \mid \langle i, a, Q \rangle . m \triangleright P) \mid R_2) \setminus \vec{b}^1) \mid R_4) \setminus \vec{b}^2 \dots \mid R_n) \setminus \vec{b}^m \end{aligned}$$

for some  $R_i$  any of which could be missing and for some  $\vec{b}^j$ , possibly missing as well. If  $\alpha = \tau$ , then  $m_{R/S}$  will contain the pair of memory stacks that has been changed by the transition. Intuitively, the notation  $m_{R/S}$  is useful to extract the memory stack(s) modified by a forward transition from  $R$  to  $S$ .

Two transitions are *coinitial* if they have the same source process and *cofinal* if they have the same target process. We say that two coinitial transitions  $t_1 = R \xrightarrow{i_1:\alpha_1} S_1$  and  $t_2 = R \xrightarrow{i_2:\alpha_2} S_2$  are *concurrent* if  $m_{R/S_1} \cap m_{R/S_2} = \emptyset$ , that is, if the transitions modify disjoint memories in  $R$ .

The square lemma [11, Lemma 8] says that moreover, given two such concurrent transitions, there exists two cofinal and concurrent transitions  $t'_1 = S_1 \xrightarrow{i_2:\alpha_2} S$  and  $t'_2 = S_2 \xrightarrow{i_1:\alpha_1} S$ . The name of the lemma comes from this picture:



Moreover, the traces  $\theta_1 = t_1; t'_1$  and  $\theta_2 = t_2; t'_2$  are *equivalent* [11, Definition 9]. This allows one to define *equivalence classes on transitions*:  $t_1$  in  $\theta_1$  is equivalent to  $t'_2$  in  $\theta_2$  if  $\theta_1$  is equivalent to  $\theta_2$  and  $t_1$  and  $t'_2$  have the same index. Then in the trace  $t_1; t'_1$  we are now allowed to say that  $t_1$  is concurrent to  $t'_1$ .

In a trace  $t_1; t_2$  we have that  $t_1$  is concurrent to  $t_2$  iff  $t_1$  is not a cause of  $t_2$ . This follows from a case analysis using the definitions of concurrency and causality. Thanks to trace equivalence, we also have that in a trace  $t_1; \dots; t_n$  either  $t_1$  is a cause of  $t_n$  or the two transitions are concurrent. Those intuitions are enough for us to carry on our development, but a complete treatment of concurrency and causality in the trace of a CCS process [8] can give better insight to the curious reader.

The definitions of concurrency for forward coinitial traces and of causality for forward traces can easily be ‘flipped’ into definitions of concurrency for backward cofinal traces, and of causality for backward traces.

## C Proofs and Auxiliary Materials

In this section we detail the proofs missing from the main text, and introduce some intermediate definitions and lemmas that are necessary for the proofs.

### C.1 Operations on Identified Configurations Structures (Sect. 2.1)

Our main goal here is to prove that the operations of Definition 5 preserve  $\mathcal{I}$ -structures (Lemma 40). The product and coproduct (used to define the nondeterministic choice below) have particular roles, since they have a direct representation in the categorical world.

The structures we considered are *full* w.r.t. the sets of labels and identifiers, i.e. the labeling and identifying functions are surjective. This only impacts the relabelling and reidentifying operations, where we have to additionally require that  $\ell'$  and  $m'$  are surjective.

We redefine the nondeterministic choice of Definition 5 by first defining the coproduct on  $\mathcal{I}$ -structures and then using relabeling and reidentifying to get rid of the extra indices in the label and the identifier of events. It is easy to check that the two definitions of nondeterministic choice are equivalent, but working with the one from Definition 5 is easier and simpler.

► **Definition 37.** *We redefine nondeterministic choice using coproduct as follows:*

**The coproduct** of  $\mathcal{I}_1$  and  $\mathcal{I}_2$  is  $\mathcal{I}_1 \pm \mathcal{I}_2 = (E, C, L, \ell, I, m)$ , where

- $E = \{\{1\} \times E_1\} \cup \{\{2\} \times E_2\}$  with  $\pi_1 : E \rightarrow \{1, 2\}$  and  $\pi_2 : E \rightarrow E_1 \cup E_2$ ,
- $C = \{\{i\} \times x \mid x \in C_i\}$ ,
- $L = \{\{1\} \times L_1\} \cup \{\{2\} \times L_2\}$ ,
- $\ell(e) = (i, \ell_i(\pi_2(e)))$  for  $\pi_1(e) = i$ ,
- $I = \{\{1\} \times I_1\} \cup \{\{2\} \times I_2\}$ ,
- $m(e) = (i, m_i(\pi_2(e)))$  for  $\pi_1(e) = i$ .

and with the expected injections  $\iota_i : \mathcal{I}_i \rightarrow \mathcal{I}_1 + \mathcal{I}_2$ .

**The nondeterministic choice** of  $\mathcal{I}_1$  and  $\mathcal{I}_2$  is  $\mathcal{I}_1 + \mathcal{I}_2 = (\mathcal{I}_1 + \mathcal{I}_2)[\ell'/\ell][m'/m]$  where

- $\ell'(e) = a$  if  $\ell(e) = (j, a)$ ,  $j \in \{1, 2\}$ ,
- $m'(e) = i$  if  $\ell(e) = (j, i)$ ,  $j \in \{1, 2\}$ .

► **Lemma 38.** *The product and coproduct of  $\mathcal{I}$ -structures is the product and coproduct in  $\mathbb{D}$ .*

In the categorical setting, the product and coproduct on labeled configuration structures can be obtained by a straightforward enrichment of the un-labeled configuration structures [37, Propositions 11.2.2 and 11.2.3]. In a similar vein, we obtain the extension of those operations on identified (labeled) configuration structures directly.

**Proof.** The proof for the product and the coproduct follows the same pattern: assume  $\mathcal{I}$  is the result of applying the operation to  $\mathcal{I}_1$  and  $\mathcal{I}_2$ , and observe that

1.  $\mathcal{I}$  is an  $\mathcal{I}$ -structure. This follows from the fact that the structure underlying  $\mathcal{I}$ ,  $\mathcal{F}(\mathcal{I}) = \mathcal{C}$  is the product or coproduct of the configurations underlying  $\mathcal{F}(\mathcal{I}_1) = \mathcal{C}_1$  and  $\mathcal{F}(\mathcal{I}_2) = \mathcal{C}_2$ , hence, that it is a valid configuration structure. As  $m_1$  and  $m_2$  are such that Collision Freeness is enforced, then by construction  $m$  enforces it too and  $\mathcal{I} = \mathcal{C} \oplus m$  is an  $\mathcal{I}$ -structure.

2. The morphisms  $\gamma_i$  (for the product) and  $\iota_i$  (for the coproduct) extends the corresponding morphisms on configuration structure with valid morphisms on identifiers.

Let us suppose that  $\mathcal{C}$  is the product and let  $\alpha_1 : \mathcal{C} \rightarrow \mathcal{C}_1$  be one of the projections. Let us write  $\mathcal{I} = \mathcal{C} \oplus m = (\mathcal{C}, m, I)$  and  $\mathcal{I}_1 = \mathcal{C}_1 \oplus m_1 = (\mathcal{C}_1, m_1, I_1)$ . From the definition of product on  $\mathcal{I}$ -structures,  $I = I_1 \times_{\star} I_2$  is the product on the sets on identifiers and let  $p_1 : I \rightarrow I_1$  be one of the projections, defined as  $p_1(i_1, i_2) = i_1$ , for  $i_1 \neq \star$ . We can now write  $\gamma_1 = (\alpha_1, p_1)$  the projection on  $\mathcal{I}$ -structures, which is indeed a morphism: suffices to verify that  $p_1(m(e)) = m_1(\alpha_1(e))$ , for all events  $e$ , which follows from the definition of  $m$ . A similar argument holds for the coproduct and the injections.

3. The universal properties follow easily by lifting the universal property of the underlying configuration structures in  $\mathbb{D}$ . ◀

The restriction of the operations of Definition 5 to configuration structures are standard [35, 36], except for postfixing. We prove below that the restriction of this operation to configuration structures is correct.

► **Lemma 39.** *The postfixing of a label  $a$  to an event structure  $\mathcal{C}_1 = (E_1, C_1, L_1, \ell_1)$ , defined as  $\mathcal{C}_1 :: (a) = (E, C, L, \ell)$  where*

- $E = E_1 \cup \{e\}$ , for  $e \notin E_1$ ,
- $L = L_1 \cup \{a\}$ ,
- $C = C_1 \cup \{x \cup \{e\} \mid x \in C_1 \text{ is maximal and finite}\}$ ,
- $\ell = \ell_1 \cup \{e \mapsto a\}$ ,

*is a configuration structure.*

**Proof.** Looking back at Definition 1, we simply have to prove that the four axioms of configuration structures are respected by  $\mathcal{C}_1 :: (a)$ , knowing that they are respected by  $\mathcal{C}_1$  by assumption.

