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The "non triviality" of a $\Phi 4 4$ model II The construction of the solution

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The “non triviality” of a Φ_4^4 model **II**

The construction of the solution

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April 23, 2020

Abstract

Under the global title “*The non triviality of a Φ_4^4 model*” we show in the form of three separate articles the existence and uniqueness of a solution to a Φ_4^4 non linear renormalized system of equations of motion in Euclidean space. This system represents a non trivial model which describes the dynamics of the Φ_4^4 Green’s functions in the Axiomatic Quantum Field Theory (AQFT) framework.

The present paper is the second part called “**II** - *The construction of the Φ_4^4 solution*” . It completes the first one (called “**I** *The new mapping \mathcal{M}^* and the Φ_4^4 -iteration*”), by the proofs of the necessary statements given in **I** for the construction of the unique non trivial solution of the model.

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1 Introduction

Under the global title “*The non triviality of a Φ_4^4 model*” we show in the form of two separate articles the existence and uniqueness of a solution to a Φ_4^4 non linear renormalized system of equations of motion in Euclidean space. This system represents a non trivial model which describes the dynamics of the Φ_4^4 Green’s functions in the Axiomatic Quantum Field Theory (AQFT) framework.

The present paper is the complement of “**I**” [20], in the following sense: apart from the theorems 2.1 and 2.2 which are direct consequences of previously published results, it contains the proofs of all propositions and theorems of section 3 and 4 of “**I**”.

In a more qualitative way, we construct this solution by the following five steps ensured in terms of the seven included Appendices:

1. In Appendix 2.1 we justify the non triviality of the subset $\Phi_R \in B_R$ (cf. def. 3.1 of [20]) by choosing as a representative element of it, the so called fundamental sequence H_{T0} , which carries all the characteristic physical and mathematical informations about the subset Φ_R .

Inspired by the Φ_0^4 solution [18], [19] it is defined as starting point of the Φ_4^4 -iteration (cf. def. 3.1 of [20]) and constitutes an approximate solution of the dynamical system of equations in 4-dimensions (cf. theorem 4.1 of [20] of local contractivity that we demonstrate in Appendix 2.7).

2. In Appendix 2.2 we establish the equivalence between the so called new mapping \mathcal{M}^* and the initial mapping- \mathcal{M} defined by the infinite system of dynamic integral equations 1.1 presented in [20]. As we explained in [20] we constructed \mathcal{M}^* in such a way that: a) On one hand, the splitting, signs, bounds and limits of the Green’s functions and of the renormalization parameters are preserved by all the orders of the Φ_4^4 -iteration (cf. theorem 3.2 of stability of [20]) b) On the other hand being contractive in a closed neighbourhood of the fundamental sequence, it provides the Φ_4^4 unique solution as limit of the convergent Φ_4^4 - iteration (cf. Appendix 2.7).
3. In Appendices 2.3, 2.4 2.5 we present the proofs of the signs, bounds and limits at infinity of the “global terms” B^{n+1} A^{n+1} and D_n respectively. These properties constitute the necessary tools for the proof of the stability of Φ_4^4 -iteration inside Φ_R (theorem 3.2 of [20] which is established in Appendix 2.6 via the proofs of propositions 3.5 and 3.6 (of [20])).
4. Finally, in Appendix 2.7, as we noticed before, we show the theorem of the local contractivity of \mathcal{M}^* inside a closed ball with center the fundamental sequence.

For the reader’s convenience the proof of each theorem or proposition of “**I**” is preceded by the precise name of the statement, and the number of Appendix as it

is announced in “**I**”. Moreover the corresponding number of the related definitions or formulas are reminded along the lines of the associated proofs.

2 Tha proofs of [20]

1 The non triviality of Φ_R

APPENDIX 2.1 *Proof of theorem 3.1 of “**I**” [20]*

Theorem 2.1 *The subset Φ_R is a nontrivial subset of \mathcal{B}_R*

We consider the fundamental sequence H_{T0} (cf. definition 2.5 of “**I**” [20]) and verify successively the properties of Φ_R (cf. definition 3.1 (of “**I**” [20])). Precisely:

1.

$$\forall (q, \Lambda) \in (\mathcal{E}_{(q)}^4 \times \mathbb{R}^{+*})$$

$$H_{T0}^2 = (q^2 + m^2)(1 + \delta_{10}(q, \Lambda)\Delta_F)$$

with:

$$\delta_{10}(q, \Lambda)\Delta_F = \frac{-\rho_0 + \Lambda\delta_{3,\min}([N_3] - [N_3]_{(q^2+m^2)=0})\Delta_F}{1 + \rho_0 + \Lambda|a_0|}$$

We verify:

$$\lim_{(q^2+m^2)=0} H_{T0}^2(q, \Lambda)\Delta_F(q) = 1 \quad (\text{or} \quad \lim_{(q^2+m^2)=0} \delta_{10}(q, \Lambda)\Delta_F = 0)$$

and in view of the logarithmic asymptotic behaviour of the Φ_4^4 operation

$$\text{namely: } [N_3]\Delta_F \sim_{q \rightarrow \infty} \log(q^2 + m^2)$$

$$H_{T0}^2(q, \Lambda) \leq (q^2 + m^2)^{(1+\pi^2/18)}$$

and

$$H_{\min}^2(q) \leq H_{T0}^2(q, \Lambda), \quad \text{with } H_{\min}^2(q) = q^2 + m^2$$

(2.1)

(cf. properties (3.53) of [20]).

2. For every $n = 2k + 1, k \geq 1$ and $\forall (q, \Lambda) \in (\mathcal{E}_{(q)}^{4n} \times \mathbb{R}^{+*})$ we verify (recurrently) that the functions

$$H_{T0}^4 = -\delta_{3,\min}(\Lambda) \prod_{l=1,2,3} H_{T0}^2(q_l)\Delta_F(q_l)$$

$$H_{T0}^{n+1}(q, \Lambda) = \frac{\delta_{n,\min}(\Lambda)C_{T0}^{n+1}(q, \Lambda)}{3\Lambda n(n-1)}; \quad (2.2)$$

(where $\{\delta_{n,\min}\}_{n \geq 3}$ is the splitting sequence of definition 2.3, of “**I**”[20]), have the structure of (G.R. Φ .C’s) and belong to the class $\mathcal{A}_{4n}^{(\alpha_n \beta_n)}$ of Weinberg functions with corresponding asymptotic indicatrices given as follows:

$$\forall S \subset \mathcal{E}_{(q)}^{4n}$$

$$\alpha_n(S) = \left\{ \begin{array}{l} -(n-3) \text{ if } S \not\subset \text{Ker } \lambda_n \\ 0 \text{ if } S \subset \text{Ker } \lambda_n \\ \beta_{n(S)} = \nu_{(n)} = 2n \quad \forall S \subset \mathcal{E}_{(q)}^{4n} \end{array} \right\} \quad (2.3)$$

(cf. properties (3.54) of [20]).

3. The properties 3.55, 3.56, 3.57 of “**I**” [20] are automatically satisfied by the tree type definition of H_{T0} in terms of the the splitting sequence $\{\delta_{n,min}\}_{n \geq 3}$ of definition 2.3 of “**I**”:

$$\begin{aligned} \text{with } \delta_{3,min}(\Lambda) \underset{q \rightarrow \infty}{\sim} \Lambda \quad (\delta_{3,min} = \frac{6\Lambda}{1 + 9\Lambda(1 + 6\Lambda^2)}) \\ \lim_{\Lambda \rightarrow 0} \frac{\delta_{3,min}(\Lambda)}{\Lambda} = 6, \quad \text{and} \end{aligned} \quad (2.4)$$

$$\begin{aligned} \delta_{n,min}(q, \Lambda) \underset{q \rightarrow \infty}{\sim} \Lambda \quad \text{and} \quad \lim_{\Lambda \rightarrow 0} \frac{\delta_{n,min}(\Lambda)}{\Lambda} = 3n(n-1) \\ \delta_{n,min}(\Lambda) < \delta_{n,max}(\Lambda) \end{aligned} \quad (2.5)$$

The signs and bounds 3.58 3.59 of [20] are also ensured recurrently by the H_{T0}^{n+1} Green’s functions in view of the splitting-tree structure (cf. definition 2.2).

4. Finally, we trivially obtain that H_{T0} also verifies the property 4 of Φ_R (cf. equations 3.60 , 3.61 and 3.62 of “**I**” [20], of the renormalization constants)

$$\gamma_0 = 1; \quad a_0 = -\delta_{3,min}[N_3]_{(q^2+m^2)=0} \quad (2.6)$$

$$\rho_0 = \Lambda \delta_{3,min} \left[\frac{\partial}{\partial q^2} [N_3]_{(q^2+m^2)=0} \right] \quad \blacksquare \quad (2.7)$$

2 The new mapping \mathcal{M}^*

APPENDIX 2.2 *Proof of proposition 3.1 of [20] (“ \mathcal{M}^* is equivalent to \mathcal{M} ” of [20])*

i) We first notice that as far as the renormalization parameters a , ρ and γ are concerned the corresponding definitions of the mapping \mathcal{M}^* are identical to those appearing precisely in the definition (1.1) eqs 1.6 of \mathcal{M} of [20](modulo the particular properties in Φ_R cf. also Remarks 3.1 of [20]).

