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Nearest Neighbor Control For Practical Stabilization of Passive Nonlinear Systems

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Abstract

This paper studies static output feedback stabilization of continuous-time (incrementally) passive nonlinear systems where the control actions can only be chosen from a discrete (and possibly finite) set of points. For this purpose, we are working under the assumption that the system under consideration is large-time norm observable and the convex hull of the realizable control actions contains the target constant input (which corresponds to the equilibrium point) in its interior. We propose a nearest-neighbor based static feedback mapping from the output space to the finite set of control actions, that is able to practically stabilize the closed-loop systems. Consequently, we show that for such systems with m -dimensional input space, it is sufficient to have $m + 1$ elements (other than the zero element for general passive systems or the target constant input for incrementally passive systems). Furthermore, we present constructive algorithms to design such $m + 1$ input points that satisfy the conditions for practical stability using our proposed nearest-neighbor control.

Key words: Nonlinear passive systems; finite control set; output feedback; binary control; practical stabilization.

1 Introduction

The design of feedback control systems, where the space of input or output variables is a continuum or is discretized through the use of logarithmic or uniform quantizer, is well-studied in literature. This continuum assumption on the admissible input/output space has allowed us to analyze the stability and asymptotic behavior of the closed-loop systems through various approaches. However when there are design constraints imposed on the systems such that the input or output space is constrained to finite discrete sets, there are only a few control design approaches and control systems analyses that are applicable. Examples of such systems' design constraints are, for instance, the design of the power take-off systems of the Ocean Grazer wave energy converter (WEC), where the device can only actuate a fixed set of constant actuator systems [1,32] or a fixed configuration of constant thruster systems in space rockets. In these examples, the control input can only be assigned from a finite set of discrete values.

A number of control design methods and control systems analyses, where binary input or minimal information is considered, have been discussed, among many others, in [11,18] for linear systems case and in [8,9,15] for the networked control systems settings. As these papers consider the use of binary input values per input dimension, the stabilization of an m -dimensional input-output system implies that there should be at least 2^m admissible input values and the stabilizing control law must dynamically assign one of these values as control input at every time instances. In this paper, we shall focus on designing control laws with a minimal set of discrete control values whose cardinality is at most $m + 1$.

Let us consider nonlinear systems described by

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \quad (1)$$

where the state $x(t) \in \mathbb{R}^n$ and the input and output

signals $u(t), y(t) \in \mathbb{R}^m$. The functions f , g , and h are assumed to be continuously differentiable, $f(0) = 0$, $g(x)$ is full-rank for all x , and $h(0) = 0$. As we consider limited actuation / information transmission, the control input u can only take values from a finite discrete set $\mathcal{U} := \{u_0, u_1, u_2, \dots, u_p\}$ with $u_i \in \mathbb{R}^m$ for each $i = 0, \dots, p$. For such systems with a finite actuation set \mathcal{U} , assuming we have a stabilizing output feedback law $y \mapsto F(y)$ (when \mathcal{U} is continuum), two relevant questions for its stabilization are: a) how to map $F(y)$ to an element in \mathcal{U} ?; and b) how to determine the minimal cardinality of \mathcal{U} ? By addressing these questions, with generic output maps and nonlinear dynamics, our aim is to design a mapping $\phi : \mathbb{R}^m \rightarrow \mathcal{U}$, with \mathcal{U} be discrete (and possibly minimal), such that $u = \phi(F(y)) \in \mathcal{U}$ practically stabilizes Σ .

The question of designing the quantization mapping $\phi : \mathbb{R}^m \rightarrow \mathcal{U}$ has been addressed in various forms in literature. Since the input can only take the available values in the discrete set \mathcal{U} , the quantizer ϕ , in some sense, defines the partition of the output space with respect to \mathcal{U} . For instance, when $\mathcal{U} := \{-N, -N+1, \dots, N-1, N\}^m$, the partition may be a regular grid in \mathbb{R}^m and the existing work of [10,28,5,12,15,9] are directly applicable at the cost of some stability margin / level of accuracy $\epsilon > 0$. In some of the cases, the desired accuracy can be achieved by certain time-varying feedback law depending on the available quantization levels [21,27]. It remains a question on how to partition the output space when the set \mathcal{U} does not have particular structure as typically assumed in the standard quantizers (which is either logarithmic or uniform). In this paper, we address this by showing that, we can define a simple static mapping that maps $F(y)$ to the nearest element in \mathcal{U} which (practically) stabilizes the system. Dynamical systems where the inputs are computed from solving an optimization problem, and are discontinuous appear in different applications [2]. Passivity is a useful tool to analyze the overall system in such cases, and hence we restrict ourselves to passive systems in this work.

