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Maxim Kazarian, Ricardo Uribe-Vargas. Characteristic Points, Fundamental Cubic Form and Euler Characteristic of Projective Surfaces. *Moscow Mathematical Journal*, 2020, 20 (3), pp.511-530. 10.17323/1609-4514-2020-20-3-511-530 . hal-02568225v2

**HAL Id: hal-02568225**

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Submitted on 20 May 2020

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# Characteristic Points, Fundamental Cubic Form and Euler Characteristic of Projective Surfaces

By Maxim Kazarian\*, and Ricardo Uribe-Vargas†

## Abstract

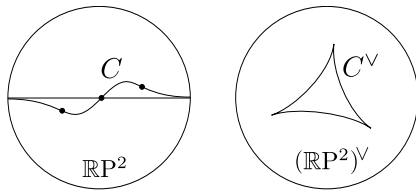
We define local indices for projective umbilics and godrons (also called cusps of Gauss) on generic smooth surfaces in projective 3-space. By means of these indices, we provide formulas that relate the algebraic numbers of those characteristic points on a surface (and on domains of the surface) with the Euler characteristic of that surface (resp. of those domains). These relations determine the possible coexistences of projective umbilics and godrons on the surface. Our study is based on a “fundamental cubic form” for which we provide a simple expression.

**Keywords.** Differential geometry, surface, front, singularity, parabolic curve, flecnodal curve, index, projective umbilic, quadratic point, godron, cusp of Gauss.

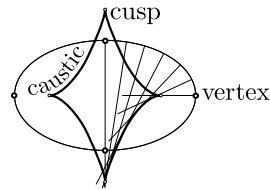
**MSC.** 53A20, 53A55, 53D10, 57R45, 58K05

## 1 Introduction

**1.1 Counting Characteristic Points.** The classical Möbius theorem asserts that a non-contractible curve  $C$  embedded in the projective plane has at least three *inflections* (points where the curve has unusual tangency with its tangent line). Its *dual curve*, denoted  $C^\vee \subset (\mathbb{RP}^2)^\vee$ , consists of the tangent lines to  $C$ . *The higher contact of  $C$  with its tangent line at an inflection is expressed as a cusp of  $C^\vee$*  (Fig. 1).



**Fig. 1.** Möbius Theorem.



**Fig. 2.** Four-vertex Theorem.

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The classical 4-Vertex Theorem (S. Mukhopadhyaya, 1909) asserts that an embedded closed curve in Euclidean plane has at least 4 *vertices* (points where the curve has unusual tangency with its osculating circle). The *caustic* (or *evolute*) of a curve  $C$  of Euclidean plane is the envelope of its normal lines. *The higher contact of  $C$  with its osculating circle at a vertex is expressed as a cusp of the caustic* (Fig. 2).

Mukhopadhyaya also proved the statement that a convex closed plane curve has at least 6 *sextactic points* (where the curve has unusual tangency with its osculating conic).

The recent activity on variations and generalisations of these theorems (cf. [5, 10, 14, 4, 9]) was in part stimulated by the works of V. I. Arnold who presented these results as special cases of general theorems of contact and symplectic topology, in which singularity theory plays an important rôle (cf. [1, 2, 3]).

In this paper, we provide a “global counting” of projective umbilics and godrons (both defined below) of generic smooth surfaces of  $\mathbb{RP}^3$  (and  $\mathbb{R}^3$ ), and we state some coexistence relations:

A generic smooth surface of  $\mathbb{RP}^3$  (or  $\mathbb{R}^3$ ) consists of three parts, may be empty: an open *elliptic domain* at which the second fundamental form  $Q$  is definite (the Gaussian curvature  $K$  being positive); an open *hyperbolic domain* where the form  $Q$  is indefinite ( $K$  being negative); and a *parabolic curve*  $P$  where  $Q$  is degenerate ( $K = 0$ ). The two lines on which  $Q$  vanish at a given hyperbolic point are called *asymptotic lines* of the surface at that point. At the parabolic points there is a unique (but double) asymptotic line.

Each of the above three parts contains characteristic isolated points: a *godron* (or *cusp of Gauss*) is a parabolic point at which the unique (but double) asymptotic line is tangent to the parabolic curve; a *hyperbolic* or an *elliptic projective umbilic*<sup>1</sup> (or *node*) is a point where the surface is approximated by a quadric up to order 3. We also call *hyperbonodes* the hyperbolic projective umbilics and *ellipnodes* the elliptic ones.

Thus ellipnodes and hyperbonodes of surfaces are the analogues of sextactic points of curves.

Each godron of a generic surface has an intrinsic index with value  $-1$  or  $+1$  (cf. [10, 17]). Below, we characterise the hyperbonodes and ellipnodes as the singular points of an intrinsic field of (triples of) lines, which ascribes also to them an index.

A godron, a hyperbonode or an ellipnode is said to be *positive* (or *negative*) if its index is positive (resp. negative). Let us state our main result.

Given a generic smooth compact surface  $S$  of  $\mathbb{RP}^3$ , let  $H$  be a connected component of the hyperbolic domain and  $E$  a connected component of the elliptic domain.

Write  $\#e(E)$ ,  $\#g(E)$ ,  $\#h(H)$  and  $\#g(H)$  for the respective algebraic numbers of ellipnodes in  $E$ , godrons on  $\partial E$ , hyperbonodes in  $H$  and godrons on  $\partial H$ .

**Theorem 1.** *For a generic surface  $S$  of  $\mathbb{RP}^3$  the following three equalities hold*

- (a)  $\#h(H) = \chi(H)$ ;
- (b)  $\#g(\partial H) = 2\chi(H)$ ;
- (c)  $\#e(E) - \#g(\partial E) = 3\chi(E)$ .

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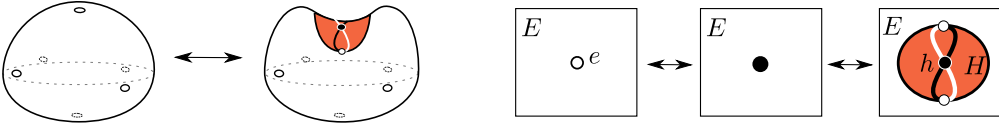
<sup>1</sup>Projective umbilics are also called *quadratic points* (see [9]).

Summing up the three equalities of Theorem 1 over all connected components of both hyperbolic and elliptic domains, we obtain the following relation derived first in [10].

**Corollary 1.** *For a generic surface  $S$  of  $\mathbb{RP}^3$  the sum of the algebraic numbers of ellipnodes and hyperbonodes on  $S$  is thrice the Euler characteristic of  $S$ :*

$$\#e(S) + \#h(S) = 3\chi(S).$$

**Example.** A surface diffeomorphic to a sphere has at least 6 projective umbilics. For example, an ovaloid of a cubic surface has exactly 6 positive ellipnodes [10]. Consider a local continuous deformation that produces a small hyperbolic island  $H$  (Fig. 3-left). By Theorem 1 a and b,  $H$  has one positive hyperbonode and its boundary  $\partial H$  has two positive godrons (in the simplest case). Then, by Theorem 1 c, the elliptic domain has five positive ellipnodes (in the simplest case). The local transition is described in Fig. 3-right.



**Fig. 3.** Left: an ovaloid and its deformation. Right: a generic local transition ([16]).

**1.2 Fundamental Cubic Form.** To prove Theorem 1 we characterise the projective umbilics as the singular points of the field of zeroes of a “fundamental cubic form”.

