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# Potential theory, Maxwell Equations and the Lorentz Force.

G. M. Stephan

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1

## Résumé

We show that Maxwell equations and the Lorentz force can be expressed in terms of the spatial and temporal derivatives of the electromagnetic potential.

## 1 Introduction.

Maxwell equations are the basis of electromagnetism. They express the relationships between electromagnetic fields and their sources. There are several ways to write these equations [1] but all formulations distinguish fields from sources. The fields can be expressed as functions of the spatial and temporal derivatives of the electromagnetic potential. Sources are electrical charges and currents. The first aim of this study is to show that sources can also be expressed in terms of these derivatives. In other words, we show that Maxwell equations can be expressed with the potential derivatives only. The second objective is to show that the Lorentz force is obtained as a direct application of these new expressions for sources.

The theory starts with a 4-potential  $A^i$  at each event  $M$  in Minkowski's spacetime.  $M$  is defined by its coordinates  $x^k$ . The 16 partial derivatives  $\partial A^i/\partial x^k$  are the components of the gradient tensor  $D(A^i)$ . The antisymmetric part of  $D(A^i)$  is the usual electromagnetic tensor. The Lagrangian density  $\mathcal{L}$  at  $M$  is proportional to the determinant of  $D(A^i)$ . It allows the calculation of the induction tensor. The antisymmetric part of this tensor contains the usual electromagnetic induction. The symmetric part contains the sources. An application of Euler-Lagrange equations gives the second pair of Maxwell equations. In the last section we use the new expressions for the charge and current densities to deduce the Lorentz force. The theory is very simple and does not need any specific form of the Lagrangian which confers a wide generality to it.

Associated to this subject is the question of the preeminence of potentials and fields which arises in standard textbooks on classical electromagnetism : In ref.[2] , electric and magnetic fields are deduced from a 4-potential. In refs.[3, 4] , the 4-potential is deduced from the fields. In quantum theory, it is the potential which is more fundamental as illustrated by the Aharonov-Bohm effect[5, 6]. Recently, two articles[7, 8] have been published where the authors develop Richard Feynman's idea of introducing potentials before fields[9]. The theory which is presented here brings the proof that the potential is also more fundamental than the fields in classical electromagnetism.

## 2 Electromagnetic tensors.

### 2.1 Electromagnetic potential and its derivatives .

The aim of this section is to describe the notations and to give the relations between the components of the different tensors and the pseudovectors which appear in Maxwell equations.

From the beginning, it is important to stress the fact that we are dealing with quantities, scalars, vectors and tensors having a dimensioned physical reality. These quantities can be represented in direct (or real space), or in inverse (or reciprocal) space. A vector can be represented by its covariant or contravariant components and both quantities are related by the metric tensor. A 2x2 tensor has 4 representations following those of its constitutive vectors.

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One starts with the contravariant components of the electromagnetic 4-potential vector which are usually noted  $A^i$  ( $i = 0, 1, 2, 3$ ) in real space. The scalar potential is  $A^0 = \phi/c$  and the set  $\vec{A} = (A^x, A^y, A^z)$  represents the vector potential. An event  $M$  in real Minkowski's spacetime is defined by its coordinates  $x^k = (ct, x, y, z)$  and to each event corresponds a 4-potential vector :  $A^i = A^i(M)$ . The coordinates are defined in the cartesian frame spanned by the normalized basis vectors  $(\vec{e}_t, \vec{e}_x, \vec{e}_y, \vec{e}_z)$ . All the theory which is described here is *local* : the point  $M$  is surrounded by a volume which is as small as we want. There are quantities, like fields, which are defined at  $M$  and densities which are defined around  $M$ .

In order to obtain the corresponding covariant components  $A_i$  in the reciprocal space, we use the  $(+, -, -, -)$  convention for the metric tensor  $[\eta_{mn}]$  and one has the relation :  $A_i = \eta_{im} A^m$  written with Einstein's summation convention.

The 16 partial derivatives  $a_k^i = \partial A^i / \partial x^k$  are the components of the tensor  $[a_k^i]$  which we write in matrix form :

$$[a_k^i] = \begin{bmatrix} \frac{\partial(\phi/c)}{c\partial t} & \frac{\partial A^x}{c\partial t} & \frac{\partial A^y}{c\partial t} & \frac{\partial A^z}{c\partial t} \\ \frac{\partial(\phi/c)}{\partial x} & \frac{\partial A^x}{\partial x} & \frac{\partial A^y}{\partial x} & \frac{\partial A^z}{\partial x} \\ \frac{\partial(\phi/c)}{\partial y} & \frac{\partial A^x}{\partial y} & \frac{\partial A^y}{\partial y} & \frac{\partial A^z}{\partial y} \\ \frac{\partial(\phi/c)}{\partial z} & \frac{\partial A^x}{\partial z} & \frac{\partial A^y}{\partial z} & \frac{\partial A^z}{\partial z} \end{bmatrix} = \begin{bmatrix} (\phi/c)_{,t} & A^x_{,t} & A^y_{,t} & A^z_{,t} \\ (\phi/c)_{,x} & A^x_{,x} & A^y_{,x} & A^z_{,x} \\ (\phi/c)_{,y} & A^x_{,y} & A^y_{,y} & A^z_{,y} \\ (\phi/c)_{,z} & A^x_{,z} & A^y_{,z} & A^z_{,z} \end{bmatrix} \quad (1)$$

The compressed notation  $\partial A^i / \partial x^k \equiv A^i_{,k}$  is introduced for brevity.  $[a_k^i]$  is the representation of the partial derivatives in direct space. Other representations exist in the inverse space under the form  $[a^k_i]$  or in mixtures of direct and inverse spaces (forms  $[a^{ki}]$  or  $[a_{ki}]$ ). We will essentially use the form  $[a_k^i]$  because Maxwell equations are expressed in real space.  $k$  and  $i$  respectively label the line and the column index for a reason which will appear in eq.(22).

The covariant form  $[a_{ki}]$  will be used as a mathematical tool to split  $[a_k^i]$  into two parts where the first is related to the electromagnetic tensor.