Passive systems have received attentions in many different research fields as they are able to model physical phenomena exhibited by almost all thermo-chemo-electromechanical systems [31,25]. In this regard, most of the aforementioned systems carry natural energy properties that can be related to passivity. In particular, such systems are to be said passive if the rate of change of the systems's "stored energy" never exceed the power supplied by the environment through their external ports. There are different classes of passivity. For example, incremental passivity and differential passivity. These variations of passivity, along with the "original" passivity notion have been shown to be useful for control design purposes [17,20]. We refer interested readers to the various expositions on passive systems in [26,25,19,30].

The second question of finding the minimal set \mathcal{U} for feedback stabilization has also received considerable attention. One question regarding this matter is on the minimal cardinality of the set \mathcal{U} . As an example, consider the work of [23]. In this paper, a discrete-time linear system, under some appropriate setting, is stabilizable if the number of bits per sample (rate of communication) is greater than the intrinsic entropy of the system. Similar results are available for continuous-time linear systems setting [7,6]. With certain passivity structure in the dynamics, Σ is practically stabilizable by using binary control for each input dimension which directly translates to $2^m + 1$ elements in \mathcal{U} , e.g., $\mathcal{U} = \{0\} \cup \{-1, 1\}^m$, see [8,15].

As a relaxation of these results, and dealing with rather generic class of multi-input multi-output passive nonlinear systems, we show in this paper that such practical stabilization can be achieved by simply using $m + 1$ elements in \mathcal{U} , in addition to $\{0\}$ or the required constant input u^* when the system is required to track a desired constant reference y^* . When quantization effect is of a particular concern, the interconnection of passive systems and quantizers has been studied for the past decade in various different contexts. For instance, the practical stability analysis of passive systems in a feedback loop with a quantizer using an adapted circle criterion for nonsmooth systems is presented in [16].

In summary, the main contributions of this paper are as follows. Firstly, we propose nearest-neighbor based control laws and analyze the stability of the closed-loop systems when the input u can only be taken from the finite discrete set \mathcal{U} . Secondly, we provide algorithmic procedure to construct minimal discrete sets that are able to practically stabilize the systems by mean of nearest-neighbor based control law. Furthermore, the corresponding bounds related to the proposed control algorithms are computed.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries on set-valued dynamics resulting from the use of nonsmooth control laws and on convex polytopes; and formulate the control problem. Our main results are presented in Section 3 and generalized in Section 4, where we study practical stabilization of (incrementally) passive nonlinear systems under the nearest neighbor control (NNC) approach. Some simple design on the minimal action set along with their construction procedures and properties associated to the NNC approach are provided and analyzed in Section 5. Finally, some concluding remarks are provided in Section 6.

2 Preliminaries and Problem Formulation

Notation: For a vector in \mathbb{R}^n , or a matrix in $\mathbb{R}^{m \times n}$, we denote the Euclidean norm and the corresponding

induced norm by $\|\cdot\|$. For a signal $z : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$, the essential supremum norm of z over an interval $I \subset \mathbb{R}_{\geq 0}$ is denoted by $\|z\|_I$. For any $c \in \mathbb{R}^n$, the set $\mathbb{B}_\epsilon(c) \subset \mathbb{R}^n$ is defined as, $\mathbb{B}_\epsilon(c) := \{\xi \in \mathbb{R}^n \mid \|\xi - c\| \leq \epsilon\}$. For simplicity, we write $\mathbb{B}_\epsilon(0)$ as \mathbb{B}_ϵ . The inner product of two vectors $\mu, \nu \in \mathbb{R}^m$ is denoted by $\langle \mu, \nu \rangle$. For a given set $\mathcal{S} \subset \mathbb{R}^m$, and a vector $\mu \in \mathbb{R}^m$, we let $\langle \mu, \mathcal{S} \rangle := \{\langle \mu, \nu \rangle \mid \nu \in \mathcal{S}\}$. For a discrete set \mathcal{U} , its cardinality is denoted by $\text{card}(\mathcal{U})$. The convex hull of vertices from a discrete set \mathcal{U} is denoted by $\text{conv}(\mathcal{U})$. The interior of a set $S \subset \mathbb{R}^n$ is denoted by $\text{int}(S)$. A unit vector whose i -th element is 1 and the other elements are 0 is denoted by e_i . A vector whose entries are 1 is denoted by $\mathbb{1}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class \mathcal{K} if it is continuous, strictly increasing, and $\gamma(0) = 0$. We say that $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class \mathcal{K}_∞ if γ is of class \mathcal{K} and unbounded.