Similarly as the second fundamental form describes the quadratic deviation of a surface from its tangent plane, the fundamental cubic form describes the cubic deviation of the surface from its quadratic part. Let us introduce our notations.

**Monge form.** To give an expression of the fundamental cubic form, we identify the affine chart of the projective space,  $\{[x : y : z : 1]\} \subset \mathbb{RP}^3$  with  $\mathbb{R}^3$ , and present the germs of surfaces at the origin in Monge form  $z = f(x, y)$  with  $f(0, 0) = 0$  and  $df(0, 0) = 0$ .

We shall express the partial derivatives of  $f$  with numerical subscripts:

$$f_{ij}(x, y) := \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x, y) \quad \text{and} \quad f_{ij} := f_{ij}(0, 0).$$

Take the Taylor expansion  $f = Q + C + \dots$ , where  $Q$  is the quadratic part and  $C$  the cubic part. The discriminant of the quadratic form  $Q$  (multiplied by 4) is given by the Hessian

$$H = f_{20}(x, y)f_{02}(x, y) - f_{11}^2(x, y)$$

(which is positive at the elliptic points and negative at the hyperbolic points, so that the parabolic curve is given by the equation  $H = 0$ ).

**Formula for the Fundamental Cubic Form.** The fundamental cubic form of a surface is the homogeneous degree 3 form on the tangent space given by

$$W = 4HC - Q dH$$

where  $dH$  is the linear part of the Taylor expansion of the function  $H$ .

In the hyperbolic domain, the zeroes of the form  $W$  define a field  $\tau$  of lines whose singular points are the hyperbonodes; while in the elliptic domain, the zeroes of  $W$  define a field  $\tau$  of triples of lines whose singular points are the ellipnodes.

**1.3 Expressions for the Indices.** Take the asymptotic lines as coordinate axes. At a hyperbonode  $h$ , the index of the field  $\tau$  of lines is given by (Proposition 2):

$$\text{ind}_h(\tau) = 1 \cdot \text{sign} \left( 4f_{11}^2 f_{40} f_{04} - (2f_{11} f_{31} - 3f_{21}^2)(2f_{11} f_{13} - 3f_{12}^2) \right). \quad (1)$$

At an ellipnode  $e$  for which  $Q = \frac{\alpha}{2}(x^2 + y^2)$ , and the cubic terms are absent, the index of the field  $\tau$  of triples of lines is given by the expression (Proposition 3):

$$\text{ind}_e(\tau) = \frac{1}{3} \cdot \text{sign} \left( (f_{31} - 3f_{13})(f_{13} - 3f_{31}) - (f_{40} - 3f_{22})(f_{04} - 3f_{22}) \right).$$

**Remark (on a cross-ratio invariant).** At a hyperbonode  $h$  the tangent lines to the flecnodal curves and the asymptotic lines define a cross-ratio invariant, that we note  $\rho(h)$ . The asymptotic lines at  $h$  have 4-point contact with the surface. We say that  $h$  has *parity*  $\sigma(h) = +1$  if both asymptotic lines locally lie on the same side of the surface and  $\sigma(h) = -1$  otherwise.

Taking the asymptotic lines as coordinate axes,  $\rho(h)$  and  $\sigma(h)$  are given by ([18]):

$$\begin{aligned} \rho(h) &= 1 - \frac{(3f_{21}^2 - 2f_{11}f_{31})(3f_{12}^2 - 2f_{11}f_{13})}{4f_{11}^2 f_{40} f_{04}}, \\ \sigma(h) &= \text{sign}(f_{40} f_{04}). \end{aligned}$$

From these expressions and equality (1) we get the formula

$$\text{ind}_h(\tau) = \text{sign}(\rho(h)\sigma(h)).$$

A similar formula holds for the ellipnodes.

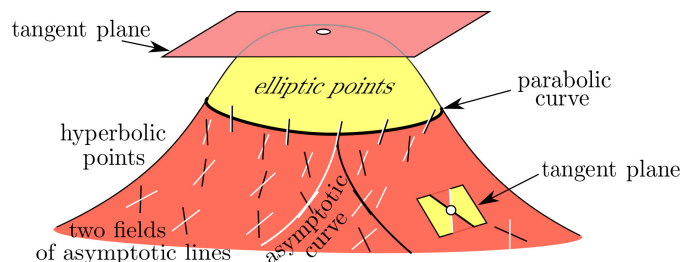
**Organisation of the paper.** In § 2, we recall some basic properties of generic smooth surfaces related to the characteristic points. In § 3, we give a formal definition of the fundamental cubic form and the proofs of its properties. In § 4, we extend the statement of Poincaré-Hopf theorem to the case of “multivalued fields of lines” on a surface with boundary. In § 5, we prove Theorem 1. In § 6, we compute the index of the field of (triples of) lines  $\tau$  at hyperbonodes, ellipnodes and godrons (as boundary singular points).

**Acknowledgments.** We are grateful to J.J. Nuño Ballesteros who communicated to us reference [8]. M. Kazarian appreciates the support of Russian Science Foundation, project 16-11-10316. R. Uribe-Vargas is grateful to Laboratory Solomon Lefschetz UMI2001 CNRS, Universidad Nacional Autonoma de México.

## 2 Properties of surfaces related to the characteristic points

Let us mention some basic features of generic smooth surfaces related to the characteristic points.

**2.1 Contact with Lines.** There are several characterisations of the different kinds of points of smooth surfaces. Let us describe the classification of points of generic surfaces according to their contact with lines. The points of a surface in 3-space are characterised by *asymptotic lines* (tangent lines of the surface with more than 2-point contact). A point is called hyperbolic (resp. elliptic) if there is two distinct asymptotic lines (resp. no asymptotic line). The *parabolic curve* consists of the points where there is a unique (but double) asymptotic line. The *flecnodal curve* is the locus of hyperbolic points where an asymptotic line admits more than 3-point contact with the surface.



**2.2 Flecnodal Curve, Hyperbonodes and Godrons.** An *asymptotic curve* is an integral curve of a field of asymptotic lines.

Through each hyperbolic point of a surface in oriented space  $\mathbb{R}P^3$  there pass two asymptotic curves; one of them is *left* (the first 3 derivatives form a negative frame) and the other is *right* (the frame of the first 3 derivatives is positive). *The flecnodal curve consists of the inflections of the asymptotic curves.* The branch formed by the inflections of the left (right) asymptotic curves is called *left (right) flecnodal curve*.

A *hyperbonode* is an intersection point of the left and right flecnodal curves (Fig. 4).

Moreover, a *godron* is a point of tangency of the flecnodal and parabolic curves. It locally separates the left and right branches of the flecnodal curve (Fig. 5).

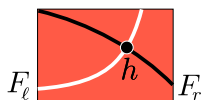


Fig. 4. A hyperbonode.

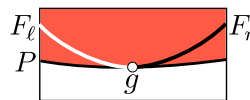
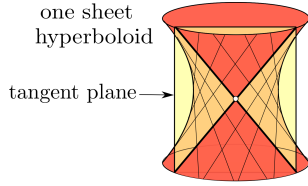


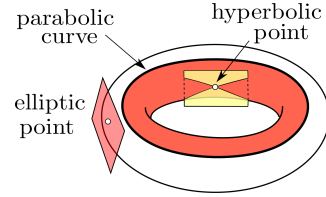
Fig. 5. Flecnodal curve at a godron.