The metric tensor  $[\eta_{mn}]$  is used as a lowering operator to obtain the corresponding covariant tensor :

$$[a_{ki}] = \left[ \sum_m a_k^m \eta_{mi} \right] = \begin{bmatrix} (\phi/c)_{,t} & -A^x_{,t} & -A^y_{,t} & -A^z_{,t} \\ (\phi/c)_{,x} & -A^x_{,x} & -A^y_{,x} & -A^z_{,x} \\ (\phi/c)_{,y} & -A^x_{,y} & -A^y_{,y} & -A^z_{,y} \\ (\phi/c)_{,z} & -A^x_{,z} & -A^y_{,z} & -A^z_{,z} \end{bmatrix}$$

This tensor is divided into its symmetric and antisymmetric parts :

$$[s_{ki}] = \frac{1}{2} ([a_{ki}] + [a_{ik}]) \quad , \quad (2)$$

$$[f_{ki}] = \frac{1}{2} ([a_{ki}] - [a_{ik}]) \quad . \quad (3)$$

Covariant and contravariant tensors keep their symmetry or antisymmetry property in a coordinate change[13].

This property disappears for a mixed tensor.

The antisymmetric part of  $[a_{ki}]$  is :

$$[f_{ki}] = \frac{1}{2} \begin{bmatrix} 0 & -A^x_{,t} - (\phi/c)_{,x} & -A^y_{,t} - (\phi/c)_{,y} & -A^z_{,t} - (\phi/c)_{,z} \\ (\phi/c)_{,x} + A^x_{,t} & 0 & A^x_{,y} - A^y_{,x} & A^x_{,z} - A^z_{,x} \\ (\phi/c)_{,y} + A^y_{,t} & A^y_{,x} - A^x_{,y} & 0 & A^y_{,z} - A^z_{,y} \\ (\phi/c)_{,z} + A^z_{,t} & A^z_{,x} - A^x_{,z} & A^z_{,y} - A^y_{,z} & 0 \end{bmatrix}$$

The corresponding mixed tensor is :

$$[f_k^i] = \left[ \sum_m f_{km} \eta^{mi} \right] = \frac{1}{2} \begin{bmatrix} 0 & A^x_{,t} + (\phi/c)_{,x} & A^y_{,t} + (\phi/c)_{,y} & A^z_{,t} + (\phi/c)_{,z} \\ (\phi/c)_{,x} + A^x_{,t} & 0 & -A^x_{,y} + A^y_{,x} & -A^x_{,z} + A^z_{,x} \\ (\phi/c)_{,y} + A^y_{,t} & -A^y_{,x} + A^x_{,y} & 0 & -A^y_{,z} + A^z_{,y} \\ (\phi/c)_{,z} + A^z_{,t} & -A^z_{,x} + A^x_{,z} & -A^z_{,y} + A^y_{,z} & 0 \end{bmatrix}$$

The components of the electromagnetic field are defined from the components of  $[f_k^i]$ . Below are the usual equations which condense these definitions :

$$\begin{aligned} \vec{E} &:= -\frac{\partial \vec{A}}{\partial t} - \text{grad} \phi \\ \vec{B} &:= \text{curl} \vec{A} \end{aligned} \quad (4)$$

In the following we will keep in mind that the fields which are used in Maxwell equations are pseudovectors which are defined in the direct space. We use the special notation  $E^x, E^y, E^z, B^x, B^y, B^z$  showing that they are components of a  $[f_k^i]$  tensor. This precaution will be useful later.

The electromagnetic tensor writes :

$$[f_k^i] = \frac{1}{2} \begin{bmatrix} 0 & -E^x/c & -E^y/c & -E^z/c \\ -E^x/c & 0 & B^z & -B^y \\ -E^y/c & -B^z & 0 & B^x \\ -E^z/c & B^y & -B^x & 0 \end{bmatrix} \quad (5)$$

At the risk of being redundant, we should again insist on the fact that the sets  $(E^x, E^y, E^z)$  and  $(B^x, B^y, B^z)$  are the components of pseudovectors originating from the mixed tensor  $[f_k^i]$ . Elements  $f_k^i$  are defined in the real space.

The preceding formulas are not new : they belong to the basic knowledge of electromagnetism. This is not the case for the symmetric part of  $[a_{ki}]$  :

$$[s_{ki}] = \frac{1}{2} \begin{bmatrix} 2(\phi/c)_{,t} & -A_{,t}^x + (\phi/c)_{,x} & -A_{,t}^y + (\phi/c)_{,y} & -A_{,t}^z + (\phi/c)_{,z} \\ (\phi/c)_{,x} - A_{,t}^x & -2A_{,x}^x & -A_{,x}^y - A_{,y}^x & -A_{,x}^z - A_{,z}^x \\ (\phi/c)_{,y} - A_{,t}^y & -A_{,y}^x - A_{,x}^y & -2A_{,y}^y & -A_{,y}^z - A_{,z}^y \\ (\phi/c)_{,z} - A_{,t}^z & -A_{,z}^x - A_{,x}^z & -A_{,z}^y - A_{,y}^z & -2A_{,z}^z \end{bmatrix} \quad (6)$$

We have named this tensor the mass part because its Lagrangian is associated to the local density of mass (or matter) energy. This energy is complementary to the field energy related to  $[f_{ki}]$ .  $[s_{ki}]$  has been ignored in textbooks[3, 2] or in specialized literature. We have shown[10] that it can be used to describe electromagnetic particles in the following way : being symmetric,  $[s_{ki}]$  can be diagonalized and if symmetry properties are invoked, one obtains a Helmholtz equation whose solutions  $A^i(\ell, m, n)$  give a classification of these particles following the value of 3 integers  $(\ell, m, n)$ . When expressions for  $A^i(\ell, m, n)$  and its derivatives are included into the tensors, one obtains[11] a complete description of the particle  $(\ell, m, n)$  at each event  $M$ . A particle is thus represented in spacetime by a vector field  $A^i$  and a tensor field  $[a_k^i]$ .

Two remarks are in order :

- 1- There is no scale associated to the potential : the formalism can thus be applied to the inside of an electron or to a galaxy, and
- 2- The potential is an extensive quantity and thus  $A^i$  can result from the sum of two or more different entities. For instance it can describe an electron embedded in a microwave field, or an electron crossing a slit in a self-interference experiment.

There is absolutely no reason to reduce  $[a_k^i]$  to its antisymmetric part  $[f_k^i]$  and both parts contribute to Maxwell equations. If  $[s_k^i]$  is neglected, which is the case in classical electromagnetism, it is necessary to replace it by phenomenological quantities. These are charges and currents : they are perfectly adapted to our scale but become useless if one wants to understand the femtoscopic world.

