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# Interpretation of the Five Dimensional Quantum Propagation of a Spinless Massless Particle

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We consider a five dimensional (5D) space-time with a space-like fifth dimension. We implement a quantum formalism by path integrals, and postulate that all the physical information on a 5D massless particle propagation is provided by the statistics over null paths in this 5D space-time. If the 5D metric is independent of the fifth coordinate, then the propagation problem can be reduced to four dimensions by foliation along the fifth coordinate, and we obtain a formulation of 4D Quantum Mechanics. If the 5D metric is independent of time, we foliate along the time coordinate, and obtain a formulation of 4D Statistical Mechanics. If the 5D metric is independent of both time and the fifth coordinate, then Quantum and Statistical Mechanics are pictures of the same 5D reality. We also discuss the foliation of a proper space dimension, the Klein-Gordon equation, and a 5D Special Relativity, completing our interpretation of the 5D geometry.

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## §1. Introduction

In 1921, Kaluza<sup>1)</sup> proposed a five dimensional (5D) geometrical framework for the unification of the gravitational and the electromagnetic interactions. Following the model of the 4D General Relativity, Kaluza discussed both the field equations and the particle propagation along 5D time-like geodesics, trying to establish the role of the fifth dimension for the traditional 4D physics. Kaluza<sup>1)</sup> proposed that the fifth dimension be proportional to electric charge. Several years later, Klein studied quantum aspects of the 5D propagation,<sup>2),3)</sup> and noticed that the assumption of a compact and planckian fifth dimension leads to the quantization of electric charge, while addressing the question of why the fifth dimension is not experimentally observable.<sup>3)</sup> The Kaluza-Klein approach to the field theory has later been devoted an extensive amount of literature,<sup>4),5)</sup> not the same has happened to the theory of the 5D particle propagation, in spite of its fundamental importance. In fact, Kaluza<sup>1)</sup> immediately recognized that his 5D particle propagation theory cannot apply to the electron and other elementary particles.<sup>6)</sup> In 1984, Gegenberg and Kunstatter<sup>7)</sup> were led to the conclusion that, indeed, it is seemingly impossible to naturally describe the propagation of charged light particles (i.e., with rest mass less than the Planck mass) within a classical or quantum 5D Kaluza-Klein framework. In String Theory this problem seems solved by giving up the concept of point particle. Here we ask ourselves whether this problem is deeply rooted in the Kaluza-Klein interpretation of the 5D geometry, and search for alternatives.

In the attempt to construct a *scale-invariant gravity*, Wesson<sup>5),8)</sup> proposed an interpretation of a 5D geometry where the fifth dimension, not necessarily compact, is proportional to mass (i.e.,  $x^5 = Gm/c^2$ , where  $G$  is the Newton constant, and  $c$  is the velocity of light). This idea is known in the literature as *Kaluza-Klein gravity*. Investigating 5D time-like geodesics in this new framework, Wesson was led to predict the existence of an exotic fifth force, yet to be discovered experimentally. Recently, Seahra and Wesson<sup>9)</sup> approached the problem of the anomalous fifth force for null geodesics. Their study is limited to 5D manifolds with metrics conformally conjugated to that of the Kaluza-Klein theory in the absence of electromagnetic fields. [However, they do not require the fifth dimension be compact.] Furthermore, the metric depends on the fifth coordinate *exclusively* through the conformal factor<sup>10)</sup> which is required to depend *only* on the fifth coordinate. Then, writing the 4D *on-shell* constraint  $p^\mu p_\mu = -m^2 c^2$  [ $\mu = 0, 1, 2, 3$ , and the 4D metric is  $diag(-1, 1, 1, 1)$ ] as  $p^A p_A = 0$  ( $A = 0, 1, 2, 3, 5$ ), where  $p^5 = p_5 = mc$ , inspired Seahra and Wesson to pursue and demonstrate the idea that a 5D massless particle propagating along a 5D null geodesic can be seen as a 4D massive particle propagating along a 4D time-like geodesic. Then, they show that for the 5D null geodesic propagation, the anomalous fifth force can be removed by a reparametrization, which in turn introduces ambiguities in defining the 4D proper time of the corresponding 4D time-like geodesic. Thus, it seems that Seahra and Wesson have found an appropriate, anomaly-free 4D interpretation of the 5D null geodesics in the absence of electromagnetic fields (and imposing other constraints mentioned above). However, a close look to their theory, reveals several consistency problems. First, the fact that the fifth dimension

is proportional to mass is inconsistent with the 5D interpretation of the on-shell constraint. Consider the simple case of a 5D massless particle following a null geodesic in a flat, Minkovski-like 5D metric. Assume that the particle has constant 4D mass. By the interpretation of the fifth dimension, the particle is moving in a hyperplane of constant  $x^5$ . By the interpretation of the 4D on-shell constraint, the particle has constant momentum along the fifth dimension, and thus, unless  $p^5 = 0$  (i.e.,  $m = 0$ , or the particle is a 4D photon), the fifth coordinate changes with time. Second, choosing a particular proper time for the 5D null geodesics as seen in four dimensions breaks 5D covariance, and this is inconsistent with the claim of a 5D gravity. While it is remarkable that 5D null geodesics can be regarded as 4D time-like geodesics, it is not clear *why* an observer would use such an interpretation.

In the attempt to find the proper relation between the 4D and the 5D particle propagation, this paper discusses a new interpretation of the 5D space-time geometry. We give a new 4D interpretation to 5D null propagation, and we apply it to the case of weak fields. To address the propagation problem in the presence of electromagnetic fields, in both classical and quantum regimes, we equip the 5D geometry with a quantum principle for 5D spinless and massless particle propagation. The geodesic propagation is then obtained by taking the classical limit.

An important idea for our work is the distinction between *active*, *pasive* and *inertial* mass.<sup>11)</sup> In short, active mass is the source of gravitational field, and passive mass is the object the gravitational field acts upon.<sup>12)</sup> We adopt the Equivalence Principle of General Relativity which identifies the inertial and the passive mass. The equivalence between the passive and the active mass is required by the Action-Reaction Principle of Newtonian Mechanics. In General Relativity, we may consider the active and the passive mass as distinct concepts, whose comparison is not necessarily meaningful.<sup>13)</sup> We also distinguish active and passive electric charge.

Another important concept for our work is that of path integral introduced to physics by Feynman<sup>14)-16)</sup> in 1948. Here we consider path integrals to be coordinate-free quantities. Given two points 1 and 2 in a space-time manifold, a sum over all paths from 1 to 2 depends on the choice of the points 1 and 2, but it is independent of coordinates. However, in the practice of calculating path integrals, global coordinates as a consistent labeling of all points in the space-time manifold are extremely useful. In 4D quantum physics, path integrals are now very powerful tools. Feynman<sup>14)-16)</sup> rewrote the quantum principles of 4D Quantum and Statistical Mechanics in the same language of path integrals. In this paper, we implement a 5D path integral quantum principle to describe the massless and spinless 5D propagation. We then show that 4D Quantum and Statistical Mechanics can emerge as different particular interpretations of the 5D geometry endowed with this Feynman quantum principle. Thus, in five dimensions, we obtain not only the unification of electromagnetic and gravitational interactions, but also a unification of 4D Quantum and 4D Statistical Mechanics. Perhaps, this is not surprising given that 4D mechanics and 4D statistics have been developing now for many decades side by side, borrowing each other ideas and formalism both in the quantum mechanical and in the quantum field theoretical frameworks, and being just a Wick rotation away.

We start with a 5D space-time having a space-like fifth dimension. We do not request the fifth dimension be compact, and, in 5D context, we do not restrict the transformations of coordinates to be cylindrical [i.e.,  $y^\mu = y^\mu(x^\nu)$ ,  $\mu, \nu, \dots = 0, 1, 2, 3$ , and  $y^5 = y^5(x^5)$ ]. Thus, in principle, the fifth dimension is observable both in the field equations and particle propagation. It is the purpose of this paper to discuss how the fifth dimension is revealed to an electrically uncharged 4D observer that perceives geometrically only the first four dimensions, but not the fifth. In four dimensions, the fifth dimension is not manifested as a geometrical entity, but rather through its consequences.<sup>17)</sup>

Consider the situation where a 5D spinless and massless quantum particle, called a *5D photon*, propagates in a 5D curved space-time. We formulate the setup of the propagation problem introducing a path integral quantum principle. *A zero rest mass particle is created at a point 1 of a 5D space-time with fixed metric, and then annihilated at a point 2 in the future cone of 1. We assume that the existence of this particle does not alter the space-time geometry. Maximal information on the particle's quantum propagation is obtained from the number of all null paths between points 1 and 2.* We call this setup *5D Quantum Optics*. We discuss breaking of 5D covariance when a 4D observer measures a 5D photon and assigns it a 4D physical picture for 5D metrics independent of some coordinates. In the case of translational symmetry along the fifth coordinate, our procedure has formal similarities to the traditional Kaluza-Klein dimensional reduction. In contrast, the spontaneous compactification of String Theory is a proper physical phenomenon; a macroscopic observer detects all 11 (or 10) dimensions of the space-time up to quantum effects due to the small diameters of the compact dimensions.

The outline of the paper is as follows. We proceed with presenting the general case of the 5D Quantum Optics (Sec. 2). In Section 3, we introduce the particular situation where the 5D metric is independent of the fifth space-like coordinate. We first shortly discuss the field equations. Then, in Sec. 3.1, we show that the 5D null path integral is equivalent to a 4D path integral over time-like paths with specified length. Thus, we construct a microcanonical ensemble for our 4D Quantum Mechanics from the microcanonical ensemble of the 5D Quantum Optics. In Sec. 3.2 we introduce the corresponding 4D canonical ensemble.<sup>18)</sup> Then, we make the connection between the nonrelativistic limit of our formulation of Quantum Mechanics and Feynman's. We conclude that our Quantum Mechanics is anomaly-free and equivalent to the traditional Feynman formulation. In Sec. 4, we investigate another particular case of the 5D Quantum Optics which we interpret as a formulation of Statistical Mechanics. The state of thermal equilibrium implies that macroscopic observables do not change in time. Presumably, this can be achieved for a 5D metric which is time-independent. In this case, we reformulate the 5D Quantum Optics as a Statistical Mechanics for a single spinless particle which belongs to an ensemble of non-interacting particles. It seems unavoidable that our path integral formulation of Statistical Mechanics needs the Ergodic Principle in order to establish relations with experiment. Also, our Statistical Mechanics describes only ensembles of discernable particles, unless correction factors are introduced 'by hand'. If the 5D metric is

independent of both time and the fifth coordinate, Quantum and Statistical Mechanics make different pictures of the same 5D reality. In Sec. 5, we present how the Klein-Gordon equation appears from the 5D formalism. Section 6 discusses the main aspects of the 5D Special Relativity, and makes further connections with traditional 4D physics. In Sec. 7 we discuss the case where the 5D metric is independent on a proper space dimension (i.e.,  $x^3$ ) and then conclude our work.