**Example 1.** A one sheet hyperboloid is infinitely degenerate: the asymptotic lines, at every point, are part of the surface. Therefore every point is a hyperbonode (Fig. 6).

**Example 2.** A generic torus (non-symmetric) is a more typical example. Its exterior part is elliptic and the interior one is hyperbolic. The parabolic curve consists of two closed curves that separate the hyperbolic domain from the elliptic one (Fig. 7).



**Fig. 6.** A one sheet hyperboloid.



**Fig. 7.** A generic torus.

**2.3 Positive and Negative Godrons.** A godron is said to be *positive* or of *index +1* (resp. *negative* or of *index -1*) if, at the neighbouring parabolic points, the half-asymptotic lines directed to the hyperbolic domain point towards (resp. away from) the godron :



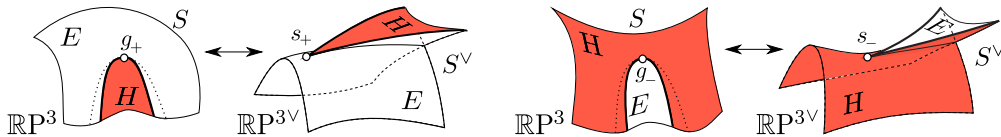
**Fig. 8.** A positive and a negative godron.

Many characterisations and several local and global properties of positive and negative godrons (and swallowtails) are geometrically described in [17].

**2.4 Tangential Map, Godrons and Swallowtails.** The *tangential map* of a smooth surface  $S$ ,  $\tau_S : S \rightarrow (\mathbb{RP}^3)^\vee$ , associates to each point of  $S$  its tangent plane at that point. The image  $S^\vee$  of  $\tau_S$  is called the *dual surface* of  $S$ .

It is known (cf. [12]) that under the tangential map of  $S$  the *parabolic curve* of  $S$  corresponds to the *cuspidal edge* of  $S^\vee$ , a *godron* corresponds to a *swallowtail point*, and the *elliptic (hyperbolic) domain* of  $S$  to the *elliptic (resp. hyperbolic) domain* of  $S^\vee$ .

A swallowtail point of a generic front is said to be *negative (positive)* if, locally, its self-intersection line is contained in the hyperbolic (resp. elliptic) domain. *The dual of a surface at a positive (negative) godron is a positive (resp. negative) swallowtail.* (see Fig. 9)



**Fig. 9.** Duality godron  $\leftrightarrow$  swallowtail for positive and negative godrons.

Thus godrons of surfaces of  $\mathbb{RP}^3$  “are the analogs” of inflections of curves of  $\mathbb{RP}^2$ .

**2.5 Local Transitions and Characteristic Points.** If the surface depends on one real parameter (say the time) the configuration formed by the characteristic points, and the parabolic and flecnodal curves may change.

Ellipnodes, hyperbonodes and godrons are crucial in the transitions of the parabolic curve of evolving smooth surfaces, and in the transitions of wave fronts occurring in generic 1-parameter families [16]: *every Morse transition of the parabolic curve takes place at an ellipnode which is replaced by a hyperbonode (or the opposite). At such transition two godrons of equal signs born or die* (Fig. 10-left).



**Fig. 10.** Examples where an ellipnode is replaced by a hyperbonode or vice-versa.

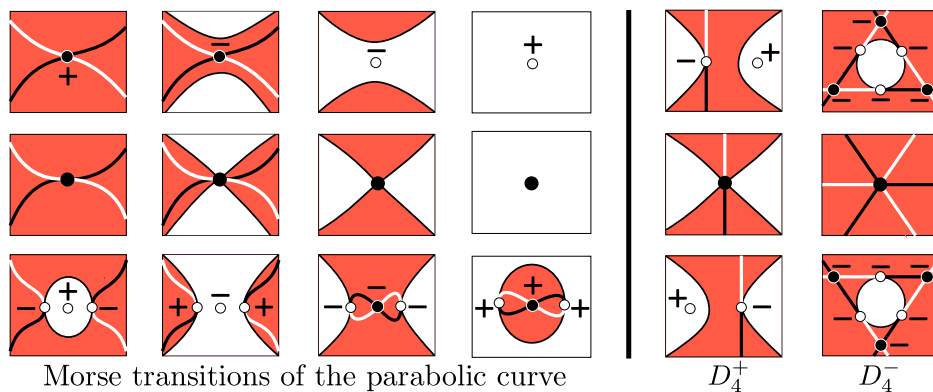
Since the planes tangent to  $S$  at the points of the flecnodal curve form the flecnodal curve of the dual surface  $S^\vee$ , and the projective dual to a hyperbonode (resp. ellipnode) is again a hyperbonode (resp. ellipnode) [15, 16], we get that

*Every  $A_3$ -transition of a wave front, where two swallowtail points born or die, takes place at an ellipnode which is replaced by a hyperbonode (or the opposite). The two involved swallowtails have equal signs* (Fig.10-right).

**Finding the signs.** Corollary 1 implies that in both, a Morse transition of the parabolic curve of an evolving smooth surface and an  $A_3$ -transition of an evolving wave front, the involved ellipnode and hyperbonode have equal signs.

Using Theorem 1 one easily gets the signs of the godrons, ellipnodes and hyperbonodes that take part in generic local transitions. For example, the hyperbonode of Fig. 10-left is negative because it appears after the Euler characteristic of the hyperbolic domain decreases by 1, and, for the same reason, both godrons are positive.

In the same way, we get the signs of the characteristic points for the local transitions occurring in generic 1-parameter families of smooth surfaces (found in [16]) in which both, godrons and projective umbilics, are involved (Fig. 11).



**Fig. 11.** The signs of godrons and projective umbilics involved in generic local transitions.



**2.6 On Normal Forms.** Although tangential singularities had been studied in the 19th Century by Salmon, Cayley, Zeuthen et al (cf. [12]), the normal forms of jets of generic surfaces at different kinds of points was done in the 1980's [7, 11], while the normal forms of the jets (up to order 5) of surfaces at points that appear in generic 1- and 2-parameter families of smooth surfaces were just published in 2017 [13].

The Landis-Platonova normal form for the 4-jet at a godron is equivalent to

$$z = \frac{y^2}{2} - x^2y + \frac{\rho x^4}{2} \quad (\rho \neq 1), \quad (\mathcal{P})$$

where  $\rho > 1$  corresponds to positive godrons and  $\rho < 1$  to the negative ones.

According to Landis-Platonova's Theorem [7, 11] and Ovsienko-Tabachnikov's Theorem [9], the 4-jet of a surface at a hyperbonode can be sent by projective transformations to the respective normal forms

$$z = xy + \frac{1}{3!}(ax^3y + bxy^3) + \frac{1}{4!}(x^4 \pm y^4) \quad (ab \neq \pm 1), \quad (\text{L-P})$$

$$z = xy + \frac{1}{3!}(x^3y \pm xy^3) + \frac{1}{4!}(Ix^4 + Jy^4) \quad (IJ \neq \pm 1), \quad (\text{O-T})$$

To encompass both normal forms we shall consider the "prenormal" form

$$z = xy + \frac{1}{3!}(ax^3y + bxy^3) + \frac{1}{4!}(Ix^4 + Jy^4) \quad (IJ \neq ab).$$

where the genericity conditions on the parameter values  $(a, b, I, J)$  are imposed in order to avoid the moment of creation/annihilation of two hyperbonodes.