### 3 Maxwell Equations.

Maxwell equations write :

$$\begin{aligned} \text{div} \vec{B} &= 0 & \text{curl} \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \text{div} \vec{D} &= \rho & \text{curl} \vec{H} &= \frac{\partial \vec{D}}{\partial t} + \vec{j} \end{aligned} \quad (7)$$

The first couple displays the relation between electric  $\vec{E}$  and magnetic  $\vec{B}$  fields. The second couple links the inductions  $\vec{D}$  and  $\vec{H}$  to the sources  $\rho$  (charge density) and  $\vec{j}$  (current density). The pseudovectors  $\vec{E}, \vec{B}, \vec{D}$  and  $\vec{H}$  are all expressed in the real space (they should be elements of tensors of the form  $t_k^i$ ). Inductions are defined from a Lagrangian  $\mathcal{L}$ . Lagrangians are quantities which are used in optics, in mechanics and more generally, in any physics domain. The Lagrangian of a system **must** be invariant in a coordinate change. The invariants of a  $2 \times 2$  tensor are the coefficients of its characteristic polynomial.

In the following, we will associate a Lagrangian to the determinant of a mixed tensor of the form  $[A_k^i]$ . In matrix notation the formula to transform such a tensor from a first to a second coordinates system is indeed  $[\overline{T}_k^i] = M [T_k^i] M^{-1}$  where  $M$  is the transformation matrix. The determinant is invariant in such

a transformation.

The Lagrangian corresponding to the tensor of derivatives  $\|a_k^i\|$  is proportional to the determinant :

$$\mathcal{L} \propto \|a_k^i\| = \begin{vmatrix} (\phi/c)_{,t} & A_{,t}^x & A_{,t}^y & A_{,t}^z \\ (\phi/c)_{,x} & A_{,x}^x & A_{,x}^y & A_{,x}^z \\ (\phi/c)_{,y} & A_{,y}^x & A_{,y}^y & A_{,y}^z \\ (\phi/c)_{,z} & A_{,z}^x & A_{,z}^y & A_{,z}^z \end{vmatrix}$$

This expression will not be explicitly used in the following. However, it shows that the induction term which is the derivative of  $\mathcal{L}$  with respect to the term  $A_k^i$  is the determinant of the minor relative to  $A_k^i$ . For instance :

$$\frac{\partial \mathcal{L}}{\partial((\phi/c)_{,t})} \propto \begin{vmatrix} A_{,x}^x & A_{,x}^y & A_{,x}^z \\ A_{,y}^x & A_{,y}^y & A_{,y}^z \\ A_{,z}^x & A_{,z}^y & A_{,z}^z \end{vmatrix}$$

The dimension of  $\mathcal{L}$  is that of a density of energy in 4-dimensional space :  $[\mathcal{L}] = \text{M L}^{-2} \text{T}^{-2}$ . The dimension of the induction is that of  $\partial \mathcal{L} / \partial A_k^i$  which is  $\text{Q L}^{-2} \text{T}^{-1}$ .

### 3.1 First couple.

Let us express the first couple of Maxwell equation with the potential derivatives (4) :

$$\text{div} \vec{B} = 0 \quad \overrightarrow{\text{curl}} \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The first equation is an identity because  $\vec{B}$  is a curl and the divergence of a curl is identically zero. The second equation is also an identity. The  $x$  component writes :

$$\partial E^z / \partial y - \partial E^y / \partial z = -\partial B^x / \partial t \quad (8)$$

The identity is verified by replacing  $E^y$ ,  $E^z$  and  $B^x$  by their expressions (4). The first couple of Maxwell equations is nicely expressed by the identity[12] :

$$\frac{\partial f_{k\ell}}{\partial x^m} + \frac{\partial f_{\ell m}}{\partial x^k} + \frac{\partial f_{mk}}{\partial x^\ell} = 0 \quad (9)$$

### 3.2 Second couple.

#### 3.2.1 Inductions.

The second couple of Maxwell's equations links electric and magnetic inductions  $\vec{D}$  and  $\vec{H}$  with sources  $\rho$  and  $\vec{j}$  :

$$\text{div} \vec{D} = \rho \quad \overrightarrow{\text{curl}} \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j} \quad (10)$$

An element  $\mathcal{L}'^k_i$  of the tensor  $[\mathcal{L}'^k_i]$  is obtained from the derivative of the Lagrangian with respect to the element  $a_k^i$ .  $[\mathcal{L}'^k_i]$  is the induction tensor. Its developed form is :

$$[\mathcal{L}'^k_i] = \left[ \frac{\partial \mathcal{L}}{\partial a_k^i} \right] = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,t}} & \frac{\partial \mathcal{L}}{\partial A_{,t}^x} & \frac{\partial \mathcal{L}}{\partial A_{,t}^y} & \frac{\partial \mathcal{L}}{\partial A_{,t}^z} \\ \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,x}} & \frac{\partial \mathcal{L}}{\partial A_{,x}^x} & \frac{\partial \mathcal{L}}{\partial A_{,x}^y} & \frac{\partial \mathcal{L}}{\partial A_{,x}^z} \\ \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,y}} & \frac{\partial \mathcal{L}}{\partial A_{,y}^x} & \frac{\partial \mathcal{L}}{\partial A_{,y}^y} & \frac{\partial \mathcal{L}}{\partial A_{,y}^z} \\ \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,z}} & \frac{\partial \mathcal{L}}{\partial A_{,z}^x} & \frac{\partial \mathcal{L}}{\partial A_{,z}^y} & \frac{\partial \mathcal{L}}{\partial A_{,z}^z} \end{bmatrix} \quad (11)$$

Note that the elements of this tensor are defined in the reciprocal space. An element  $\mathcal{L}'^k_i$  is commonly named the *canonical momentum* corresponding to  $A_k^i$ .

Now the corresponding covariant tensor  $[\mathcal{L}'_{ki}]$  is split into its symmetric and antisymmetric parts which are then transformed into mixed tensors[14]. One obtains the separation of  $[\mathcal{L}'^k_i]$  into two parts :