## §2. 5D Quantum Optics

We consider a 5D space-time with metric  $h_{AB}$ , ( $A, B, \dots = 0, 1, 2, 3, 5$ ) having the signature  $diag(-1, 1, 1, 1, 1)$ . All transformations of coordinates are allowed, like in the case of 4D gravity. [We do not restrict to considering only cylindrical transformations of coordinates – i.e.,  $y^\mu = y^\mu(x^\nu)$ ,  $\mu, \nu, \dots = 0, 1, 2, 3$ , and  $y^5 = y^5(x^5)$ .]

Consider a 5D massless particle propagation between two measurement events 1 and 2 (with 2 in the future cone of 1) in the 5D space-time. We implement a quantum formalism by path integrals and postulate that all physical information about the particle propagation can be obtained from the number of all null paths between 1 and 2

$$\mathcal{R}(2, 1) = \sum_{\substack{\text{all null paths} \\ \text{between 1 and 2}}} 1 \equiv \int_{\substack{\text{all null paths} \\ \text{between 1 and 2}}} [d^5x].$$

$\mathcal{R}(2, 1)$  assumes equal probability of realization for every 5D null path between 1 and 2, and will be treated like a microcanonical sum.<sup>19)</sup> We notice that  $\mathcal{R}(2, 1)$  is positively defined and conformally invariant. It also satisfies the following selfconsistency relation which results from its geometrical meaning<sup>20)</sup>

$$\mathcal{R}(2, 1) = \int d^5x^{(3)} \sqrt{|h|} \mathcal{R}(2, 3) \mathcal{R}(3, 1).$$

By  $\int d^5x^{(3)} \sqrt{|h|}$ , we denote the volume integration over all points 3 in the future cone of 1 and in the past cone of 2. The points 1 and 2 would be special *observable* events of creation and annihilation of a particle, but in this theory we do not request them to be geometrical elements in space-time for the simplicity of the 5D geometry. If the points 1 and 2 have no geometrical meaning,  $\mathcal{R}(2, 1)$  is not invariant to an arbitrary change of coordinates.

## §3. Quantum Mechanics

We assume that  $h_{AB}$  does not depend on  $x^5$  and foliate the 5D space-time along the fifth coordinate. That is, we see the static 5D space-time as a 4D space-time *evolving* in the fifth coordinate<sup>21)</sup>

$$ds_5^2 = h^{AB} dx_A dx_B = g^{\mu\nu} (dx_\mu + N_\mu dx_5) (dx_\nu + N_\nu dx_5) + N^2 dx_5^2.$$

$N$  is the lapse, and  $N_\mu$  is the shift of the 4D foliation. In matrix form, we have

$$h^{AB} = \begin{pmatrix} g^{\mu\nu} & N^\mu \\ N^\nu & N^\rho N_\rho + N^2 \end{pmatrix} \Leftrightarrow h_{AB} = \begin{pmatrix} g_{\mu\nu} + \frac{N_\mu N_\nu}{N^2} & -\frac{N_\mu}{N^2} \\ -\frac{N_\nu}{N^2} & \frac{1}{N^2} \end{pmatrix}.$$

With the notation

$$N_\mu = -\frac{q}{c^2} A_\mu, \\ \frac{1}{N} = \Phi,$$

where  $q$  is a parameter to be discussed in Sec. 3.2, we get the so-called Kaluza-Klein parameterization of the 5D metric

$$h_{AB} = \begin{pmatrix} g_{\mu\nu} + \frac{q^2}{c^4} \Phi^2 A_\mu A_\nu & \frac{q}{c^2} \Phi^2 A_\mu \\ \frac{q}{c^2} \Phi^2 A_\nu & \Phi^2 \end{pmatrix}. \quad (3.1)$$

We now proceed to discuss the field equations. The case where the 5D space-time manifold is not Ricci flat introduces a 5D energy-momentum tensor which has to be motivated from the experimental point of view. In this paper we restrict to 5D Ricci flat space-time manifolds; i.e., where  $R_{AB}^5 = 0$ . The field equations for the Kaluza-Klein parametrization of the  $x^5$ -independent metric [given by Eq. (3.1)], and some of their exact solutions can be found in Ref.<sup>5)</sup> From our subsequent results, the *physical* 4D metric providing the trajectories of particles is not  $g_{\mu\nu}$ , but  $\tilde{g}_{\mu\nu} = g_{\mu\nu}/\Phi^2$ . Anticipating this, we are interested to see how the field equations look under the aforementioned 4D conformal transformation. Straightforward calculations yield

$$\begin{aligned} \tilde{R}_{\mu\nu}^4 - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R}^4 &= \frac{q^2}{2c^4} \tilde{T}_{\mu\nu}^{EM} + \tilde{T}_{\mu\nu}^\Phi, \\ \tilde{\nabla}^\mu \tilde{F}_{\mu\nu} &= -3 \frac{\tilde{\nabla}^\mu \Phi}{\Phi} \tilde{F}_{\mu\nu}, \\ \frac{1}{\Phi} \tilde{\square} \Phi &= \frac{q^2}{4c^4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{6} \tilde{T}_{\mu\nu}^\Phi \tilde{g}^{\mu\nu}, \end{aligned} \quad (3.2)$$

where  $\tilde{R}_{\mu\nu}^4$  and  $\tilde{\nabla}_\mu$  are, respectively, the Ricci tensor and the covariant derivative of the metric  $\tilde{g}_{\mu\nu}$ ,  $\tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , and

$$\tilde{\nabla}^\mu \equiv \tilde{g}^{\mu\nu} \tilde{\nabla}_\nu, \quad \tilde{\square} \equiv \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu, \quad \tilde{F}^{\mu\nu} \equiv \tilde{F}_{\alpha\beta} \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta},$$

$$\tilde{T}_{\alpha\beta}^{EM} = \tilde{g}_{\beta\mu} \tilde{F}_{\alpha\lambda} \tilde{F}^{\lambda\mu} + \frac{1}{4} \tilde{g}_{\alpha\beta} \tilde{F}_{\mu\lambda} \tilde{F}^{\mu\lambda},$$

$$\tilde{T}_{\mu\nu}^\Phi \equiv \frac{1}{\Phi} \left[ \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Phi - \frac{2}{\Phi} (\tilde{\nabla}_\mu \Phi) (\tilde{\nabla}_\nu \Phi) \right] - \frac{1}{\Phi} \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \left[ \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \Phi - \frac{2}{\Phi} (\tilde{\nabla}_\alpha \Phi) (\tilde{\nabla}_\beta \Phi) \right].$$

Equations (3.2) represent the Einstein and the Maxwell equations in an interpretation of *induced* matter.<sup>22)</sup> The empty 5D space-time seems to contain active matter when it is given 4D physical meaning.  $A_\mu$ , which so far was just an abstract field proportional to the shift of the 4D foliation, becomes now the electromagnetic field *on* a 4D manifold with the metric  $\tilde{g}_{\mu\nu}$ . The terms in the RHS of the Einstein equations are interpreted as induced sources of gravitational field, and the RHS of the Maxwell equations is interpreted as induced 4-current density of *active* electrical charge. Under the 4D conformal transformation, the sources of gravitational field completely separate into electromagnetic sources and  $\Phi$ -sources; this is not the case for the untransformed field equations (see<sup>5)</sup>). The term  $\tilde{T}_{\mu\nu}^\Phi$  depends exclusively on  $\Phi$  and  $g_{\mu\nu}$ , it is independent of  $A_\mu$ , and characterizes the distribution of induced active matter in the 4D foliated manifold.

In traditional 5D Kaluza-Klein theory, the parameter  $q$  would be chosen  $q = 4c^2\sqrt{\pi G}$  such that the above Einstein equations are similar to their 4D formulation. Here instead, we will use the equations of particle propagation (see Sec. 3.2) to interpret the parameter  $q$  as *passive* specific electrical charge.

Equations (3.2) and, consequently, their 4D interpretation are not invariant to a general 5D transformation of coordinates. For them to hold, we restrict to cylindrical transformations (i.e.,  $y^\mu = y^\mu(x^\nu)$ , and  $y^5 = y^5(x^5)^{23}$ ); noncylindrical ones possibly *mix* the gravitational and the electromagnetic fields. However, consider the situation of a 4D observer in a frame where Eqs. (3.2) represent an adequate description of reality. It is not obvious how the 4D observer would perform a certain noncylindrical transformation of coordinates since this transformation essentially belongs to a 5D geometry. For the 4D observer, noncylindrical transformations are not just plain 4D geometrical transformations like those of the General Relativity with the metric  $\tilde{g}_{\mu\nu}$ .

We now give 4D physical interpretation to the sum over 5D null paths introduced in Sec. 2. This interpretation also breaks 5D covariance, and will not hold if all possible transformations of coordinates are applied; we again restrict to cylindrical transformations. Consider the 5D null path element

$$ds_5^2 = h_{AB}dx^A dx^B = g_{\mu\nu}dx^\mu dx^\nu + \Phi^2 \left( dx^5 + \frac{q}{c^2} A_\mu dx^\mu \right)^2 = 0, \quad (3.3)$$

and solve for  $dx^5$

$$dx^5 = \pm \frac{1}{\Phi} \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} - \frac{q}{c^2} A_\rho dx^\rho \equiv \pm \sqrt{-\tilde{g}_{\mu\nu}dx^\mu dx^\nu} - \frac{q}{c^2} A_\rho dx^\rho. \quad (3.4)$$

We note that if  $dx^5 = 0$  and  $q = 0$  ( $q$  has meaning of specific charge; see Sec. 3.2), Eq. (3.4) yields 4D null paths describing 4D photons.

If  $\tilde{g}_{\mu\nu}$  and  $A_\mu$  are independent of  $x^5$ , integrating (3.4) between two events 1 and 2 of the 5D space-time yields

$$\int_1^2 dx^5 = \Delta x^5 = \int_1^2 \left( \pm \sqrt{-\tilde{g}_{\mu\nu}dx^\mu dx^\nu} - \frac{q}{c^2} A_\rho dx^\rho \right) \equiv D_\pm^{21}. \quad (3.5)$$

Notice that 5D covariance is lost in Eq. (3.5). The RHS of (3.5) belongs exclusively to a 4D space-time manifold for which we define the infinitesimal time-like distance

as

$$ds_{4\pm} = \pm \sqrt{-\tilde{g}_{\mu\nu} dx^\mu dx^\nu} - \frac{q}{c^2} A_\rho dx^\rho.$$

The LHS of (3.5) is a constant that depends only on the initial and final points of the particle propagation. Therefore, counting null paths between events 1 and 2 in the 5D manifold is equivalent in this case with counting 4D time-like paths of length  $D_\pm^{21} = \Delta x^5$  between the 4D projections  $1^4$  and  $2^4$  of the 5D events 1 and 2, respectively.