- Finiteness is satisfied because  $\mathcal{C}_1$  is a configuration structure, and every configuration in  $\{x \cup \{e\} \mid x \in C_1 \text{ is maximal and finite}\}$  is finite.
- For Coincidence Freeness, we only have to check the configurations containing  $e$ , otherwise it follows from  $\mathcal{C}_1$  being a configuration structure. Given  $y \in \{x \cup \{e\} \mid x \in C_1 \text{ is maximal and finite}\}$ , there exists  $y'$  such that  $y = y' \cup \{e\}$ . Given two events  $e_1, e_2$  in  $y$  such that  $e_1 \neq e_2$ , if  $e_1 = e$  or  $e_2 = e$ , then  $y'$  is the configuration we are looking for. Otherwise, it follows from  $y'$  being a configuration in the configuration structure  $\mathcal{C}_1$ .
- In Finite Completeness, let  $X$  be a subset of configurations. If  $\forall x \in X, e \notin x$ , then the result follows from  $\mathcal{C}_1$  being a configuration structure. Otherwise, there exists  $x' \in X$  such that  $e \in x'$ , hence we know that  $x'$  is maximal. Now, assume there is  $y \in C$  finite such that  $\forall x \in X, x \subseteq y$ . As  $x'$  is finite and maximal and  $y$  is finite, then  $x' = y$ . Suppose by contradiction that  $\bigcup X \notin C$ . It implies that  $\exists y' \in X$  with  $e' \in y'$  such that  $e' \notin x'$ . We reach a contradiction: as  $y' \subseteq y, e' \in y$ , and as  $x' \subseteq y$  and  $e' \notin x'$ , we obtain that  $x' \subsetneq y$ , which contradicts the maximality of  $x'$ . Hence, for all  $x \in X, x \subseteq x'$ , and  $\bigcup X = x' = y$ .
- For Stability let us consider  $x$  and  $y$  such that  $x \cup y \in C$ . If  $e \notin x \cup y$  the result follows from the stability of  $\mathcal{C}_1$ . Otherwise assume, that  $e \in x$ , but then  $x$  is maximal, and for  $x \cup y \in C$  to be the case it must be that either  $y = \emptyset$  or  $y \subseteq x$ , and  $x \cap y \in C$  holds trivially. ◀

► **Lemma 40.** *The operations of Definition 5 (relabeling, reidentifying, restriction, prefixing, postfixing, non-deterministic choice and product), coproduct, as well as the parallel composition (Definition 7) preserve  $\mathcal{I}$ -structures.*

**Proof.** Let us note that 1. the restriction of the relabeling, restrictions, prefixing, postfixing, non-deterministic choice, coproduct and product to configuration structures preserve configuration structures, as their are adaptations of the usual operations [35, 36], or proven correct in Lemma 39, 2. any configuration structure endowed with a valid identifying function (i.e., such that no two events in the same configuration have the same identifier, cf. Collision Freeness) is a valid  $\mathcal{I}$ -structure.

**For the relabeling,** since this operation does not change anything but the labels, we have nothing to prove.

**For the reidentifying,** since the function  $m'$  is supposed to make the resulting  $\mathcal{I}$ -structure respect the Collision Freeness condition, there is nothing to prove.

**For the restriction,** note that this operation only removes events in configurations and keeps the identifying function intact. Hence if the initial structure has a valid identifying function, then the identifying function of the new structure is a valid one by assumption.

**For the prefixing,** since  $i$  is a fresh identifier, Collision Freeness is trivially preserved.

**For the postfixing,** we know that the underlying configuration of  $\mathcal{I}_1 :: (a, i)$ ,  $\mathcal{C}_1 :: (a)$  is a valid configuration structure by Lemma 39. And since  $i$  is a fresh identifier, Collision Freeness is trivially preserved.

**For the coproduct,** it follows trivially from Lemma 38.

**For the product,** it follows trivially from Lemma 38.

**For the nondeterministic choice,** note that Collision Freeness holds only if  $I_1 \cap I_2 = \emptyset$ . We assume this to be the case, as we can always apply a reidentifying operation when it is not, to guarantee that nondeterministic choice is well defined. Moreover the two definitions of nondeterministic choice from Definition 5 and Definition 37 coincide, which follows trivially.

**For the parallel composition,** we need to first observe that this operation is defined in terms of product, restriction, relabeling, and reidentifying along  $m'$ , and we know that those operations are correct provided  $m'$  respects the Collision Freeness condition. In other terms, given  $\mathcal{I}_1, \mathcal{I}_2$  and their product  $\mathcal{I}_3 = (E_3, C_3, L_3, \ell_3, I_3, m_3)$ , we only need to prove that  $\mathcal{I}_3[m'/m_3]$  is a valid  $\mathcal{I}$ -structure, and it follows from a case analysis. Given a configuration  $x \in C_3$ , as and two events  $e, e' \in x$ , we can reason by case analysis on the first and second projections of both events, and these are the possible cases:

- $\pi_2(e) = \star$  and  $\pi_2(e') = \star$ . In this case, looking at the definition of the product in  $\mathcal{I}$ -structures,  $m'(e) = (m_1(\pi_1(e)), \star)$  and  $m'(e') = (m_1(\pi_1(e')), \star)$ . If  $m'(e) = m'(e')$ , then  $m_1(\pi_1(e)) = m_1(\pi_1(e'))$  in the configuration  $\pi_1(x)$  in  $\mathcal{I}_1$ . But that's a contradiction, since  $\pi_1(e)$  and  $\pi_1(e')$  are in the same configuration in  $\mathcal{I}_1$  and the identifying function of  $\mathcal{I}_1$  is valid.
- $\pi_1(e) = \star$  and  $\pi_1(e') = \star$ . This case is similar as the previous one, except that it uses that the identifying function of  $\mathcal{I}_2$  is valid.
- $\pi_1(e) \neq \star$ ,  $\pi_2(e) = \star$  and  $\pi_2(e') \neq \star$  (with either  $\pi_1(e') = \star$  or  $\pi_1(e') \neq \star$ ). If  $m'(e) = \perp_{(m_1(\pi_1(e)), \star)}$ , then clearly  $m'(e) \neq m'(e')$  as  $\pi_2(e') \neq \star$ . If  $m'(e) = m_1(\pi_1(e))$ , then we know that  $\forall e_2 \in E_2, m_2(e_2) \neq m_1(\pi_1(e))$ , and it is impossible that  $m'(e') = m'(e)$ , as  $\pi_2(m'(e')) = m_1(\pi_1(e))$  cannot be.
- $\pi_1(e) \neq \star$ ,  $\pi_2(e) \neq \star$  and  $\pi_2(e') \neq \star$  (again with either  $\pi_1(e') = \star$  or  $\pi_1(e') \neq \star$ ). If  $m'(e) = \perp_{m_3(e)}$ , then having  $m'(e) = m'(e')$  would imply  $m'(e') = \perp_{m_3(e')}$  and  $m_3(e) = m_3(e')$ , which would contradict that product is well-defined. Otherwise,  $e$  is a synchronization or a fork (cf. Sync. or Fork in Definition 7), and it follows that  $m_1(\pi_1(e)) = m_2(\pi_2(e))$  and  $m'(e) = m_1(\pi_1(e))$ . We prove by case analysis that  $m'(e) = m'(e')$  can not be the case:
  - If  $\pi_1(e') = \star$ , then  $m'(e') = \perp_{m_3(e')}$  as  $\pi_1(e)$  prevents from applying Extra  $\star.2$ .
  - If  $\pi_1(e') \neq \star$ ,  $m_3(e') = m_3(e)$ , which would contradict the correctness of the product.
- $\pi_1(e) \neq \star$ ,  $\pi_2(e) \neq \star$  and  $\pi_2(e') = \star$  is similar to above, except that it uses Extra  $\star.1$ .
- Finally, if  $\pi_1(e) \neq \star$ ,  $\pi_2(e) \neq \star$ ,  $\pi_1(e') \neq \star$ ,  $\pi_2(e') \neq \star$ , then  $m'(e) = m(e')$  would require to have applied either Sync. or Fork or Error to both cases, and both situations would contradict the correctness of the product. ◀



## C.2 Properties of Memory Encodings and Operational Correspondence (Sections 2.3 and 2.5)

Our goal here is to show that there is an operational correspondence between  $R$  and  $\lceil R \rceil$  (Lemma 19, Sect. C.2.2), and this requires intermediate lemmas (Lemmas 43–45) about the encoding of memories and their relation to maximal events. To prove those, we start by exhibiting some useful properties of memory encoding (Sect. C.2.1).

### C.2.1 Properties of Memory Encodings

We assume given reachable processes  $R$  and  $S$  and we write  $O_R$  for the origin of  $R$ , and  $\lceil R \rceil$  as  $(E_R, C_R, \ell_R, I_R, m_R)$  and similarly for  $S$ .

To prove interesting properties about the encoding of memory, we first need this small technical lemma.

► **Lemma 41.** *For every reversible thread  $m \triangleright P$  of a reachable process  $R$ , and for all  $i \in \mathsf{l}(m)$ ,  $i$  occurs once in  $m$ .*

**Proof.** We prove it by induction on the structure of  $m$ .

- if  $m = \emptyset$ , then it is obvious.
- if  $m = \Upsilon.m'$ , then by induction hypothesis, for all  $i \in \mathsf{l}(m')$ ,  $i$  occurs only once in  $m'$ , and since no identifier occur in  $\Upsilon$ ,  $i$  occurs only once in  $m$ .
- if  $m = \langle j, \lambda, Q \rangle.m'$  then there exists  $S$  such that  $m' \triangleright S \xrightarrow{j:\lambda} \langle j, \lambda, Q \rangle.m' \triangleright P$  since  $R$  is reachable and  $\langle j, \lambda, Q \rangle.m' \triangleright P$  is a thread in it. By induction we know that for all  $i \in \mathsf{l}(m')$ ,  $i$  occurs once in  $m'$ . We reason on the derivation tree of the transition that adds the memory event  $\langle j, \lambda, Q \rangle$  and we have that for any such transition, the rule  $\text{act.of}$  of Fig. 2 is applied as axiom at the top of the derivation tree. By the side condition of the rule  $\text{act.}$ , we know that  $j \notin \mathsf{l}(m')$ , hence that for all  $i \in \mathsf{l}(m)$ ,  $i$  occurs once in  $m$ . ◀

Note that the property above holds for reversible threads, and not for RCCS processes in general: we actually *want* memory events to sometimes share the same identifiers. Indeed, two memory events need to have the same identifiers if they result from a synchronization (i.e., the application of the  $\text{syn.rule}$  of Fig. 2) or a fork (i.e., the application of the Distribution of Memory rule of structural equivalence, Definition 9).

► **Lemma 42 (Uniqueness of identifiers).** *For all  $e_1, e_2 \in E_R$ ,  $m_R(e_1) = m_R(e_2)$  implies  $e_1 = e_2$ .*

**Proof.** We proceed by structural induction on  $R$ . From Lemma 41 the only interesting case is the parallel composition, i.e.  $R = R_1 \mid R_2$ . From the definition of parallel composition of  $\mathcal{I}$ -structures (Definition 7), it follows that  $m_R(e_1) = m_R(e_2)$  implies  $e_1 = e_2$ . ◀

► **Lemma 17 (Memories give posets).** *For all  $R$ , letting  $x$  be the maximal configuration in  $\lceil R \rceil$  (Definition 2),  $(\lceil R \rceil, \subseteq)$  is a partially ordered set (poset) with maximal element  $x$ .*

**Proof.** We proceed by induction on  $R$ .

**If  $R$  is  $m \triangleright P$ ,** we prove that  $\lceil m \triangleright P \rceil$  is a partially ordered set (poset) with one maximal element by induction on  $m$ . The base case, if  $m$  is  $\emptyset$ , is trivial, since  $\lceil \emptyset \rceil = \mathbf{0}$  is a poset with one maximal element,  $\emptyset$ . If  $m$  is  $\Upsilon.m'$ , then it follows by induction hypothesis, since  $\lceil \Upsilon.m' \rceil = \lceil m' \rceil$ . If  $m$  is  $\langle i, \alpha, P \rangle.m'$ , then by induction hypothesis,  $\lceil m' \rceil$  is a poset with one maximal element, and the postfixing construction used to define  $\lceil \langle i, \alpha, P \rangle.m' \rceil$ , detailed in Definition 5, preserves that property.