$[\mathcal{L}'^k_i] = [\mathbb{D}^k_i] + [\mathbb{S}^k_i]$ . The first part is directly linked to the usual induction tensor, it corresponds to the antisymmetric part of  $[\mathcal{L}'_{ki}]$ . The second part corresponds to the symmetric part of  $[\mathcal{L}'_{ki}]$  and will be named the source tensor. Expressions of these tensors are :

$$[\mathbb{D}^k_i] = \frac{1}{2} \begin{bmatrix} 0 & \frac{\partial \mathcal{L}}{\partial A^x_t} + \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,x}} & \frac{\partial \mathcal{L}}{\partial A^y_t} + \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,y}} & \frac{\partial \mathcal{L}}{\partial A^z_t} + \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,z}} \\ \frac{\partial \mathcal{L}}{\partial A^x_t} + \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,x}} & 0 & \frac{\partial \mathcal{L}}{\partial A^y_t} - \frac{\partial \mathcal{L}}{\partial A^x_y} & \frac{\partial \mathcal{L}}{\partial A^z_t} - \frac{\partial \mathcal{L}}{\partial A^x_z} \\ \frac{\partial \mathcal{L}}{\partial A^y_t} + \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,y}} & \frac{\partial \mathcal{L}}{\partial A^x_y} - \frac{\partial \mathcal{L}}{\partial A^y_x} & 0 & \frac{\partial \mathcal{L}}{\partial A^z_t} - \frac{\partial \mathcal{L}}{\partial A^y_z} \\ \frac{\mathcal{L}}{\partial A^z_t} + \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,z}} & \frac{\partial \mathcal{L}}{\partial A^x_z} - \frac{\partial \mathcal{L}}{\partial A^y_z} & \frac{\partial \mathcal{L}}{\partial A^y_z} - \frac{\partial \mathcal{L}}{\partial A^x_z} & 0 \end{bmatrix} \quad (12)$$

and ;

$$[\mathbb{S}^k_i] = \frac{1}{2} \begin{bmatrix} 2 \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,t}} & \frac{\partial \mathcal{L}}{\partial A^x_t} - \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,x}} & \frac{\partial \mathcal{L}}{\partial A^y_t} - \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,y}} & \frac{\partial \mathcal{L}}{\partial A^z_t} - \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,z}} \\ -\frac{\partial \mathcal{L}}{\partial A^x_t} + \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,x}} & 2 \frac{\partial \mathcal{L}}{\partial A^x_x} & \frac{\partial \mathcal{L}}{\partial A^y_x} + \frac{\partial \mathcal{L}}{\partial A^x_y} & \frac{\partial \mathcal{L}}{\partial A^z_x} + \frac{\partial \mathcal{L}}{\partial A^x_z} \\ -\frac{\partial \mathcal{L}}{\partial A^y_t} + \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,y}} & \frac{\partial \mathcal{L}}{\partial A^y_x} + \frac{\partial \mathcal{L}}{\partial A^x_y} & 2 \frac{\partial \mathcal{L}}{\partial A^y_y} & \frac{\partial \mathcal{L}}{\partial A^z_y} + \frac{\partial \mathcal{L}}{\partial A^y_z} \\ -\frac{\mathcal{L}}{\partial A^z_t} + \frac{\partial \mathcal{L}}{\partial(\phi/c)_{,z}} & \frac{\partial \mathcal{L}}{\partial A^x_z} + \frac{\partial \mathcal{L}}{\partial A^y_z} & \frac{\partial \mathcal{L}}{\partial A^y_z} + \frac{\partial \mathcal{L}}{\partial A^x_z} & 2 \frac{\partial \mathcal{L}}{\partial A^z_z} \end{bmatrix} \quad (13)$$

Electric and magnetic inductions are given by the derivatives of the Lagrangian with respect to the components of the fields  $\vec{E}/c$  and  $\vec{B}$ . One applies the chain rule and relations (4) to obtain :

$$\begin{aligned} \mathbb{D}_x &= \frac{\partial \mathcal{L}}{\partial(E^x/c)} = -\frac{\partial \mathcal{L}}{\partial(\phi/c)_{,x}} - \frac{\partial \mathcal{L}}{\partial A^x_t} & \mathbb{D}_y &= \frac{\partial \mathcal{L}}{\partial(E^y/c)} = -\frac{\partial \mathcal{L}}{\partial(\phi/c)_{,y}} - \frac{\partial \mathcal{L}}{\partial A^y_t} \\ \mathbb{D}_z &= \frac{\partial \mathcal{L}}{\partial(E^z/c)} = -\frac{\partial \mathcal{L}}{\partial(\phi/c)_{,z}} - \frac{\partial \mathcal{L}}{\partial A^z_t} \\ H_x &= \frac{\partial \mathcal{L}}{\partial B^x} = \left( \frac{\partial \mathcal{L}}{\partial A^z_y} - \frac{\partial \mathcal{L}}{\partial A^y_z} \right) & H_y &= \frac{\partial \mathcal{L}}{\partial B^y} = \left( \frac{\partial \mathcal{L}}{\partial A^x_z} - \frac{\partial \mathcal{L}}{\partial A^z_x} \right) & H_z &= \frac{\partial \mathcal{L}}{\partial B^z} = \left( \frac{\partial \mathcal{L}}{\partial A^x_y} - \frac{\partial \mathcal{L}}{\partial A^y_x} \right) \end{aligned} \quad (14)$$

We have used the special notation  $(\mathbb{D}_x, \mathbb{D}_y, \mathbb{D}_z)$  in order to make the distinction with the usual displacement vector  $(D_x, D_y, D_z)$  which appears in Maxwell equations. These components are defined from the derivatives of  $\mathcal{L}$  with respect to the electric field. The dimension of  $D_x$  is a density of dipoles in 4-space : Q L/(L<sup>4</sup>) while that of  $\mathbb{D}_x$  is Q L<sup>-2</sup> T<sup>-1</sup>. Both quantities are proportional ( $\mathbb{D}_x = cD_x, \dots$ ).

We have also used an lower index notation  $\mathbb{D}_x, \mathbb{D}_y, \dots$  to stress the fact that the components of the induction pseudo-vectors are those of a type  $[\mathbb{D}^k_i]$  tensor :

$$[\mathbb{D}^k_i] = \frac{1}{2} \begin{bmatrix} 0 & -\mathbb{D}_x & -\mathbb{D}_y & -\mathbb{D}_z \\ -\mathbb{D}_x & 0 & H_z & -H_y \\ -\mathbb{D}_y & -H_z & 0 & H_x \\ -\mathbb{D}_z & H_y & -H_x & 0 \end{bmatrix} \quad (15)$$

Again, this tensor is defined in the reciprocal space. The corresponding expression in the real space is :

$$[\mathbb{D}^i_k] = \left[ \sum_{n,m} \eta_{kn} \mathbb{D}^n_m \eta^{mi} \right] = \frac{1}{2} \begin{bmatrix} 0 & \mathbb{D}^x & \mathbb{D}^y & \mathbb{D}^z \\ \mathbb{D}^x & 0 & H^z & -H^y \\ \mathbb{D}^y & -H^z & 0 & H^x \\ \mathbb{D}^z & H^y & -H^x & 0 \end{bmatrix} \quad (16)$$

The relations between the pseudo vectors  $\mathbb{D}^x, \dots, H^x \dots$  defined in the real space and those defined in the reciprocal space are :

$$\mathbb{D}^x = -\mathbb{D}_x \quad , \quad \mathbb{D}^y = -\mathbb{D}_y \quad , \quad \mathbb{D}^z = -\mathbb{D}_z \quad (17)$$

$$H^x = H_x \quad , \quad H^y = H_y \quad , \quad H^z = H_z \quad (18)$$

In the following we will use these notations to write Maxwell's equations in both spaces.