### 3.1. Microcanonical Ensemble of Quantum Mechanics

The 5D null path integral  $\mathcal{R}(2, 1)$  is equivalent to the following 4D path integral over time-like paths with length  $D = \Delta x^5$

$$R_\pm(D, 2^4, 1^4) = \sum_{\substack{\text{all time-like paths of} \\ \text{length } D \text{ between } 1^4 \text{ and } 2^4}} 1 \equiv \int_{D_\pm^{21}=D} [d^4x].$$

$R_\pm(D, 2^4, 1^4)$  is positively defined and satisfies a selfconsistency relation resulting from its geometrical meaning

$$R_\pm(D, 2^4, 1^4) = \int d^4x^{(3)} \sqrt{|\tilde{g}|} \int_{-\infty}^{\infty} dD' R_\pm(D', 2^4, 3^4) R_\pm(D - D', 3^4, 1^4). \quad (3.6)$$

Event  $2^4$  must be in the future cone of  $1^4$ . By  $3^4$  we denoted any intermediary 4D event which is in the future cone of  $1^4$  and in the past cone of  $2^4$ . By the notation  $\int d^4x^{(3)}$  we understand integrating over the set of all points  $3^4$  satisfying the aforementioned condition. Consistency with Eq. (3.5) demands that the distance along a path between  $1^4$  and  $2^4$  in a future cone is minus the distance along the same path between  $2^4$  and  $1^4$  in a past cone

$$D_\pm^{21} = -D_\pm^{12}. \quad (3.7)$$

We call a 4D loop made from one path in future cone and another in past cone a *quantum loop*; see Fig. 1. The microcanonical sum over quantum loops,  $R_\pm(D, 1^4, 1^4)$ , is invariant to gauge transformations of  $A_\mu$  (see Sec. 3.2), and plays a special role in this formalism.

### 3.2. Canonical Ensemble of Quantum Mechanics

Not only that path integrals are hard to calculate, but constrained path integrals must be even harder. We eliminate the constraint in  $R_\pm(D, 2^4, 1^4)$  by a Fourier transform with respect to  $D$

$$K_\pm(\lambda^{-1}, 2^4, 1^4) = \int_{-\infty}^{\infty} dD e^{iD\lambda^{-1}} R_\pm(D, 2^4, 1^4) = \int [d^4x] e^{iD_\pm^{21}\lambda^{-1}}.$$

This Fourier transform is justified by the translation symmetry of the 5D space-time manifold along the fifth coordinate. Writing  $D_\pm^{21}$  explicitly [see Eq. (3.5)] yields

$$K_\pm(\lambda^{-1}, 2^4, 1^4) = \int [d^4x] \exp \left[ i\lambda^{-1} \int_1^2 \left( \pm \sqrt{-\tilde{g}_{\mu\nu} dx^\mu dx^\nu} - \frac{q}{c^2} A_\rho dx^\rho \right) \right]. \quad (3.8)$$

$K_{\pm}$  provides a complete quantum description of a 5D particle propagation, as seen in four dimensions. Performing a Fourier transformation in both hands of (3.6) yields a selfconsistency relation for  $K_{\pm}$

$$K_{\pm}(\lambda^{-1}, 2^4, 1^4) = \int d^4x^{(3)} \sqrt{|\tilde{g}|} K_{\pm}(\lambda^{-1}, 2^4, 3^4) K_{\pm}(\lambda^{-1}, 3^4, 1^4).$$

We now proceed to compare the nonrelativistic limit of this path integral formulation of Quantum Mechanics to that proposed by Feynman.<sup>14),15)</sup> For doing so, we assume that our 5D manifold has the topology of the flat space-time, and that  $x^A$  are pseudocartesian coordinates (i.e., all coordinates  $x^A$  take values on the whole real axis). Also, we consider that the metric  $\tilde{g}_{\mu\nu}$  describes a weak gravitational field ( $i, j = 1, 2, 3$ )

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -1 - \frac{2V}{mc^2} & 0 \\ 0 & \delta_{ij} \end{pmatrix},$$

where  $|2V/(mc^2)| \ll 1$ ,  $V$  corresponding to the Newtonian gravitational potential ( $V \propto m$ ). We expand the nonrelativistic limit of  $D_- \lambda^{-1}$  in  $2V/(mc^2)$ , and keep only the first order

$$D_-^2 \lambda^{-1} \approx -\lambda^{-1} c(t_2 - t_1) + \lambda^{-1} c \int_1^2 dt \left[ \frac{1}{2c^2} \left( \frac{d\vec{x}}{dt} \right)^2 - \frac{q}{c^3} \vec{A} \frac{d\vec{x}}{dt} - \frac{q}{c^2} A_0 - \frac{V}{mc^2} \right], \quad (3.9)$$

where  $ct \equiv x^0$ . If we choose  $\lambda$  to be the Compton wavelength of a particle with (passive) mass  $m$  (i.e.,  $\lambda^{-1} = mc/\hbar$ ), we obtain that the RHS of (3.9) is  $[-mc^2(t_2 - t_1) + S(m, mq)]/\hbar$ , where by  $S(m, mq)$  we denote the traditional nonrelativistic mechanical action of a particle with mass  $m$  and electrical charge  $mq$ . This identification gives physical meaning to our abstract formalism. Thus,  $\lambda^{-1}$  is essentially the passive rest mass of a 4D particle,<sup>24)</sup> and  $q$  is the particle's passive specific electrical charge.  $D_-$  (i.e.,  $\Delta x^5$ ) is essentially the particle's mechanical action, and conjugated by Fourier transform to  $\lambda^{-1}$ . Similar calculations yield  $D_+ \lambda^{-1} \approx [mc^2(t_2 - t_1) - S(m, -mq)]/\hbar$ . Thus, if we say that  $D_- \lambda^{-1}$  describes a particle propagating forward in time, then  $D_+ \lambda^{-1}$  describes an antiparticle propagating backwards in time. Note that  $D_+ \lambda^{-1}$  can also be written as  $D_+ \lambda^{-1} \approx [-(-m)c^2(t_2 - t_1) + S(-m, mq)]/\hbar$ , and interpreted as describing a particle with mass  $-m$  and charge  $mq$  propagating forward in time.

We now recognize that the nonrelativistic limit of  $K_{\pm}$  is proportional to a Feynman path integral. We can show, following the same steps as Feynman,<sup>14),15)</sup> that the nonrelativistic limit of  $K_{\pm}$  is the propagator of a Schrödinger equation

$$\pm \frac{\hbar}{i} \frac{\partial K_{\pm}}{\partial t} = \frac{1}{2m} \left[ \frac{\hbar}{i} \nabla - \frac{mq}{c} \vec{A} \right]^2 K_{\pm} \mp (mq) A_0 K_{\pm} + V K_{\pm} + mc^2 K_{\pm}.$$

We emphasize that this Schrödinger equation results from the metric  $\tilde{g}_{\mu\nu}$ , and not from the original 4D metric  $g_{\mu\nu}$ . The fields observable by the propagation of a 4D

quantum particle are only  $\tilde{g}_{\mu\nu}$  and  $A_\mu$ .  $\Phi$  is a scalar field unobservable in Quantum and Classical Mechanics.

We now resume the analysis of the relativistic theory. Consider a gauge transformation of  $A_\rho$  which must not change the observables of the canonical ensemble

$$A_\rho = A'_\rho + \partial_\rho A.$$

From (3.8), this gauge transformation implies a local  $U(1)$  transformation of  $K_\pm$

$$K_\pm(\lambda^{-1}, 2^4, 1^4) = \exp \left\{ -i\lambda^{-1} \frac{q}{c^2} [A(2^4) - A(1^4)] \right\} \\ \times \int [d^4x] \exp \left[ i\lambda^{-1} \int_1^2 \left( \pm \sqrt{-\tilde{g}_{\mu\nu} dx^\mu dx^\nu} - \frac{q}{c^2} A'_\rho dx^\rho \right) \right].$$

Equivalently, we can write a 5D noncylindrical coordinate transformation such that the transformed 5D metric contains the gauge transformed electromagnetic field  $A'_\rho$  [see Eq. (3.3)]

$$y^\mu = x^\mu, \quad (3.10) \\ y^5 = x^5 - \frac{q}{c^2} A(x^\mu) + (const.).$$

As a result of this noncylindrical transformation, the new mechanical action  $D'^{21}_\pm = \Delta y^5$  relates to the old mechanical action  $D^{21}_\pm = \Delta x^5$  as

$$D^{21}_\pm = D'^{21}_\pm - \frac{q}{c^2} [A(2^4) - A(1^4)].$$

Thus, the actions of the gauge transformation of  $A_\rho$ , of the local  $U(1)$  transformation, and of the noncylindrical change of coordinates (3.10) on  $K_\pm$  are equivalent.

In general,  $D^{21}_\pm$  is not gauge invariant, but can be made gauge invariant if the 5D events 1 and 2 have the same 4D projection (i.e., if  $1^4 = 2^4$ ). In particular,  $D^{21}_\pm$  is gauge invariant if its integration path is a quantum loop. Thus, the canonical sum over quantum loops is gauge invariant, and can be written as<sup>25)</sup>

$$K_\pm(\lambda^{-1}, 1^4, 1^4) = \int d^4x^{(3)} \sqrt{|\tilde{g}|} K_\pm(\lambda^{-1}, 3^4, 1^4) K_\pm(\lambda^{-1}, 1^4, 3^4),$$

or using (3.7),

$$K_\pm(\lambda^{-1}, 1^4, 1^4) = \int d^4x^{(3)} \sqrt{|\tilde{g}|} K_\pm(\lambda^{-1}, 3^4, 1^4) K_\pm^*(\lambda^{-1}, 3^4, 1^4), \quad (3.11)$$

where \* is symbol for complex conjugation. The quantum kernel (or the propagator)  $K_\pm(\lambda^{-1}, 1^4, 1^4)$  from one point of the 4D space-time to itself is normalizable to 1. Equation (3.11) represents a formula for the quantum probability, and stands at the very core of Quantum Mechanics. It can be further generalized for wavefunctions. The generalization is rather straightforward, and we will not present it here. See Feynman<sup>14), 15)</sup> for a discussion on how to construct mathematical formulae holding

for wavefunctions when formulae for the quantum kernel are provided.

We now shortly discuss the classical limit of the path integral formulation in the canonical ensemble. The most substantial contribution to the canonical sum over paths is given by the paths with stationary length. Such paths are the trajectories of Classical Mechanics satisfying Hamilton's Principle

$$\delta D_{\pm} = 0.$$

We consider  $D_{\pm}$  to be the action of a massive particle propagating in a 4D manifold with the metric  $\tilde{g}_{\mu\nu}$ . A suitable choice of parametrization for the geodesics as observed in four dimensions is

$$d\tau^2 = -\tilde{g}_{\mu\nu} dx^{\mu} dx^{\nu}.$$

With this parametrization, the Euler-Lagrange equations turn out to be

$$\dot{x}^{\mu} \tilde{\nabla}_{\mu} \dot{x}^{\tau} \pm \frac{q}{c^2} \tilde{g}^{\tau\sigma} F_{\sigma\rho} \dot{x}^{\rho} = 0, \text{ where } \dot{x}^{\mu} = \frac{dx^{\mu}}{d\tau}, \quad (3.12)$$

which are the expected classical equations of motion of a massive particle with specific charge  $q$ , in gravitational and electromagnetic fields. If we rewrite (3.3) as

$$\left( \dot{x}^5 + \frac{q}{c^2} A_{\mu} \dot{x}^{\mu} \right)^2 = 1, \quad (3.13)$$

then (3.12) and (3.13) represent the 5D null geodesic equations of the metric  $h_{AB}$  written in a parametrization that breaks 5D covariance. A 5D massless particle propagation corresponds to a 4D massive particle propagation.