In the case of ellipnodes we shall consider the prenormal form

$$z = \frac{1}{2}(x^2 + y^2) + \frac{1}{3!}(ax^3y + bxy^3) + \frac{1}{4}cx^2y^2 + \frac{1}{4!}(Ix^4 + Jy^4).$$

where the corresponding genericity condition is  $(a - 3b)(b - 3a) \neq (I - 3c)(J - 3c)$ .

### 3 Fundamental cubic form

In this section we introduce an important local projective invariant of a surface, the so called fundamental cubic form. The second fundamental form describes the quadratic deviation of a surface from its tangent plane. In the same way, the fundamental cubic form describes the cubic deviation of the surface from its quadratic part.

Let  $S$  be a smooth surface generically embedded to the affine 3-space. Assume that it is given in Monge form  $z = f(x, y)$ . Consider the Taylor expansion of  $f$ ,

$$f(x, y) = Q(x, y) + C(x, y) + \dots,$$

where  $Q$  is homogeneous of degree 2 and  $C$  of degree 3. Observe that linear changes of the coordinates  $x$  and  $y$  preserve the homogeneous forms regarded as forms on the

tangent plane. We discuss the ambiguity in the definition of these forms for more general affine changes of coordinates in the ambient space.

**Second Fundamental Form.** It is easy to see that  $Q$  provides a correctly defined quadratic form on the tangent plane taking values on the “normal line”  $\nu_p S = T_p \mathbb{R}P^3 / T_p S$ . It is called *second fundamental form* of the surface. However, if we treat this form as taking values in real numbers, it is defined up to a factor only. It is sign definite, indefinite or degenerate if the surface is respectively elliptic, hyperbolic or parabolic at the corresponding point.

The action of affine changes on the cubic part of the Taylor expansion is more complicated. Namely, the cubic term  $C$  provides a cubic form on the tangent plane up to a factor and up to a summand of the form  $QL$  where  $L$  is linear. Theorem 2 below claims that, given  $Q$ , the family of cubic forms  $C + QL$  for all possible choices of  $L$  has a canonical representative that we call *fundamental cubic form*. This means that the quadratic form  $Q$  produces a splitting of the space of cubic forms in two parts:

**Canonical splitting of the space of cubic forms.** Given a nondegenerate quadratic form  $Q$  on an abstract 2-dimensional vector space  $V \approx \mathbb{R}^2$ , the “convolution with the inverse of  $Q$ ” is an operation on the homogeneous forms which lowers the degree by two. In coordinates, if  $Q = ax^2 + 2bxy + cy^2$ , this operation acts as the second order differential operator

$$\Lambda := c \frac{\partial^2}{\partial x^2} - 2b \frac{\partial^2}{\partial x \partial y} + a \frac{\partial^2}{\partial y^2}. \quad (2)$$

Observe that  $\left(\frac{1}{4(ac-b^2)}\right) \Lambda Q = 1$  and that this “convolution operation” determines the surjective linear map  $\lambda_Q : \text{Sym}^3 V^* \rightarrow \text{Sym}^1 V^*$  given by  $\lambda_Q(C) = \Lambda C$ .

**Splitting Lemma.** *The form  $Q$  determines a canonical splitting of the 4-dimensional space of cubic forms as a direct sum of two 2-dimensional subspaces,*

$$\text{Sym}^3 V^* = U_Q^+ \oplus U_Q^-, \quad (3)$$

where  $U_Q^+$  consists of the cubic forms multiples of  $Q$  (i.e. written as  $QL$  with  $L$  linear) and  $U_Q^-$  is the kernel of the surjective linear map  $\lambda_Q : \text{Sym}^3 V^* \rightarrow \text{Sym}^1 V^*$ .

*Proof.* To prove that  $U_Q^+$  and  $U_Q^-$  are transversal, it is sufficient to check this fact for  $Q$  in a normal form, one for the hyperbolic case and one for the elliptic one.

In the hyperbolic case, we choose coordinates such that  $Q = xy$ , then  $U_Q^+$  is spanned by  $x^2y$  and  $xy^2$ , and  $U_Q^-$  is spanned by  $x^3$  and  $y^3$ .

In the elliptic case, we choose coordinates such that  $Q = \pm(x^2 + y^2)$ . Then  $U_Q^+$  is spanned by  $x^3 + xy^2$  and  $x^2y + y^3$  while  $U_Q^-$  is spanned by  $x^3 - 3xy^2$  and  $3x^2y - y^3$ .

Therefore the splitting (3) holds true for any nondegenerate quadratic form  $Q$ .  $\square$

Now we assume  $V$  is the tangent space to the surface at a non-parabolic point :

**Fundamental Cubic Form.** The fundamental cubic form  $W$  of a surface (FCF) is the homogeneous degree 3 form (on the tangent plane  $V$ ) obtained as the projection of the form  $C$  to the space  $U_Q^-$  along the space  $U_Q^+$ .

This definition leads to the following explicit formula:

**Theorem 2** (proved below). *If  $f(x, y) = Q(x, y) + C(x, y) + \dots$  with  $Q$  non degenerate, the FCF is given by*

$$\widehat{W} = C - Q \frac{dH}{4H}, \quad (4)$$

where  $H(x, y) = f_{20}(x, y)f_{02}(x, y) - f_{11}^2(x, y)$  and  $dH$  is the linear part of  $H$ .

In the parabolic case (i.e. when  $Q$  is degenerate) there is no splitting because in that case  $U_Q^- = U_Q^+$ . [For example, if  $Q = cy^2$ , then  $\Lambda = c\partial_x^2$  so that  $U_Q^-$  is spanned by  $\{xy^2, y^3\}$ , coinciding thus with  $U_Q^+$ .] It follows that our definition and expression (4) are not applicable at the parabolic points. But since the cubic form is defined up to a factor, we rescale the form (4) to

$$W = 4HC - QdH \quad (5)$$

extending it continuously (and unambiguously) to the parabolic points. Thus we consider (5) as the coordinate expression of the fundamental cubic form for all points.

**Remark.** In an abstract vector space  $V \simeq \mathbb{R}^2$ , the canonical representative of the cubic forms  $C + QL$  for all possible choices of  $L$ , defined up to a factor, is the form

$$W = 4H_Q C - 2Q\Lambda C,$$

where  $H_Q = 4(ac - b^2)$  is the Hessian of the quadratic form  $Q$ .

**Projective Umbilics or Nodes.** A non parabolic point of a generic surface (i.e.,  $Q$  is nondegenerate) is called *projective umbilic* or *node* (*ellipnode* or *hyperbonode*) if the cubic form  $C$  is divisible by  $Q$ , that is,  $C = QL$  for some linear function  $L$ .

**Theorem 3.** *Let  $W$  be the fundamental cubic form of a smooth surface.*

1. *The lines of zeroes of the form  $W$  are well defined tangent lines on the surface;*
2. *The form  $W$  vanishes at the projective umbilics (ellipnodes or hyperbonodes);*
3. *At every elliptic point which is not an ellipnode, the form  $W$  has three distinct real lines of zeroes;*
4. *At every hyperbolic point which is not a hyperbonode, the form  $W$  has one real line of zeroes;*
5. *At the parabolic points different from godrons the form  $W$  has a double zero line which is also a double zero line of  $Q$ , and a simple zero line tangent to the parabolic curve;*
6. *At every godron the form  $W$  has the triple zero line tangent to the parabolic curve.*

*Proof.* Item 1 follows because the form  $W$  is well defined up to a factor.