In passing, one should note that the splitting  $[\mathcal{L}'^k_i] = [\mathbb{D}^k_i] + [\mathbb{S}^k_i]$  allows the study of special cases where one of the tensors nullifies in some regions of space while the other still exists. An illustration is the Aharonov-Bohm effect[5] which shows that a potential can exist in a region of space even in the absence of any field ( $f_k^i = 0, D_k^i = 0$ ). One sees that in such a situation it is the source tensor  $[\mathbb{S}^i_k]$  which can change the phase of the electron when it crosses this region.

### 3.2.2 Euler-Lagrange equations.

Among the fundamental principles which fix the dynamics of a physical system[15] are :

1- The principle of least action which leads to Euler-Lagrange equations :

$$\sum_k \frac{\partial}{\partial x^k} \left( \frac{\partial \mathcal{L}}{\partial A_{,k}^i} \right) - \frac{\partial \mathcal{L}}{\partial A^i} = 0 \quad (19)$$

2- The symmetry principle which leads to conservation laws (Noether's theorem) :

$$\frac{\partial}{\partial x^k} \left( \frac{\partial \mathcal{L}}{\partial A_{,k}^i} A_{,\ell}^i \right) = 0 \quad (20)$$

These equations introduce the fundamental quantities  $\partial \mathcal{L} / \partial A_{,k}^i$  which form the complete induction tensor (11). A part of it contains the usual electric and magnetic inductions, the other part will contain the sources.

Now we will use Euler-Lagrange's equations (19) to find the relations between the source terms in Maxwell's equations and the derivatives of the potential. The Lagrangian density  $\mathcal{L}$  does not depend explicitly on the potentials (on its derivatives only) and equation (19) reduces to the first term. It introduces the tensor  $[\mathcal{L}'^k_i]$  whose elements have been written before :

$$\mathcal{L}'^k_i = \frac{\partial \mathcal{L}}{\partial a_{,k}^i} \quad (21)$$

This tensor is defined in the reciprocal space and the summation-derivation operation  $\sum_k \partial / \partial x^k$  corresponds to a contraction over the  $k$  index which is conform to the tensorial dimensionality of (19) which is that of a covariant vector.

When  $\partial \mathcal{L} / \partial A^i = 0$ , equation (19) can be written in matrix form :

$$\left( \frac{\partial}{c \partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) [\mathcal{L}'^k_i] = (0, 0, 0, 0) \quad (22)$$

This equation groups 4 equations, and is the set of the second couple of Maxwell's equations in reciprocal space. Now we use the splitting of  $[\mathcal{L}'^k_i]$  into its two parts and write eq.(22) in compressed notation :

$$(\partial) [\mathcal{L}'^k_i] = (0) \quad \text{or} \quad (\partial) ([\mathbb{D}^k_i]) = -(\partial) ([\mathbb{S}^k_i]) \quad (23)$$

Expressions for  $\mathbb{D}^k_i$  and  $\mathbb{S}^k_i$  are given by eqs. (12) and (13). In the following we skip the factor 1/2 before  $\mathbb{D}^k_i$  and  $\mathbb{S}^k_i$  which simplifies in eq.(23).

The first term  $(\partial) (\mathbb{D}^k_i)$  is computed first :

$$\begin{aligned} (\partial)([\mathbb{D}^k_i]) &= \left( \frac{\partial}{c \partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{bmatrix} 0 & -cD_x & -cD_y & -cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{bmatrix} \\ &= \left( -c \operatorname{div} \vec{D}_t, \left( -\overrightarrow{\operatorname{curl}} \vec{H} - \frac{\partial \vec{D}_t}{\partial t} \right) \right) \end{aligned} \quad (24)$$

$\vec{D}_t = (D_x, D_y, D_z)$  is the symbol for the induction vector in the reciprocal space.

The resulting 4-vector has a time component  $\operatorname{div} \vec{D}_t$  et 3 space components  $(-\overrightarrow{\operatorname{curl}} \vec{H} - \frac{\partial \vec{D}_t}{\partial t})$ . These are the induction components in Maxwell's equations.

The r.h.s. term  $(\partial) ([S^k_i])$  in eq.(23) is computed now :

$$\begin{aligned}
(\partial) [S^k_i] &= (\partial) \begin{bmatrix} 2 \frac{\partial \mathcal{L}}{\partial(\phi/c),t} & \frac{\partial \mathcal{L}}{\partial A^x_t} - \frac{\partial \mathcal{L}}{\partial(\phi/c),x} & \frac{\partial \mathcal{L}}{\partial A^y_t} - \frac{\partial \mathcal{L}}{\partial(\phi/c),y} & \frac{\partial \mathcal{L}}{\partial A^z_t} - \frac{\partial \mathcal{L}}{\partial(\phi/c),z} \\ -\frac{\partial \mathcal{L}}{\partial A^x_t} + \frac{\partial \mathcal{L}}{\partial(\phi/c),x} & 2 \frac{\partial \mathcal{L}}{\partial A^x_x} & \frac{\partial \mathcal{L}}{\partial A^y_x} + \frac{\partial \mathcal{L}}{\partial A^x_y} & \frac{\partial \mathcal{L}}{\partial A^z_x} + \frac{\partial \mathcal{L}}{\partial A^x_z} \\ -\frac{\partial \mathcal{L}}{\partial A^y_t} + \frac{\partial \mathcal{L}}{\partial(\phi/c),y} & \frac{\partial \mathcal{L}}{\partial A^y_x} + \frac{\partial \mathcal{L}}{\partial A^x_y} & 2 \frac{\partial \mathcal{L}}{\partial A^y_y} & \frac{\partial \mathcal{L}}{\partial A^z_y} + \frac{\partial \mathcal{L}}{\partial A^y_z} \\ -\frac{\mathcal{L}}{\partial A^z_t} + \frac{\partial \mathcal{L}}{\partial(\phi/c),z} & \frac{\partial \mathcal{L}}{\partial A^z_x} + \frac{\partial \mathcal{L}}{\partial A^x_z} & \frac{\partial \mathcal{L}}{\partial A^z_y} + \frac{\partial \mathcal{L}}{\partial A^y_z} & 2 \frac{\partial \mathcal{L}}{\partial A^z_z} \end{bmatrix} \\
&= \left( \frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial(\phi/c),t} & -\frac{\partial \mathcal{L}}{\partial(\phi/c),x} & -\frac{\partial \mathcal{L}}{\partial(\phi/c),y} & -\frac{\partial \mathcal{L}}{\partial(\phi/c),z} \\ -\frac{\partial \mathcal{L}}{\partial A^x_t} & \frac{\partial \mathcal{L}}{\partial A^x_x} & \frac{\partial \mathcal{L}}{\partial A^x_y} & \frac{\partial \mathcal{L}}{\partial A^x_z} \\ -\frac{\partial \mathcal{L}}{\partial A^y_t} & \frac{\partial \mathcal{L}}{\partial A^y_x} & \frac{\partial \mathcal{L}}{\partial A^y_y} & \frac{\partial \mathcal{L}}{\partial A^y_z} \\ -\frac{\mathcal{L}}{\partial A^z_t} & \frac{\partial \mathcal{L}}{\partial A^z_x} & \frac{\partial \mathcal{L}}{\partial A^z_y} & \frac{\partial \mathcal{L}}{\partial A^z_z} \end{bmatrix} \quad (25)
\end{aligned}$$