ii) Now, we identify the two images of $H^{2'}$ in [20] and write for example: on the left the corresponding definition of the mapping \mathcal{M}^* equations (3.65) and on the right hand side the image by \mathcal{M} equations (1.6):

$$\begin{aligned} (q^2 + m^2)(1 + \delta'_1 \Delta_F) &= -\frac{\Lambda}{(\tilde{\gamma} + \tilde{\rho})} \{ [N_3^{(3)} H^4] - \tilde{a} H^2 \Delta_F \} + \frac{(q^2 + m^2) \tilde{\gamma}}{(\tilde{\gamma} + \tilde{\rho})} \\ \text{or} \\ \delta'_1 \Delta_F &= \frac{-\tilde{\rho} - \Lambda \{ [N_3^{(3)} H^4] - \tilde{a} H^2 \Delta_F \} \Delta_F}{(\tilde{\gamma} + \tilde{\rho})} \quad \blacksquare \end{aligned} \quad (2.8)$$

iii) In an analogous way $\forall n \geq 3$, $(q, \Lambda) \in \mathcal{E}_{(q)}^{4n} \times \mathbb{R}^+$, taking into account \mathcal{M} and by using the “splitting” property in Φ_R : $H^{n+1} = \frac{\delta_n(q, \Lambda)C^{n+1}}{3\Lambda n(n-1)}$, we have:

$$H^{n+1}(q, \Lambda) = \frac{1}{(\tilde{\gamma} + \tilde{\rho})} \{ [A^{n+1} + B^{n+1} + C^{n+1}](q, \Lambda) + \Lambda \tilde{a} H^{n+1}(q, \Lambda) \} \text{ or}$$

$$H^{n+1} = \frac{1}{(\tilde{\gamma} + \tilde{\rho})} [A^{n+1} + B^{n+1} + \Lambda \tilde{a} H^{n+1}] + \frac{H^{n+1} 3\Lambda n(n-1)}{\delta_n(\tilde{\gamma} + \tilde{\rho})}$$

or

$$\delta_n \{ (\tilde{\gamma} + \tilde{\rho}) H^{n+1} - [A^{n+1} + B^{n+1} + \Lambda \tilde{a} H^{n+1}] \} = H^{n+1} 3\Lambda n(n-1)$$

and finally

$$\delta'_n(q, \Lambda) = \frac{3\Lambda n(n-1)}{(\tilde{\gamma} + \tilde{\rho}) - D_n(H) - \Lambda \tilde{a}} \quad \text{with } D_n(H) = \frac{B^{n+1} + A^{n+1}}{H^{n+1}}$$

■

(2.9)

3 The properties of the global terms $B_{(\nu)}^{n+1}$ and $A_{(\nu)}^{n+1}$ and $D_{(\nu)}^{n+1}$

APPENDIX 2.3 Proof of Proposition 3.2 of [20]

Let $H_\nu \in \Phi_R$.

- We start with $n=3$.

By the hypothesis $H_\nu \in \Phi_R$ the properties i) “opposite sign property” (using $H_\nu^4 < 0$) ii) the axiomatic field theory properties, euclidean invariance and iii) asymptotic momentum behaviour at infinity are directly obtained thanks to the hypothesis $H_\nu \in \Phi_R$ (precisely the corresponding properties of $H_\nu^4 < 0$).

In order to obtain iv a) We apply the splitting of H_ν^4 , then the sign and bound $H_{(\nu, min)}^4$ and obtain the bound $|B_{(\nu, min)}^4|$. Precisely:

$$\forall \text{ fixed } (\tilde{q}, \tilde{\Lambda}) \in (\mathcal{E}_{(\tilde{q})}^{12} \times]0, 0.05])$$

$$|B_\nu^4| \geq |B_{\nu, min}^4|$$

where:

$$|B_{\nu, min}^4| = \frac{9\Lambda}{2} I_G^0(q_2, q_3) \delta_{3, min} \prod_{i=1}^3 \Delta_F(q_i) H_{(\nu)}^2(q_i) \quad \text{with}$$

$$I_G^0(q_2, q_3) = \int R_G^{(0)} [H_\nu^2(k + \sum_{i=2}^3 q_i) [\Delta_F(k + \sum_{i=2}^3 q_i)]^2 \Delta_F(k)] d^4 k$$

(2.10)

(For the precise momentum assignment of the integral cf. figure 1).

- Now, for $n \geq 5$ the properties i) “opposite sign property” ii) the axiomatic field theory properties and euclidean invariance and iii) asymptotic momentum behaviour at infinity are again easily established thanks to the hypothesis $H_\nu \in \Phi_R$.

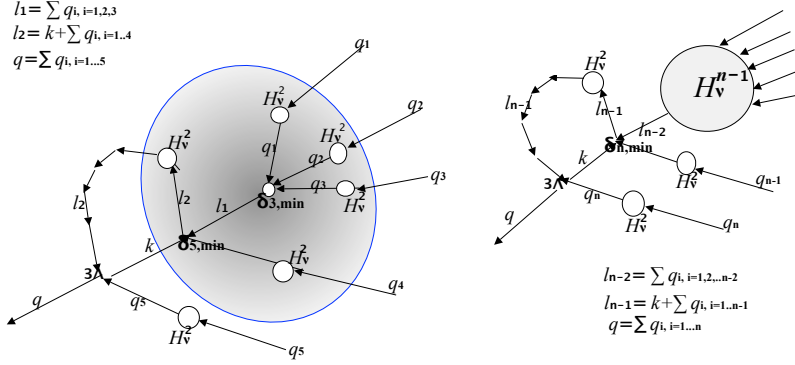


Figure 2: On the left we give the graphical representation of the term $3\Lambda[N_2 H_{\nu,min}^6][N_1 H_\nu^2]$ of order ν). In the same figure a zoom of the bubble-function $H_{\nu,min}^6$ after application of the minimization and the two splittings of $H_{\nu,min}^6$ and $H_{\nu,min}^4$ is displayed. On the right we display the analogous integration procedure for $3\Lambda[N_2 H_{\nu,min}^{n+1}][N_1 H_\nu^2]$.

Proof of Lemma 3.1 Following the proven bound 3.71 and the hypothesis $H_\nu \in \Phi_R$ we write:

$$\begin{aligned}
\frac{|B_{\nu,min}^6|}{|H_{\nu,max}^6|} &= \frac{15\Lambda}{2} I_G^0(q_{(4)}) \frac{\delta_{5,min} |H_\nu^4| \Delta_F(\sum_{i=1,2,3} q_i) \prod_{l=4,5} H_\nu^2(q_l) \Delta_F(q_l)}{\delta_{5,max} |H_\nu^4| \Delta_F(\sum_{i=1,2,3} q_i) \prod_{l=4,5} H_\nu^2(q_l) \Delta_F(q_l)} \\
&= \frac{15\Lambda}{2} I_G^0(q_{(4)}) \frac{\delta_{5,min}}{\delta_{5,max}} \quad \text{with (cf.fig.2)} \\
I_G^0(q_{(4)}) &= \int R_G^{(0)} [H_\nu^2(k + \sum_{i=1}^4 q_i) [\Delta_F(k + \sum_{i=1}^4 q_i)]^2 \Delta_F(k)] d^4 k
\end{aligned} \tag{2.12}$$

Moreover by analogy, when $n = 3$ we have:

$$\begin{aligned}
\frac{|B_{\nu,min}^4|}{|H_{\nu,max}^4|} &= \frac{9\Lambda}{2} I_G^0(q_{(2)}) \frac{\delta_{3,min}}{\delta_{3,max}} \quad \text{with (cf.fig.1)} \\
I_G^0(q_{(2)}) &= \int R_G^{(0)} [H_\nu^2(k + \sum_{i=1}^2 q_i) [\Delta_F(k + \sum_{i=1}^2 q_i)]^2 \Delta_F(k)] d^4 k
\end{aligned} \tag{2.13}$$

Now, we take into account the positivity and the logarithmic asymptotic behaviour with respect to the external momenta of both integrands together with the inclusion relation of the momentum subspaces $S(q_{(2)}) \subset S(q_{(4)})$,

so that we could write:

$$\begin{aligned}
& R_G^{(0)}[H_\nu^2(k + \sum_{i=1}^4 q_i)[\Delta_F(k + \sum_{i=1}^4 q_i)]^2 \Delta_F(k)] \\
& \geq R_G^{(0)}[H_\nu^2(k + \sum_{i=1}^2 q_i, q_3 = 0, q_4 = 0)[\Delta_F(k + \sum_{i=1}^2 q_i)]^2 \Delta_F(k)] \\
& = R_G^{(0)}[H_\nu^2(k + \sum_{i=1}^2 q_i)[\Delta_F(k + \sum_{i=1}^2 q_i)]^2 \Delta_F(k)] \\
& \text{or} \\
& I_G^0(q_{(4)}) \geq I_G^0(q_{(2)})
\end{aligned} \tag{2.14}$$

On the other hand we verify (by application of definition 2.3 of splitting parameters of [20]), that the inequality:

$$5 \frac{\delta_{5,min}}{\delta_{5,max}} > 3 \frac{\delta_{3,min}}{\delta_{3,max}} \tag{2.15}$$

holds under the weak condition: $\forall \Lambda \leq 0.5$. It then follows that

$$\frac{|B_{\nu,min}^6|}{|H_{\nu,max}^6|} \geq \frac{|B_{\nu,min}^4|}{|H_{\nu,max}^4|} \quad \blacksquare \tag{2.16}$$

- Then, we show recurrently that: \forall fixed $(\tilde{q}, \tilde{\Lambda}) \in (\mathcal{E}_{(q)}^{4n} \times]0, 0.05])$ and $n \geq 7$ the sequence:

$$\left\{ \tilde{\delta}_{n,\nu}^B \right\}_{n=2k+1, k \geq 3} = \frac{|B_{\nu,min}^{n+1}|}{n(n-1)|H_{\nu,max}^{n+1}|} \tag{2.17}$$

increases with increasing n . In other words we prove that:

$$\frac{|B_{\nu,min}^{n+1}|}{n(n-1)|H_{\nu,max}^{n+1}|} \geq \frac{|B_{\nu,min}^{n-1}|}{(n-2)(n-3)|H_{\nu,max}^{n-1}|} \tag{2.18}$$