If  $R$  is  $R' \setminus a$ , then it trivially follows by induction hypothesis.

If  $R$  is  $R_1 \mid R_2$ , then by induction hypothesis we get that  $\lceil R_1 \rceil$  and  $\lceil R_2 \rceil$  are both posets with a maximal configuration. We also know by Lemma 42 that events in  $\lceil R_1 \rceil$  and  $\lceil R_2 \rceil$  have disjoint identifiers (which does not imply that identifiers occurring in  $\lceil R_1 \rceil$  and in  $\lceil R_2 \rceil$  need to be disjoint). Looking at the definition of parallel composition for  $\mathcal{I}$ -structures (Definition 7), we may observe that  $\lceil R \rceil = \lceil R_1 \mid R_2 \rceil = \lceil R_1 \rceil \mid \lceil R_2 \rceil$  consists of the structure  $(\lceil R_1 \rceil \times \lceil R_2 \rceil)[m'/m][\ell'/\ell] \perp_{\perp}$  for a certain  $m'$  and  $\ell'$ .

We show that there exists more than one maximal configurations in  $\lceil R_1 \rceil \times \lceil R_2 \rceil$  and that all but one are removed by the restriction.

We show this by first showing that there exists more than one maximal configurations in  $\lceil R_1 \rceil \times \lceil R_2 \rceil$ , denoted here with  $\mathcal{I}$ . From the definition of product (Definition 5), we have that there exists  $y_1, \dots, y_n$  maximal configurations in  $\mathcal{I}$  such that  $\pi_1(y_i)$  and  $\pi_2(y_i)$  are maximal in  $\mathcal{I}_1$  and  $\mathcal{I}_2$ , respectively.

A second step is then to show that the restriction keeps only one maximal configuration. Let  $y_i$  and  $y_j$  be two maximal configurations in  $\lceil R_1 \rceil \times \lceil R_2 \rceil$ . As they are maximal it implies that  $y_i \cup y_j$  is not a configuration in  $\mathcal{I}$ , for  $i \neq j \leq n$ . In turn, this implies that there exists  $e_i \in y_i$  and  $e_j \in y_j$  such that  $\pi_1(e_i) = \pi_1(e_j)$  and  $e_i \neq e_j$ , as otherwise  $y_i \cup y_j$  would be defined. Here we assume that  $\pi_1(e_i) = \pi_1(e_j)$  but we could also take  $\pi_2(e_i) = \pi_2(e_j)$  and the argument still holds.

Let us now take  $d$  an event in  $\lceil R_1 \rceil$  and take  $e_1, \dots, e_m$  the subset of events in  $E$  such that  $\pi_1(e_i) = d$ . The restriction in the parallel composition of  $\lceil R_1 \rceil \mid \lceil R_2 \rceil$  keeps only one such event  $e_i$  and removes the rest: all the other events are reidentified with  $\perp_k$  since no two events can have the same identifier in the same component of the event. Therefore, from all maximal configurations  $y_1, \dots, y_m$  such that  $e_i \in y_i$ ,  $i \leq m$ , only one remains.

By applying the argument above to all events in  $\lceil R_1 \rceil$  (and  $\lceil R_2 \rceil$ ), we have that the restriction removes all but one  $y_i$ , which is then the maximal configuration in  $\lceil R_1 \rceil \mid \lceil R_2 \rceil$ . ◀

## C.2.2 Operational Correspondence

► **Lemma 43.** *If  $R \equiv S$ , then there exists an isomorphism  $f$  between  $\lceil R \rceil$  and  $\lceil S \rceil$ , with  $f_L = \text{id}$  and  $f_m = \text{id}$ .*

**Proof.** Looking back at Definition 9, there are only limited ways for  $R$  and  $S$  to be structurally equivalent (i.e., in the  $\equiv$  relation), and we review them one by one. Before that, let us observe that the sets of identifiers and of labels occurring in the memories of  $R$  and  $S$  are identical.

In  $\alpha$ -Conversion, it should be noted that the memory is left untouched, so their encodings are equal.

For Distribution of Memory, we have that

$$\lceil m \triangleright (P \mid Q) \rceil = \lceil m \rceil$$

and

$$\begin{aligned} \lceil (\gamma.m \triangleright P) \mid (\gamma.m \triangleright Q) \rceil &= \lceil (\gamma.m \triangleright P) \rceil \mid \lceil (\gamma.m \triangleright Q) \rceil \\ &= \lceil \gamma.m \rceil \mid \lceil \gamma.m \rceil \\ &= \lceil m \rceil \mid \lceil m \rceil \end{aligned}$$

The construction of the isomorphism between  $\lceil m \rceil$  and  $\lceil m \rceil \mid \lceil m \rceil$  maps  $e$  and  $(e, e)$ : let  $e$  be an event in  $\lceil m \rceil$ . Note that the only event  $e'$  in  $\lceil m \rceil \mid \lceil m \rceil$  such that  $\pi_1(e') = e$  is  $(e, e)$ .

Indeed, suppose  $(e, e'')$  is in  $[m] \mid [m]$ , for  $e'' \neq e$ . Then, by Lemma 42,  $m(e) \neq m(e'')$ , since they are both events in  $[m]$ . But by the definition of parallel composition of  $\mathcal{I}$ -structures (Definition 7),  $(e, e'')$  should have been re-identified with  $\perp_k$  and then discarded. Similarly,  $(e, \star)$  can not be an event in  $[m] \mid [m]$ , since it should have been re-identified with  $\perp_k$  and then discarded. Hence, we can map  $e$  in  $[m]$  and  $(e, e)$  in  $[m] \mid [m]$  and obtain our isomorphism. It follows from the definition of parallel composition of  $\mathcal{I}$ -structures that the functions  $f_L$  and  $f_m$  in the isomorphism are indeed the identities.

For Scope of Restriction, we have that have  $[m \triangleright (P \setminus a)] = [m] = [(m \triangleright P) \setminus a]$ . ◀

► **Lemma 44.** *The event introduced in the postfixing of a memory event to an identified structure is maximal in the resulting identified structure.*

**Proof.** The proof is immediate: the event introduced occurs only in the finite maximal configurations of the identified structure, and hence there cannot be any other event that causes it. If the maximal configuration is infinite, then the event introduced by the postfixing is not in it, and there cannot be any other event causing it. ◀

Furthermore, the maximality of an event can be ‘preserved’ by parallel composition:

► **Lemma 45.** *For all identified structure  $\mathcal{I}_1 = (E_1, C_1, L_1, \ell_1, I_1, m_1)$  with  $e_1 \in E_1$  a maximal event in it, and for all identified configuration  $\mathcal{I}_2 = (E_2, C_2, L_2, \ell_2, I_2, m_2)$  such that  $m_1(e_1) \notin I_2$ ,  $(e_1, \star)$  is maximal in  $\mathcal{I}_1 \mid \mathcal{I}_2$ .*

**Proof.** The definition of parallel composition of identified configuration structures (Definition 7) should make it clear that the only event in  $\mathcal{I}_1 \mid \mathcal{I}_2$  whose first projection is  $e_1$  is  $(e_1, \star)$ , since  $m_1(e_1) \notin I_2$ : all the other pairings of events from  $E_2$  with  $e_1$  being reidentified with  $\perp_k$  and subsequently removed. Now, suppose for the sake of contradiction that  $(e_1, \star)$  is not maximal in  $\mathcal{I}_1 \mid \mathcal{I}_2$ . It means that there is a maximal configuration  $x$  in  $\mathcal{I}_1 \mid \mathcal{I}_2$  and an event  $e \in x$  such that  $(e_1, \star) <_x e$ .

By the definition of product of  $\mathcal{I}$ -structures (Definition 5),  $\pi_1(x) \in C_1$ , and we prove that  $\pi_1(x)$  is maximal in  $\mathcal{I}_1$  by contradiction. Suppose  $\pi_1(x)$  is not maximal in  $\mathcal{I}_1$ , then there exists  $z \in C_1$  such that  $\pi_1(x) \subsetneq z$ . Assume  $z$  is maximal in  $\mathcal{I}_1$  (if it is not, then take  $z'$  the maximal configuration such that  $z \subseteq z'$ , which always exists), and note that  $e_1 \in z$ , since  $e_1 \in \pi_1(x)$  and  $\pi_1(x) \subset z$ . By Stability, since  $z \cup \pi_1(x) = z$  is a configuration in  $C_1$ ,  $z \cap \pi_1(x) = z \setminus \pi_1(x)$  also is, and note that for all events  $e'$  in  $z \setminus \pi_1(x)$ , we have that  $e_1 \not\prec_{z \setminus \pi_1(x)} e'$ . Since  $e' \in z \setminus \pi_1(x)$ ,  $e' \in z$  and  $e_1 \not\prec_z e'$ , we have that  $z$  is a maximal configuration in  $\mathcal{I}_1$  where  $e_1$  is not maximal: a contradiction. Hence, we know that  $\pi_1(x)$  is maximal in  $\mathcal{I}_1$ .

Now, we prove that  $\pi_1(e) \neq \star$ . For the sake of contradiction, suppose that  $\pi_1(e) = \star$ . Then, observe that the underlying configuration structure  $\mathcal{F}(\mathcal{I}_1 \mid \mathcal{I}_2)$  also have this configuration  $x$  with  $(e_1, \star) <_x e$ . Without loss of generality, we can assume that  $e$  is an immediate cause of  $(e_1, \star)$  [2, Definition 20], that is, that there is no  $e''$  in  $x$  such that  $(e_1, \star) <_x e'' <_x e$ . We can now use a small proposition connecting events in products with the order in their original configuration [2, Proposition 3] to get that it must be the case that either  $\pi_1(e_1, \star) <_{\pi_1(x)} \pi_1(e)$  or  $\pi_2(e_1, \star) <_{\pi_2(x)} \pi_2(e)$ . But since  $\pi_2(e_1, \star) = \star \notin \pi_2(x)$ , and since we assumed  $\pi_1(e) = \star \notin \pi_1(x)$ , both scenario are impossible, so we reached a contradiction, and it must be the case that  $\pi_1(e) \neq \star$ .