The second expression is obtained after simplification by eq.(22). Equating each component of the 4-vector of eq.(23) gives :

$$\text{div} \vec{D}_t = \frac{\partial}{c\partial t} \frac{\partial \mathcal{L}}{(\phi/c),t} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{A^x_t} - \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{A^y_t} - \frac{\partial}{\partial z} \frac{\partial \mathcal{L}}{A^z_t} \quad (26a)$$

$$\frac{\partial D_x}{\partial t} + [\vec{\text{curl}} \vec{H}]_x = -\frac{\partial}{c\partial t} \frac{\partial \mathcal{L}}{(\phi/c),x} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{A^x_x} + \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{A^y_x} + \frac{\partial}{\partial z} \frac{\partial \mathcal{L}}{A^z_x} \quad (26b)$$

The two remaining equations along the  $y$  and  $z$  axis are obtained from circular permutations of  $x, y, z$  and  $X, Y, Z$ .

We use these equations to introduce the following new 4-vectors in spacetime :

$$\vec{\mathcal{L}}_t = \left( \frac{\partial \mathcal{L}}{(\phi/c),t}, -\frac{\partial \mathcal{L}}{A^x_t}, -\frac{\partial \mathcal{L}}{A^y_t}, -\frac{\partial \mathcal{L}}{A^z_t} \right) \quad (27a)$$

$$\vec{\mathcal{L}}_x = \left( \frac{\partial \mathcal{L}}{(\phi/c),x}, -\frac{\partial \mathcal{L}}{A^x_x}, -\frac{\partial \mathcal{L}}{A^y_x}, -\frac{\partial \mathcal{L}}{A^z_x} \right) \quad (27b)$$

$$\vec{\mathcal{L}}_y = \left( \frac{\partial \mathcal{L}}{(\phi/c),y}, -\frac{\partial \mathcal{L}}{A^x_y}, -\frac{\partial \mathcal{L}}{A^y_y}, -\frac{\partial \mathcal{L}}{A^z_y} \right) \quad (27c)$$

$$\vec{\mathcal{L}}_z = \left( \frac{\partial \mathcal{L}}{(\phi/c),z}, -\frac{\partial \mathcal{L}}{A^x_z}, -\frac{\partial \mathcal{L}}{A^y_z}, -\frac{\partial \mathcal{L}}{A^z_z} \right) \quad (27d)$$

One sees that the r.h.s. of eqs(26a,26b) are all 4-divergences of these vectors :

$$\begin{aligned}
c \text{div} \vec{D}_t &= \text{div} \vec{\mathcal{L}}_t \\
\frac{\partial D_x}{\partial t} + [\vec{\text{curl}} \vec{H}]_x &= -\text{div} \vec{\mathcal{L}}_x
\end{aligned}$$

These divergences define the source terms :

$$\rho := \frac{1}{c} \text{div} \vec{\mathcal{L}}_t \quad (28a)$$

$$j_x := -\text{div} \vec{\mathcal{L}}_x \quad j_y := -\text{div} \vec{\mathcal{L}}_y \quad j_z := -\text{div} \vec{\mathcal{L}}_z \quad (28b)$$

The lower indices  $x, y, z$  label the components of the covector  $\vec{j}_i = (j_x, j_y, j_z)$  in the reciprocal space. These equations can be written in matrix form :

$$\left( \frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{bmatrix} 0 & -cD_x & -cD_y & -cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{bmatrix} = (c\rho, j_x, j_y, j_z) \quad (29)$$

Finally, Maxwell equations in the direct space are obtained after transforming the covariant quadrivector  $(c\rho, j_x, j_y, j_z)$  (with the metric tensor) into its contravariant counterpart  $(c\rho, -j^x, -j^y, -j^z)$  and the pseudovector  $\vec{D}_t$  into  $\vec{D}$ . These operations give the desired result :

$$\begin{aligned}
\text{div} \vec{D} &= \rho \\
[\vec{\text{curl}} \vec{H}]^x &= \frac{\partial D^x}{\partial t} + j^x \quad (x \text{ component})
\end{aligned}$$



Maxwell equations can be written in matrix form :

$$\left( \frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{bmatrix} 0 & cD^x & cD^y & cD^z \\ cD^x & 0 & H^z & -H^y \\ cD^y & -H^z & 0 & H^x \\ cD^z & H^y & -H^x & 0 \end{bmatrix} = (c\rho, -j^x, -j^y, -j^z) \quad (30)$$

One can use the above formulae to verify the continuity equation :

$$\operatorname{div} \vec{j} = -\frac{\partial \rho}{\partial t} \quad (31)$$

This is described in the Annex.

## 4 Lorentz force.

The usual expression for the Lorentz force  $\vec{F}$  is :

$$\vec{F} = q \left( \vec{E} + \vec{v} \wedge \vec{B} \right) \quad (32)$$

Where  $q$  is the elementary charge,  $\vec{E}$  the electric field,  $\vec{B}$  the magnetic field and  $\vec{v}$  the charge velocity. In the following, the corresponding expression for the density of charge will be obtained from the preceding definitions of the source terms in 4-space.