By using the corresponding proven bound 3.71 $|B_{\nu,min}^{n+1}|$ (resp. $|B_{\nu,min}^{n-1}|$), and proposition 3.6 for $|H_{\nu,max}^{n+1}|$ (resp. for $|H_{\nu,max}^{n-1}|$), the inequality 2.18 is equivalent to the following one:

$$\frac{\delta_{n,min} \tilde{\mathcal{T}}_n I_G^0(q_{(n-1)})}{\delta_{n,max} \mathcal{T}_n} \geq \frac{\delta_{(n-2),min} \tilde{\mathcal{T}}_{n-2} I_G^0(q_{(n-3)})}{\delta_{(n-2),max} \mathcal{T}_{n-2}} \tag{2.19}$$

where: (Reminder)

$$\begin{aligned}
\mathcal{T}_n &= \frac{(n-3)^2}{48} + \frac{(n-3)}{3} + 1 \\
\tilde{\mathcal{T}}_n &= \frac{(n-3)^2}{48} + \frac{(n-3)}{3}
\end{aligned} \tag{2.20}$$

Notice that the integrals $I_G^0(q_{(n-1)})$, $I_G^0(q_{(n-3)})$ are defined by analogy to 2.12 and 2.13 (see also the right part of figure 2) with corresponding logarithmic asymptotic coefficients which verify : $\beta_{(n,\nu)} = \beta_{(1,\nu)} n \geq \beta_{(n-2,\nu)} = \beta_{(1,\nu)}(n-2) \quad \forall S \subset \mathcal{E}_{(q)}^{4n}, S(q_{(n-3)}) \subset S(q_{(n-1)})$.

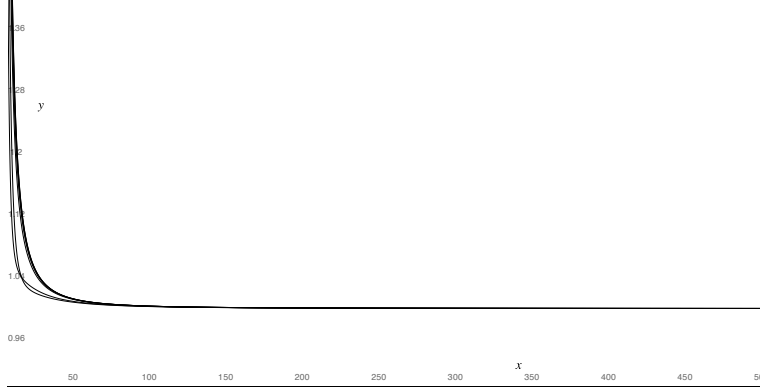


Figure 3: For the values of n ($= x$ continuous) in the interval $]7, 500]$ f_{d_0} decreases continuously always from values bigger than 1 up to the limit value of 1

So that finally, it remains to show that:

$$f_{d_0} \equiv \frac{\delta_{n,\min} \tilde{\mathcal{T}}_n \delta_{(n-2),\max} \mathcal{T}_{n-2}}{\delta_{n,\max} \mathcal{T}_n \delta_{(n-2),\min} \tilde{\mathcal{T}}_{n-2}} \geq 1$$

or

$$\frac{[1 + 3\Lambda(n-2)(n-3)][1 + n(n-1)d_0](n-3)^2 \left[\frac{(n-5)^2}{48} + \frac{(n-5)}{3} + 1 \right]}{1 + 3\Lambda n(n-1)[1 + (n-2)(n-3)d_0](n-5)^2 \left[\frac{(n-3)^2}{48} + \frac{(n-3)}{3} + 1 \right]} \geq 1 \quad (2.21)$$

Of course the tedious numerical calculations show clearly that the difference between numerator and denominator is always positive and goes to zero by positive values for sufficiently large n . For different values of d_0 figure 3 represents graphically the curves of $f_{d_0}(n)$ which for n ($= x$ continuous) in the interval $]7, 500]$ decreases continuously always from values bigger than 1 up to the limit value of 1. This completes the proof of the proposition. ■

APPENDIX 2.4 Proof of Proposition 3.3 of [20] (cf. figure 4)

Let $H_\nu \in \Phi_R$. In an analogous way, by application of the splittings signs and bounds of $|H_{\nu,\max}^6|$ and $|H_{(\nu,\max)}^4|$ we establish the upper bound $|A_{\nu,\max}^4|$. The convergence of the double integral due to the renormalization procedure allows us to apply Fubini's theorem and integrate first with respect to the k four-momentum to obtain the following: (cf. fig 5 with our comments).

$$|A_{\nu,\max}^4| = \int R_G I_{G,\nu}^{(3)}(k) [H_{(\nu)}^2(l_1) [\Delta_F(l_1)]^2 \Delta_F(k+q_1) d^4k \\ \times \Lambda \delta_{(5,\max)} \delta_{3,\max} \prod_{i=1}^3 \Delta_F(q_i) H_{(\nu)}^2(q_i)] \quad (2.22)$$

where $I_{G\nu}^{(3)}$ represents the result of the first integration with respect to k_1 .

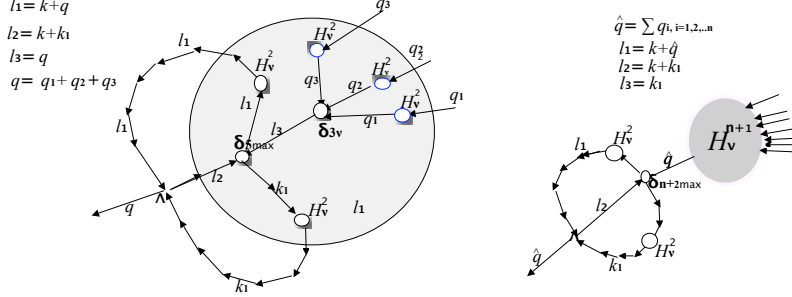


Figure 4: On the left the global term $A_\nu^4 = \Lambda[N_3^{(5)} H_\nu^6]$ (Φ^4 -operation of order ν) is graphically represented with a zoom of the bubble H_ν^6 -point function after the application of maximization and the splittings to the H_ν^6 and H_ν^2 -point functions thanks to the hypothesis $H_\nu \in \Phi_R$. On the right we represent graphically the analogous double-loop integration procedure of $A_\nu^{n+1} = \Lambda[N_3^{(n+2)} H_\nu^{n+3}]$ (Φ^4 -operation of order ν). Despite the quadratic divergence of the double integral with respect to k and k_1 , the overall asymptotic behaviour (decreasing as n increases) is, as expected, the same as that of the $R.\Phi.C. H_\nu^{n+1}$.

The positivity of the two integrands in equations 2.10 and 2.22 allows the comparison of the two integrals (cf. also the figures 1 and 5). More precisely the faster decrease of the propagator $[\Delta_F(l_1)]^2$ in the integrand of equ. 2.22 in comparison with the propagator $\Delta_F(k + \sum_{i=2}^3 q_i)^2$ in the integrand of equ. 2.10 enables us to obtain the following inequality:

$$I_G^0(q_2, q_3) \geq \int R_G^{(0)} I_{G,\nu}^{(3)}(k) [H_{(\nu)}^2(l_1) [\Delta_F(l_1)]^2 \Delta_F(k + q_1) d^4k \quad (2.23)$$

On the other hand taking into account definition 2.3 of [20] of the splitting parameters, we verify that:

$$\frac{9}{2} \delta_{3,min} > \delta_{(5,max)} \delta_{3,max} \quad \forall \Lambda \leq 0.05 \quad (2.24)$$

So finally:

$$|B_{\nu,min}^4| - |A_{\nu,max}^4| > 0 \quad \forall \Lambda \leq 0.05 \quad (2.25)$$

■

The properties i) “good sign property” ii) the axiomatic field theory properties and euclidean invariance and iii) asymptotic momentum behaviour at infinity are directly obtained thanks to the hypothesis $H_\nu \in \Phi_R$.

iv. a) In order to prove the upper bound $|A_{\nu,max}^{n+1}|$ we use analogous arguments as for the lower bound $|B_{\nu,min}^{n+1}|$, and in particular exactly as for the upper bound $|A_{\nu,max}^4|$.

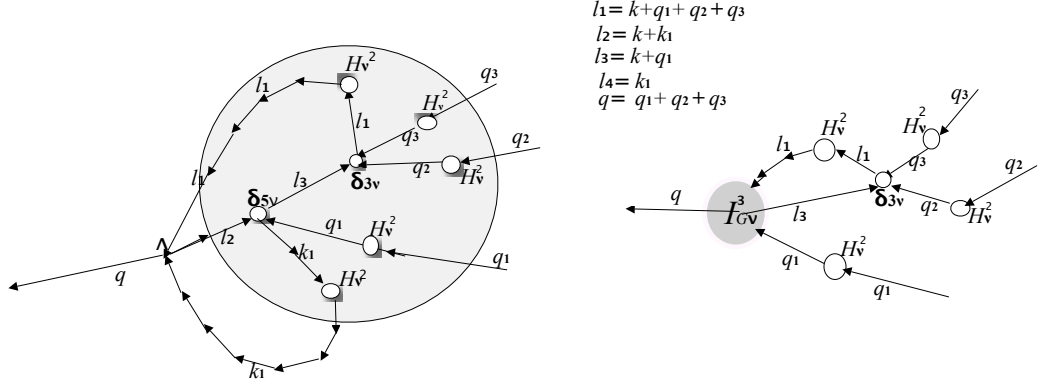


Figure 5: Graphical representation of the global term $A_\nu^4 = \Lambda[N_3^{(5)} H_\nu^6]$ (Φ^4 -operation of order ν). On the left hand side the figure displays a zoom on the bubble H^6 - point function after the application of the splittings to the H^6 and H^4 -point functions. On the right hand side graph the vertex-ball $I_{G\nu}^{(3)}$ represents the result of the integration with respect to k_1 .