The 5D manifold has a Killing field  $\xi^A = (0, 0, 0, 0, 1)$  due to the invariance of the metric  $h_{AB}/\Phi^2$  to translations along  $x^5$ . Thus, there is a conserved quantity along a 5D null geodesic

$$\bar{m} \equiv \frac{h_{AB}}{\Phi^2} \xi^A \frac{dx^B}{d\sigma} = \frac{dx_5}{d\sigma},$$

with  $\sigma$  being the affine parameter of the 5D null geodesic of the metric  $h_{AB}/\Phi^2$ .  $\bar{m}$  can be thought as the fifth component of the 5D photon's 5-momentum. We associate the existence of  $\bar{m}$  to the fact that the passive mass is a constant of motion in 4D Classical Mechanics.

#### §4. Statistical Mechanics

We now assume that the metric  $h_{AB}$  is independent of  $x^0$ . Following the same steps of the foliation along  $x^5$ , we parameterize the 5D metric similar to the Kaluza-Klein form

$$h_{AB} = \begin{pmatrix} -\phi^2 & -\frac{Q}{c^2} \phi^2 a_{\nu} \\ -\frac{Q}{c^2} \phi^2 a_{\mu} & G_{\mu\nu} - \frac{Q^2}{c^4} \phi^2 a_{\mu} a_{\nu} \end{pmatrix}, \quad (4.1)$$

where now  $\mu, \nu, \dots = 1, 2, 3, 5$ . We can get the field equations for the case of a 5D Ricci flat manifold in this parameterization of the 5D metric by the formal substitution  $A_\mu \rightarrow a_\mu$ ,  $\tilde{G}_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu}$  and  $\Phi^2 \rightarrow -\phi^2$  in Eqs. (3-2). We note that  $a_\mu$  is similar to the electromagnetic field  $A_\mu$ .

We look for an interpretation of the sum over null paths. Solving the null path element in  $dx^0$  and then integrating yields

$$\Delta x^0 \equiv \int_1^2 dx^0 = \int_1^2 \left( \pm \frac{1}{\phi} \sqrt{G_{\mu\nu} dx^\mu dx^\nu} - \frac{Q}{c^2} a_\rho dx^\rho \right) \equiv d_{\pm}^{21}. \quad (4-2)$$

$\Delta x^0$  plays role of action in Statistical Mechanics. Therefore, in this context,  $x^0$  cannot be interpreted as coordinate time. We think of it as *physical time*, intrinsically related to the propagation phenomenon as mechanical action relates to the particle propagation in Quantum Mechanics. An essential difference from the case of Quantum Mechanics is that every observable phenomenon takes a positive amount of time; i.e.,  $\Delta x^0 > 0$ . Mathematically, we request that  $\Delta x^0$ , as a functional of paths, is bounded from below.<sup>26)</sup> This implies that we have only one choice of distance on the foliated 4D manifold [c.f., Eq. (4-2)]

$$dS_{4+} = \frac{1}{|\phi|} \sqrt{G_{\mu\nu} dx^\mu dx^\nu} - \frac{Q}{c^2} a_\rho dx^\rho = \sqrt{\tilde{G}_{\mu\nu} dx^\mu dx^\nu} - \frac{Q}{c^2} a_\rho dx^\rho.$$

The observable 4D metric in Statistical Mechanics is not  $G_{\mu\nu}$ , but  $\tilde{G}_{\mu\nu} \equiv G_{\mu\nu}/\phi^2$ .  $\phi$  is unobservable in Statistical Mechanics, as  $\Phi$  is unobservable in Quantum Mechanics. Denote the projections of the 5D events 1 and 2 to the 4D Riemannian manifold by  $1^4$  and  $2^4$ , respectively. Similar to the case of Quantum Mechanics, the  $\mathcal{R}(2, 1)$  sum over 5D null paths is equivalent to a sum over paths from  $1^4$  to  $2^4$  in a 4D manifold, having a certain length  $d_{\pm}^{21} = d$ . For the Statistical Mechanics interpretation to hold, we restrict to cylindrical transformations of coordinates determined by the foliation along  $x^0$  [i.e.,  $y^\mu = y^\mu(x^\nu)$ , and  $y^0 = y^0(x^0)$ ]. We formally keep the range of values for the lengths of paths,  $d$ , to be the whole real axis, instead we consider that all functions of  $d$  vanish under the integral for values of the argument lower than a certain threshold  $d_{\min}$ .

#### 4.1. Microcanonical Ensemble of Statistical Mechanics

$\mathcal{R}(2, 1)$  is equivalent to a path integral in a 4D Riemannian manifold with the infinitesimal distance  $dS_{4+}$

$$\rho_+(d, 2^4, 1^4) = \sum_{\substack{\text{all paths of length } d \\ \text{between } 1^4 \text{ and } 2^4}} 1 \equiv \int_{d_+=d} [d^4 x].$$

$\rho_+(d, 2^4, 1^4)$  is positively defined and must satisfy the following selfconsistency relation deriving from its geometrical meaning

$$\rho_+(d, 2^4, 1^4) = \int d^4 x^{(3)} \sqrt{|\tilde{G}|} \int_{-\infty}^{\infty} dd' \rho_+(d', 2^4, 3^4) \rho_+(d - d', 3^4, 1^4). \quad (4-3)$$

Event  $3^4$  is not constrained by causal structure in this case. We define the thermodynamic entropy as

$$S_+(G_{\mu\nu}, d) = k_B \ln \int d^4x \sqrt{|\tilde{G}|} \rho_+(d, 1^4, 1^4),$$

where  $k_B$  is the Boltzmann constant.  $S_+(G_{\mu\nu}, d)$  is 4D covariant, and invariant to gauge transformations of  $a_\mu$ .

#### 4.2. Canonical Ensemble of Statistical Mechanics

We introduce the canonical ensemble quite similarly to the case of Quantum Mechanics. However, since  $d_+$  is bounded from below, a Laplace transform will be more appropriate than a Fourier transform

$$\begin{aligned} k_+(\Lambda^{-1}, 2^4, 1^4) &= \int_{-\infty}^{\infty} dd e^{-d\Lambda^{-1}} \rho_+(d, 2^4, 1^4) \\ &= \int [d^4x] \exp \left[ -\Lambda^{-1} \int_1^2 \left( \sqrt{\tilde{G}_{\mu\nu}} dx^\mu dx^\nu - \frac{Q}{c^2} a_\rho dx^\rho \right) \right]. \end{aligned}$$

We expect that  $\Lambda^{-1}$  denotes a physical concept characterizing thermal equilibrium. We get a selfconsistency relation for  $k_+$  by applying a Laplace transform on both sides of (4.3)

$$k_+(\lambda^{-1}, 2^4, 1^4) = \int d^4x^{(3)} \sqrt{|\tilde{G}|} k_+(\lambda^{-1}, 2^4, 3^4) k_+(\lambda^{-1}, 3^4, 1^4).$$

Similar to the case of Quantum Mechanics, we find a noncylindrical transformation of coordinates which is equivalent to a gauge transformation of  $a_\rho$ . If the points 1 and 2 have the same 4D projection, then  $k_+(\Lambda^{-1}, 1^4, 2^4 = 1^4)$  (i.e., the path integral over 4D loops) is invariant to gauge transformations of  $a_\rho$ . We call a 4D loop a *statistical loop*. By definition, a statistical loop indicates *Equilibrium* Statistical Mechanics. In 5D, the particle is measured at different moments of time 1 and 2 to be at the same 4D position. The definition of the Massieu function for the canonical ensemble is

$$\Psi_+(G_{\mu\nu}, \Lambda^{-1}) = k_B \ln \int d^4x \sqrt{|\tilde{G}|} k_+(\Lambda^{-1}, 1^4, 1^4).$$

$\Psi_+(G_{\mu\nu}, \Lambda^{-1})$  is 4D covariant, and invariant to gauge transformations of  $a_\mu$ .

We now proceed to make the connection with Feynman's formulation of Statistical Mechanics. We first shortly review Feynman's version.<sup>15),16)</sup> Consider a time independent problem of Nonrelativistic Quantum Mechanics for a single spinless particle. If  $E_a$  and  $\Phi_a$  are the eigenvalues and the corresponding eigenvectors of the Hamiltonian operator, then the quantum kernel between two points  $1^4$  and  $2^4$  (with  $2^4$  in the future cone of  $1^4$ ) can be written as

$$K(\lambda^{-1}; \vec{x}_2, t_2; \vec{x}_1, t_1) = \sum_a \Phi_a^*(\vec{x}_2) \Phi_a(\vec{x}_1) e^{iE_a(t_2-t_1)/\hbar}.$$

On the other hand, from the traditional formalism of Statistical Mechanics, the canonical density of states for one particle is

$$\hat{k}(\beta; \vec{x}) = \sum_a \Phi_a^*(\vec{x}) \Phi_a(\vec{x}) e^{-\beta E_a}, \quad (4.4)$$

where  $\beta = 1/(k_B T)$  with  $T$  being Kelvin temperature. The striking similarity of the last two formulae suggested Feynman that they may come from similar formalisms. He wrote  $\hat{k}$  as

$$\hat{k}(\vec{x}_2, u_2; \vec{x}_1, u_1) = \sum_a \Phi_a^*(\vec{x}_2) \Phi_a(\vec{x}_1) e^{-E_a(u_2 - u_1)/\hbar},$$

where  $u_2 - u_1 \equiv \beta\hbar$ ;  $u$  is a new coordinate with dimension of time and physical meaning of temperature to minus one. It was then easy to guess a path integral formulation for Statistical Mechanics<sup>27)</sup>

$$\hat{k}(\vec{x}_2, \beta\hbar; \vec{x}_1, 0) = \int [d^3x] \exp \left\{ -\frac{1}{\hbar} \int_0^{\beta\hbar} du \left[ \frac{m}{2} \left| \frac{d\vec{x}}{du} \right|^2 + v(\vec{x}) \right] \right\},$$

where  $v$  is an interaction potential, and for calculating thermodynamically relevant quantities one must consider  $\vec{x}_1 = \vec{x}_2$  [c.f., Eq. (4.4)].

There is no doubt that this formulation of Statistical Mechanics holds. However, it turns out to be impossible to understand beyond formal level. Two issues arise. First, the Kelvin temperature makes a poor coordinate because of the existence of absolute zero which gives a preferred origin. Second, the particular way the above path integral depends on  $\beta$  makes impossible to express a microcanonical ensemble for Statistical Mechanics in terms of path integrals, as we expect. These are the main reasons why a new formulation of Statistical Mechanics in terms of path integrals may be more appropriate. In the remaining of this section, we develop our new theory in the *nonrelativistic* limit (i.e.,  $|d\vec{x}/dx^5| \ll 1$ ) looking for a Schrödinger-like equation that will facilitate physical interpretation.