Item 2 follows because at the projective umbilics the form  $C$  lies, by definition, in the kernel of the projection to  $U_Q^-$  along  $U_Q^+$ .

At a parabolic point we have  $H = 0$  so that  $W = Q dH$ , which implies immediately the statements of item 1 as well as of items 5 and 6.

To prove items 3 and 4, it is sufficient to check them for  $Q$  in a normal form:

In the elliptic case, we choose coordinates such that  $Q = \pm(x^2 + y^2)$ . Then  $U_Q^-$  is spanned by  $x^3 - 3xy^2$  and  $3x^2y - y^3$  and assertion 3 follows from the fact that any nonzero combination of the forms  $x^3 - 3xy^2$  and  $3x^2y - y^3$  has three real zero lines.

In the hyperbolic case, we choose coordinates such that  $Q = xy$ . Then  $U_Q^-$  is spanned by  $x^3$  and  $y^3$  and assertion 4 follows from the fact that any nonzero combination of the forms  $x^3$  and  $y^3$  has one real zero line. Theorem 3 is proved.  $\square$

**Remark.** The fundamental cubic form of a surface is actually a known and well studied object in affine differential geometry, see, e.g., [8]. It is defined by

$$\mathfrak{C}(X, Y, Z) = (\nabla_X h)(Y, Z), \quad (6)$$

where  $\nabla$  is the connection on the (tangent bundle of the) surface defined by its canonical Blaschke structure associated with the affine embedding, and  $h$  is the quadratic fundamental form with a normalisation that differ from our form  $Q$  by a factor, see details in [8]. Namely, for a surface of the form  $z = f(x, y)$  we have explicitly

$$\begin{aligned} h &= H^{-1/4} Q, \\ \mathfrak{C} &= H^{-1/4} \left( C - \frac{1}{4} Q d \log H \right). \end{aligned}$$

We see the forms  $h$  and  $\mathfrak{C}$  agree with our respective forms  $Q$  and  $W$  up to a factor. The statements 2–4 of Theorem 3 are also known, see [8, Sect. II.11].

Observe, however, that our approach to the definition of the fundamental cubic form is completely different from that one of affine differential geometry. It is important to notice also that, up to a factor, *the quadratic and the cubic fundamental forms are well defined local invariants of the surface which come from the projective structure of the ambient space*, rather than the affine one. Besides, the asymptotic behavior of the field of zeroes of the cubic form on the parabolic line subject to statements 5–6 of Theorem 3 has not been studied before, to our knowledge. In fact, the definition of the cubic form (6) assumes the nondegeneracy of  $h$  and is not applicable to parabolic points, in contrast to our definition of the (normalised) form  $W$ .

### 3.1 Proof of Theorem 2

Notice that if  $f(x, y) = Q(x, y) + C(x, y) + \dots$  and  $H_f$  denotes its Hessian function

$$H_f(x, y) = f_{20}(x, y)f_{02}(x, y) - f_{11}^2(x, y),$$

then at the origin we have  $H_f(0,0) = H_Q = 4(ac - b^2)$ ; and its differential depends only on  $Q$  and  $C$ :  $dH_f(0,0) = dH_{Q+C}$ . Therefore in the following lemma, which characterises the cubic forms divisible by  $Q$ , we consider the functions  $f(x,y) = Q(x,y) + C(x,y)$ .

**Lemma 1.** *If  $f(x,y) = Q(x,y) + C(x,y)$  where  $Q$  is a non degenerate quadratic form and  $C = QL$  with  $L$  linear (i.e. the cubic form  $C$  is divisible by  $Q$ ) then at the origin*

$$dH_{Q+QL} = 4H_Q L, \quad (7)$$

where  $dH_{Q+QL}$  is the linear part of the function  $H_{Q+QL}$  and  $H_Q = 4(ac - b^2)$  is the Hessian of the non degenerate quadratic form  $Q$ .

*Proof.* If  $f = Q + QL$  with  $Q(x,y) = ax^2 + 2bxy + cy^2$  and  $L(x,y) = ux + vy$ , then

$$H_f(x,y) = (2a(L+1) + 2uQ_x)(2c(L+1) + 2vQ_y) - (2b(L+1) + vQ_x + uQ_y)^2.$$

Since  $L = Q_x = Q_y = 0$  at the origin, the linear part of the Hessian at the origin is given by  $dH_f = 4^2(ac - b^2)ux + 4^2(ac - b^2)vy$ , that is  $dH_f = 4H_Q L$ .  $\square$

Lemma 1 suggests the canonical representative  $W$  of the cubic forms  $C + QL$  for all possible choices of  $L$  should satisfy  $dH_{Q+W} = 0$ . A simple calculation proves the

**Lemma 2.** *Let  $f = Q + \widehat{C}$  with  $Q$  non degenerate. Then at the origin we have*

$$\Lambda \widehat{C} = \frac{1}{2} dH_{Q+\widehat{C}}, \quad (8)$$

where  $\Lambda$  is the operator (2) determined by  $Q$ .

**Proof of Theorem 2.** From the Lemmas 1 and 2 we get  $(1/4H)QdH = (1/2H_Q)Q\Lambda C$ . Then the linearity of the map  $\lambda_Q$  applied to  $\widehat{W}$  (see (4)) provides the equality

$$\Lambda(C - (1/2H_Q)Q\Lambda C) = \Lambda C - (1/2H_Q)\Lambda(Q\Lambda C). \quad (9)$$

So relation (8) applied to  $\widehat{C} = Q\Lambda C$  and then relation (7) applied to  $L = \Lambda C$  provide

$$(1/2H_Q)\Lambda(Q\Lambda C) = (1/2H_Q)\frac{1}{2}dH_{Q+Q\Lambda C} = (1/4H_Q)4H_Q\Lambda C. \quad (10)$$

Finally, equalities (9) and (10) imply that  $\widehat{W}$  is annihilated by the operator  $\Lambda$ .

The definition of the FCF implies that a change of  $C$  by a summand of the form  $QL$  (i.e., lying in  $U^+$ ) does not change its image under the projection. This proves Theorem 2, in the non-parabolic case, because the splitting (3) is defined intrinsically.  $\square$

## 4 Poincaré-Hopf theorem for multivalued line fields

The classical Poincaré-Hopf theorem claims that the sum of indices of singular points of a vector field  $v$  on a compact smooth  $n$ -dimensional manifold equals the Euler characteristic of that manifold. This equality holds true also for a manifold with boundary if the field is transverse to the boundary and directed, say, outside the manifold at every its boundary point. The index is defined as the degree of the mapping  $S^{n-1} \rightarrow S^{n-1}$  where the source sphere is the boundary of a small ball on the manifold centred at a singular point of the field, and the mapping is given by  $x \mapsto v(x)/\|v(x)\|$ . In the case of a surface ( $n = 2$ ) the index can be treated also as the number of rotations of the vector  $v(x)$  about the origin while the point  $x$  makes one turn rotation around the singular point of the field.