We apply the splitting by the hypothesis $H_\nu \in \Phi_R$. For simplicity on the left of the figure 4 we give the configuration of the analogous splitting of A_ν^4 . For the double integration we use the momentum prescription we defined on the right of the figure 4. We also notice that despite the quadratic divergence of the double integral with respect to k and k_1 , the overall asymptotic behaviour (decreasing as n increases) is as expected the same as that of H_ν^{n+1} 's. So:

$$\begin{aligned}
|\Lambda[N_3^{(n+2)} H_\nu^{n+3}]| &\leq \Lambda \int R_G^{(3)} [|H_{(\nu, max)}^{n+3}| \prod_{i=1,2,3} \Delta_F(l_i)] d^4 k_1 d^4 k \\
&\leq \Lambda \delta_{(n+2, max)} \mathcal{T}_{n+2} \Delta_F(\sum_{i=1}^n q_i) |H_{(\nu, max)}^{n+1}(q(n))| \\
&\times \int R_G^{(2)} \prod_{i=2,3} [H_\nu^2(l_i) [\Delta_F(l_i)]^2 \Delta_F(k_1 + k)] d^4 k_1 d^4 k \\
&\quad \text{(notation in fig: 4 } l_3 = k_1) \quad \blacksquare
\end{aligned} \tag{2.26}$$

Remark 2.1

By application of the previous result for the case of $A_{\nu, max}^6$ and $A_{\nu, max}^4$ together with the hypothesis $H_\nu \in \Phi_R$ for $|H_{\nu, max}^6|$ and $|H_{\nu, max}^4|$, we easily obtain that:

$$\frac{|A_{\nu, max}^6|}{|H_{\nu, max}^6|} \leq \frac{|A_{\nu, max}^4|}{|H_{\nu, max}^4|} \quad \forall \Lambda \in]0, 0.05] \tag{2.27}$$

This inequality constitutes the first step of the decreasing behaviour of the sequence $\left\{ \tilde{\delta}_n^A \right\}_{n=2k+1, k \geq 2}$ that we show below.

iv.b) We show that \forall fixed $(\tilde{q}, \tilde{\Lambda}) \in (\mathcal{E}_{(q)}^{4n} \times]0, 0.05])$ the sequence:

$$\left\{ \tilde{\delta}_n^A \right\}_{n=2k+1, k \geq 2} = \frac{|A_{\nu, max}^{n+1}|}{n(n-1)|H_{\nu, max}^{n+1}|} \quad (2.28)$$

decreases with increasing n . In other words we prove that:

$$\frac{|A_{\nu, max}^{n+1}|}{n(n-1)|H_{\nu, max}^{n+1}|} \leq \frac{|A_{\nu, max}^{n-1}|}{(n-2)(n-3)|H_{\nu, max}^{n-1}|} \quad (2.29)$$

By using the previous result of $|A_{\nu, max}^{n+1}|$ the previous inequality 2.29 is equivalent to the following:

$$\frac{\delta_{(n+2, max)} \mathcal{T}_{n+2} \Delta_F(\sum_{i=1}^n q_i) \int [I_G^{(n)}]}{n(n-1)} \leq \frac{\delta_{(n, max)} \mathcal{T}_n \Delta_F(\sum_{i=1}^{n-2} q_i) \int [I_G^{(n-2)}]}{(n-2)(n-3)} \quad (2.30)$$

Now, due to the total external momentum dependence of the “external” and “internal propagators” inside of each of the two integrands $I_G^{(n)}$ and $I_G^{(n-2)}$, we have:

$$\Delta_F(\sum_{i=1}^n q_i) \int [I_G^{(n)}] d^4 k_1 d^4 k \leq \Delta_F(\sum_{i=1}^{n-2} q_i) \int [I_G^{(n-2)}] d^4 k_1 d^4 k \quad (2.31)$$

So that finally we have to verify that the following continuous function of n f_{d_1} is smaller than 1:

$$f_{d_1}(n) \equiv \frac{\delta_{n+2, max} \mathcal{T}_{n+2} (n-2)(n-3)}{\delta_{n, max} \mathcal{T}_n n(n-1)} \leq 1$$

or

$$\frac{(n+1)(n+2)[1 + n(n-1)d_0][\frac{(n-1)^2}{48} + \frac{(n-1)}{3} + 1](n-2)(n-3)}{n(n-1)[1 + (n+1)(n+2)d_0][\frac{(n-3)^2}{48} + \frac{(n-3)}{3} + 1]n(n-1)} \leq 1 \quad (2.32)$$

By giving to the numerical constant $d_0 = 3\Lambda 10^{(-1)}$ different values in the interval $[0.02, 0.45]$ and after numerical calculations we can find that the difference between the denominator and numerator is always positive.

For the values of n ($= x$ continuous) in the interval $]7, 200]$ (cf.figure 6) the function $f_{d_1}(n)$ increases continuously (with positive values always smaller than 1) up to the limit value of 1.

APPENDIX 2.5 Proof of Proposition 3.4 ([20])

i) By application of the hypothesis $H_\nu \in \Phi_R$, of the signs of H_ν^{n+1} -functions and the sign properties established before of B_ν^{n+1} and A_ν^{n+1} , the expressions 3.83 3.84 ([20]) are trivially verified.

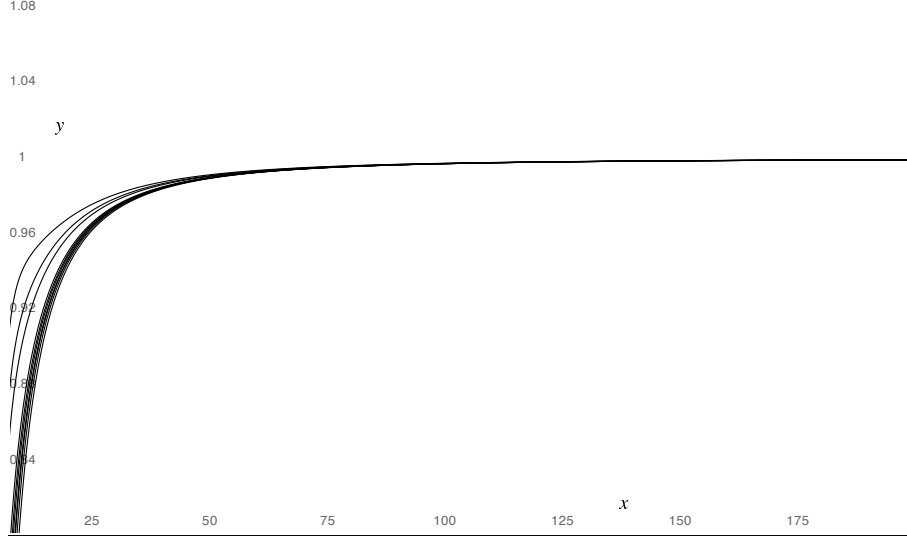


Figure 6: For the values of $n (= x \text{ continuous})$ in the interval $]7, 200]$ the function $f_{d_1}(n)$ increases continuously (with positive values always smaller than 1) up to the limit value of 1.

ii) Following the hypothesis $H_\nu \in \Phi_R$, the splitting properties 3.56, 3.57 (of [20]) yield recurrently at every fixed values of q ,the behavior of every $H_\nu \in \Phi_R$ with respect to the coupling constant as $(-\Lambda)^{(n-1)/2}$. But, B_ν^{n+1} (resp. A_ν^{n+1}) behaves by definition as $(-\Lambda)^{(n+1)/2}$ (resp. as $(-\Lambda)^{(n+3)/2}$), so finally:

$$\forall \text{ fixed } (\tilde{q}, n), \quad \lim_{\Lambda \rightarrow 0} D_{n,\nu}(\Lambda) = 0 \quad (2.33)$$

iii) By application of the previous results 2.16 2.27 and 2.25 we obtain:

$$D_{5,\nu,\min} \geq D_{3,\nu,\min} \geq 0 \quad (2.34)$$

Moreover the upper and lower bounds $B_{\nu,\min}^{n+1}, B_{\nu,\max}^{n+1}, A_{\nu,\max}^{n+1}$, together with the increase of the sequence $\{\tilde{\delta}_{n,\nu}^B\}_{n \geq 5}$ (resp. the decrease of $\{\tilde{\delta}_n^A\}_{n \geq 5}$) established previously in appendices 2.3 and 2.4 respectively, allow us to obtain directly the upper and lower bounds of $D_{n,\nu}$.

In particular the increase property of $D_{n,\nu,\min}/n(n-1)$ (cf. 3.88 of [20]) is obtained in a similar way to that of proof of 2.34:

$$\frac{D_{n,\nu,\min}}{n(n-1)} = (\tilde{\delta}_{n,\nu}^B - \tilde{\delta}_{n,\nu}^A) \geq (\tilde{\delta}_{n-2,\nu}^B - \tilde{\delta}_{n-2,\nu}^A) = \frac{D_{n-2,\nu,\min}}{(n-2)(n-3)} \quad (2.35)$$

iv) Finally the existence of the limit at infinity (3.89) of [20] is directly ensured.

These results complete the proof of proposition 3.4

■

4 The proof of the stability (Theorem 3.2 of [20])

In order to simplify the notations we often omit the arguments (q, Λ) and (q^2) .