Since  $\pi_1(x)$  is maximal in  $\mathcal{I}_1$ , and since  $\pi_1(e), e_1 \in \pi_1(x)$ , we reached a contradiction:  $\pi_1(e)$  is not caused by  $e_1$ , but that cannot be since  $e_1$  is not maximal. Hence, there is no such  $e \in x$ , and  $(e_1, \star)$  is maximal in  $\mathcal{I}_1 \mid \mathcal{I}_2$ . ◀

We can now use Lemmas 43–45 to prove our main result for this section. We use an extra notation here, we write  $d < e$  if for every  $x \in C$  such that  $d, e \in x$ , we have  $d <_x e$ .

► **Lemma 19** (Operational Correspondence). *For all  $R$  and  $S$ , writing  $(E_R, C_R, \ell_R, I_R, m_R)$  for  $\lceil R \rceil$  and similarly for  $S$ , if  $R \xrightarrow{i:\alpha} S$  or  $S \xrightarrow{i:\alpha} R$ , then there exists  $e \in E_S$  maximal in  $\lceil S \rceil$  with  $m_S(e) = i$  s.t.  $\lceil R \rceil \cong \lceil S \rceil \upharpoonright_{\{e\}}$ . For all  $R$  and  $e$  a maximal event in  $\lceil R \rceil$ , there is a transition  $R \xrightarrow{m_R(e):\ell_R(e)} S$  with  $\lceil S \rceil \cong \lceil R \rceil \upharpoonright_{\{e\}}$ .*

**Proof.** We can rephrase the lemma as follows:

1. For all transitions  $R \xrightarrow{i:\alpha} S$ ,  $\lceil R \rceil \cong \lceil S \rceil \upharpoonright_{\{e\}}$  with  $e$  maximal in  $\lceil S \rceil$  and  $m_S(e) = i$ .
2. For all transitions  $R \xrightarrow{i:\alpha} S$ ,  $\lceil S \rceil \cong \lceil R \rceil \upharpoonright_{\{e\}}$  with  $e$  maximal in  $\lceil R \rceil$  and  $m_R(e) = i$ .
3. For any maximal event  $e$  in  $\lceil R \rceil$ , there is a transition  $R \xrightarrow{m_R(e):\ell_R(e)} S$  with  $\lceil S \rceil \cong \lceil R \rceil \upharpoonright_{\{e\}}$ .

We prove the three items separately.

1. We proceed by case on  $\alpha$  in the transition  $R \xrightarrow{i:\alpha} S$ :

If  $\alpha = a$  For the transition  $R \xrightarrow{i:a} S$  to take place, it must be the case that  $R$  contains a thread  $T = m \triangleright a.P + Q$  for some  $m$ ,  $P$  and  $Q$ , and that  $T \xrightarrow{i:a} \langle i, a, Q \rangle.m \triangleright P$ . Additionally, the  $\text{par.L}$ ,  $\text{par.R}$ ,  $\text{res.}$  and  $\equiv$  rules of Fig. 2, gives us that it must be the case that

$$R \equiv ((R_{n-1} \cdots ((R_3 \mid ((R_1 \mid T) \mid R_2) \setminus \vec{b}^1) \mid R_4) \setminus \vec{b}^2 \cdots) \mid R_n) \setminus \vec{b}^m$$

for some  $R_i$  any of which (and its corresponding  $\mid$  constructor) could be missing and for some  $\vec{b}^j$ , any of which (along with their  $\setminus$  constructor) could be missing as well. Hence, the transition can be written as

$$R \xrightarrow{i:a} \underbrace{((R_{n-1} \cdots ((R_3 \mid ((R_1 \mid \langle i, a, Q \rangle.m \triangleright P) \mid R_2) \setminus \vec{b}^1) \mid R_4) \setminus \vec{b}^2 \cdots) \mid R_n) \setminus \vec{b}^m}_{=S'}$$

with  $S' \equiv S$ . By Lemma 44, we know that there is a maximal event  $e_1$  in  $\lceil \langle i, a, Q \rangle.m \triangleright P \rceil$  that has for identifier  $i$ . We show that this event can be ‘traced through’  $\lceil S' \rceil$ , using four arguments:

- From the  $\text{par.L}$  or  $\text{par.R}$  rule in Fig. 2 we have that  $i \notin I(R_1)$ . Using Lemma 45, it follows that there exists a maximal event  $e_2$  in  $\lceil R_1 \mid \langle i, a, Q \rangle.m \triangleright P \rceil$  such that  $\pi_2(e_2) = e_1$ , and from the definition of the parallel composition of  $\mathcal{I}$ -structures (Definition 7) that its identifier is  $i$ .
- We can use the same reasoning to prove that there is a maximal event  $e_3$  in  $\lceil (R_1 \mid \langle i, a, Q \rangle.m \triangleright P) \mid R_2 \rceil$  such that  $\pi_1(e_3) = e_2$  and such that its identifier is  $i$ .
- Since  $\lceil R_3 \mid ((R_1 \mid \langle i, a, Q \rangle.m \triangleright P) \mid R_2) \setminus \vec{b}^1 \rceil = \lceil R_3 \mid ((R_1 \mid \langle i, a, Q \rangle.m \triangleright P) \mid R_2) \rceil$ , it is trivial that  $e_3$  is a maximal event with identifier  $i$  in it.
- Using those three arguments repeatedly, and ‘skipping’ them if a parallel composition or a restriction is ‘missing’, we can ‘trace’ the maximal event with identifier  $i$  in  $\lceil S' \rceil$ , that we write  $e$ .

But we still need to find a maximal event whose identifier is  $i$  in  $\lceil S \rceil$ . Since  $S' \equiv S$ , Lemma 43 gives us that  $\lceil S' \rceil \cong \lceil S \rceil$ . Let us write  $f$  this isomorphism, we know that it is such that  $f_L = \text{id}$ , and hence that we can use Remark 36 to study only the part of  $f$  that

maps events, that we also write  $f$ . Now, we want to prove that  $f(e)$  is maximal in  $[S]$ . To do so, let us use the ‘if’ part of the upcoming Lemma 46<sup>4</sup> to obtain a l&o-p bijection between the maximal configurations of  $[S']$  and the maximal configuration of  $[S]$ , that we also write  $f$ . Now, let’s suppose that  $f(e)$  is not maximal in  $[S]$ . Then, there exists a maximal configuration  $x_m$  in  $[S]$  and an event  $e'$  in  $x_m$  such that  $f(e) <_{x_m} e'$ . But since  $f$  is an isomorphism, and since  $f$  is order-preserving, we have that it must be the case that  $e <_{f^{-1}(x_m)} f^{-1}(e')$ , which contradicts the maximality of  $e$  in  $S'$ .

Hence,  $f(e)$  is maximal in  $[S]$ , and since by Lemma 43  $f_m = \text{id}$ ,  $m_S(e) = i$ . It is obvious that this event is the only one that was added to the encoding of  $R$ , so that  $[R] = [S] \upharpoonright_{\{e\}}$ .

**If  $\alpha = \tau$**  Then it must be the case that  $R$  has two threads,  $T = m \triangleright a.P + Q$  and  $T' = m' \triangleright \bar{a}.P' + Q'$  for some  $m, m', a, P, P', Q$ , and  $Q'$ , and that

$$R = (\dots (R_3 \mid (((R_1 \mid T) \mid R_2) \setminus \vec{b}^1 \mid R_4)) \setminus \vec{b}^2 \dots \mid ((R'_1 \mid T') \mid R'_2) \setminus \vec{c}^1 \dots \mid R_n) \setminus \vec{b}^m$$

for some  $R_i, R'_i$  any of which (and its corresponding  $\mid$  constructor) could be missing and for some  $\vec{b}^j, \vec{c}^k$ , any of which (along with their  $\setminus$  constructor) could be missing as well. Then the transition becomes

$$R \xrightarrow{i:\tau} (\dots ((R_3 \mid (R_1 \mid ((i, a, Q).m \triangleright P) \mid R_2)) \setminus \vec{b}^1 \mid R_4) \setminus \vec{b}^2 \dots \mid (R'_1 \mid ((i, \bar{a}, Q').m' \triangleright P') \mid R'_2) \setminus \vec{c}^1 \dots \mid R_n) \setminus \vec{b}^m \equiv S.$$

We use the same reasoning as in the case above where  $\alpha = a$  to deduce that there is a maximal event in  $[((i, a, Q).m \triangleright P)]$ , let us name it  $e_1$ , and one in  $[((i, \bar{a}, Q').m' \triangleright P')]$ , let us name it  $e'_1$ , that both have identifier  $i$ . Using the same argument as in the first case, we can ‘trace’ in parallel  $e_1$  and  $e'_1$ , until the  $\mathcal{I}$ -structures that hold their ‘descendants’ are put in parallel: at that point, by the definition of parallel composition of  $\mathcal{I}$ -structures (Definition 7), it should be clear that a single maximal event resulting from their composition will emerge, that it will be labeled  $\tau$  and have identifier  $i$ . Finally, using again the same argument as in the first case, this event resulting from the synchronization of our two maximal events will still be maximal in  $[S]$ , and hence we can conclude that there exists a maximal event in  $[S]$  labeled  $\tau$  whose identifier is  $i$ .

**2.** If  $R \xrightarrow{i:\alpha} S$ , then by the Loop Lemma [11, Lemma 6],  $S \xrightarrow{i:\alpha} R$ . By **1.**, we have that  $[S] \cong [R] \upharpoonright_{\{e\}}$  with  $e$  maximal in  $[R]$  and  $m_R(e) = i$ , which is what we wanted to show.

**3.** For any reachable process  $R$ , from Definition 10 and the remark below it [11, Lemma 10], we have that there exists a *forward-only* trace  $O_R \Longrightarrow R$ . We consider without loss of generality the following trace:  $\emptyset \triangleright O_R = R_0 \xrightarrow{1:\alpha_1} R_1 \dots \xrightarrow{k:\alpha_k} R_k = R$ , for some  $k, \alpha_1, \dots, \alpha_k$ .