In this section, we extend, following the idea of [6], the statement of the Poincaré-Hopf theorem to the case of “multivalued fields of lines” on a surface.

***k*-Valued Line Fields.** A *k*-valued line field  $\tau$  on a surface is a correspondence that associates to a point on a surface an unordered *k*-tuple of pairwise distinct non oriented tangent lines. We assume that this *k*-tuple of tangent lines depends continuously on the point of the surface and is defined for all but finitely many points on the surface. We refer to the points where the *k*-valued line field is not defined as the *singular* points of this field.

If a point of the surface follows a loop, the continuity of the multivalued line field along this loop leads to a (cyclic) permutation of its lines.

***Fractional Index.*** The *fractional index* of a singular point of a *k*-valued line field is the rational number  $p/q$  such that each line of the field comes to its initial position with the same orientation and makes  $p$  turns of rotations while the point of the base makes  $q$  turns around the singular point of the surface in positive direction. It is an element of  $(1/2k)\mathbb{Z}$ .

In order to apply this definition, one needs to fix a choice of the orientation of the surface in a neighbourhood of a singular point of the field. However, the actual value of the fractional index is independent of this choice and the equality holds independently of whether the surface is orientable or not. Indeed, a change of the orientation of the surface changes orientations of both the source and the target circles of the mapping defining the index, thus preserving its value.

**Proposition 1.** *For any closed surface and a multivalued field of lines on it, the sum of fractional indices of the singular points of the field is equal to the Euler characteristic of the surface.*

This extension of the Poincaré-Hopf theorem can be proved, for example, by passing to a suitable ramified covering surface such that the field becomes uni-valued and oriented on the covering surface, and by applying the usual Poincaré-Hopf theorem to the covering surface.

**Remark.** Even for a single-valued line field Proposition 1 is not formally equivalent to Poincaré-Hopf theorem if the field is not oriented; the index being half-integer.

**Triviality condition for surfaces with boundary.** The statement of Proposition 1 can be extended also to the case of a surface with boundary. We say that a multivalued direction field on a surface is *trivialised along the boundary* if the boundary contains no singular point and either the directions of the field are never tangent to the boundary or if at each point of the boundary one of the branches of the field is tangent to the boundary. Then the equality of Proposition 1 holds as well.

## 5 Proof of Theorem 1

**Proof of Theorem 1a.** Let  $S$  be a compact surface in the projective 3-space and  $H$  be one of the connected components of its hyperbolic domain. Take for  $\tau$  the field of zeroes of the cubic fundamental form. By Theorem 3, we have  $k = 1$ .

The line field  $\tau$  extends continuously to the boundary  $\partial H$ . The extended field is not transverse to the boundary, but the triviality condition formulated above is satisfied since the extended field is tangent to the parabolic curve  $\partial H$  at *each* of its points (including the godrons). The internal singular points of  $\tau$  are exactly the hyperbonodes and the local computations (Proposition 2 below) show that

$$\text{ind}_h \tau = \pm 1.$$

We obtain immediately the equality

$$\#h(H) = \chi(H). \quad \square$$

**Fractional index of singular points on the boundary.** Let us also extend the equality of Proposition 1 to the case when the multivalued field of directions has singular points on the boundary. Assume the triviality condition of the multivalued line field holds along the boundary, except at a finite number of its points. Take one of these points and pick a small disk centred at this point. Write  $\gamma$  for the part of the boundary of the disk lying in the surface, and  $A$  and  $B$  for the endpoints of  $\gamma$ . Since  $A$  and  $B$  belong to the part of the boundary of the surface where the field is trivialised, there is a homeomorphism of the tangent planes at the points  $A$  and  $B$  that identifies both values of the direction field and the (cooriented) tangent direction of the boundary. This identification provides a multivalued direction field along the closed path  $\gamma/(A \sim B)$ .

**Border Fractional Index.** Define the *fractional index of the multivalued field of lines at a singular point on the boundary* as the index of the obtained closed path  $\gamma/(A \sim B)$ .

For example, using this definition, the fractional indices of the 2-valued fields of lines of Fig. 12 along the closed paths  $\gamma/(A \sim B)$  are respectively  $1/2$  and  $-1/2$ .

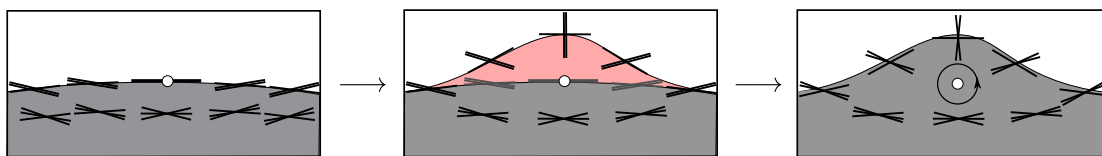
This definition is justified by the following theorem (with its proof).

**Theorem 4.** *For any multivalued direction field on a surface with boundary, the sum of indices of all singular points, both internal and lying on the boundary, is equal to the Euler characteristic of the surface.*



**Fig. 12.** Boundary singular points with fractional indices  $1/2$  (left) and  $-1/2$  (right).

*Proof.* Let us extend the surface by attaching a collar along the boundary which is a narrow strip (Fig. 13-centre). The field can be extended to the collar in such a way that it becomes trivialised along the (new) boundary, without singularities inside the collar, and the boundary singularity of the original field becomes an internal singularity for the extended field (Fig. 13-right). Thus, the situation of Proposition 1 is applied and leads to the equality of Theorem 4.



**Fig. 13.** Computation of the index at a boundary singular point.

**Remark.** The statement of Theorem 4 is applicable, for example, in the case of a vector field which is not necessarily transverse to the boundary. A point of the boundary is ‘singular’ for the field if it is an isolated point where the field is tangent to the boundary. The index of such point is half-integer. For example, the vector field  $v = \partial_x$  on the unit disk  $D = x^2 + y^2 \leq 1$  has no internal singular points, but it has two boundary singular points of indices  $1/2$  both, which gives  $\chi(D) = 1/2 + 1/2 = 1$ .

**Proof of Theorem 1b.** The only singularities of the 2-valued field of asymptotic lines, on a connected component  $H$  of the hyperbolic domain, are the godrons.

Consider the normal form  $(\mathcal{P})$  near a godron:  $f(x, y) = y^2/2 - x^2y + \rho x^4/2$ . The asymptotic lines are the zeroes of the second fundamental form

$$Q = (-y + 6\rho x^2) dx^2 - 4x dx dy + dy^2.$$

**Fact 1b.** Near the godron the asymptotic lines are never parallel to the  $y$ -axis (because the coefficient of  $dy^2$  in  $Q$  is constant) and the sectors where  $Q$  takes positive (or negative) values are symmetric with respect to the  $y$ -axis:  $Q_{(x,y)}(v_x, v_y) = Q_{(-x,y)}(-v_x, v_y)$ .

At the parabolic points near a positive godron  $g^+$ , the half-asymptotic lines directed to the hyperbolic domain point towards  $g^+$  (see Fig. 8). So at a hyperbolic point  $A$  near  $g^+$  with  $x > 0$  we have two close half-lines of zeroes that determine a thin sector directed to  $H$  and to  $g^+$  (Fig. 12-left). Fact 1b implies that after moving on  $\gamma$  to a hyperbolic point  $B$  with  $x < 0$ , near  $g^+$ , the considered sector is directed away from  $g^+$  and towards the elliptic side. Therefore the fractional index of the field at  $g^+$  equals  $1/2$ .