1 Signs and bounds

APPENDIX 2.6 *Proof of Propositions 3.5 and 3.6 (of [20])*

A) *Proof of proposition 3.5*

1. Let $H_{\nu-1} \in \Phi_R$. For the positivity and asymptotic behaviour at infinity of the two point function H_ν^2 , we proceed as follows: the mapping \mathcal{M}^* yields:

$$H_\nu^2 = (q^2 + m^2)(1 + \delta_{1,\nu}\Delta_F)$$

$$\text{with: } \delta_{1,\nu}\Delta_F(q^2) = \frac{-\tilde{\rho} - \Lambda\{[N_3^{(3)}H_{\nu-1}^4] - \tilde{a}H_{\nu-1}^2\Delta_F\}\Delta_F(q^2)}{(\tilde{\gamma}_{\nu-1} + \tilde{\rho}_{\nu-1})} \quad (2.36)$$

Then the positive signs of the dominant terms in the numerator ensured by the recurrence hypothesis $H_{\nu-1} \in \Phi_R$, $\forall \Lambda \leq 0,05$ precisely:

$$-\Lambda[N_3^{(3)}H_{\nu-1}^4] > 0, \quad -\tilde{a}H_{\nu-1}^2 > 0 \quad (2.37)$$

(due to the negative sign of $H_{\nu-1}^4$) and the two positive terms of the denominator

$$\tilde{\gamma}_{\nu-1} > 0, \quad \tilde{\rho}_{\nu-1} > 0 \quad (2.38)$$

The same terms 2.11 yield the increasing logarithmic behaviour at large momenta so the upper bounds at every order ν and at infinity are obtained (cf. 3.53 or 3.90 of [20]).

2. As far as the 4-point function H_ν^4 is concerned we apply again the definition (3.2 of [20]) of the mapping \mathcal{M}^* on $H_{\nu-1} \in \Phi_R$ and write

$$H_\nu^4 = -\delta_{3,(\nu)} \prod_{l=1,2,3} [H_\nu^2 \Delta_F(q_l, \Lambda)]$$

Here by the definition 3.2 of \mathcal{M}^* ([20])

$$\delta_{3,(\nu)} = \frac{6\Lambda}{(\tilde{\gamma} + \tilde{\rho})_{(\nu-1)} + D_{3,(\nu-1)} + \Lambda|a_{(\nu-1)}|} \quad (2.39)$$

By using proposition 3.4 of [20], of the positivity of $D_{3,(\nu-1)}$ and definition 2.3 of [20] for the positive minimal and maximal values of the renormalisation parameters, we verify the upper and lower positive bounds of $\delta_{3,(\nu)}$ in Φ_R . Then, taking into account the preceding result of the H_ν^2 -point function, the negative sign and bounds of H_ν^4 in Φ_R , $\forall \nu$ and at infinity are obtained, (automatically the properties 3.55 or 3.92 of [20]) so the proof of proposition 3.5 is completed. ■

B) Proof of proposition 3.6

Taking into account the previous results of H_ν^2, H_ν^4 as starting point we apply, for every $n \geq 5$, a recursive procedure: we suppose that for all the previous H_ν^{m+1} , ($m \leq n - 2$) Green's functions the "good sign" and upper and lower bounds (properties (3.99), (3.100), (3.101) of proposition (3.6) of [20]) are satisfied. We then apply this hypothesis on the recurrent definition of the sum-tree term C_ν^{n+1} in terms of all previous Green's functions and the "good sign" and upper and lower bounds and limits at infinity (cf. 3.94, 3.95, 3.96, 3.97 of [20]), of the latter are trivially obtained.

For the signs and bounds of every H_ν^{n+1} we proceed exactly in an analogous way as we did for the H_ν^4 function: we first consider the splitting functions using the mapping \mathcal{M}^* :

$$\delta_{n,\nu} = \frac{3\Lambda n(n-1)}{(\tilde{\gamma}_{(\nu-1)} + \tilde{\rho}_{(\nu-1)} + D_{n,(\nu-1)}(H_{(\nu-1)}) - \Lambda \tilde{a}_{(\nu-1)})} \quad (2.40)$$

$$\text{with } D_{n,(\nu-1)}(H_{(\nu-1)}) = \frac{|B_{(\nu-1)}^{n+1}| - |A_{(\nu-1)}^{n+1}|}{|H_{(\nu-1)}^{n+1}|}$$

In view of the hypothesis $H_{\nu-1} \in \Phi_{\mathbb{R}}$ we apply propositions 3.2 (on $|B_{(\nu-1)}^{n+1}|$) respectively 3.3 (on $|A_{(\nu-1)}^{n+1}|$), 3.4 (on $D_{n,(\nu-1)}$) and the corresponding bounds and limits of the renormalization parameters, so that the bounds and limits of $\delta_{n,\nu}$ 3.57 or 3.98, of [20] are established.

Now, we use the previous results about the properties of C_ν^{n+1} and $\delta_{n,\nu}$ and by application of the definition 3.2 of of the mapping \mathcal{M}^* ("splitting property") ($\forall n \geq 5$) the signs 3.99, the bounds 3.100, 3.101 of [20] are recurrently verified, so the proof of proposition 3.6 is completed. ■

The proof of theorem 3.2 of [20]

Finally, in order to complete the proof of theorem 3.2 we apply the previous results of proposition 3.5 concerning the H_ν^2 and H_ν^4 Green's functions and obtain the properties of the renormalization parameters 3.60, 3.61, 3.62 of [20] at every order ν .

So, the stability of the Φ_4^4 -iteration under the action of \mathcal{M}^* inside the subset Φ_R is ensured. ■

5 Proof of the local contractivity of the mapping \mathcal{M}^* or the convergence of the Φ_4^4 iteration to a unique non trivial solution inside $Sr(0)$ (cf. theorem.4.1 of [20])

APPENDIX 2.7 By application of the definition 2.10 of the norm \mathcal{N} , in [20] the contractivity criteria are equivalent to the following conditions: $\forall \Lambda \in]0, 0.04]$

$$(I) \sup_{(n,q,\Lambda)} \left\{ \frac{|H_\nu^{n+1} - H_{T0}^{n+1}|}{M_n}; \frac{|\frac{\partial^{(0,1)}}{\partial q^2} N_3(H_\nu^4 - H_{T0}^4)|}{\hat{M}_{(3)}^{(0,1)}}; \frac{|N_2(H_\nu^{n+1} - H_{T0}^{n+1})|}{\hat{M}_{(n,2)}}; \frac{|\gamma_\nu - \gamma_0|}{\mathcal{N}_\gamma} \right\} \leq k^{(0)}(\Lambda)r_0 \quad \text{with } k^{(0)}(\Lambda) < 1$$

and

$$(II) \sup_{(n,q,\Lambda)} \left\{ \frac{|H_\nu^{n+1} - H_{\nu-1}^{n+1}|}{M_n}; \frac{|\frac{\partial^{(0,1)}}{\partial q^2} N_3(H_\nu^4 - H_{\nu-1}^4)|}{\hat{M}_3^{(0,1)}}; \frac{|N_2(H_\nu^{n+1} - H_{\nu-1}^{n+1})|}{\hat{M}_{(3,2)}}; \frac{|\gamma_\nu - \gamma_{\nu-1}|}{\mathcal{N}_\gamma} \right\} \leq K^\nu(\Lambda)\|H_{\nu-1} - H_{\nu-2}\|; \quad \text{with } K^\nu(\Lambda) < 1 \quad \text{and } k^{(0)} + K^\nu < 1 \quad (2.41)$$

1. Proof of (I)

We first obtain the corresponding bounds for $\nu = 1$. We start from $n = 1, n = 3$ and generalize recurrently for every $n \geq 5$. Then we apply the same procedure for every $\nu \geq 2$.

a) Let $n = 1$

$$\frac{|H_1^2 - H_{T0}^2|}{M_1} \leq \frac{\Lambda |N_3(H_{max}^4 - H_{min}^4)| \Delta_F}{M_1} \quad (2.42)$$

By using definition 4.1 of the ball $S_{r(0)}(H_{T0})$, the norm definition 2.10 and proposition 3.5 of **I** [20], we finally obtain:

$$\frac{|H_1^2 - H_{T0}^2|}{M_1} \leq k_{1,1}^{(0)} r(0)$$

$$\text{with: } k_{1,1}^{(0)} = \frac{6\Lambda^2(q^2 + m^2)^{\pi^2/54}(1 + 6\Lambda^2(q^2 + m^2)^{\pi^2/54})}{(q^2 + m^2)(1 + 6(q^2 + m^2)^{\pi^2/54})} \quad (2.43)$$

For $n = 3$ we have:

$$H_1^4(q\Lambda) = -\delta_{3,(1)}(q, \Lambda) \prod_{\ell=1,2,3} H_{(1)}^2(q\ell, \Lambda) \Delta_F(q\ell); \quad (2.44)$$

so

$$\frac{|H_1^4 - H_{T0}^4|}{M_3} \leq \frac{\{\delta_{3,max} - \delta_{3,min}\}}{M_3} \prod_{\ell=1,2,3} H_{(max)}^2(q\ell, \Lambda) \Delta_F(q\ell) + 3\delta_{3,max} \frac{|H_1^2 - H_{T0}^2|}{M_3} \prod_{\ell=1,2} H_{(max)}^2(q\ell, \Lambda) \Delta_F(q\ell) \quad (2.45)$$

And again by the definition of norms, of $r(0)$ and the previous result for $n = 1$ we obtain:

$$\begin{aligned} \frac{|H_1^4 - H_{T0}^4|}{M_3} &\leq \Lambda r(0)(1 + (18)^2 \Lambda^2) \quad \text{i.e. } k_{1,3}^{(0)} = \Lambda(1 + (18)^2 \Lambda^2) \\ \text{and } k_{1,3}^{(0)} &< 1 \text{ for } \Lambda \leq 0.1 \end{aligned} \quad (2.46)$$

Now for every $n \geq 5$ and $\bar{n} \leq n - 2$ we suppose that we have established an analogous inequality, namely:

$$\begin{aligned} \frac{|H_1^{\bar{n}+1} - H_{T0}^{\bar{n}+1}|}{M_{\bar{n}}} &\leq r(0)k_{1,\bar{n}}^{(0)} \\ \text{with } k_{1,\bar{n}}^{(0)} &< k_{1,3}^{(0)} \text{ if } \Lambda \leq 0.1 \text{ and we show that } k_{1,n}^{(0)} < k_{1,3}^{(0)} \\ \text{without any supplementary condition on } \Lambda. \end{aligned} \quad (2.47)$$