For this purpose we will start with the partial derivative of a Lagrangian  $\mathcal{L} = \mathcal{L}(A^i, A_{,k}^i)$  with respect to a coordinate  $x^\ell$  :

$$0 = \frac{\partial \mathcal{L}}{\partial x^\ell} = \sum_i \frac{\partial \mathcal{L}}{\partial A^i} \frac{\partial A^i}{\partial x^\ell} + \sum_{ik} \frac{\partial \mathcal{L}}{\partial A_{,k}^i} \frac{\partial A_{,k}^i}{\partial x^\ell} \quad (33)$$

We use Euler-Lagrange equation (19) to replace  $\partial \mathcal{L} / \partial A^i$  and the relation :  $\partial A_{,k}^i / \partial x^\ell = \partial A_{,\ell}^i / \partial x^k$  to obtain[16] :

$$0 = \sum_i \left( \sum_k \frac{\partial \mathcal{L}'^k_i}{\partial x^k} \right) A_{,\ell}^i + \sum_{ik} \mathcal{L}'^k_i \frac{\partial A_{,\ell}^i}{\partial x^k} \quad (34)$$

Dimensions of this equation are those of a density of force in 4-space. The null total results from the balance between action (first term) and reaction (second term).

One introduces now the splitting of the two tensors  $\mathcal{L}'^k_i$  and  $A_{,\ell}^i$  into their symmetric and antisymmetric parts :

$$[A_{,\ell}^i] = [f_{,\ell}^i] + [s_{,\ell}^i] \quad \text{et} \quad [\mathcal{L}'^k_i] = [\mathbb{D}^k_i] + [\mathbb{S}^k_i] \quad (35)$$

One obtains 4 terms, each of them describes a force :

$$\begin{aligned} \sum_i \left( \sum_k \frac{\partial \mathcal{L}'^k_i}{\partial x^k} A_{,\ell}^i \right) &= \sum_i \left( \left( \sum_k \frac{\partial \mathbb{D}^k_i}{\partial x^k} \right) f_{,\ell}^i \right) + \sum_i \left( \left( \sum_k \frac{\partial \mathbb{S}^k_i}{\partial x^k} \right) s_{,\ell}^i \right) + \\ &+ \sum_i \left( \left( \sum_k \frac{\partial \mathbb{S}^k_i}{\partial x^k} \right) f_{,\ell}^i \right) + \sum_i \left( \left( \sum_k \frac{\partial \mathbb{D}^k_i}{\partial x^k} \right) s_{,\ell}^i \right) \end{aligned} \quad (36)$$

It is not the purpose of this article to study each term. However one is tempted to associate each of them to one of the four fundamental forces which govern the behavior of elementary particles : The first term gives the Lorentz force, the second originates from the matter (symmetric) tensors and should give the gravitational force and the two other terms could represent the strong and weak forces.

Now we insert expressions (25) and (5) and we make use of the antisymmetry of  $f_{,\ell}^i$  ( $f_{,\ell}^i = -f_{,\ell}^i$ ) to develop

the first term of (36) :

$$\begin{aligned}
& - \sum_i \left( \sum_k \left( \frac{\partial \mathbb{D}_i^k}{\partial x^k} f^i \right) \right) = \\
& - \frac{1}{4} \left( \frac{\partial}{c \partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{bmatrix} 0 & \mathbb{D}^x & \mathbb{D}^y & \mathbb{D}^z \\ \mathbb{D}^x & 0 & H^z & -H^y \\ \mathbb{D}^y & -H^z & 0 & H^x \\ \mathbb{D}^z & H^y & -H^x & 0 \end{bmatrix} \begin{bmatrix} 0 & -E^x/c & -E^y/c & -E^z/c \\ -E^x/c & 0 & -B^z & B^y \\ -E^y/c & B^z & 0 & -B^x \\ -E^z/c & -B^y & B^x & 0 \end{bmatrix} \\
& = -\frac{1}{4} (c\rho, j_x, j_y, j_z) \begin{bmatrix} 0 & -E^x/c & -E^y/c & -E^z/c \\ -E^x/c & 0 & -B^z & B^y \\ -E^y/c & B^z & 0 & -B^x \\ -E^z/c & -B^y & B^x & 0 \end{bmatrix} \equiv (f^t, f^x, f^y, f^z)
\end{aligned} \tag{37}$$

Including the factor 4 inside  $f^i$ , one obtains the final result :

$$\begin{aligned}
f^t &= j_x E^x/c + j_y E^y/c + j_z E^z/c \\
f^x &= \rho E^x + j_y B^z - j_z B^y \\
f^y &= \rho E^y + j_z B^x - j_x B^z \\
f^z &= \rho E^z + j_x B^y - j_y B^x
\end{aligned} \tag{38}$$

These are the expressions of the Lorentz force density. Their structure is the same but they are much more general than eqs.(32) which can be obtained after proper integrations over spacetime in any particular case.

## 5 Conclusion.

We have shown in this article that Maxwell equations and the Lorentz force can be obtained using only the concept of a 4-potential  $A^i$  in spacetime  $x^k$  together with the principles of least action (Euler-Lagrange equations) and of symmetry (Noether's theorem). The theory is conceptually and technically very simple. There are two fundamental tensors :

- The gradient  $D(A^i) = [A_k^i]$  of this potential, built with the 16 partial derivatives  $A_k^i = \partial A^i / \partial x^k$ .
- The induction tensor  $[\mathcal{L}'_i^k]$ , built with the derivatives of the Lagrangian density  $\mathcal{L}$  with respect to these  $A_k^i$ .

A simple manipulation of these tensors shows that fields and electromagnetic inductions are the elements of their antisymmetric part while the sources are deduced from the symmetric part.

These tensors can be used at any scale and the above synthesis of the inductions and the sources extends our preceding study of electromagnetic particles[10]. Here we demonstrated the synthesis of field (described by  $[f_k^i]$ ) and matter (described by  $[s_k^i]$ ), which unifies the concepts of waves and particles. The situation is more complicated here because both tensors  $D(A^i)$  and  $[\mathcal{L}'_i^k]$  interact. This interplay gives rise to several forces one of which is the Lorentz force on which we focussed our attention here. Interpretation of the other forces can shed light on gravitational forces. The theory is *local* and the involved quantities are essentially densities : applications to practical cases have to proceed through integrations over spacetime.

The potential appears to be the fundamental paradigm. It acts by following the fundamental principles governing natural phenomena. This finding is a confirmation of Richard Feynman's intuition that potentials are more fundamental than fields.