Exactly in the same way, one proves that the fractional index of the 2-valued field of asymptotic lines at a negative godron equals  $-1/2$  (see Fig. 12-right).



Since the Euler characteristic of  $H$  equals the sum of indices at the godrons on the boundary  $\partial H$  (by Theorem 4), the algebraic sum of godrons on  $\partial H$  equals  $2\chi(H)$ .  $\square$

One also can prove the equality  $\#g(\partial H) = 2\chi(H)$  by using the double covering of the asymptotic directions (it is explained and used in [17]).

**Proof of Theorem 1c.** Let  $E$  be one of the connected components of the elliptic domain. We are going to apply Theorem 4 to the line field  $\tau$  of zeroes of the fundamental cubic form on  $E$ . By Theorem 3,  $\tau$  is a 3-valued line field that has singularities at the ellipnodes (which are internal points of the elliptic domain) and at the godrons (which are boundary points of that elliptic domain). Thus, to obtain the Euler characteristic of  $E$ , we have to sum the contribution of the indices of the ellipnodes in  $E$ , and that of the indices of the godrons lying on the boundary  $\partial E$ .

**Fact 1c.** The local computations for the ellipnodes (Proposition 3 below) show that

$$\text{ind}_e\tau = \pm\frac{1}{3}. \quad (11)$$

To find the local index of  $\tau$  at a godron, notice that the field  $\tau$  extends continuously to the boundary  $\partial E$ , but the extended field does not satisfy the triviality condition: two of the directions glue together at the boundary points and the third one becomes tangent to the boundary. This degeneracy can be resolved by moving slightly from the boundary point to a close internal point of  $E$ . Applying a small such modification we obtain a field satisfying the necessary conditions on the boundary of  $E$ . The fractional index of  $\tau$  at a godron is then computed using the above definition (Fig. 12).

**Fact 2c.** The local computations at the godrons on  $\partial E$  (Proposition 4 below) show that *the index at a positive godron equals  $-1/3$  and at a negative godron equals  $1/3$ :*

$$\text{ind}_{g^+}\tau = -\frac{1}{3}, \quad \text{ind}_{g^-}\tau = \frac{1}{3}. \quad (12)$$

Theorem 4 together with equalities (11) and (12) imply the relation

$$\#e(E) - \#g(\partial E) = 3\chi(E). \quad \square$$

## 6 Local indices at hyperbonodes, ellipnodes and godrons

We shall compute the local index of the multivalued line field  $\tau$ , defined by the fundamental cubic form  $W = 4HC - QdH$ , at generic hyperbonodes, ellipnodes and godrons. To perform these computations (by hand), we only need to find the relevant terms of  $C$ ,  $Q$ ,  $H$  and  $dH$ .

**6.1 Local Index at a Hyperbonode.** Write the surface in Monge form  $z = f(x, y)$ .

**Proposition 2.** *The local index of a hyperbonode  $h$ , taking the asymptotic lines as coordinate axes, equals*

$$\text{ind}_h(\tau) = 1 \cdot \text{sign} \left( 4f_{11}^2 f_{40} f_{04} - (2f_{11} f_{31} - 3f_{21}^2)(2f_{11} f_{13} - 3f_{12}^2) \right).$$

**Corollary.** For the above normal forms of Landis-Platonova and of Ovsienko-Tabachnikov we get the respective expressions of the index

$$\text{ind}_h(\tau) = \pm 1 - ab \quad \text{and} \quad \text{ind}_h(\tau) = IJ \mp 1.$$

**Remark 1.** If at  $h$  we take the diagonals  $y = \pm x$  as asymptotic lines and we assume the cubic terms of  $f$  are missing for the chosen affine coordinate system, then

$$\text{ind}_h(\tau) = 1 \cdot \text{sign}((f_{40} + 3f_{22})(f_{04} + 3f_{22}) - (f_{31} + 3f_{13})(f_{13} + 3f_{31})). \quad (13)$$

**Notation.** In order to get not so long lines along the proofs, we shall replace the partial derivatives  $f_{ij}$  with constants  $\alpha, u, v, a, b, I, J$ , etc, just for the calculations.

*Proof of Proposition 2.* Let us locally express the surface in Monge form

$$f(x, y) = \alpha xy + \frac{1}{2}ux^2y + \frac{1}{2}vxy^2 + \frac{1}{3!}(ax^3y + bxy^3) + \frac{1}{4!}(Ix^4 + Jy^4).$$

The explicit expressions of the relevant terms for  $H$ ,  $dH$ ,  $Q$  and  $C$  are

$$\begin{aligned} H &\approx -\alpha^2 - 2\alpha(ux + vy) + \dots; \\ dH &\approx -(2\alpha u + 2(u^2 + \alpha a)x + uvv + \dots)dx - (2\alpha v + 2(v^2 + \alpha b)y + uvx + \dots)dy; \\ Q &\approx \frac{1}{2}(uy + axy + \dots)dx^2 + (\alpha + ux + vy + \dots)dxdy + \frac{1}{2}(vx + bxy + \dots)dy^2; \\ C &= \frac{(ay + Ix)dx^3}{3!} + \frac{(u + ax)dx^2dy}{2} + \frac{(v + by)dxdy^2}{2} + \frac{(bx + Jy)dy^3}{3!}. \end{aligned}$$

Then the expression for  $4HC$  up to terms of order 1 in  $x, y$  is

$$4HC \approx -\frac{2}{3}\alpha^2(ay + Ix)dx^3 + \varphi(x, y)dx^2dy + \psi(x, y)dxdy^2 - \frac{2}{3}\alpha^2(bx + Jy)dy^3,$$

where  $\varphi(x, y) = -2\alpha^2(u + ax) + 4\alpha u(ux + vy)$  and  $\psi(x, y) = -2\alpha^2(v + by) + 4\alpha v(ux + vy)$ .

The corresponding expression for  $QdH$ , up to its first order terms, is

$$QdH \approx -\alpha u^2y dx^3 + \varphi(x, y)dx^2dy + \psi(x, y)dxdy^2 - \alpha v^2x dy^3.$$

Therefore, the fundamental cubic form  $W = 4HC - QdH$  is given by

$$W \approx -\frac{\alpha}{3} \left( 2\alpha Ix + (2\alpha a - 3u^2)y \right) dx^3 - \frac{\alpha}{3} \left( (2\alpha b - 3v^2)x + 2\alpha Jy \right) dy^3. \quad (14)$$

Given a point that makes a positive turn on a very small circle around the origin, the line of zeroes of (14) makes a positive turn if and only if the image of our small circle by the map  $(x, y) \mapsto (2\alpha Ix + (2\alpha a - 3u^2)y, (2\alpha b - 3v^2)x + 2\alpha Jy)$  makes a positive turn around  $(0, 0)$ ; that is, if and only if this map preserves the orientation. Since its determinant is equal to  $4\alpha^2 IJ - (2\alpha a - 3u^2)(2\alpha b - 3v^2)$ , we get that

$$\text{ind}_h(\tau) = 1 \cdot \text{sign} \left( 4f_{11}^2 f_{40} f_{04} - (2f_{11} f_{31} - 3f_{21}^2)(2f_{11} f_{13} - 3f_{12}^2) \right). \quad \square$$

**6.2 Local Index at an Ellipnode.** Writing the surface in Monge form  $z = f(x, y)$  with  $Q = \frac{\alpha}{2}(x^2 + y^2)$  and assuming the cubic terms of  $f$  are missing (for the chosen affine coordinate system) we get the

**Proposition 3.** *The local index of  $\tau$  at an ellipnode  $e$  for which  $Q = \frac{\alpha}{2}(x^2 + y^2)$ , equals*

$$\text{ind}_e(\tau) = \frac{1}{3} \cdot \text{sign}((f_{31} - 3f_{13})(f_{13} - 3f_{31}) - (f_{40} - 3f_{22})(f_{04} - 3f_{22})) .$$

(Compare this formula with expression (13) of Remark 1.)