2.1 Passive systems and observability notions

The central object of this paper is the nonlinear control systems Σ given in (1). The fundamental property that we associate with Σ is that, it is *passive*, i.e., for all pairs of input and output signals u, y , we have $\int_0^T \langle y(t), u(t) \rangle dt > -\infty$ for all $T > 0$; see [33,30,25] for some primary references on passive systems. By the well-known Hill-Moylan conditions, the passivity of Σ implies that there exists a positive definite storage function $H : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that $\langle \nabla H(x), f(x) \rangle \leq 0$ and $\langle \nabla H(x), g(x) \rangle = h^\top(x)$. Without loss of generality, we assume that the storage function H is *proper*, i.e. all level sets of H are compact.

Using the passivity assumption on Σ , it is immediate to see that $u \equiv 0$ implies that all level sets of H are positively invariant. More precisely, for any $c > 0$, if $H(x(0)) \leq c$ then $H(x(t)) \leq c$ for all $t \geq 0$. In other words, if we initialize the state of Σ such that $x(0) \in \Omega_c := \{\xi \mid H(\xi) \leq c\}$ with $u \equiv 0$ then $x(t) \in \Omega_c$ for all $t \geq 0$. We will use this property later to establish the practical stability of our closed-loop systems in conjunction with the following observability notion from [13]. System (1) has the *large-time initial-state norm observability* property if there exist $\tau > 0, \gamma, \chi \in \mathcal{K}_\infty$ such that the solution x of (1) satisfies

$$\|x(t)\| \leq \gamma(\|y\|_{[t, t+\tau]}) + \chi(\|u\|_{[t, t+\tau]})$$

for all $x(0), u, t \geq 0$. In particular, we will use the large-time initial-state norm observability property for the autonomous system (with $u = 0$):

$$\left. \begin{aligned} \dot{x} &= f(x) \\ y &= h(x). \end{aligned} \right\} \quad (2)$$

In this case, large-time initial-state norm observability

of (2) implies

$$\exists \tau > 0, \gamma \in \mathcal{K}_\infty \text{ such that, for each } x(0) \in \mathbb{R}^n, \\ \|x(t)\| \leq \gamma(\|y\|_{[t, t+\tau]}), \quad \forall t \geq 0. \quad (3)$$

We note that in the standard passivity-based control literature, the notion of zero-state observability or zero-state detectability is typically assumed for establishing the convergence of the state to zero in the Ω -limit set. However, these notions cannot be used to conclude the boundedness of the state trajectories given the bound on the output trajectories. Therefore, instead of using these notions, we will use the above large-time initial-state norm observability for deducing the practical stability based on the information on y in the Ω -limit set.

Remark 1 If the dynamics in system (2) are linear, that is, $\dot{x} = Ax$, $y = Cx$, and the pair (A, C) is observable, then one can quantify γ in (3) using the observability Gramian. In particular, if for $\tau > 0$

$$W_\tau(t) = \int_t^{t+\tau} e^{A^\top(s-t)} C^\top C e^{A(s-t)} ds$$

then $x(t) = (W_\tau(t))^{-1} \int_t^{t+\tau} e^{A^\top(s-t)} C^\top y(s) ds$, for each $t \geq 0$, and $\tau > 0$, which in particular yields

$$\|x(t)\| \leq \|(W_\tau(t))^{-1}\| \int_t^{t+\tau} \|e^{A^\top(s-t)} C^\top\| ds \sup_{s \in [t, t+\tau]} |y(s)|$$

for each $t \geq 0$, and any $\tau > 0$.

2.2 Stabilization problem with limited control

We are interested in feedback stabilization of the system Σ described in (1) using the output measurements. The key element of our problem is that the input u can only take values in a discrete set, which is finite. Thus, the objective is to find a reasonable way to map the outputs (taking values in \mathbb{R}^m) to a finite set such that the closed-loop system is stable in some appropriate sense. More formally, we address the following problem:

Practical output-feedback stabilization with limited control (POS-LC): For a given system Σ as in (1) and for a given ball \mathbb{B}_ϵ with $\epsilon > 0$, determine the minimal number p of elements $u_i \in \mathbb{R}^m$, and describe $\phi : \mathbb{R}^m \rightarrow \mathcal{U} := \{u_0, \dots, u_p\}$ such that the closed-loop system of (1) with $u = \phi(y)$ satisfies $x(t) \rightarrow \mathbb{B}_\epsilon$ as $t \rightarrow \infty$ for all initial conditions $x(0) \in \mathbb{R}^n$.