Using **1.**, we have that  $[R_j] \upharpoonright_{\{e_j\}} = [R_{j-1}]$ , for  $j \leq k$  and  $m(e_j) = j$ . Therefore we can construct a bijection  $h$  between events  $e_j$  in  $[R]$  and transitions  $t_j : R_{j-1} \xrightarrow{j:\alpha_j} R_j$  with  $m(e_j) = j$ .

Let  $e$  be a maximal event in  $[R]$  and let  $h(e) = R_{j-1} \xrightarrow{j:\alpha_j} R_j$ , where

$$R_0 \xrightarrow{1:\alpha_1} R_1 \dots R_{j-1} \xrightarrow{j:\alpha_j} R_j \xrightarrow{j+1:\alpha_{j+1}} R_{j+1} \dots R_{k-1} \xrightarrow{k:\alpha_k} R_k = R$$

<sup>4</sup> This part of the lemma is *not* proved using the lemma we are currently proving, but the ‘and only if’ part does, hence the choice of postponing it after this current lemma.

As  $e$  is maximal, by Definition 18, there exists no event  $e'$  in  $[R]$  such that  $e < e'$ . From  $[R_{i+1}] \upharpoonright_{\{e_{i+1}\}} = [R_i]$ , for  $i \leq k$ , we have that  $[R_i] \subseteq [R_{i+1}]$  and by induction,  $[R_i] \subseteq [R_k]$ , for all  $i \leq k$ . Then  $[R_{j+i}] \subseteq [R_k]$  and we have that  $e_i \not\leq e$ , for all  $j+1 \leq i \leq k$ . It implies that  $e_i$  is concurrent with  $e$ , for all  $j+1 \leq i \leq k$ .

We use now concurrency in traces, trace equivalences and the  $m_{R/S}$  notations from Appendix B. Consider two consecutive transitions  $t_j = R_j \xrightarrow{j+1:\alpha_{j+1}} R_{j+1}$  and  $t_{j+1} = R_{j+1} \xrightarrow{j+2:\alpha_{j+2}} R_{j+2}$ . Using the bijection  $h$  defined above between transitions and events, we have that there exists two event  $e_j, e_{j+1}$  in  $[R_{j+2}]$  such that  $h(e_j) = t_j$  and  $h(e_{j+1}) = t_{j+1}$ . Suppose that  $e_j$  is concurrent to  $e_{j+1}$ . By an induction on the structure of memories in  $[R_{j+2}]$ , we have that  $m_{R_j/R_{j+1}} \cap m_{R_{j+1}/R_{j+2}} = \emptyset$  which implies that there exists a transition  $t'_j = R_j \xrightarrow{j+2:\alpha_{j+2}} R_{j+2}$  such that  $t_j$  is concurrent with  $t'_j$ .

Recall that  $e_i$  is concurrent with  $e$ , for all  $j+1 \leq i \leq k$  in our trace. We can now use the trace equivalence [11, Definition 9] reminded in Appendix B, and use the previous remark repeatedly to re-organize our original trace as follows:

$$R_0 \xrightarrow{1:\alpha_1} R_1 \cdots R_{j-1} \xrightarrow{j+1:\alpha_{j+1}} R'_{j+1} \cdots R'_{j-1} \xrightarrow{k:\alpha_k} R'_k \xrightarrow{j:\alpha_j} R$$

The core idea being that we can ‘postpone’ the transition that will create the event  $e$  in the encoding of  $R$ , by flipping it with the other transitions, and have it become the last transition that leads to the same  $R$ .

We can now use the Loop Lemma [11, Lemma 6] to ‘reverse’ the last transition and get that  $R \xrightarrow{j:\alpha_j} R'_k$ . Using 2. on this transition, we have that  $[R'_k] = [R] \upharpoonright_{\{e\}}$ . ◀

### C.3 Proof for Sect. 3.2

► **Theorem 26** (Collapsing B&F and SB&F). *If  $R_1$  and  $R_2$  are without auto-concurrency, then they are B&F iff they are SB&F.*

**Proof.** Let  $R_1, R_2$  be two reversible processes without auto-concurrency. We want to show that

1.  $(R_1, R_2, f) \in \mathcal{R}'$  for  $\mathcal{R}'$  a B&F implies that  $(R_1, R_2) \in \mathcal{R}$  for  $\mathcal{R}$  a SB&F.
2.  $(R_1, R_2) \in \mathcal{R}$  for  $\mathcal{R}$  a SB&F implies that there exists  $f$  a bijection between  $l(R_1)$  and  $l(R_2)$  such that  $(R_1, R_2, f) \in \mathcal{R}'$  is a B&F.

For 1., there is nothing to prove, since we are dropping a requirement, the existence of the bijection.

For 2., we first note that there exists forward-only traces from  $\emptyset \triangleright O_{R_1}$  to  $R_1$  [11, Lemma 10]. We pick one,  $\theta_1$ , and construct the bijection  $f$  between  $l(R_1)$  and  $l(R_2)$  using it. We start by letting  $(\emptyset \triangleright O_{R_1}, \emptyset \triangleright O_{R_2}, \emptyset) \in \mathcal{R}'$ . Then, assuming  $\theta_1$  starts with  $\emptyset \triangleright O_{R_1} \xrightarrow{i:\alpha} R'_1$ , since  $(R_1, R_2) \in \mathcal{R}$ , we know there are  $R'_2$  and  $j$  such that  $\emptyset \triangleright O_{R_2} \xrightarrow{j:\alpha} R'_2$  and  $(R'_1, R'_2) \in \mathcal{R}$ , and we let  $(R'_1, R'_2, \{i \rightarrow j\}) \in \mathcal{R}'$ . We iterate this construction until we obtain a bijection  $f$  and let  $(R_1, R_2, f) \in \mathcal{R}'$ .

Now, we need to show that  $(R_1, R_2, f) \in \mathcal{R}'$  is a valid relation, and for this we must show that it can accommodate the forward and backward transitions. That the bijection  $f$  can be extended in case of forward transitions from  $R_1$  or  $R_2$  is obvious, using  $\mathcal{R}$ . The more difficult part of the proof concerns the backward transition, which requires to show that  $f$  is ‘right’. Indeed, if  $R_1$  or  $R_2$  does a backward transition to a term  $S$ , and if  $S$  is paired with

$S'$  in  $\mathcal{R}$ , then we need to prove that the identifiers of the backward transitions leading to  $S$  and  $S'$  are ‘matched’, i.e., in bijection in  $f$ .

Stated formally, we want to show that for any  $(R_1, R_2) \in \mathcal{R}$  with  $f$  constructed as above, then for all  $R_1 \xrightarrow{i:\alpha} S_1$  and  $R_2 \xrightarrow{j:\alpha} S_2$  such that  $(S_1, S_2) \in \mathcal{R}$ ,  $\{i \mapsto j\} \in f$ . We show this by contradiction: suppose that we have two pairs of indices  $\{i \mapsto j\}$  and  $\{i' \mapsto j'\}$  in  $f$ , and that  $R_1 \xrightarrow{i':\alpha} S_1$  and  $R_2 \xrightarrow{j:\alpha} S_2$  such that  $(S_1, S_2) \in \mathcal{R}$ .

First, observe that all four indices  $i, i', j$  and  $j'$ , are associated to the same label  $\alpha$ : due to the way  $f$  was constructed, following  $\mathcal{R}$ , it cannot be the case that two transitions with different labels have their identifiers paired. Secondly, since  $R_2 \xrightarrow{j:\alpha} S_2$  and  $\{i \mapsto j\}$  is in  $f$ , it must be the case that there is a transition in  $\theta_1$  with the identifier  $i$  by construction of  $f$ . Let us denote  $t_i : S_1 \xrightarrow{i:\alpha} R_1'$  and  $t_{i'} : S_1 \xrightarrow{i':\alpha} R_1$ .

Now, let us put the remarks on trace of Appendix B to good use. Observe that  $i$  and  $i'$  cannot be concurrent in  $\theta_1$ . If they were, then in the encoding of  $R_1$ , there would be two events—the one introduced by  $t_i$  and the one introduced by  $t_{i'}$ —that are concurrent, with the same label, but that is not possible since  $R_1$  is without auto-concurrency. Since two transitions are either concurrent or causal, it follows that either  $i < i'$  or  $i' < i$ <sup>5</sup>.

- As the cause cannot backtrack before the effect it follows that the case  $i' < i$  is not possible.
- Then  $i < i'$ . We now reason by cases on the order of the traces  $t_j$  and  $t_{j'}$ : either  $j < j'$  or  $j' < j$ .
  - If  $j < j'$  then  $R_1$  must have already backtracked on  $j'$ , which implies that there is no  $j'$  in  $f$  as we assumed.
  - The case  $j' < j$  is not possible. Note that  $i < i'$  implies an order in which the pair of indices are added to the bijection, in this case the pair  $\{i \mapsto j\}$  occurs before  $\{i' \mapsto j'\}$  and therefore it cannot be that  $j' < j$ .

From this reasoning, we get that for any  $R_1 \xrightarrow{i:\alpha} S_1$  and  $R_2 \xrightarrow{j:\alpha} S_2$  such that  $(S_1, S_2) \in \mathcal{R}$ ,  $\{i \mapsto j\} \in f$ . We still need to show that  $(S_1, S_2, f \setminus \{i \mapsto j\}) \in \mathcal{R}'$  is a ‘valid’ relation. We only need to show that  $f \setminus \{i \mapsto j\}$  matches the ‘right’ indices in case of a backward transition, but we can simply iterate the previous reasoning until we reach  $(\emptyset \triangleright O_{R_1}, \emptyset \triangleright O_{R_2}, \emptyset) \in \mathcal{R}'$ . This concludes this direction of the proof. ◀

## C.4 Lemmas and Proofs for Sect. 3.3

Our goal here is to prove Theorems 29 and 30, which give as an immediate corollary our main result (Corollary 31) and an interesting remark (Corollary 49). We first (Sect. C.4.1) identify isomorphisms of  $\mathcal{I}$ -structures with l&o-p functions (Lemma 46) and then connect our previous formalism with the encoding of memories (Lemma 48). Then, using this connection and the operational correspondence detailed in Sect. 2.5, Sect. C.4.2 concludes by proving our two main theorems.