Consider  $x^5 = cu$ ; so we build a  $u$  coordinate with dimension of time (with physical meaning of mechanical action in Quantum Mechanics). We define a new mass  $M$ , for which  $\Lambda$  plays role of Compton wavelength

$$\Lambda = \frac{\hbar}{Mc}. \quad (4.5)$$

We use the same 5D metric from Sec. 3.2 where we made the connection with traditional Quantum Mechanics. It corresponds to the situation of weak gravitational field (i.e.,  $\tilde{g}_{ij} = \delta_{ij}$ ,  $\tilde{g}_{0i} = 0$ , and  $\tilde{g}_{00} = -1 - 2V/(mc^2)$ , with  $|2V/(mc^2)| \ll 1$ ). In addition, we assume that the electromagnetic field ( $A_0, A_i$ ) is also weak (i.e.,  $|qA_0/c^2| \ll 1$ , and  $|qA_i/c^2| \ll 1$ ), and  $\Phi^2 \approx 1$ . Identifying the two alternative parametrizations of the 5D metric given by Eqs. (3.1) and (4.1), straightforward calculations in the first order in  $2V/(mc^2)$ ,  $qA_0/c^2$ , and  $qA_i/c^2$  yield

$$\begin{aligned} \phi^2 &\approx 1 + 2V/(mc^2), & a_5 &\approx -qA_0/Q, & a_i &\approx 0, \\ G_{ij} &\approx g_{ij}, & G_{5j} &\approx qA_j/c^2, & G_{55} &\approx 1. \end{aligned} \quad (4.6)$$

With Eqs. (4.6), the first order of the *nonrelativistic* expansion (i.e.,  $|d\vec{x}/du| \ll c$ ) of  $d_+$  becomes

$$d_+ \approx c(u_2 - u_1) + c \int_1^2 du \left[ \frac{1}{2c^2} \left| \frac{d\vec{x}}{du} \right|^2 + \frac{q}{c^3} \frac{d\vec{x}}{du} \vec{A} + \frac{q}{c^2} A_0 - \frac{V}{mc^2} \right]. \quad (4.7)$$

We need an interpretation of the nonrelativistic  $u$  in terms of the relativistic frame just to clarify what we mean by breaking the 4D covariance in the case of Statistical Mechanics. In the case of Quantum Mechanics, the nonrelativistic time comes from the relativistic coordinate time. The coordinate time takes the place of proper time under the assumption that, in the nonrelativistic limit, it makes a good parameter of the time-like path we consider. The nonrelativistic approximation  $|d\vec{x}/dt| \ll c$  may hold everywhere along a quantum event, with the exception of the special point where the path in the future light cone is continued by path in the past light cone. A similar interpretation of the nonrelativistic  $u$  for Statistical Mechanics is not appropriate because the nonrelativistic approximation would not hold everywhere along a statistical event (there are no causal cones in this case). This calls for a different interpretation of the nonrelativistic  $u$ . The nonrelativistic  $u$  comes from the relativistic proper time of the statistical loops. We consider the proper time  $u'$  of a statistical loop to vary along every loop from  $u'_1$  to  $u'_2$  ( $u'_1 < u'_2$ ). By breaking 4D covariance in the Riemannian manifold we understand changing coordinates from  $(\vec{x}, cu)$  to  $(\vec{x}, cu')$ . This transformation opens 4D loops into 3D loops and makes it possible for the nonrelativistic approximation  $|d\vec{x}/du'| \ll c$  to hold everywhere for some statistical loops. We give the nonrelativistic  $u$  the physical meaning of the proper time  $u'$  which for our version of Statistical Mechanics has interpretation of physical time.

The nonrelativistic approximation of the canonical sum over paths  $k_+$  is

$$k_+(\Lambda^{-1}, 2, 1) = e^{-\Lambda^{-1}c(u_2-u_1)} \int [d^3x] \exp \left\{ -\frac{1}{\hbar} \int_1^2 du l_{\text{nr}} \left( \frac{d\vec{x}}{du}, \vec{x}, u \right) \right\},$$

where

$$l_{\text{nr}} \left( \frac{d\vec{x}}{du}, \vec{x}, u \right) = \frac{M}{2} \left| \frac{d\vec{x}}{du} \right|^2 + \frac{Mq}{c} \vec{A} \frac{d\vec{x}}{du} + (Mq)A_0 - \frac{M}{m}V,$$

and  $k_+$  is the propagator of a Schrödinger-like equation

$$-\hbar \frac{\partial k_+}{\partial u} = \frac{-1}{2M} \left[ -\hbar \nabla + \frac{Mq}{c} \vec{A} \right]^2 k_+ + (Mq)A_0 k_+ - \frac{M}{m}V k_+ + Mc^2 k_+. \quad (4.8)$$

In order to make the physical interpretation of  $M$  and  $\Lambda$  transparent, consider (4.8) in the case where  $\vec{A}$ ,  $A_0$  and  $V$  vanish

$$\frac{\partial k_+}{\partial u} = \frac{c\Lambda}{2} \nabla^2 k_+ - \frac{c}{\Lambda} k_+. \quad (4.9)$$

We interpret (4.9) as a Fokker-Planck equation for Brownian motion which is the microscopical substrate of Thermodynamics. Therefore,  $\Lambda = 2D/c$ , where  $D$  is

the diffusion constant of the Brownian motion.<sup>28)</sup> The theory of Brownian motion predicts  $D = 1/(\beta\zeta)$ , where  $\zeta$  is a drag coefficient (i.e., from a friction force  $F_f = -\zeta dx/du$  in the Langevin equation; see for example<sup>29)</sup>). The drag coefficient  $\zeta$  can also be written as  $\zeta = m\gamma$ , where  $\gamma$  is called friction constant, and  $\gamma^{-1}$  is called *friction time*. Combining  $D = 1/(\beta\zeta)$  and  $\Lambda = 2D/c$  yields

$$\Lambda^{-1} = \beta\zeta c/2. \quad (4.10)$$

We know from the traditional Statistical Mechanics that the concept of temperature (i.e.,  $\beta$ ), and not the one of diffusion (i.e.,  $\beta\zeta$ ) characterizes thermal equilibrium. Therefore, for a specific canonical ensemble,  $\zeta$  must be a constant.<sup>30)</sup> With this in mind,  $\Lambda^{-1}$  does characterize thermal equilibrium, as anticipated in the introduction of this section.

From (4.5) and (4.10) we get an explicit expression for the mass  $M$  of a *quantum* particle described by Eq. (4.8)

$$M = \frac{\hbar\gamma}{2k_B T} m. \quad (4.11)$$

We consider (4.8) not only for the canonical path integral but also for *wavefunctions*. For particles described by such wavefunctions, the initial and final positions of propagation always coincide [c.f., Eq. (4.4)]. A measurement of position yields one of the eigenvalues of the position operator. The probability of measuring such an eigenvalue is given by the coefficient of the corresponding eigenvector in the expansion of the wavefunction in the position eigenbasis. (Only wavefunctions with all such coefficients being positive are given physical meaning as corresponding to real particles.) The wavefunction is reduced in the process of measurement such that, after the measurement, it equals the eigenvector corresponding to the measured eigenvalue. Brownian motion can be thought as a set of such measurements in which the particle's wavefunction is reduced in every measurement process.<sup>31)</sup> A *space-time* diagram of Brownian motion is depicted in Fig. 2. Since space-time diagrams do not belong to quantum physics, but rather to its classical limit (where quantum fluctuations tend to zero), this picture corresponds to the limit of small temperatures. The universe line has vertical segments (solid lines) corresponding to the particle propagation interrupted by horizontal segments (dotted lines) corresponding to the perturbations of the measurements. Alternatively, one can consider the space and time picture of the Newtonian Mechanics provided by the Langevin equation, which delivers the same physics in a different interpretation. The temperature-dependent stochastic force of the Langevin picture of the Brownian motion corresponds to the fact that a 4D observer permanently measures the statistical particle.

We now consider Eq. (4.8) in the case where all operators in the RHS are  $u$  independent. The propagator of the differential equation between  $1^4 = (\vec{x}, 0)$  and  $2^4 = (\vec{x}, cu)$  can be written as

$$k_+(A^{-1}, \vec{x}, u) = \sum_a \phi_a^*(\vec{x}) \phi_a(\vec{x}) e^{-e_a u/\hbar}, \quad (4.12)$$

where

$$e_a \phi_a = \frac{-1}{2M} \left[ -\hbar \nabla + \frac{Mq}{c} \vec{A} \right]^2 \phi_a + (Mq) A_0 \phi_a - \frac{M}{m} V \phi_a + M c^2 \phi_a,$$

resembling the setup of the traditional Statistical Mechanics. One essential question remains. We have to find a physical interpretation of  $u$  in terms of 4D experimentally accessible quantities. None of the results of the traditional Statistical Mechanics contains a mysterious  $u$  parameter. The fact that the operators in the RHS of (4.8) are now  $u$ -independent (i.e.,  $x_5$ -independent) implies that the statistical particle has a well-defined passive mechanical mass. How do we measure  $u$  for such a physical system under the conditions that we cannot perceive the fifth coordinate? Our answer is that  $u$  has to be proportional with the momentum along the fifth coordinate  $u \propto m$ .<sup>32)</sup> We can get  $m$  by measuring the mass of the whole thermodynamical ensemble as if it were a *single* mechanical particle and then divide by the number of particles in the ensemble. This is justified by the fact that an ensemble of  $N$  noninteracting particles propagating a distance  $u$  along the fifth coordinate is equivalent, in terms of suitably defined observables, to a single particle propagating a distance  $Nu$  (Ergodic Principle). Since  $u$  has meaning of physical time, playing role of action in Statistical Mechanics, we request  $u$  to be *quantized* for a particle with definite mass (mechanical action is thought to be quantized in Quantum Mechanics). We postulate that for the nonrelativistic limit of our Statistical Mechanics, a *quanta of physical time* for a particle with mass  $m$  is double its friction time<sup>33)</sup>

$$u = 2m/\zeta = 2/\gamma.$$

Consider now the case of a particle propagating a single quanta of physical time. (The case of many quanta propagation can be reduced to this using the Ergodic Principle.) Assuming that  $V$ ,  $\vec{A}$  and  $A_0$  all vanish,  $e_a$  and  $E_a$  scale with  $Mc^2$  and  $mc^2$ , respectively, and we obtain

$$e_a u / \hbar = E_a \beta,$$

which shows equivalence between our new formulation of Statistical Mechanics and Feynman's, for an ensemble of free particles; compare Eqs. (4.4) and (4.12). If  $V$ ,  $\vec{A}$  and  $A_0$  depend only on the elementary charge  $e$ , Planck's constant  $\hbar$ , and the speed of light  $c$ , and no other energy scale than  $Mc^2$  is introduced by additional dimensional constants, we always have that

$$e_a u / \hbar \propto \beta.$$

The most important contribution to the canonical sum over paths in Statistical Mechanics is given by paths with local minimum length. These trajectories can be interpreted in the context of a new 4D mechanics which we call *Zero Kelvin Mechanics*. This formalism would eventually prove to be a useful approximation in

the limit  $\Lambda \rightarrow 0$ . The fundamental equation of this mechanics is

$$\delta d_+ = 0, \quad (4.13)$$

where  $d_+$  is given by Eq. (4.2), and has meaning of physical time. The extremum condition of the propagation time between two points in space is known to physics as the Fermat Principle, which is the founding principle of the Geometrical Optics. We get the geometrical physics of light rays from the generalized Fermat Principle (4.13) if we remember that 4D photons satisfy  $dx^5 = 0$ , and also demand  $Q = 0$  [ $qA_0 = 0$  and  $a_5(x^\mu) \neq 0$  imply  $Q = 0$ ].