By using the definition of $S_{r(0)}(H_{T0})$, the norm definition, the splitting properties, the bounds of the tree terms, and the recursion we have successively:

$$\begin{aligned} \frac{|H_1^{n+1} - H_{T0}^{n+1}|}{M_n} &\leq \frac{\{\delta_{n,max} - \delta_{n,min}\}|C_{max}^{n+1}|}{3\Lambda n(n-1)M_n} + \frac{\delta_{n,max}|C_1^{n+1} - C_{T0}^{n+1}|}{3\Lambda n(n-1)M_n} \\ &< \frac{r(0)(n-3)^2}{48n(n-1)} \left\{ \frac{|H_{max}^{n-1}|}{M_{n-2}} + k_{1,n-2}^{(0)} \frac{H_{(max)}^2}{M_1} + 2k_{1,1}^{(0)} \frac{|H_{max}^{n-1}|}{M_{n-2}} \right\} \end{aligned} \quad (2.48)$$

In the last formula we used again the result of ref. [15, c] on the number \mathcal{T}_n of different partitions inside the tree terms as we did in proposition 3.6. Now we note that for every n we have:

$$\frac{|H_{max}^{n+1}|}{M_n} < \frac{|H_{max}^{n-1}|}{M_{n-2}} \quad (2.49)$$

As a matter of fact by application of proposition 3.6 and the norm definition we can write:

$$\frac{|H_{max}^{n+1}|}{M_n} < \frac{(n-3)^2 |H_{max}^{n-1}|}{48n(n-1)M_{n-2}} < \frac{|H_{max}^{n-1}|}{48M_{n-2}} \quad (2.50)$$

It then follows that:

$$\frac{|H_{max}^{n+1}|}{M_n} < \frac{(n-3)^2 |H_{max}^{n-1}|}{48n(n-1)M_{n-2}} < \frac{|H_{max}^4|}{48M_3} < \Lambda \quad (2.51)$$

From these results and the recurrent hypothesis

$$k_{1,n-2}^{(0)} < k_{1,3}^{(0)} \quad (2.52)$$

we have:

$$\begin{aligned} \frac{|H_1^{n+1} - H_{T0}^{n+1}|}{M_n} &< \frac{r(0)}{48} \{\Lambda + k_{1,3}^{(0)} + 2k_{1,1}^{(0)}\} \\ \text{or } \frac{|H_1^{n+1} - H_{T0}^{n+1}|}{M_n} &< r(0)k_{1,n}^{(0)} \end{aligned} \quad (2.53)$$

with:

$$k_{1,n}^{(0)} = \frac{k_{1,3}^{(0)}}{16} \quad (2.54)$$

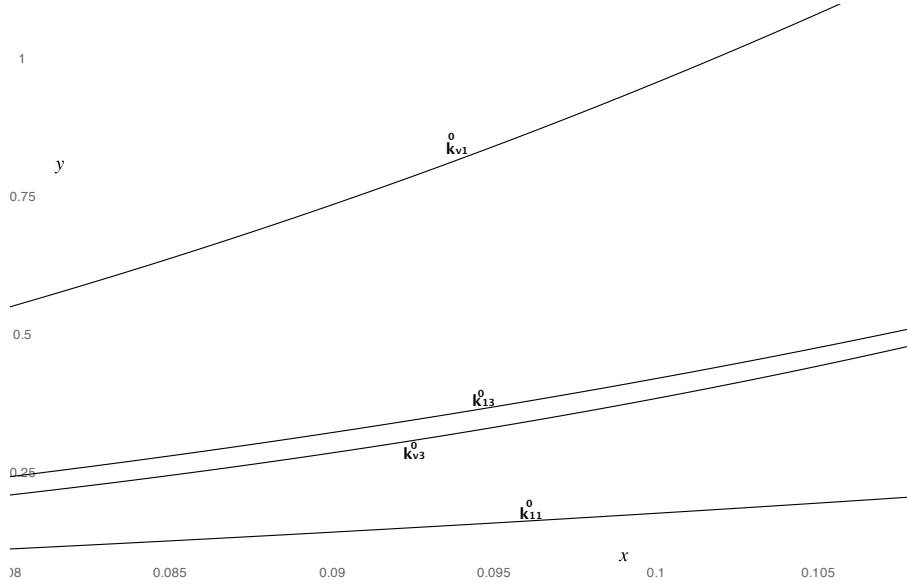


Figure 7: The stronger condition to require for the coupling constant comes from H_ν^2 precisely: $\Lambda \leq 0.101$ with $k_{\nu,1} = 0.9925$ while $k_{1,3} = 0.4189$ and corresponding values $k_{\nu,3} = 0.40$ and $k_{1,1} = 0.1846$.

b) In the case of $\nu \geq 2$ we follow an analogous procedure and find similar results. We just notice that for $n = 1$ the condition imposed on Λ , in order that $k_{\nu,1}^{(0)} < 1$, is stronger than the one of every $k_{\nu,n}^{(0)} < 1$, with $n \geq 3$ (cf. figure 7).

As a matter of fact at every order $\nu \geq 2$ of the Φ_4^4 -iteration the contributions coming from the values of the renormalization constants $\tilde{\gamma}$, $\tilde{\rho}$, \tilde{a} become nontrivial.

Then as before by taking into account the norm definition, and the definition of $S_{r(0)}(H_{T0})$ and $r(0)$ we first have:

$$\begin{aligned} |\tilde{\gamma}_{\nu-1} - \tilde{\gamma}_0| &\leq r(0)(1 + 9\Lambda(1 + 6\Lambda^2)) \\ |\tilde{\rho}_{\nu-1} - \tilde{\rho}_0| &\leq \Lambda r(0) \hat{M}_3^{r(0,1)} \\ |\tilde{a}_{\nu-1} - \tilde{a}_0| &\leq \Lambda r(0) \hat{M}_3^{r(0,1)} \end{aligned} \quad (2.55)$$

and finally (after some trivial estimations):

$$\begin{aligned} \frac{|H_\nu^2 - H_{T0}^2|}{M_1} &\leq k_{\nu,1} r(0) \quad \text{with } k_{\nu,1}(\Lambda) = 48\Lambda^2(1 + 10\Lambda) \\ \text{and } k_{\nu,1}(\Lambda) &< 1 \text{ for } \Lambda \leq 0.1 \end{aligned} \quad (2.56)$$

Now as before, $\forall n \geq 3$ we estimate the following bounds:

$$\begin{aligned} \frac{|H_\nu^4 - H_{T0}^4|}{M_3} &\leq \Lambda r(0)(1 + 144\Lambda^2(1 + 10\Lambda)) \\ \text{which means: } k_{\nu,3}^{(0)} &= \Lambda(1 + 144\Lambda^2(1 + 10\Lambda)) \\ \text{and } k_{\nu,3}^{(0)} &< 1 \quad \text{for } \Lambda \leq 0.135 \quad (\text{cf. figure 7}) \end{aligned} \quad (2.57)$$

and similar results for:

$$\frac{|N_3(H_\nu^4 - H_{T0}^4)|}{\hat{M}_3^{r(0,1)}} \quad \text{and} \quad \frac{|\frac{\partial}{\partial q^2} N_3(H_\nu^4 - H_{T0}^4)|}{\hat{M}_3^{r(0,1)}} \quad (2.58)$$

Moreover we find again recurrently, using the same arguments as when $\nu = 1$ that for all $n \geq 5$:

$$k_{\nu,n}^{(0)} < k_{\nu,3}^{(0)} < k_{\nu,1}^{(0)} = 48\Lambda^2(1 + 10\Lambda) \quad \text{independent of } \nu \quad (2.59)$$

Conclusion:

$$\begin{aligned} \forall \nu \geq 2, \|H_\nu - H_{T0}\| &\leq k^{(0)}(\Lambda)r(0) \quad \text{where} \\ k^{(0)}(\Lambda) &= 48\Lambda^2(1 + 10\Lambda) < 1 \quad \forall \Lambda \leq 0.1 \end{aligned}$$

■

$$(2.60)$$

2. Proof of (II)

The first step $\nu = 2$ being easily verified, we suppose that for all $\bar{\nu} \leq \nu - 1$ the corresponding inequality 2.41 is verified.

a) For $n = 1$ we write:

$$\begin{aligned} \frac{|H_\nu^2 - H_{\nu-1}^2|}{M_1} &\leq \frac{|\tilde{\rho}_{\nu-1} - \tilde{\rho}_{\nu-2}| |\tilde{\gamma}_{\nu-1} + 2(\tilde{\rho}_{\nu-1} + \tilde{\rho}_{\nu-2})| + |\tilde{\gamma}_{\nu-1} - \tilde{\gamma}_{\nu-2}| \tilde{\rho}_{\nu-1}}{|\tilde{\gamma}_0 + \tilde{\rho}_0|^2 |M_1|} \\ &\quad + \frac{\Lambda |N_3 H_{\nu-1}^4 (\tilde{\gamma}_{\nu-2} + \tilde{\rho}_{\nu-2}) - N_3 H_{\nu-2}^4 (\tilde{\gamma}_{\nu-1} + \tilde{\rho}_{\nu-1})| \Delta_F}{|(\tilde{\gamma}_{\nu-1} + \tilde{\rho}_{\nu-1})(\tilde{\gamma}_{\nu-2} + \tilde{\rho}_{\nu-2})| |M_1|} \\ &\quad + \frac{\Lambda |N_3 H_{\nu-1}^4 (\tilde{\gamma}_{\nu-2} + \tilde{\rho}_{\nu-2}) - N_3 H_{\nu-2}^4 (\tilde{\gamma}_{\nu-1} + \tilde{\rho}_{\nu-1})|_{(q^2+m^2)=0} H_{\nu-1}^2 \Delta_F}{|(\tilde{\gamma}_{\nu-1} + \tilde{\rho}_{\nu-1})(\tilde{\gamma}_{\nu-2} + \tilde{\rho}_{\nu-2})| |M_1|} \\ &\quad + \Lambda \frac{|H_{\nu-1}^2 - H_{\nu-2}^2| |N_3 H_{\nu-2}^4|_{q^2+m^2=0} \Delta_F}{|(\tilde{\gamma}_{\nu-1} + \tilde{\rho}_{\nu-1})(\tilde{\gamma}_{\nu-2} + \tilde{\rho}_{\nu-2})| |M_1|} \end{aligned} \quad (2.61)$$