Below, we abuse the notation by writing  $\llbracket R \rrbracket = (\llbracket O_R \rrbracket, x_R)$ , for the encoding of a reversible process into a pair made of the CCS encoding of its origin and a particular configuration in it, that we call the address of  $R$  [2]. The formal connection made in Lemma 48 will justify this abuse, and the context should always make it clear which encoding we are referring to.

<sup>5</sup> Remember that we write  $i < k$  for  $t_i < t_k$ .

### C.4.1 On Connecting Formalisms

We first establish a connection between isomorphisms (in Category theory) and l&o-p bijections (that are used to define (H)HPB). Then, we construct a bridge between  $[R]$  and our previous formalism [2].

► **Lemma 46.** *Letting  $x_1^m$  and  $x_2^m$  be the unique maximal configurations in  $[R_1]$  and  $[R_2]$ , there is an isomorphism  $f$  between  $[R_1]$  and  $[R_2]$  with  $f_L = \text{id}$  iff there exists a l&o-p bijection between  $x_1^m$  and  $x_2^m$ .*

**Proof.** Let  $f : [R_1] \rightarrow [R_2]$  be an isomorphism: since, by hypothesis,  $f_L = \text{id}$ , we can use Remark 36 to take  $f$  to be fully determined by its function between events  $f_E$ , that we also write  $f$ . Then  $f : x_1^m \rightarrow x_2^m$  is by definition a label-preserving bijection, as instantiating Definition 32 with  $f_L = \text{id}$  precisely gives the condition of Definition 21. For two events  $e, e'$  in  $[R_1]$ , if  $e <_{x_1^m} e'$  it follows that for all  $x_1$  in  $[R_1]$  such that  $e' \in x_1$  then  $e \in x_1$ . From  $f$  an isomorphism we have that  $f^{-1} : [R_2] \rightarrow [R_1]$  is a well defined morphism. As  $f^{-1}$  preserves configurations it follows that for all  $x_2$  in  $[R_2]$ , there exists  $x_1$  in  $[R_1]$  such that  $f(x_1) = x_2$ . Then for all  $x_2$ , such that  $f(e') \in x_2$  it implies that  $e' \in f^{-1}(x_2)$  and  $e \in f^{-1}(x_2)$  and finally,  $f(e) \in x_2$ . Then  $f(e) <_{x_2^m} f(e')$ , as for all  $x_2, x_2 \subseteq x_2^m$ , since  $[R_2]$  is a poset (Lemma 17). Similarly we proceed for the concurrent events, and obtain that  $f$  is an order-preserving bijection on top of being label-preserving, hence that it is l&o-p.

For the reverse direction, let  $f : x_1^m \rightarrow x_2^m$  be a l&o-p bijection. All events in  $[R_1]$  are present in  $x_1^m$ , since it is the maximal element in  $[R_1]$ , and therefore  $f : [R_1] \rightarrow [R_2]$  is a bijection on events. As  $f$  is label-preserving, it is the identity on the labels, and we can study only this mapping on events thanks to Remark 36. Remains to show that  $f$  preserves configurations, that is, for all  $x_1 \subseteq x_1^m$ ,  $f(x_1) = \{f(e) \mid e \in x_1\}$  is a configuration in  $[R_2]$ . In other words,  $f(x_1) = x_2$  must satisfies the properties of Definition 1.

- Finiteness follows from the fact that all configurations in the encoding of an RCCS memory are finite.
- For Coincidence Freeness let us first note that there is a unique and finite maximal configuration  $x_2^m$  in which all distinct events are either causal or concurrent. Let us consider  $e_2 \neq e'_2$  two events in  $x_2$ . Since  $e_2 \neq e'_2$  then either **1.**  $e_2 \leq_{x_2^m} e'_2$  (or  $e_2 \leq_{x_2^m} e'_2$  but this is similar), or **2.**  $e_2 \text{ co}_{x_2^m} e'_2$ . As  $x_2^m$  is the unique maximal configuration, the relations above hold for  $x_2$  as well. In the case **1.**, as  $e_2 \neq e'_2$  then  $e'_2 \not\leq_{x_2} e_2$ . We apply the definition of causality (Definition 2) to obtain the configuration  $z \subseteq x_2$  where  $e_2 \in z$  and  $e'_2 \notin z$  as required by Coincidence Freeness. In the case **2.**, we use the definition of concurrency, which implies that  $e_2 \not\leq_{x_2} e'_2$  and  $e'_2 \not\leq_{x_2} e_2$ . Then there exists  $z \subseteq x_2$  such that  $e_2 \in z$  and  $e'_2 \notin z$  and moreover, there exists  $z' \subseteq x_2$  such that  $e'_2 \in z'$  and  $e_2 \notin z'$ .
- To show Finite Completeness, it suffices to note that there exists a unique maximal configuration  $x_2^m$  which can be an upper bound for the union of any subset of configurations in  $[R_2]$ .
- To show Stability we have to show that  $\forall x_2, x'_2 \in C_2$  such that  $\exists x_1, x'_1 \in C_1$ , with  $x_1 = f^{-1}(x_2)$  and  $x'_1 = f^{-1}(x'_2)$ ,  $x_2 \cup x'_2 \in C_2 \implies x_2 \cap x'_2 \in C_2$ . We will show that  $x_2 \cap x'_2 = f(x_1 \cap x'_1)$  is a configuration in  $[R_2]$  by exploiting the causality and concurrency relations in  $x_1 \cap x'_1$ . As  $[R_1]$  and  $[R_2]$  are both posets by Lemma 17, we can w.l.o.g consider  $x'_1 \subseteq x_1$  and  $x_1 = x_1^m$ . We can assume, for  $i \in \{1, 2\}$ , that  $[R_i] \neq \mathbf{0}$  (otherwise the property would be trivial). Hence, there is a backward transition  $R_i \xrightarrow{j:\alpha} S_i$ . Now, by the second item of Lemma 19 we have that there exists a maximal event  $e_1^1$  in  $x_1$  and from  $[R_1] \setminus \{e_1^1\} \subseteq [S_1]$ ,  $x_1 \setminus e_1^1$  is a configuration in  $[R_1]$ . We can re-use the same



reasoning for  $x_1 \setminus \{e_1\}$  in  $\lceil S_1 \rceil \upharpoonright_{\{e_1^2\}}$  with  $e_1^2$  maximal and  $\lceil R_1 \rceil \upharpoonright_{\{e_1^1, e_1^2\}} \subset \lceil S_1 \rceil \upharpoonright_{\{e_1^1\}} \subset \lceil S_1 \rceil$ . We have then a sequence of events  $e_1^1, \dots, e_1^n$  such that  $x_1 \setminus \{e_1, \dots, e_n\} = x_1 \cap x_1'$ .

Let us consider  $e_2 \neq e_2'$  two events in  $x_2$  and as  $f$  is a bijection, let  $e_1 \neq e_1'$  be two events in  $f^{-1}(x_2) = x_1$  such that  $f^{-1}(e_2) = e_1$  and  $f^{-1}(e_2') = e_1'$ . If  $e_2 \neq e_2'$  then either  $e_2 \leq_{x_2^m} e_2'$  or  $e_2 \text{ co}_{x_2^m} e_2'$ . It follows that either  $e_1 \leq_{x_1^m} e_1'$  or  $e_1 \text{ co}_{x_1^m} e_1'$ , respectively, since  $f^{-1}$  is l&o-p as well. As  $x_1^m, x_2^m$  are the unique maximal configurations the relations above hold for  $x_2$  and  $x_1$  as well.

We therefore have that  $f(e_1^1)$  is the maximal event in  $x_2$ . Again we use the second item of Lemma 19 and as above we obtain the sequence of events  $f(e_1^1), \dots, f(e_1^n)$  such that  $f(x_1) \setminus f(e_1), \dots, f(e_n) = f(x_1 \cap x_1')$  is a configuration in  $\lceil R_2 \rceil$ . ◀

For the reader familiar with event structures, a configuration  $x$  defines an event structure  $(x, \leq_x, \ell)$ . The construction below mirrors the transformation from an event structure to a configuration structure [37].

► **Definition 47** (Generation of a  $\mathcal{I}$ -structure from a configuration). *Given  $\mathcal{I} = (E, C, L, \ell, I, m)$ , for  $x \in C$ , the  $\mathcal{I}$ -structure generated by  $x$  is  $x \downarrow = (x, \{y \mid y \in C, y \subseteq x\}, \{a \mid \exists e \in x_R, \ell(e) = a\}, \ell \upharpoonright_x, \{i \mid \exists e \in x_R, m(e) = i\}, m \upharpoonright_x)$ .*

► **Lemma 48.** *The  $\mathcal{I}$ -structure  $\lceil R \rceil$  is isomorphic to  $x_R \downarrow$ , where  $\llbracket R \rrbracket = (\llbracket O_R \rrbracket, x_R)$  is the encoding previously defined [2].*

**Proof.** As  $R$  is reachable there exists a forward-only trace [11, Lemma 10]  $\theta : O_R \xrightarrow{i_1:\alpha_1} \dots \xrightarrow{i_n:\alpha_n} R$ . We reason then by induction on the trace  $\theta$ . As the base case is similar, we only treat the inductive case.

By induction, we have that the result holds for  $R$ , that is let  $h : \lceil R \rceil \rightarrow x_R \downarrow$  be an isomorphism. As both  $\lceil R \rceil$  and  $x_R \downarrow$  result from encoding the same process  $R$ , we can consider w.l.o.g. that  $h_L = \text{id}$  and  $h_m = \text{id}$ . We use an argument similar to Lemma 46 to show that instead of an isomorphism, we can reason on  $h : x^m \rightarrow x_R$  as a l&o-p bijection, where  $x^m$  is the maximal configuration of  $\lceil R \rceil$ , and  $x_R$  is the maximal configuration in  $x_R \downarrow$ .