Generally, the Lagrangian corresponding to the  $d_+$  functional of paths is

$$l_+(\dot{x}^\mu, x^\mu) = \sqrt{\tilde{G}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} - \frac{Q}{c^2}a_\rho\dot{x}^\rho, \text{ where } \dot{x}^\rho = \frac{dx^\rho}{d\tau}.$$

The Euler-Lagrange equation of the Lagrangian  $l_+$  with the choice of parametrization  $d\tau = \sqrt{\tilde{G}_{\mu\nu}dx^\mu dx^\nu}$  is

$$\dot{x}^\mu \tilde{\nabla}_\mu \dot{x}^\tau + \frac{Q}{c^2} \tilde{G}^{\tau\sigma} f_{\sigma\rho} \dot{x}^\rho = 0, \quad (4.14)$$

where  $\tilde{\nabla}_\mu$  is the covariant derivative of the metric  $\tilde{G}^{\mu\nu}$ , and  $f_{\sigma\rho} = \partial_\sigma a_\rho - \partial_\rho a_\sigma$ . Writing the 5D null path element as

$$\left( \dot{x}^0 + \frac{Q}{c^2} a_\mu \dot{x}^\mu \right)^2 = 1, \quad (4.15)$$

we have Eqs. (4.14) and (4.15) representing the 5D null geodesic equations in a parametrization that breaks 5D covariance. From the fact that the 5D manifold has a Killing field  $\Xi^A = (1, 0, 0, 0, 0)$  due to the invariance of the metric to translations along  $x^0$ , there is a conserved quantity along the 5D null geodesic

$$\bar{M} \equiv \frac{h_{AB}}{\phi^2} \xi^A \frac{dx^B}{d\Sigma} = \frac{dx_0}{d\Sigma},$$

with  $\Sigma$  being the affine parameter of the 5D null geodesic of  $h_{AB}/\phi^2$ . The existence of  $\bar{M}$  as a constant of motion corresponds to the fact that  $M$  (i.e., the Kelvin temperature) is a *constant of motion* in the limit that statistical fluctuations go to zero in the canonical ensemble of Statistical Mechanics.

In contrast to traditional mechanics, this new theory does not have causal cones. Besides the nonrelativistic approximation (i.e.,  $|d\vec{x}/du| \ll c$ ), corresponding to particles close to being at rest, we may also consider a *superrelativistic* approximation (i.e.,  $|d\vec{x}/du| \gg c$ ) for particles in the relativistic domain of the traditional mechanics.<sup>34)</sup>

## §5. The Klein-Gordon Equation

In this section, we investigate the problem of counting 5D null paths for the metric  $h_{AB} = \text{diag}(-1, 1, 1, 1, 1)$ . Firstly, we generalize the sum over 5D null paths

to a sum over 5D paths with a certain length  $\mathcal{D}$

$$\mathcal{R}(\mathcal{D}, 2, 1) = \int_{\mathcal{D}=\text{const.}} [d^5x].$$

Secondly, we perform a Wick rotation of the time coordinate  $x^0 \rightarrow ix^0$ . This eliminates the causal structure of the 5D Lorentzian manifold and transforms it into a 5D Riemannian manifold. Thirdly, we perform a Laplace transform of the path integral  $\mathcal{R}(\mathcal{D}, 2, 1)$  with respect to  $\mathcal{D}$ , which we denote  $\mathcal{K}(\mathcal{L}^{-1}, 2, 1)$

$$\mathcal{K}(\mathcal{L}^{-1}, 2, 1) = \int [d^5x] \exp\left(-\mathcal{L}^{-1} \int_1^2 \sqrt{\delta_{AB} dx^A dx^B}\right).$$

$\mathcal{K}$  satisfies the following selfconsistency relation<sup>35)</sup>

$$\mathcal{K}(\mathcal{L}^{-1}, 2, 1) = \int d^5x^{(3)} \mathcal{K}(\mathcal{L}^{-1}, 2, 3) \mathcal{K}(\mathcal{L}^{-1}, 3, 1). \quad (5.1)$$

We can derive a differential equation for the canonical propagator  $\mathcal{K}(\mathcal{L}^{-1}, 2, 1)$  using (5.1) for two points 2 and 3 close to each other (i.e.,  $2 \equiv x^A$  and  $3 \equiv x^A - \eta^A$ , with  $\delta_{AB} \eta^A \eta^B$  small)

$$\mathcal{K}(\mathcal{L}^{-1}, x^A, 1) = \int d^5\eta \mathcal{K}(\mathcal{L}^{-1}, x^A - \eta^A, 1) \mathcal{A} \exp\left(-\mathcal{L}^{-1} \sqrt{\delta_{AB} \eta^A \eta^B}\right).$$

We expand  $\mathcal{K}(\mathcal{L}^{-1}, x^A - \eta^A, 1)$  in series with respect to  $\eta^A$  up to the second order, and remark that the linear term in  $\eta^A$  vanishes by integration

$$\begin{aligned} \mathcal{K}(\mathcal{L}^{-1}, x^A, 1) &= \mathcal{K}(\mathcal{L}^{-1}, x^A, 1) \left[ \mathcal{A} \int d^5\eta \exp\left(-\mathcal{L}^{-1} \sqrt{\delta_{AB} \eta^A \eta^B}\right) \right] + \\ &+ \delta^{AB} \partial_A \partial_B \mathcal{K}(\mathcal{L}^{-1}, x^A, 1) \left[ \mathcal{A} \int d^5\eta \delta_{AB} \eta^A \eta^B \exp\left(-\mathcal{L}^{-1} \sqrt{\delta_{AB} \eta^A \eta^B}\right) \right]. \end{aligned}$$

We choose the normalization constant  $\mathcal{A}^{-1} = \int d^5\eta \exp\left(-\mathcal{L}^{-1} \sqrt{\delta_{AB} \eta^A \eta^B}\right)$  so that the zeroth order of the expansion in  $\eta^A$  in the RHS cancels exactly with the LHS. An inverse Wick rotation of the resulting partial differential equation for  $\mathcal{K}$  yields

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \mathcal{K} + \partial^5 \partial_5 \mathcal{K} = 0,$$

where  $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ . Finally, a Fourier transformation with respect to  $x^5$  yields the Klein-Gordon equation

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \mathcal{K} - \left(\frac{mc}{\hbar}\right)^2 \mathcal{K} = 0.$$

The Klein-Gordon equation is obtained in the literature from a Lagrangian quadratic in  $dx^\mu/d\tau$ <sup>14), 36), 37)</sup> which does not have a geometrical meaning.

### §6. Local Structure of the 5D Space-Time. 5D Special Relativity

The 5D space-time has the status of a space-time in General Relativity. In the case of Quantum Mechanics, the relation between local 5D causality and local 4D causality is described by Fig. 3. Only events on the surface of the same 5D causal cone can be connected by a classical trajectory. The surface of the local 5D causal cone is isomorphic with a de Sitter space-time with zero cosmological constant. Projected in four dimensions, it becomes the surface and the interior of the 4D local causal cone. A succession of 4D events inside the local 4D cone corresponds to a succession of 5D events on the surface of the local 5D cone with  $\Delta x^5$  being either positive or negative (i.e., positive or negative particle mass). The local geometry of the 5D space-time is Minkovski-like and its isometry group is  $O(4,1)$ . This group is known as the de Sitter group or, for other purposes, as the quantum number group of the hydrogen atom. The irreducible representations of  $O(4,1)$  have been completely classified.<sup>38)</sup>  $O(4,1)$  can be described by  $SO(4,1)$  and two discrete transformations: time reversal  $T$  (i.e.,  $T$  transforms  $x^0 \rightarrow -x^0$ ) and  $CP$ , where  $P$  is parity (i.e.,  $P$  transforms  $\vec{x} \rightarrow -\vec{x}$ ), and  $C$  transforms  $x^5 \rightarrow -x^5$ . Applying  $C$  in the context of Quantum Mechanics corresponds to changing the direction of foliation from  $x^5$  to  $-x^5$ , which is equivalent to changing the sign of the shift of the foliation  $N_\mu = -qA_\mu/c^2$ . Therefore,  $C$  is equivalent to a specific charge conjugation  $q \rightarrow -q$ .  $C$  can also be thought as leaving  $q$  unchanged, instead transforming the mass  $m$  of the quantum particle to  $-m$ .

The foliated physical interpretation breaks the  $O(4,1)$  symmetry into  $O(3,1) \otimes U(1)$  for Quantum Mechanics, and into  $O(4) \otimes D(1)$  for Statistical Mechanics, where  $D(1)$  denotes the dilatation group in one dimension. In contrast with the symmetry breaking in the Quantum Field Theory, these are not proper physical phenomena, instead they are due to how the 4D observer foliates the 5D manifold to four dimensions.

It is both natural and useful to introduce a Special Relativity for the 5D Optics. Since this requests a 5D Minkovski-like metric, there is no gravitational or electromagnetic interaction in this theory. The only type of interaction is collision. There are two types of particles in 4D Special Relativity: photons with null 4-momentum, and massive particles with 4-momentum of magnitude  $mc$ . In this theory, we only have 5D photons (i.e., on-shell particles) with null 5-momentum

$$p^A = (p^\mu, mc).$$

We postulate conservation of 5-momentum in a collision process. An example of a phenomenon that gains understanding from these considerations is particle-antiparticle pair creation  $\gamma \rightarrow X + \bar{X}$ . (We ignore effects related to the charges and the spins of the particles.) A 4D photon  $\gamma$  with wavevector  $\vec{k}$  and energy  $\hbar\omega$  collides with another 4D photon with arbitrary small energy. Conservation of 5-momentum writes

$$(\hbar\omega/c, \hbar\vec{k}, 0) = (E_X/c, \vec{p}_X, m_X c) + (E_{\bar{X}}/c, \vec{p}_{\bar{X}}, m_{\bar{X}} c).$$

Therefore, we get  $m_{\bar{X}} = -m_X$ , and the conservation of 4-momentum. After collision,

the 4D photons are perceived in four dimensions as a particle-antiparticle pair (see Fig. 4). This *rotation* of a 4D photon pair into a particle-antiparticle pair cannot be described in the context of Quantum Mechanics because it implies a noncylindrical transformation of coordinates. In 5D Special Relativity, these transformations contain either a boost along  $x^5$  or a rotation of  $x^5$ .