Then using the definitions of the norm and the renormalization constants we first have:

$$\begin{aligned} |\tilde{\gamma}_{\nu-1} - \tilde{\gamma}_{\nu-2}| &\leq \|H_{\nu-1} - H_{\nu-2}\| \mathcal{N}_\gamma(\Lambda)|_{q^2=0} \\ |\tilde{\rho}_{\nu-1} - \tilde{\rho}_{\nu-2}| &\leq \Lambda \|H_{\nu-1} - H_{\nu-2}\| \hat{M}_3^{(0,1)}|_{q^2+m^2=0} \end{aligned} \quad (2.62)$$

Then, after some elementary estimations the first term of the R.H.S. of 2.61 yields:

$$\begin{aligned} \frac{\mathcal{O}_1}{M_1} &= \frac{|\tilde{\rho}_{\nu-1} - \rho_{\nu-2}| \tilde{\gamma}_{\nu-2} + |\tilde{\gamma}_{\nu-1} - \tilde{\gamma}_{\nu-2}| \tilde{\rho}_{\nu-2}}{|\tilde{\gamma}_0 + \tilde{\rho}_0|^2 M_1} \leq \\ &\leq K_1^{\nu,1}(\Lambda) \|H_{\nu-1} - H_{\nu-2}\| \end{aligned} \quad (2.63)$$

$$\text{with } K_1^{\nu,1}(\Lambda) = 12\Lambda < 1 \text{ when } \Lambda \leq 0.08$$

We take the sum of the second and third term of 2.61 and call it \mathcal{O}_2 :

By application again of the norm definition we obtain:

$$\begin{aligned} \frac{\mathcal{O}_2}{M_1} &\leq K_1^{\nu,2}(\Lambda) \|H_{\nu-1} - H_{\nu-2}\| \\ \text{here: } K_1^{\nu,2}(\Lambda) &= 12\Lambda \frac{H_{max}^2 \Delta_F |N_3 H_{max}^4| \hat{M}_3}{M_1} < 12\Lambda \\ \text{and again that means for } \Lambda &\leq 0.08 \quad K_1^{\nu,2} < 1 \end{aligned} \quad (2.64)$$

The last term of 2.61 that we call \mathcal{O}_3 yields:

$$\begin{aligned} \frac{\mathcal{O}_3}{M_1} &\leq \Lambda \frac{|H_{\nu-1}^2 - H_{\nu-2}^2| |N_3 H_{\nu-2}^4|_{q^2+m^2=0} \Delta_F}{|(\tilde{\gamma}_{\nu-1} + \tilde{\rho}_{\nu-1})(\tilde{\gamma}_{\nu-2} + \tilde{\rho}_{\nu-2})| M_1} \leq \\ &\leq K_1^{\nu,3}(\Lambda) \|H_{\nu-1} - H_{\nu-2}\| \quad \text{here: } K_1^{\nu,3}(\Lambda) = 6\Lambda^2 \end{aligned} \quad (2.65)$$

By using the corresponding bounds from 2.63, 2.64 2.65 we obtain:

$$\begin{aligned} \frac{|H_\nu^2 - H_{\nu-1}^2|}{\tilde{M}_1} &< K_1^\nu(\Lambda) \|H_{\nu-1} - H_{\nu-2}\| \\ \text{here: } K_1^\nu(\Lambda) &= K_1^{\nu,1} + K_1^{\nu,2} + K_1^{\nu,3} = 6\Lambda(4 + \Lambda) \\ \text{with } K_1^\nu &< 1 \quad \forall \Lambda \leq 0.04 \end{aligned}$$

■
(2.66)

b) Let $n = 3$ we write:

$$\frac{|H_\nu^4 - H_{\nu-1}^4|}{M_3} \leq A + B \quad (2.67)$$

where:

$$A = \frac{\delta_{3,\nu} \prod_{\ell=1,2,3} |H_\nu^2(q_\ell, \Lambda) - H_{\nu-1}^2(q_\ell, \Lambda)| \Delta_F(q_\ell)}{M_3} \quad (2.68)$$

and

$$B = \frac{|\delta_{3,\nu} - \delta_{3,\nu-1}| \prod_{\ell=1,2,3} H_{\nu-1}^2(q_\ell, \Lambda) \Delta_F(q_\ell)}{M_3} \quad (2.69)$$

i) By an analogous procedure as before we obtain:

$$A \leq K_{3,A}^\nu(\Lambda) \|H_{\nu-1} - H_{\nu-2}\|$$

$$\text{with } K_{3,A}^\nu(\Lambda) = \frac{3K_1^\nu}{\tilde{D}_{3,\min}} \prod_{l=2,3} \frac{H_{(max)}^2(q_l, \Lambda)}{M_1(q_l)} \quad (2.70)$$

$$\text{and } \tilde{D}_{3,\min} = 1 + \rho_0 + \Lambda|a_0| + 0, 18\Lambda \quad (2.71)$$

Now by application of the norm definition of M_1 and definition of H_{max}^2 (Proposition 3.5 of [20]), we obtain a best estimate of the ratio $\frac{H_{max}^2}{M_1}$:

$$\frac{H_{max}^2}{M_1} \leq 1 - \frac{6(q^2 + m^2)^{\pi^2/54}(1 - \Lambda^2)}{1 + 6(q^2 + m^2)^{\pi^2/54}} \quad (2.72)$$

and for sufficiently large q^2 : $\frac{H_{max}^2}{M_1} \sim 6\Lambda^2$

and so:

$$K_{3,A}^\nu(\Lambda) < 108\Lambda^4 K_1^\nu < 0.7K_1^\nu \quad (\forall \Lambda < 0.08) \quad (2.73)$$

ii) As far as the term B of the r.h.s. of 2.64 is concerned we use the same arguments as before and we obtain:

$$\frac{|\delta_{3,\nu} - \delta_{3,\nu-1}| \prod_{\ell=1,2,3} H_{\nu-1}^2(q_\ell) \Delta_F(q_\ell)}{M_3} \leq (B.1) + (B.2) + (B.3) \quad (2.74)$$

with explicitly:

ii.1)

$$(B.1) \leq \prod_{\ell=1,2,3} \frac{H_{\nu-1}^2(q_\ell) |\tilde{\gamma}_{\nu-1} - \tilde{\gamma}_{\nu-2}| + |\tilde{\rho}_{\nu-1} - \rho_{\nu-2}| + |\tilde{a}_{\nu-1} - \tilde{a}_{\nu-2}|}{M_1(q_\ell) \tilde{D}_{3,\nu-1} \tilde{D}_{3,\nu-2}}$$

Here we used the definitions of the mapping \mathcal{M}^*

$$\tilde{D}_{3,\nu-1} = \tilde{\gamma}_{\nu-1} + \tilde{\rho}_{\nu-1} + D_{3,\nu-1} - \Lambda \tilde{a}_{\nu-1}$$

(and the analogous expression for $\tilde{D}_{3,\nu-2}$), so

$$(B.1) \leq K_{3,B.1}^\nu(\Lambda) \|H_{\nu-1} - H_{\nu-2}\|$$

with $K_{3,B.1}^\nu \leq (6\Lambda)^3(1+9\Lambda)$ for sufficiently large q^2
and for small q^2 $K_{3,B.1}^\nu(\Lambda) < 1, \forall \Lambda \leq 0.05$

(2.75)

ii.2) and:

$$(B.2) \leq 6\Lambda \prod_{\ell=1,2,3} H_{\nu-1}^2(q_\ell) \Delta_F(q_\ell) \frac{|H_{\nu-1}^4 - H_{\nu-2}^4| |B_{\nu-2}^4 - A_{\nu-2}^4|}{M_3 |H_{\nu-1}^4| |H_{\nu-2}^4| \tilde{D}_{3,\nu-1} \tilde{D}_{3,\nu-2}}$$

(2.76)

$$\text{where } \frac{|B_{\nu-2}^4 - A_{\nu-2}^4|}{|H_{\nu-2}^4|} = D_{3,\nu-2} \quad (2.77)$$

$$\text{so } (B.2) \leq K_{3,B.2}^\nu(\Lambda) \|H_{\nu-1} - H_{\nu-2}\|$$

$$\text{where } K_{3,B.2}^\nu = \frac{D_{3,max}}{\tilde{D}_{3,min}} = \frac{9\Lambda(1+6\Lambda^2)}{(1+\rho_0+\Lambda|a_0|+0,18\Lambda)}$$

and $K_{3,B.2}^\nu(\Lambda) < 1, \forall \Lambda \leq 0.1$ (very weak condition)

(2.78)

ii.3) Moreover:

$$(B.3) < \frac{9\Lambda}{M_3 \tilde{D}_{3,\nu-1}} \{ |N_2(H_{\nu-1}^4 - H_{\nu-2}^4)| H_{max}^2 \Delta_F \}$$

$$+ \frac{9\Lambda}{M_3 \tilde{D}_{3,\nu-1}} \{ |N_2 H_{max}^4| |H_{\nu-1}^2 - H_{\nu-2}^2| \Delta_F \}$$

(2.79)