Let  $R \xrightarrow{i:\alpha} S$  be an RCCS transition. We have to show that  $\lceil S \rceil \cong x_S \downarrow$ , where  $\llbracket S \rrbracket = (\llbracket O_R \rrbracket, x_S)$ . By the operational correspondence between  $R$  and  $(\llbracket O_R \rrbracket, x_R)$  [2, Lemma 6] there exists an event  $e$  in  $\llbracket O_R \rrbracket$  such that

$$R \xrightarrow{i:\alpha} S \implies (\llbracket O_R \rrbracket, x_R) \xrightarrow{e} (\llbracket O_R \rrbracket, x_R \cup \{e\}), \text{ with } \ell(e) = \alpha \text{ and } \llbracket S \rrbracket = (\llbracket O_R \rrbracket, x_R \cup \{e\}).$$

By the operational correspondence between  $R$  and  $\lceil R \rceil$  (Lemma 19) there exists a memory event  $e_m$  such that  $\lceil R \rceil = \lceil S \rceil \upharpoonright_{\{e_m\}}$  with  $e_m$  maximal in  $\lceil S \rceil$ ,  $\ell_S(e_m) = \alpha$  and  $m_S(e_m) = i$ . As there is only one maximal configuration in  $\lceil S \rceil$  and as  $\lceil R \rceil = \lceil S \rceil \upharpoonright_{\{e_m\}}$ , it follows that  $x^m \cup \{e_m\}$  is the maximal configuration in  $\lceil S \rceil$ .

We have then to show that if  $h : x^m \rightarrow x_R$  is l&o-p bijection then so is  $h' = h \cup \{e_m \mapsto e\} : x^m \cup \{e_m\} \rightarrow x_R \cup \{e\}$ . The label preserving part follows from  $h_L = \text{id}$  and  $\ell_S(e_m) = \ell(e) = \alpha$ . To show that  $h'$  is order preserving we exploit again the operational correspondences between the encoding of RCCS terms and RCCS, and between the encoding of memories and RCCS. As a corollary from [2, Lemma 6] we get that transitions are concurrent in the forward-only trace  $\theta : O_R \xrightarrow{i_1:\alpha_1} \dots \xrightarrow{i_n:\alpha_n} R \xrightarrow{i:\alpha} S$  iff the corresponding events are concurrent in  $\llbracket S \rrbracket$ , that is in  $x_S$ . Similarly, from Lemma 19, we get that transitions are concurrent in the forward-only trace  $\theta$  iff their corresponding events are concurrent in  $\lceil S \rceil$ , and consequently in  $x^m \cup \{e_m\}$ . Finally we can conclude that  $h'$  is order preserving.

Lastly we can lift  $h'$  from a l&o-p bijection on configurations to an isomorphism on  $\mathcal{I}$ -structures by endowing the configurations with the identifier set  $I(S)$  and  $h'_m = \text{id}$ . ◀

### C.4.2 Proofs of Theorems 29 and 30

► **Theorem 29** (Equivalences).  $P_1$  and  $P_2$  are HHPB (resp. HPB) iff  $\llbracket P_1 \rrbracket$  and  $\llbracket P_2 \rrbracket$  are.

**Proof.** Let us prove the HHPB case, the other case being similar, and actually simpler.

⇒ Let  $\mathcal{R}_{\text{RCCS}}$  be a HHPB between  $P_1$  and  $P_2$  (Definition 28). We show that the following relation

$$\mathcal{R} = \{(x_1, x_2, f) \mid x_1 \in \llbracket P_1 \rrbracket, x_2 \in \llbracket P_2 \rrbracket, \exists R_1, R_2 \text{ s.t. } O_{R_1} = P_1, O_{R_2} = P_2, \\ (R_1, R_2, F) \in \mathcal{R}_{\text{RCCS}} \text{ and } \llbracket R_1 \rrbracket = (\llbracket P_1 \rrbracket, x_1), \llbracket R_2 \rrbracket = (\llbracket P_2 \rrbracket, x_2), f = \mathcal{F}(F)\}$$

is a HHPB between  $\llbracket P_1 \rrbracket$  and  $\llbracket P_2 \rrbracket$ .

We first show that for any tuple  $(x_1, x_2, f) \in \mathcal{R}$ ,  $f : x_1 \rightarrow x_2$  is a l&o-p bijection. For a tuple  $(R_1, R_2, F) \in \mathcal{R}_{\text{RCCS}}$  with  $F : [R_1] \rightarrow [R_2]$  an isomorphism, we get, by Lemma 46, that we can instead consider  $F : x_1^m \rightarrow x_2^m$  to be a l&o-p bijection between the maximal configurations  $x_1^m, x_2^m$ . The functor  $\mathcal{F}$  maps isomorphisms on identified structures to isomorphisms on configuration structures, by Lemma 35, and therefore  $f = \mathcal{F}(F) : \mathcal{F}(x_1^m) \rightarrow \mathcal{F}(x_2^m)$ .

Moreover,  $x_1$  and  $x_2$  are the maximal configurations in  $x_1 \downarrow$  and  $x_2 \downarrow$ , by Definition 47. By Lemma 48,  $[R_i] \cong x_i \downarrow$  which implies that  $\mathcal{F}(x_i^m) = x_i$ , for  $i \in \{1, 2\}$ . Thus  $f = \mathcal{F}(F)$  is well defined and indeed, a l&o-p bijection.

The rest of the proof follows the same structure as in [2, Proposition 6]. Note that  $(\emptyset, \emptyset, \emptyset) \in \mathcal{R}$ : indeed  $(\emptyset \triangleright P_1, \emptyset \triangleright P_2, \emptyset) \in \mathcal{R}_{\text{RCCS}}$  and  $\llbracket \emptyset \triangleright P_i \rrbracket = (\llbracket P_i \rrbracket, \emptyset)$ , for  $i \in \{1, 2\}$ . Let us suppose that  $(x_1, x_2, f) \in \mathcal{R}$  for  $\llbracket R_i \rrbracket = (\llbracket P_i \rrbracket, x_i)$ , for  $i \in \{1, 2\}$  and  $f : x_1 \rightarrow x_2$  a l&o-p bijection.

To show that  $\mathcal{R}$  is a HHPB we have to show that if  $x_1 \xrightarrow{e_1} y_1$  (or  $x_1 \xrightarrow{e_1} y_1$ ) then there exists  $y_2$  such that  $x_2 \xrightarrow{e_1} y_2$  (or  $x_2 \xrightarrow{e_2} y_2$  respectively) and such that  $(y_1, y_2, f') \in \mathcal{R}$  for some  $f' = f \cup \{e_1 \mapsto e_2\}$ .

Let  $x_1 \xrightarrow{e_1} y_1$ , hence by definition,  $y_1 = x_1 \cup \{e_1\}$ . From the correspondence between RCCS and their encodings [2, Lemma 6], it follows that  $R_1 \xrightarrow{i:\alpha} S_1$  such that  $\llbracket S_1 \rrbracket = (\llbracket P_1 \rrbracket, y_1)$ . As  $(R_1, R_2, F) \in \mathcal{R}_{\text{RCCS}}$  and as  $R_1 \xrightarrow{i:\alpha} S_1$ , it follows that there exists a transition  $R_2 \xrightarrow{j:\alpha} S_2$  with  $F = F' \upharpoonright_{[R_1]}$  and  $(S_1, S_2, F') \in \mathcal{R}_{\text{RCCS}}$ .

Again from the correspondence between  $R_2$  and  $\llbracket R_2 \rrbracket$  we have that  $x_2 \xrightarrow{e_2} y_2$  such that  $y_2 = x_2 \cup \{e_2\}$  and  $\llbracket S_2 \rrbracket = (\llbracket P_2 \rrbracket, y_2)$ . Then we show that  $(y_1, y_2, \mathcal{F}(F')) \in \mathcal{R}$ . The only missing argument is that  $\mathcal{F}(F') = f \cup \{e_1 \mapsto e_2\}$ . Note that, again using Lemma 46, we can consider  $F : x_1^m \rightarrow x_2^m$  and  $F' : y_1^m \rightarrow y_2^m$  to be l&o-p bijections on the maximal configurations of  $[R_1], [R_2]$  and  $[S_1], [S_2]$ , respectively. From the definition of postfixing and parallel composition, in Definitions 5 and 7, we have that  $y_1^m = x_1^m \cup \{e_1\}$  and  $y_2^m = x_2^m \cup \{e_2\}$ . Therefore  $F' = F \cup \{e_1 \mapsto e_2\}$  by which we conclude.

We treat similarly the cases where  $x_2$  does a transition, or when the transitions are backwards.

⇐ Let  $\mathcal{R}_{\text{CONF}}$  be a HHPB between  $\llbracket P_1 \rrbracket$  and  $\llbracket P_2 \rrbracket$ . We show that the following relation

$$\mathcal{R} = \{(R_1, R_2, F) \mid O_{R_1} = P_1, O_{R_2} = P_2, \llbracket R_1 \rrbracket = (\llbracket P_1 \rrbracket, x_1), \llbracket R_2 \rrbracket = (\llbracket P_2 \rrbracket, x_2), \\ \text{with } (x_1, x_2, f) \in \mathcal{R}_{\text{CONF}}, F = (f, f_m), \\ f_m(i) = j, m_1(e) = i, m_2(f(e)) = j, \text{ for all } e \in x_1\}$$

is a HHPB between  $\emptyset \triangleright P_1$  and  $\emptyset \triangleright P_2$ .

For  $(R_1, R_2, F) \in \mathcal{R}$ , let us show that  $F$  is an isomorphism between  $[R_1]$  and  $[R_2]$ . For  $f : x_1 \rightarrow x_2$  a l&o-p bijection, and for two functions  $m_1 : x_1 \rightarrow I_1$  (from  $[R_1]$ )

and  $m_2 : x_2 \rightarrow I_2$  (from  $\lceil R_2 \rceil$ ) then there exists a unique function  $f_m : I_1 \rightarrow I_2$  such that  $f_m(m_1(e)) = m_2(f(e))$ , for all  $e \in x_1$ . This follows from Collision Freeness in the definition of identified structures and from  $f$  being a bijection. We write then  $F = (f, f_m) : (x_1 \oplus m_1) \rightarrow (x_2 \oplus m_2)$ . Moreover, as in the first case, we derive that  $x_1, x_2$  are maximal in  $\lceil R_1 \rceil, \lceil R_2 \rceil$ , respectively. We use Lemma 46 to conclude that  $F : \lceil R_1 \rceil \rightarrow \lceil R_2 \rceil$  is an isomorphism.

We have that  $(\emptyset \triangleright P_1, \emptyset \triangleright P_2, \emptyset) \in \mathcal{R}$  as  $(\emptyset, \emptyset, \emptyset) \in \mathcal{R}_{\text{CONF}}$  and  $\llbracket \emptyset \triangleright P_i \rrbracket = (\llbracket P_i \rrbracket, \emptyset)$ , for  $i \in \{1, 2\}$ .