For the 5D flat metric, both the quantum and the statistical interpretations are possible. For a particle with well defined mass, at thermodynamical equilibrium (i.e., macroscopically at rest), we can write the null 5-momentum in two equivalent forms (up to a constant factor)

$$p^A \propto (E/c, 0, mc), \quad (6.1)$$

$$p^A \propto (Mc, 0, e/c). \quad (6.2)$$

If we believe that Eqs. (6.1) and (6.2) represent different pictures of the same 5D reality, we have

$$\begin{aligned} e &\propto E, \\ M &\propto m. \end{aligned} \quad (6.3)$$

Remark that the local transformation group that leaves this interpretation intact is the group of spatial rotations  $O(3)$ , because of the covariance breaking in Quantum and Statistical Mechanics. Equation (6.3) can be interpreted in the following way. Assume that we have a particle macroscopically at rest, subject to periodic measurements. It is ambiguous whether what we measure are the quantum fluctuations of a particle with passive mass  $m$ , or the Brownian motion of a particle at temperature  $T$ . If many particles of mass  $m$  formed a canonical statistical ensemble, it would have temperature  $T \propto \hbar\zeta/(2mk_B)$  [c.f., Eqs. (4.11) and (6.3)]. However, 5D Special Relativity being a classical limit (in the sense that fluctuations go to zero), this relation holds only as  $m \rightarrow \infty$  and  $T \rightarrow 0$ .

## §7. Further Discussions and Conclusions

Based on the results of Secs. 3 and 6, we are now in the position to discuss the foliation of  $h^{AB}$  along a proper space dimension,  $x^3$ . We assume that  $h^{AB}$  is independent of  $x^3$ . The mathematical structure of the formalism is similar to that of Sec. 3. For a physical interpretation, consider the situation of weak gravitational field (i.e.,  $\tilde{g}_{ij} = \delta_{ij}$ ,  $\tilde{g}_{0i} = 0$ , and  $\tilde{g}_{00} = -1 - 2V/(mc^2)$ , with  $|2V/(mc^2)| \ll 1$ ),  $A_i = 0$ ,  $|qA_0/c^2| \ll 1$ , and  $\Phi^2 \approx 1$ . We restrict to first order calculations in  $2V/(mc^2)$  and  $qA_0/c^2$ . Solving the 5D null path element in  $dx^3$  and then integrating yields

$$\Delta x^3 \equiv \Delta z \approx \pm \int_1^2 \left[ \sqrt{c^2 dt^2 - dx^2 - dy^2 - (dx^5)^2} - \frac{q}{c^2} A_0 dx^5 + \frac{V}{mc^2} c dt \right],$$

where  $ct \equiv x^0$ ,  $x \equiv x^1$ ,  $y \equiv x^2$ , and  $z \equiv x^3$ . We assume that  $(dx^{1,2,5}/dt)^2 \ll c^2$ , and we expand  $\Delta x^3$  in this *nonrelativistic* limit. This corresponds to the ultrarelativistic

limit of the traditional 4D mechanics [i.e.,  $p_{x,y}^2 \ll (E/c)^2$ , and  $m^2c^2 \ll (E/c)^2$ ]. We denote the canonical path integrals of the foliation along  $x^3$  by  $K_{z\pm}$ . They are propagators of Schrödinger-like equations

$$\begin{aligned} \pm \frac{\hbar}{i} \frac{\partial K_{z\pm}}{\partial t} = \frac{c}{2p_z} \left[ \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right)^2 + \left( \frac{\hbar}{i} \frac{\partial}{\partial y} \right)^2 + \left( \frac{\hbar}{i} \frac{\partial}{\partial x^5} + \frac{p_z}{c^2} q A_0 \right)^2 \right] K_{z\pm} \\ + \frac{p_z}{mc} V K_{z\pm} + p_z c K_{z\pm}, \end{aligned} \quad (7.1)$$

where  $p_z$  is the momentum along the z-direction and constant of motion in this ultrarelativistic approximation. Under the assumption that  $V$  and  $A_0$  are independent of  $x^5$ , a Fourier transform of (7.1) with respect to  $x^5$  yields

$$\begin{aligned} \pm \frac{\hbar}{i} \frac{\partial K_{z\pm}}{\partial t} = \frac{c}{2p_z} \left[ \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right)^2 + \left( \frac{\hbar}{i} \frac{\partial}{\partial y} \right)^2 \right] K_{z\pm} + \\ \left[ \frac{c}{2p_z} \left( mc + \frac{p_z}{c^2} q A_0 \right)^2 + \frac{p_z}{mc} V + p_z c \right] K_{z\pm}. \end{aligned} \quad (7.2)$$

We interpret (7.2) as a (2+1)D Schrödinger equation for the motion of an ultrarelativistic quantum particle with  $p_z^2 \gg m^2c^2$  transverse to the z-direction. The fields in the RHS of (7.2) must allow  $p_z$  to be an approximate constant of motion,  $p_z^2 \gg p_{x,y}^2$ , and  $p_z^2 \gg m^2c^2$ , for this approximation of the 5D Quantum Optics formalism to be valid. We believe that Eq. (7.1) could easily be tested with the current accelerator technology.<sup>39)</sup> If  $V$  and  $A_0$  in Eq. (7.1) depend on  $x^5$ , then the passive mass of the quantum particle varies with time. Thus, according to this theory, the mass variation of ultrarelativistic particles (e.g., neutrinos) probes the  $x^5$  dependence of the 5D metric.

In this paper we have investigated a 5D space-time geometry equipped with a quantum principle for 5D spinless and massless particle propagation. We have given physical interpretation to 4D foliations with corresponding symmetries. We have shown that we can regard the foliation of the fifth dimension (which escapes direct perception) as a formulation of Quantum Mechanics, while the foliation of the time dimension can be interpreted in the terms of Statistical Mechanics. The foliation along a proper space-like dimension yields a description of a spinless ultrarelativistic particle. We have derived Schrödinger-like equations for the nonrelativistic approximations of the canonical path integrals corresponding to the different foliations. We have been particularly interested in the case of weak gravitational and electromagnetic fields, but the mathematical theory of path integrals for quadratic lagrangeians in curved manifolds (see for example<sup>37),40)–42)</sup> and references therein) may allow for deriving Schrödinger-like equations in more general cases (exact solutions of the field equations). We have also introduced a 5D Special Relativity completing our interpretation of the 5D geometry. In conclusion, the interpretation of the 5D geometry with respect to 4D physical concepts depends on the particular 4D foliation of the 5D manifold. We emphasize that, in general, these interpretations are mutually exclu-

sive. For a 4D interpretation to hold, one must restrict to cylindrical transformations of coordinates (in accord to the coordinate independence).

However, a general 5D metric is not expected to possess the symmetry that is necessary in order to apply our 4D interpretations of the 5D geometry. Nevertheless, these interpretations may approximately hold as the requested symmetries approximately hold, leading to physical theories with limited validity. In fact, since *vacuum* is an abstraction never to be reached in practice, Quantum Mechanics cannot claim to be a ultimate theory; nor can Statistical Mechanics since *thermal equilibrium* is reached in an infinite amount of time.

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- 3) O. Klein, Nature **118**, 516 (1926).
- 4) For a review on quantum field aspects see M.J. Duff, *Kaluza-Klein Theory in Perspective*, hep-th/9410046, and for a review on 5D gravity see.<sup>5)</sup> In recent years, numerous new results in 5D physics bring progress in understanding the role of an extra dimension for traditional 4D concepts; e.g., C. Romero, Int. J. Mod. Phys. A **17**, 4287 (2002); L.J. Hall and Y. Nomura, Phys. Rev. D **66**, 075004 (2002); P. Halpern, Phys. Rev. D **66**, 027503 (2002); R. Emparan and H.S. Reall, Phys. Rev. Lett. **88**, 101101 (2002); Y.H. Wei and Y.C. Wang, Class. Quantum Grav. **18**, 4701 (2001); D.G.B. Edelen Class. Quantum Grav. **17**, 4715 (2000); E. Verlinde Class. Quantum Grav. **17**, 1277 (2000); K. Kleidis and D.B. Papadopoulos, Class. Quantum Grav. **15**, 2217 (1998); Gen. Rel. Grav. **29**, 275 (1997); A. Banerjee, S. Chatterjee, B. Bhui and A. Das, Class. Quantum Grav. **13**, 153 (1996), and.<sup>8),9)</sup>
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- 6) Kaluza<sup>1)</sup> acknowledges Albert Einstein for making this point.
- 7) J. Gegenberg and G. Kunstatter, Phys. Lett. A **106**, 410 (1984). This paper is primarily devoted to 5D geodesic propagation. See references therein for studies of propagation within a quantum field theoretical framework. See also A. Davidson and D.A. Owen, Phys. Lett. B **177**, 77 (1986) for a discussion of tachyon propagation in the Kaluza-Klein theory. R. Kerner, J. Martin, S. Mignemi and J.-W. van Holten, Phys. Rev. D **63**, 027502 (2000) presents a study of the geodesic deviation in Kaluza-Klein theories without scalar field.
- 8) P.S. Wesson, Astron. Astrophys. **119**, 145 (1983) is one of the early publications where Wesson discusses the idea that the fifth coordinate is proportional to mass.
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- 10) The 5D Kaluza-Klein metric (see e.g., Ref.<sup>5)</sup> or Eq. (3-1) below) is independent on the fifth coordinate.
- 11) M. Jammer, *Concepts of Mass in Classical and Modern Physics* (Harvard University Press 1961); M. Jammer, *Concepts of Mass in Contemporary Physics and Philosophy* (Princeton University Press 2000).
- 12) One may argue that it is possible to resolve the inconsistency between the interpretation of the fifth dimension and that of the 4D on-shell constraint in Wesson's gravity by using the distinction between passive and active mass. Indeed,  $p^5 = mc$  relates to passive mass, while  $x^5 = Gm/c^2$  relates to active mass. However, it is then hardly conceivable how covariance holds when a coordinate determines source terms in the Einstein equations!
- 13) Consider two massive bodies *A* and *B*, with corresponding masses  $m_A$  and  $m_B$ , subject to *mutual* gravitational interaction. In the frame of *A*,  $m_A$  appears as active mass, and

$m_B$  appears as passive mass;  $A$  acts on  $B$  with a gravitational force  $\vec{F}_{AB}$ . In the frame of  $B$ ,  $m_B$  appears as active mass, and  $m_A$  appears as passive mass;  $B$  acts on  $A$  with a gravitational force  $\vec{F}_{BA}$ . In the context of Newtonian Mechanics,  $\vec{F}_{AB}$  and  $\vec{F}_{BA}$  form an action-reaction pair. In the context of General Relativity, a relation between  $\vec{F}_{AB}$  and  $\vec{F}_{BA}$  is not meaningful, since they are forces in different frames. For extensive discussions of the concept of mass see<sup>11)</sup> and references therein.