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The transformation law of a doubly covariant tensor is :

$$\overline{S_{j\ell}} = \sum_{h=1}^n \sum_{k=1}^n \frac{\partial x^h}{\partial \bar{x}^j} \frac{\partial x^k}{\partial \bar{x}^\ell} S_{hk} \quad (39)$$

where  $S_{hk}$  is the element  $hk$  of the tensor in the first coordinate system at coordinates  $x^h x^k$  and  $\overline{S_{j\ell}}$  is the element  $j\ell$  in the second coordinate system at coordinates  $\bar{x}^j \bar{x}^\ell$ . This formula shows that if  $S_{hk} = S_{kh}$  ,  $\overline{S_{j\ell}} = \overline{S_{\ell j}}$  .

The transformation law of a mixed tensor is :

$$\overline{V_\ell^j} = \sum_{h=1}^n \sum_{k=1}^n \frac{\partial \bar{x}^j}{\partial x^h} \frac{\partial x^k}{\partial \bar{x}^\ell} V_k^h \quad (40)$$

which shows that such a tensor does not keep its symmetry or antisymmetry in a coordinate change.

- [14] The splitting of  $\left[ \mathcal{L}'^k_i \right]$  is done in the following way : : One first writes the covariant tensor  $[\mathcal{L}'_{ki}]$  whose elements are given by the operation  $\mathcal{L}'_{ki} = \sum_m \eta_{km} \mathcal{L}'^m_i$ . This operation corresponds to a change of sign for all elements of  $\left[ \mathcal{L}'^k_i \right]$  but those of the first line. One then divides  $\mathcal{L}'_{ki}$  into its symmetric ( $[S_{ki}]$ ) and antisymmetric parts ( $[D_{ki}]$ ) and uses again the metric tensor to obtain the corresponding mixed tensors.
- [15] The relativity principle is a third fundamental cornerstone of the Physics building.
- [16] The left hand side can be written with the Kronecker symbol  $\delta_\ell^k$  ;  $\partial \mathcal{L} / \partial x^\ell = \delta_\ell^k \partial \mathcal{L} / \partial x^\ell$  to give :

$$\sum_{ik} \frac{\partial}{\partial x^k} \left( \frac{\partial \mathcal{L}}{\partial A_{,k}^i} A_{,i}^k - \delta_\ell^k \mathcal{L} \right) = 0 \quad (41)$$

It is this equation which introduces the elements of the energy-momentum tensor (see ref[2] §32 ).

## 6 Annex

The continuity equation writes :

$$\operatorname{div} \vec{j} = - \frac{\partial \rho}{\partial t} \quad (42)$$

This is written with the definitions(28a) :

$$\frac{\partial}{c\partial t} \operatorname{div} \vec{\mathcal{L}}'_t + \frac{\partial}{\partial x} \operatorname{div} \vec{\mathcal{L}}'_x + \frac{\partial}{\partial y} \operatorname{div} \vec{\mathcal{L}}'_y + \frac{\partial}{\partial z} \operatorname{div} \vec{\mathcal{L}}'_z = 0 \quad (43)$$

In extended form :

$$\begin{aligned} \frac{\partial}{c\partial t} \operatorname{div} \vec{\mathcal{L}}'_t &= \frac{\partial^2}{c^2 \partial t^2} \frac{\partial \mathcal{L}}{(\phi/c)_{,t}} - \frac{\partial^2}{c\partial t \partial x} \frac{\partial \mathcal{L}}{A_{,t}^x} - \frac{\partial^2}{c\partial t \partial y} \frac{\partial \mathcal{L}}{A_{,t}^y} - \frac{\partial^2}{c\partial t \partial z} \frac{\partial \mathcal{L}}{A_{,t}^z} \\ \frac{\partial}{\partial x} \operatorname{div} \vec{\mathcal{L}}'_x &= \frac{\partial^2}{c\partial t \partial x} \frac{\partial \mathcal{L}}{(\phi/c)_{,x}} - \frac{\partial^2}{\partial x^2} \frac{\partial \mathcal{L}}{A_{,x}^x} - \frac{\partial^2}{\partial y \partial x} \frac{\partial \mathcal{L}}{A_{,x}^y} - \frac{\partial^2}{\partial z \partial x} \frac{\partial \mathcal{L}}{A_{,x}^z} \\ \frac{\partial}{\partial y} \operatorname{div} \vec{\mathcal{L}}'_y &= \frac{\partial^2}{c\partial t \partial y} \frac{\partial \mathcal{L}}{(\phi/c)_{,y}} - \frac{\partial^2}{\partial x \partial y} \frac{\partial \mathcal{L}}{A_{,y}^x} - \frac{\partial^2}{\partial y^2} \frac{\partial \mathcal{L}}{A_{,y}^y} - \frac{\partial^2}{\partial z \partial y} \frac{\partial \mathcal{L}}{A_{,y}^z} \\ \frac{\partial}{\partial z} \operatorname{div} \vec{\mathcal{L}}'_z &= \frac{\partial^2}{c\partial t \partial z} \frac{\partial \mathcal{L}}{(\phi/c)_{,z}} - \frac{\partial^2}{\partial x \partial z} \frac{\partial \mathcal{L}}{A_{,z}^x} - \frac{\partial^2}{\partial y \partial z} \frac{\partial \mathcal{L}}{A_{,z}^y} - \frac{\partial^2}{\partial z^2} \frac{\partial \mathcal{L}}{A_{,z}^z} \end{aligned} \quad (44)$$

Using Euler-Lagrange equations :

$$\begin{aligned} \frac{\partial}{c\partial t} \frac{\partial \mathcal{L}}{(\phi/c)_{,t}} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{(\phi/c)_{,x}} + \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{(\phi/c)_{,y}} + \frac{\partial}{\partial z} \frac{\partial \mathcal{L}}{(\phi/c)_{,z}} &= 0 \\ \frac{\partial}{c\partial t} \frac{\partial \mathcal{L}}{A_{,t}^x} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{A_{,x}^x} + \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{A_{,y}^x} + \frac{\partial}{\partial z} \frac{\partial \mathcal{L}}{A_{,z}^x} &= 0 \\ \frac{\partial}{c\partial t} \frac{\partial \mathcal{L}}{A_{,t}^y} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{A_{,x}^y} + \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{A_{,y}^y} + \frac{\partial}{\partial z} \frac{\partial \mathcal{L}}{A_{,z}^y} &= 0 \\ \frac{\partial}{c\partial t} \frac{\partial \mathcal{L}}{A_{,t}^z} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{A_{,x}^z} + \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{A_{,y}^z} + \frac{\partial}{\partial z} \frac{\partial \mathcal{L}}{A_{,z}^z} &= 0 \end{aligned} \quad (45)$$

one obtains the continuity equation.