In our problem formulation, both the construction of a discrete set \mathcal{U} , as well as the design of the stabilizing map ϕ constitute our control problem. Compared to the numerous works in the literature on quantized control,

our job in solving POS-LC problem is facilitated under the passivity structure, along with the appropriate observability notion. In particular, for the first of results, we will work under the following basic assumption for solving POS-LC:

- (A0) The system Σ in (1) is passive with a proper storage function H and, the corresponding autonomous system (2) is large-time initial-state norm-observable for some $\tau > 0$ and $\gamma \in \mathcal{K}_\infty$.

2.3 Set-valued analysis: Basic notions

In studying the aforementioned control problem, we recall some fundamental definitions found in the literature on differential inclusions and convex polytopes, which would be useful for analysis in later sections.

2.3.1 Regularized Differential inclusions

It turns out that a mapping which maps output from a continuum to a discrete set of control actions is essentially discontinuous (with respect to usual topology on \mathbb{R}^m). Differential equations with such state-dependent discontinuities need regularization so that the solutions are properly defined. For a discontinuous map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we can define a set-valued map $\mathcal{K}(F)$ by convexifying F as follows

$$\mathcal{K}(F(x)) := \bigcap_{\delta > 0} \overline{\text{co}}(F(x + \mathbb{B}_\delta))$$

where $\overline{\text{co}}(S)$ is the convex closure of S . The set-valued mapping $\mathcal{K}(F)$ is the Krasovskii regularization of F , and under certain regularity assumptions on F , $\mathcal{K}(F)$ is compact and convex-valued, and moreover it is upper semicontinuous.¹ For an upper semicontinuous mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, consider the differential inclusion

$$\dot{x} \in \Phi(x) \quad x(0) = x_0. \quad (4)$$

A Krasovskii solution $x(\cdot)$ on an interval $I = [0, T)$, $T > 0$ is an absolutely continuous function $x : I \rightarrow \mathbb{R}^n$ such that (4) holds almost everywhere on I . It is *maximal* if it has no right extension and it is a *global* solution if $I = \mathbb{R}_{\geq 0}$. For any upper semicontinuous set-valued map Φ such that $\Phi(\xi)$ is compact and convex for all $\xi \in \mathbb{R}^n$, the following properties have been established (see, e.g., [16, Lemma 1]): (i). the differential inclusion (4) has a solution on an interval I ; (ii). every solution can be extended to a maximal one; and (iii). if the maximal solution is bounded then it is global.

¹ A set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called *upper semicontinuous* at x if for every open set X containing $\Phi(x) \subset \mathbb{R}^n$, there exists an open set Ξ containing x such that for all $\xi \in \Xi$, $\Phi(\xi) \subset X$. Correspondingly, Φ is *upper semicontinuous* if it is upper semicontinuous at every point in \mathbb{R}^n .

2.3.2 Convex polytopes

Next, we present the definition of convex polytopes and some of their notable examples that are related to our problem. We refer to [24] and [29] for additional material on this topic. In general, there are two basic representation of convex polytopes. Firstly, the vertex representation of a convex polytope in \mathbb{R}^m , or commonly referred to as the *V-representation*, is an m -polytope defined by the convex hull of a finite set of points in \mathbb{R}^m ; i.e. for any set of points $\mathcal{U} \subset \mathbb{R}^m$, the *V-representation* of a convex polytope defined by \mathcal{U} is given by $\mathcal{P}_V(\mathcal{U}) := \text{conv}(\mathcal{U})$. Another way to define an m -polytope is by intersecting finite-number of half-spaces, commonly referred to as the *H-representation*, that is given by $\mathcal{P}_H(A, b) := \{x \in \mathbb{R}^m \mid Ax \leq b\}$. Note that both V-representation and H-representation of m -polytopes are equivalent, i.e. $\mathcal{P}_V(\mathcal{U}) = \mathcal{P}_H(A, b)$ with appropriate $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$. When it is clear from the context, we will omit the arguments in \mathcal{P}_V and \mathcal{P}_H in the rest of this paper.