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- 17) This new approach to dimensional reduction involves an anthropic principle. The idea is that the fifth dimension is not ‘small’, or hidden, or unobservable. However, it is the particular perceptive heritage of the (human) 4D observers that prevents them from directly observing the fifth dimension as a geometrical entity. Nevertheless, the fifth dimension manifests through its indirect consequences; see Secs. 3 through 7. To discuss this in more detail, let us follow the assumption of the traditional physics that the human (3+1)D space-time perception is complete. Consider that you are standing in a piazza and a pigeon is staring at you. The pigeon has sideways eyes and will stare at you with one eye, perceiving you in (2+1)D. To estimate how far it is from you, the pigeon will suddenly move its head and use its short term memory to do a triangulation. Pigeons are (3+1)D blind. If pigeons would publish, their physics would be crucially different than ours (i.e., humans). Nevertheless, I think that their physics would be correct since they are a surviving species. Arguing with humans, they would say that space-time is (2+1)D but there are extra quantities which they need to measure indirectly (i.e., not at a single glance of their (2+1)D geometry). The purpose of this work is not to feel sorry for pigeons, but rather to ask whether we are not ‘pigeons’ ourselves, and to precisely identify the extra quantities that we measure indirectly whose description can be absorbed into a 5D geometrical formalism. Indeed, this question does introduce an anthropic principle. For an incomplete list of references concerning with anthropic principles in physics see B. J. Carr and M. J. Rees, Nature **278**(5705), 605-612 (1979); G. Gale, Sci. Am. **245**(6), 154 (1981); J. Rosen, Am. J. Phys. **53**(4), 335-339 (1985); J. Rosen, Am. J. Phys. **56**(5), 415-419 (1988); and Y. V. Balashov, Am. J. Phys. **59**(12), 1069-1076 (1991) which is a review and a classification of the literature on anthropic principles.
- 18) The nomenclature of microcanonical and canonical ensembles for Quantum Mechanics has been previously used by Brody and Hughston; see D.C. Brody and L.P. Hughston, J. Math. Phys. **39**, 6502 (1998); Proc. R. Soc. Lond. A **454**, 2445 (1998). Brody and Hughston introduced the microcanonical and canonical ensembles for Quantum Mechanics reformulating the traditional theory in the geometrical language of a projective Hilbert space having a Fubini-Study metric. Here we discuss a different implementation of the microcanonical and canonical ensembles which is based on a 5D space-time geometry.
- 19) We can define a *thermodynamic* potential for the 5D manifold

$$\mathcal{S}[h_{AB}] = \int d^5x \sqrt{|h|} \mathcal{R}(1, 1).$$

$\mathcal{S}[h_{AB}]$  is the sum over the null loops in every point of the 5D space-time, and it is a 5D covariant quantity. If we request that all the paths in the sum  $\mathcal{R}(1, 1)$  pass through causally ordered 5D events, then  $\mathcal{R}(1, 1)$  is normalizable to 1, and  $\mathcal{S}[h_{AB}]$  turns out to be proportional to the volume of the 5D manifold.

- 20) The jacobian  $\sqrt{|h|}$  is irrelevant to path counting. It is introduced just formally.
- 21) C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (W.H. Freeman 1973).
- 22) The idea of the induced matter interpretation has been proposed by Kaluza.<sup>1)</sup> See also<sup>5)</sup> and references therein for an updated discussion on how induced matter can emerge from the geometry of an *empty* (i.e., Ricci flat) higher dimensional space-time.
- 23) The class of transformations which preserve Eqs. (3.2) is more general. Some noncylindrical transformations are equivalent to changes of the electromagnetic gauge (see Sec. 3.2). Cylindrical transformations, as proposed by Kaluza,<sup>1)</sup> preserve Eqs. (3.2) and the electromagnetic gauge.

- 24) In the case where  $m = 0$  (i.e.,  $\lambda^{-1} = 0$ ), the canonical path integral can be calculated exactly

$$K_{\pm}(\lambda^{-1}, 2^4, 1^4) = \int [d^4x] e^{iD_{\pm}\lambda^{-1}} = \int [d^4x] = (\text{const.}).$$

We obtain the microcanonical path integral by inverse Fourier transform

$$R_{\pm}(D, 2^4, 1^4) = (\text{const.}) \delta(D),$$

confirming that 4D particles with zero rest mass (i.e., 4D photons) propagate exclusively on 4D null paths.

- 25)  $\int d^4x^{(3)}$  denotes now the integration over *all* events  $3^4$  in the future cone of  $1^4$ .
- 26) The type of extrema of the  $d_{\pm}$  functionals is given by the quadratic form in partial derivatives of the corresponding Lagrangian with respect to the 4D velocity components (Legendre's condition). See for example M. Born and E. Wolf, *Principles of Optics* (Pergamon Press 1961), Appendix I, for how to show that the extremum of  $d_+$  (for  $\phi > 0$ ) is always a weak minimum.
- 27) This is the original formulation of the Feynman path integral for Statistical Mechanics.<sup>15),16)</sup> Whether  $u_1$  is zero or not is irrelevant for the predictions of this theory which depend only on the difference  $u_2 - u_1$ .
- 28) Naturally,  $D$  would be independent of  $u$ , as the mass  $m$  of a quantum particle is independent of time in the Schrödinger equation. (The formal analogy between the mass of a quantum particle and the diffusion constant has already been made by several authors. See for example F.W. Wiegel, *Introduction to Path-Integral Methods in Physics and Polymer Science* (World Scientific Publishing Co. 1986), Chapter V, and,<sup>40)</sup> Sec. 1.3.2.) Remark that in the case of Feynman's formulation of Statistical Mechanics, where  $u = \beta\hbar$ , we would have that  $D$  is temperature independent, which contradicts the theory of the Brownian motion.
- 29) W.T. Coffey, Yu P. Kalmykov and J.T. Waldron, *The Langevin Equation with Applications in Physics, Chemistry and Electrical Engineering*, Contemporary Chemical Physics **11** (World Scientific Publishing Co. 1996).
- 30)  $\zeta$  is similar to the  $\hbar^{-1}$  of the Quantum Mechanics.  $\zeta$  depends on temperature and on the nature of the medium where the propagation takes place.  $A$  carries these complications which this theory is unable to account for. Similarly,  $\lambda$  depends on temperature and the nature of the medium of propagation through the *effective* passive mass. The effective mass accounts for the fact that no particle propagation takes place in absolute vacuum, and thus depends on temperature and on the nature of the medium where the propagation takes place.
- 31) More explicitly, consider a statistical particle with *wavefunction*  $\delta(\vec{x} - \vec{x}_0)$  at  $u = u_0$  in the simplified case described by Eq. (4.9). As *time*  $u$  progresses, the wavefunction spreads in space like a gaussian, delocalizing the particle, until a quantum measurement at time  $u = u_1$  localizes the particle and collapses the wavefunction to  $\delta(\vec{x} - \vec{x}_1)$ .
- 32) This is consistent for 4D photons which have  $u = \Delta x^5/c = 0$ , and also  $m = 0$ .
- 33) In the limit of small drag (i.e.,  $\zeta \rightarrow 0$ ), or in the classical limit of Quantum Mechanics (i.e.,  $\lambda \rightarrow 0$ ), the duration of a single quanta of physical time tends to infinity.
- 34) In the simplified case where we drop the electromagnetic field and consider a flat 4D Riemannian metric, we have

$$l(\dot{\vec{x}}, \vec{x}, u) = Mc\sqrt{\dot{\vec{x}}^2 + c^2} = Mc|\dot{\vec{x}}| + \frac{Mc^3}{2|\dot{\vec{x}}|} + \dots$$

The first term of the expansion is the well-known optical Lagrangian in an isotropic medium. Considering the second term, it can be shown straightforwardly that this new mechanics obeys the law of inertia. Thus, in general, for minimum order approximation of the dynamics we may keep only the first order kinetic and potential terms of the Lagrangian and apply the law of inertia.

- 35) The integration over the intermediate events 3 is not constrained now by causal structure.

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- 39) As an example, consider the case where  $V = 0$  and  $A_0$  describes a perturbing electric field perpendicular to the z-direction, and periodic in time. Using the time-dependent perturbation theory of the Schrödinger equation, one can show that the kinetic energy of the ultrarelativistic particle along the direction of the electric field has a diffusive spread proportional with the square amplitude of the electric field.
- 40) D.C. Khandekar, S.V. Lawande and K.V. Bhagwat, *Path-Integral Methods and their Applications* (World Scientific Publishing Co. 1993).
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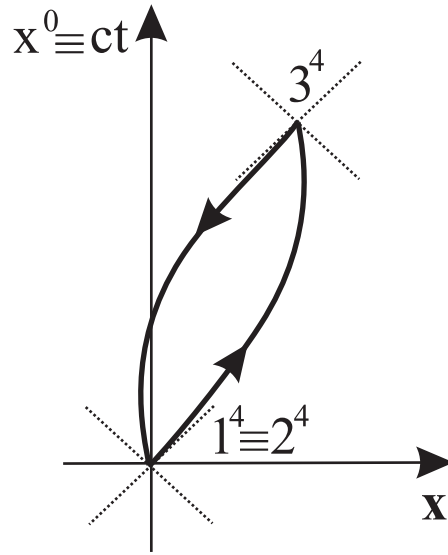


Fig. 1. Closed loop in 4D space-time: *Quantum loop*. The dotted lines represent the local causal cones of  $1^4$  and  $3^4$ .

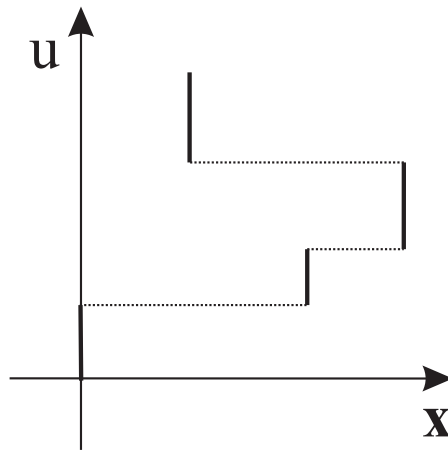


Fig. 2. *Space-time* picture of Brownian motion. The vertical solid lines represent the propagation of the particle, and the horizontal dotted lines represent the perturbations of the measurements. Since space-time diagrams do not belong to Quantum Mechanics but rather to its classical limit (where the fluctuations go to zero), this picture should be thought in the limit of small temperatures.

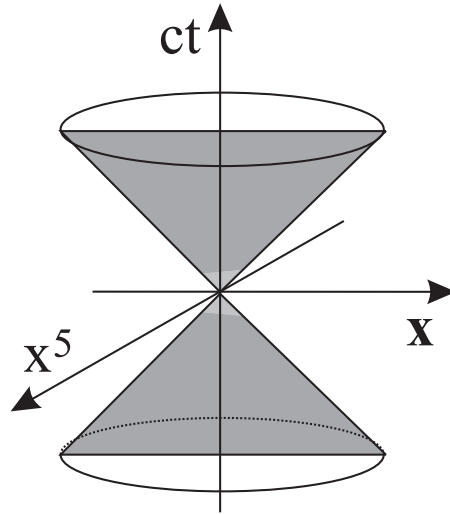


Fig. 3. 5D causal cone. Projecting its surface in four dimensions yields the surface and the interior of the 4D causal cone.

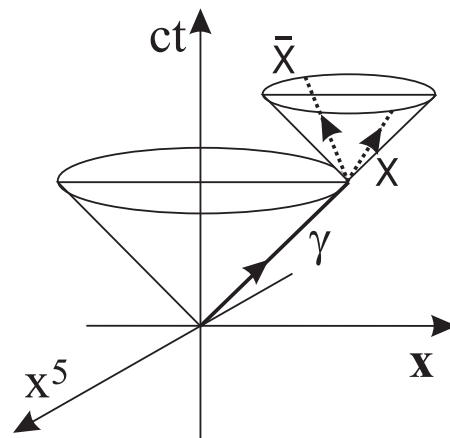


Fig. 4. Creation of particle-antiparticle pair in 5D space-time.