We suppose now that  $(R_1, R_2, F) \in \mathcal{R}$ , with  $F : \lceil R_1 \rceil \rightarrow \lceil R_2 \rceil$ . It implies that  $(x_1, x_2, f) \in \mathcal{R}_{\text{CONF}}$  for  $\llbracket R_i \rrbracket = (\llbracket P_i \rrbracket, x_i)$ ,  $i \in \{1, 2\}$  and that  $F = (f, f_m)$  with  $f : x_1 \rightarrow x_2$  l&o-p.

To show that  $\mathcal{R}$  is a HHPB we have to show that if  $R_1 \xrightarrow{i:\alpha} S_1$  (or  $R_1 \xrightarrow{\sim i:\alpha} S_1$ ) then there exists  $S_2$  such that  $R_2 \xrightarrow{j:\alpha} S_2$  (or  $R_2 \xrightarrow{\sim j:\alpha} S_2$  respectively) and such that  $(S_1, S_2, F') \in \mathcal{R}$  for some  $F'$ .

Let  $R_1 \xrightarrow{i:\alpha} S_1$ . We use again the correspondence between RCCS and their encodings [2, Lemma 6] from which we have that there exists  $e_1$  and  $y_1 = x_1 \cup \{e_1\}$  such that  $x_1 \xrightarrow{e_1} y_1$  and  $\llbracket S_1 \rrbracket = (\llbracket P_1 \rrbracket, y_1)$ . As  $(x_1, x_2, f) \in \mathcal{R}_{\text{CONF}}$  it implies that there exists  $e_2, y_2$  and  $f' = f \cup \{e_1 \mapsto e_2\}$  such that  $x_2 \xrightarrow{e_2} y_2$  and  $(y_1, y_2, f') \in \mathcal{R}_{\text{CONF}}$ .

Again, from the correspondence between RCCS and configuration structures we have that, from  $x_2 \xrightarrow{e_2} y_2$ , there exists  $S_2$  such that  $R_2 \xrightarrow{j:\alpha} S_2$  with  $\llbracket S_2 \rrbracket = (\llbracket P_2 \rrbracket, y_2)$ . From  $f' = f \cup \{e_1 \mapsto e_2\}$  it easily follows that  $F' = (f', f'_m) = (f \cup \{e_1 \mapsto e_2\}, f_m \cup \{i \mapsto j\})$ . We conclude therefore that  $(S_1, S_2, F') \in \mathcal{R}$ .

Similarly we show the cases where  $R_1$  does a backward transition, or if  $R_2$  does a forward or backward transition.  $\blacktriangleleft$

► **Theorem 30** (Equivalence (contd)).  *$P_1$  and  $P_2$  are B&F iff they are HHPB.*

**Proof.**  $\Rightarrow$   $P_1, P_2$  be two processes and let  $\mathcal{R}$  be a B&F relation between them as in Definition 24. Then for any  $(R_1, R_2, f) \in \mathcal{R}$ ,  $R_1, R_2$  are two RCCS processes and  $f : \mathsf{l}(R_1) \rightarrow \mathsf{l}(R_2)$  is a bijection on their identifiers. Letting  $\lceil R_i \rceil = (E_i, C_i, L_i, \ell_i, I_i, m_i)$ , for  $i \in \{1, 2\}$ , we show that the relation

$$\mathcal{S} = \{(R_1, R_2, F) \mid (R_1, R_2, f) \in \mathcal{R}, F(e_1) = e_2 \iff f(m_1(e_1)) = m_2(e_2)\}$$

is a HHPB relation as in Definition 28.

Let us start by observing that  $F = (F_E, F_L, F_C, F_m)$  is correctly defined as an isomorphism between  $\lceil R_1 \rceil$  and  $\lceil R_2 \rceil$  by our condition. We start by letting  $F_m = f$ , and observe that by Lemma 42, no two events can have the same identifier in  $\lceil R_1 \rceil$ , and similarly for  $\lceil R_2 \rceil$ : hence having the  $F_m$  component is enough to define the  $F_E$  component, and they are both isomorphisms. By Definition 24, no two identifiers can be paired by  $f$  unless they have the same label: hence,  $\ell_1(e) = \ell_2(F_E(e))$  iff  $F_L(\ell_1(e)) = \ell_2(F_E(e))$ , which forces  $F_L = \text{id}$ . We see that  $F$  respects the conditions of Definition 32, and that by Remark 36, we can simply write  $F$  for  $F_E$  in the following.

We prove that  $\mathcal{S}$  is a HHPB relation as in Definition 28 by induction on the processes  $R_1$  and  $R_2$ , as any process reachable from  $P_1$  has to be in  $\mathcal{R}$  (and in  $\mathcal{S}$ ) and vice versa. The base case  $(\emptyset \triangleright P_1, \emptyset \triangleright P_2, \emptyset) \in \mathcal{S}$  is trivial.

Suppose that for  $(R_1, R_2, f) \in \mathcal{R}$ , we have that  $(R_1, R_2, F) \in \mathcal{S}$ .

**For forward transitions** Now let  $R_1 \xrightarrow{i:\alpha} S_1$  for which we have that  $R_2 \xrightarrow{j:\alpha} S_2$  and  $g : \mathsf{l}(S_1) \rightarrow \mathsf{l}(S_2)$  is a bijection defined as  $f$  on  $\mathsf{l}(R_1) \subset \mathsf{l}(S_1)$  and  $g(i) = j$ . Let  $e_1 \in \lceil S_1 \rceil$  such that  $m_1(e_1) = i$  and  $e_2 \in \lceil S_2 \rceil$  such that  $m_2(e_2) = j$ . Then  $G = F \cup \{e_1 \mapsto e_2\}$  is defined by  $g(m_1(e_1)) = m_2(e_2)$ , and thus  $(S_1, S_2, G) \in \mathcal{S}$ .

We show now that  $G$  is an isomorphism between  $[S_1]$  and  $[S_2]$ .  $G$  preserves identifiers by definition; it preserves labels as  $\ell_1(e_1) = \ell_2(e_2) = \alpha$ . Lastly we have to show that it preserves configurations. We can use Lemma 46 to show instead that  $G : x_1^m \rightarrow x_2^m$  is a l&o-p function, where  $x_1^m, x_2^m$  are the maximal configurations in  $[S_1]$  and  $[S_2]$ , respectively. Also by Lemma 46 we have that  $F : y_1^m \rightarrow y_2^m$  is a l&o-p function on  $y_1^m, y_2^m$  maximal configurations in  $[R_1]$  and  $[R_2]$ , respectively. From Lemma 19,  $x_1^m \setminus \{e_1\} \in [R_1]$  and using Lemma 17 we have that  $y_1^m = x_1^m \setminus \{e_1\}$ . Therefore we have that  $G : x_1^m \rightarrow x_2^m$  is a l&o-p function on all events  $e \neq e_1$ .

Let us suppose, by contradiction, that there exists  $e$  such that  $e_1 \text{ co}_{x_1^m} e$  but that  $e_2 \text{ co}_{x_2^m} F(e)$  does not hold. Take the maximal event  $e$  with such a property. Moreover, let  $F(e) = e' <_{x_2^m} e_2$  without loss of generality.

There exists at least one sequence of events  $e'_1 \leq \dots \leq e'_n$  such that  $e \leq_{x_1^m} e'_1$  and such that  $e'_n$  is maximal. For simplicity we suppose that there is only one such sequence (the general case uses the same reasoning). From Lemma 19, we have that  $S_1 \xrightarrow{m(e'_n):\ell(e'_n)} S'_1$ . We have  $F(e) < F(e'_1) < \dots < F(e'_n)$  and  $F(e'_n)$  maximal, since  $F$  is l&o-p on all these events. As  $(S_1, S_2, g) \in \mathcal{R}$ , we have also  $(S'_1, S'_2, g') \in \mathcal{R}$ , where  $g'$  is defined as  $g$ . We apply this reasoning until we reach the process  $T_1$  where  $e$  is maximal. Then there exists  $T_2$  such that  $(T_1, T_2, g'')$  and  $g''$  is defined as  $g$  on the identifiers  $i, j$  and  $m(e)$ . Then  $e$  is maximal however  $e'$  is not and we reach a contradiction: from Lemma 19  $S_1 \xrightarrow{m(e):\ell(e)} S'_1$  but  $S_2$  cannot backtrack on  $e'$ . It implies then that  $G : x_1^m \rightarrow x_2^m$  is a l&o-p function on all events in  $x_1^m$ .

**For backward transitions** Now let  $R_1 \xrightarrow{i:\alpha} S_1$  for which we have that  $R_2 \xrightarrow{j:\alpha} S_2$  and  $g : l(S_1) \rightarrow l(S_2)$  is a bijection defined as  $f$  on  $l(S_1) \subset l(R_1)$ , and  $(S_1, S_2, g) \in \mathcal{R}$ . Then we have  $(S_1, S_2, G) \in \mathcal{S}$  with  $G$  is defined by  $g(m_1(e_1)) = m_2(e_2)$ . We only have to show that  $G$  is the restriction of  $F$  to  $[S_1]$  and a bijection. The latter follows from the former, as  $F$  is itself a bijection, and that  $G = F|_{[S_1]}$  follows from  $g$  being a restriction of  $f$  and the way  $G$  and  $F$  are defined.

$\Leftarrow$  Let  $P_1, P_2$  be two processes and let  $\mathcal{R}$  be a HHPB relation between them as defined in Definition 28. Then for any  $R_1, R_2$  two RCCS processes and  $f : [R_1] \rightarrow [R_2]$  an isomorphism such that  $(R_1, R_2, f) \in \mathcal{R}$ ,  $f$  is also a bijection on the event identifiers of the memories of  $R_1, R_2$ . The relation

$$\mathcal{S} = \left\{ (R_1, R_2, F) \mid (R_1, R_2, f) \in \mathcal{R} \text{ and } F(i) = j \iff f(e_1) = e_2, m_1(e_1) = i, m_2(e_2) = j \right\}$$

is a B&F relation as defined in Definition 24. As above, we are using the operational correspondence of Lemma 19.  $\blacktriangleleft$

**► Corollary 49.** *The relation obtained by considering only (5–6) in the definition of B&F (Definition 24) is equal to HPB on CCS terms (Definition 28).*

**Proof.** This is an immediate consequence of Theorem 30, as the proof does not mix forward and backward moves, it suffices to consider only the forward moves to obtain this result.  $\blacktriangleleft$