*Proof of Proposition 3.* Let us locally express the surface in Monge form

$$f(x, y) = \frac{\alpha}{2}(x^2 + y^2) + \frac{1}{3!}(ax^3y + bxy^3) + \frac{c}{4}x^2y^2 + \frac{1}{4!}(Ix^4 + Jy^4).$$

The explicit expressions of the relevant terms for  $H$ ,  $dH$ ,  $Q$  and  $C$  are

$$\begin{aligned} H &\approx \alpha^2 + \dots; \\ dH &\approx \alpha((a+b)y + (c+I)x + \dots)dx + \alpha((a+b)x + (c+J)y + \dots)dy; \\ Q &\approx \frac{1}{2}(\alpha + \dots)dx^2 + (\dots)dxdy + \frac{1}{2}(\alpha + \dots)dy^2; \\ C &= \frac{(ay + Ix)dx^3}{3!} + \frac{(ax + cy)dx^2dy}{2} + \frac{(by + cx)dxdy^2}{2} + \frac{(bx + Jy)dy^3}{3!}. \end{aligned}$$

Then the expression for  $4HC$  up to terms of order 1 in  $x, y$  is

$$4HC \approx 2\alpha^2 \left( \frac{1}{3}(ay + Ix)dx^3 + (ax + cy)dx^2dy + (by + cx)dxdy^2 + \frac{1}{3}(bx + Jy)dy^3 \right),$$

and the corresponding expression for  $QdH$ , up to its first order terms, is

$$QdH \approx \frac{\alpha^2}{2} \left( ((a+b)y + (c+I)x)(dx^3 + dxdy^2) + ((a+b)x + (c+J)y)(dx^2dy + dy^3) \right).$$

Therefore, the fundamental cubic form  $W = 4HC - QdH$  is given by

$$W \approx \frac{\alpha^2}{6} \left( ((I-3c)x + (a-3b)y)(dx^3 - 3dxdy^2) + ((3a-b)x + (3c-J)y)(3dx^2dy - dy^3) \right).$$

Hence, the local index of the ellipnode  $e$  is  $1/3$  multiplied by the sign of the determinant of the linear map  $(x, y) \mapsto ((I-3c)x + (a-3b)y, (3a-b)x + (3c-J)y)$ , which is equal to  $(a-3b)(b-3a) - (I-3c)(J-3c)$ . Therefore

$$\text{ind}(e) = \frac{1}{3} \cdot \text{sign}((f_{31} - 3f_{13})(f_{13} - 3f_{31}) - (f_{40} - 3f_{22})(f_{04} - 3f_{22})) . \quad \square$$

**6.3 Local Index at a Godron.** In this case, we base our computations on basic geometric properties of the three-valued line field  $\tau$  near the considered godron.

**Proposition 4.** *At a positive godron the local index of the field  $\tau$  equals  $-1/3$  and at a negative godron it is equal to  $1/3$ :*

$$\text{ind}_{g^+}(\tau) = -\frac{1}{3} \quad \text{and} \quad \text{ind}_{g^-}(\tau) = \frac{1}{3}.$$

*Proof.* Near a godron given in Landis-Platonova normal form

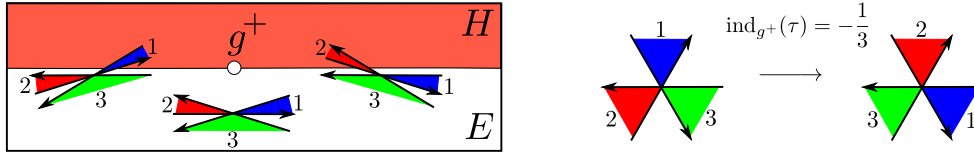
$$f(x, y) = \frac{1}{2}y^2 - x^2y + \frac{1}{2}\rho x^4,$$

we get the following expression for the fundamental cubic form

$$W = \left( -(4\rho + 8)y + (12\rho^2 - 8\rho)x^2 \right) x dx^3 + 6 \left( y + \rho x^2 \right) dx^2 dy - (6\rho x) dx dy^2 + dy^3. \quad (15)$$

**Basic observations.** By formula (15) the lines of zeroes of  $W$  are never vertical (never parallel to the  $y$ -axis) because the coefficient of  $dy^3$  is the constant 1 (near a godron the three lines are close to the horizontal). Moreover, these lines of zeroes and the sectors where  $W$  takes positive (or negative) values are symmetric with respect to the  $y$ -axis:  $W_{(x,y)}(v_x, v_y) = W_{(-x,y)}(-v_x, v_y)$ . We only need these basic observations from  $W$ .

At the parabolic points near a positive godron  $g^+$ , the half-asymptotic lines directed to the hyperbolic domain point towards  $g^+$  (see Fig. 8). Then at the elliptic points with  $x < 0$ , near  $g^+$ , we have two close half-lines of zeroes that determine a thin sector pointing to  $H$  and to  $g^+$  (sector 1 in Fig. 14). Our basic observations imply that after moving to the elliptic points with  $x > 0$ , near  $g^+$ , the considered sector points away from  $g^+$  and is adjacent to the parabolic curve from the elliptic side. Then we get the index  $-1/3$  because the sector makes a complete negative turn after three such loops around  $g^+$ .



**Fig. 14.** A positive godron contributes  $-1/3$  to the Euler characteristic of  $E$ .

At the parabolic points near a negative godron  $g^-$ , the half-asymptotic lines directed to the hyperbolic domain point away from  $g^-$  (Fig. 8). Thus at the elliptic points with  $x < 0$ , near  $g^-$ , we have two close half-lines of zeroes that determine a thin sector pointing to  $H$  and away from  $g^-$  (sector 1 in Fig. 15). Our basic observations imply that after moving to the elliptic points with  $x > 0$ , near  $g^-$ , the considered sector points to  $g^-$  and is adjacent to the parabolic curve from the elliptic side. Then we get the index  $1/3$  because the sector makes a complete positive turn after three such loops around  $g^-$ .  $\square$

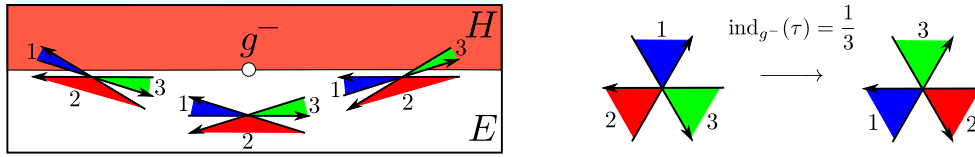


Fig. 15. A negative godron contributes  $+1/3$  to the Euler characteristic of  $E$ .

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