Notice that we have used the sign properties of B^4 and A^4 together with the definitions of the mapping. Then, by application of the norm definitions we obtain:

$$(B.3) < K_{3,B.3}^\nu(\Lambda) \|H_{\nu-1} - H_{\nu-2}\|$$

$$\text{where } K_{3,B.3}^\nu = \frac{18\Lambda H_{max}^2}{\tilde{D}_{3,min} M_1} \quad (2.80)$$

$$K_{3,B.3}^\nu(\Lambda) < 1, \forall \Lambda \leq 0.05$$

Finally inserting the results 2.70, 2.72, 2.75 and 2.77 in equation 2.64 we obtain that

$$K_3^\nu < 1 \quad \forall \Lambda \leq 0.05 \quad \blacksquare$$

c) Under weaker conditions on Λ and using the analogous procedure, (norm definitions, together with properties in Φ_R etc.) we find that:

c.i) there is a positive constant continuous function of Λ ,

$$K_\gamma^\nu(\Lambda) < 1 \quad \text{such that:} \quad (2.81)$$

$$\begin{aligned} \frac{|\tilde{\gamma}_\nu - \tilde{\gamma}_{\nu-1}|}{\mathcal{N}_\gamma} &\leq K_\gamma^\nu(\Lambda) \|H_{\nu-1} - H_{\nu-2}\|; \\ \text{with } K_\gamma^\nu(\Lambda) &= 3K_1^\nu \gamma_{max} M_1^{-2} |_{q^2=0} + K_3^\nu \gamma_{max}^2 < 1 \\ &\text{(under conditions weaker than the condition } \Lambda \leq 0.05) \end{aligned} \quad (2.82)$$

c.ii)

$$\begin{aligned} \frac{|\tilde{\rho}_\nu - \tilde{\rho}_{\nu-1}|}{M_3^{(0,1)}} &\leq \Lambda \frac{|\frac{\partial}{\partial q^2} N_3(H_\nu^4 - H_{\nu-1}^4)|}{\hat{M}_3^{(0,1)}} \leq K_\rho^\nu(\Lambda) \|H_{\nu-1} - H_{\nu-2}\|; \\ \text{with } K_\rho^\nu &= \Lambda K_3^\nu < 1 \quad \text{under weaker condition on } \Lambda \end{aligned} \quad (2.83)$$

c.iii)

$$\frac{|\tilde{a}_\nu - \tilde{a}_{\nu-1}|}{M_3^{(0,1)}} \leq K_3^\nu(\Lambda) \|H_{\nu-1} - H_{\nu-2}\| \quad (2.84)$$

c.iv) By using the basic splitting properties in Φ_R of H^4 and by analogous arguments as above we show that

$$\begin{aligned} |N_2 H_\nu^4| &\leq 6\Lambda [N_2^3] \prod_{l=1}^2 M_1(q_l) \\ \text{so that } \frac{|N_2(H_\nu^4 - H_{\nu-1}^4)|}{\hat{M}_{(3,2)}} &\leq K_3^\nu(\Lambda) \|H_{\nu-1} - H_{\nu-2}\| \end{aligned} \quad (2.85)$$

Finally:

$$\begin{aligned} \sup_{(q,\Lambda)} \left\{ \frac{|\frac{\partial}{\partial q^2} N_3(H_\nu^4 - H_{\nu-1}^4)|}{\hat{M}_3^{(0,1)}}; \frac{|N_2(H_\nu^4 - H_{\nu-1}^4)|}{\hat{M}_{(3,2)}}; \frac{|\gamma_\nu - \gamma_{\nu-1}|}{\mathcal{N}_\gamma} \right\} \\ \leq K_{3,2,\gamma}^\nu(\Lambda) \|H_{\nu-1} - H_{\nu-2}\|; \\ \text{with } K_{3,2,\gamma}^\nu(\Lambda) < 1 \quad \text{when } \Lambda \leq 0.05 \quad \blacksquare \end{aligned} \quad (2.86)$$

d) Let $n \geq 5$

We suppose that for every $\tilde{n} \leq n - 2$ the second property of 2.41 is verified in the following sense:

$\forall \Lambda \leq 0.05$ and \forall fixed ν ,

there is a strictly positive constant (continuous function of Λ) $K_{\tilde{n}}^{\nu}(\Lambda)$ such that:

$$\sup_{(q, \Lambda)} \left\{ \frac{|H_{\nu}^{\tilde{n}+1} - H_{\nu-1}^{\tilde{n}+1}|}{M_{\tilde{n}}} \right\} \leq K_{\tilde{n}}^{\nu}(\Lambda) \|H_{\nu-1} - H_{\nu-2}\| \quad \text{with} \\ K_{\tilde{n}}^{\nu}(\Lambda) < K_{\tilde{n}-2}^{\nu}(\Lambda) \leq \dots \leq K_3^{\nu}(\Lambda) \quad (2.87)$$

We show this property for $\tilde{n} = n$ by using again the definitions of the norm, the mapping \mathcal{M}^* and the properties in Φ_R .

$$\frac{|H_{\nu}^{n+1} - H_{\nu-1}^{n+1}|}{M_n} \leq A_n + B_n \quad \text{where} \\ A_n = \frac{\delta_{n, \nu} |C_{\nu}^{n+1} - C_{\nu-1}^{n+1}|}{3\Lambda n(n-1)M_n} \\ B_n = \frac{|\delta_{n, \nu} - \delta_{n, \nu-1}| |C_{\nu-1}^{n+1}|}{3\Lambda n(n-1)M_n} \quad (2.88)$$

i)

$$A_n = \frac{\delta_{n, \nu} |C_{\nu}^{n+1} - C_{\nu-1}^{n+1}|}{3\Lambda n(n-1)M_n} \leq \\ A_n \leq K_{n, A}^{\nu} \|H_{\nu-1} - H_{\nu-2}\| \quad \text{with} \\ K_{n, A}^{\nu}(\Lambda) = \frac{H_{(max)}^2}{24M_1} \left\{ \frac{K_{n-2}^{\nu} H_{(max)}^2}{M_1} + \frac{K_1^{\nu} |H_{(max)}^{n-1}|}{M_{n-2}} \right\} \quad (2.89)$$

We easily verify that under a weaker condition than $\Lambda \leq 0.05$

$$K_{n, A}^{\nu} \leq K_{n-2, A}^{\nu} < 1 \quad \blacksquare \quad (2.90)$$

ii) For the term B_n of 2.85 we use the same arguments as before and we obtain:

$$B_n = \frac{|\delta_{n, \nu} - \delta_{n, \nu-1}| |C_{\nu-1}^{n+1}|}{3\Lambda n(n-1)M_n} \leq |B_{n,1}| + |B_{n,2}| + |B_{n,3}| \quad (2.91)$$

with:

ii.1)

$$|B_{n,1}| \leq \\ \leq |C_{\nu-1}^{n+1}| \frac{|\tilde{\gamma}_{\nu-1} - \tilde{\gamma}_{\nu-2}| + |\tilde{\rho}_{\nu-1} - \rho_{\nu-2}| + |\tilde{a}_{\nu-1} - \tilde{a}_{\nu-2}|}{M_n \tilde{D}_{n, \nu-1} \tilde{D}_{n, \nu-2}} \quad (2.92)$$

In the same way we obtain :

$$|B_{n,1}| \leq K_{n,B.1}^\nu \|H_{\nu-1} - H_{\nu-2}\| \quad \text{with}$$

$$K_{n,B.1}^\nu = \prod_{l=2,3} \frac{H_{(max)}^2(q_l)(n-3)^2 |H_{max}^{n-1}| (\gamma_{max} + 2\Lambda \hat{M}_n^{(0,1)})}{M_1(q_l) 24n(n-1) M_{n-2} \tilde{D}_{3,min}} \quad (2.93)$$

Now using the obvious bound:

$$\frac{|H_{max}^{n-1}|}{M_{n-2}} \leq \prod_{l=2,3} \frac{H_{(max)}^2(q_l) |H_{max}^{n-3}|}{M_1(q_l) 24M_{n-4}} \quad (2.94)$$

and the analogous definition 2.61 of $K_{n-2,B.1}^\nu$ we have:

$$K_{n,B.1}^\nu \leq \prod_{l=2,3} \frac{H_{(max)}^2(q_l)(n-3)^2(n-2(n-3))}{M_1(q_l) 24n(n-1)(n-5)^2} K_{n-2,B.1}^\nu$$

$$\Leftrightarrow K_{n,B.1}^\nu < K_{n-2,B.1}^\nu < 1 \quad \blacksquare$$

(2.95)

ii.2) By analogy to $n = 3$

$$|B_{n,2}| \leq K_{n,B.2}^\nu \|H_{\nu-1} - H_{\nu-2}\| \quad \text{with}$$

$$K_{n,B.2}^\nu = \frac{D_{n,max}}{\tilde{D}_{n,min}} \sim \frac{D_{n-2,max}}{\tilde{D}_{n-2,min}} < 1 \quad \forall \Lambda \leq 0.05 \quad \blacksquare$$

(2.96)

ii.3) Then by an analogous to the above procedure we have:

$$|B_{n,3}| < K_{n,B.3}^\nu \|H_{\nu-1} - H_{\nu-2}\| \quad \text{with}$$

$$K_{n,B.3}^\nu = \delta_\infty [N_2^{\tilde{n}}]_{q=0} \frac{H_{max}^2}{2M_1} \left(1 + \frac{\mathcal{T}_n}{n(n-1)}\right)$$

and verify $K_{n,B.3}^\nu \sim K_{n,B.3}^{\nu-2} < 1, \quad \forall \Lambda \leq 0.05 \quad \blacksquare$

(2.97)

Finally by addition of 2.87, 2.92, 2.93 and 2.94 we obtain the proof of the recursion 2.84 and by using also the result 2.57 the proof of the contractivity criterium 2.41 follows. \blacksquare

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