One simple example of m -polytopes is the m -dimensional simplex, commonly referred to as m -simplex. For particular examples, 1-simplex is a line, 2-simplex is a triangle, and 3-simplex is a tetrahedron. The formal definition of m -simplices is given by:

Definition 2 (m -simplex) Let $\mathcal{S} := \{s_0, s_1, \dots, s_m\}$ with $s_i \in \mathbb{R}^m$, $i = 0, 1, \dots, m$ be an affinely independent set, i.e. for any $s_i \in \mathcal{S}$, the set $\tilde{\mathcal{S}}_i := \{\tilde{s} \in \mathbb{R}^m \mid \tilde{s} = s_j - s_i, \forall s_j \in \mathcal{S} \setminus \{s_i\}\}$ is linearly independent. An m -simplex \mathcal{S}_m is defined by,

$$\mathcal{S}_m = \text{conv}(\mathcal{S}) := \left\{ \sum_{i=0}^m c_i s_i \mid \sum_{i=0}^m c_i = 1, c_i \geq 0 \right\},$$

and we say that $b_{\mathcal{S}_m} = \frac{1}{m+1} \sum_{i=0}^m s_i$ is its barycenter.

Example 1 One special case of m -simplices is a *regular* m -simplex $\mathcal{S}_{m, \text{reg}}$ where all vertices have equal distances to its barycenter and, one possibly simple choice for such a simplex is

$$\mathcal{S}_{m, \text{reg}} := \text{conv} \left(\lambda \left\{ e_1, \dots, e_m, \frac{1 - \sqrt{m+1}}{m} \mathbb{1} \right\} \right) \quad (5)$$

for some $\lambda \in \mathbb{R}_{>0}$.

Another notable example of m -polytopes is the m -dimensional hypercubes: the m -cubes and the m -cross-polytopes. For a given $\lambda \in \mathbb{R}_{>0}$, an m -cube \mathcal{C}_m is given by $\mathcal{C}_m := \{x \in \mathbb{R}^m \mid -\lambda \leq x_i \leq \lambda; i = 1, \dots, m\}$, and an m -cross-polytope \mathcal{C}_m^Δ is given by $\mathcal{C}_m^\Delta := \text{conv} \{\pm \lambda e_1, \dots, \pm \lambda e_m\}$.

For our purposes, the utility of convex polytopes is seen in partitioning the output space \mathbb{R}^m into a finite number

of cells which can then be associated to a control action. In particular, given a finite set $\mathcal{S} \subset \mathbb{R}^m$ with $\text{card}(\mathcal{S}) = q$, the space \mathbb{R}^m can be partitioned into q number of cells where every cell contains all points in \mathbb{R}^m that are closer to an element of \mathcal{S} than any other element. Such cells are commonly referred to as Voronoi cells and are defined as follows.

Definition 3 Consider a countable set $\mathcal{S} \subset \mathbb{R}^m$. The Voronoi cell of a point $s \in \mathcal{S}$ is defined by

$$V_{\mathcal{S}}(s) := \{x \in \mathbb{R}^m \mid \|x - s\| \leq \|x - v\|, \forall v \in \mathcal{S} \setminus \{s\}\}.$$

Remark 4 Note that every Voronoi cell is a closed and convex polyhedron since they can always be represented by the solution of a system of linear inequalities.

3 Nearest-Neighbor Control for Passive Systems

In this section, we provide our first solution for the general passive systems when the practical stabilization of the origin is required. The motivation behind our design of these elements is to work with minimal number of elements in the set \mathcal{U} which yield the desired performance using the static output feedback only. Toward this end, the only assumption we associate with the set \mathcal{U} is the following:

- (A1) For a given set $\mathcal{U} := \{u_0, u_1, u_2, \dots, u_p\}$, with $u_0 = 0$, there exists an index set $\mathcal{I} \subset \{1, \dots, p\}$ such that the set $\mathcal{V} := \{u_i\}_{i \in \mathcal{I}} \subset \mathcal{U}$ defines the vertices of a convex polytope satisfying, $0 \in \text{int}(\text{conv}(\mathcal{V}))$.

An immediate consequence of (A1) is the following lemma, which is used in the derivation of our forthcoming main result.

Lemma 5 Consider a discrete set $\mathcal{U} \subset \mathbb{R}^m$ that satisfies (A1). Then, there exists $\delta > 0$ such that

$$V_{\mathcal{U}}(0) \subseteq \mathbb{B}_{\delta}, \quad (6)$$

that is, the following implication holds for each $\eta \in \mathbb{R}^m$

$$\|\eta\| > \delta \Rightarrow \exists v_i \in \mathcal{U} \text{ s.t. } \|v_i + \eta\| < \|\eta\|. \quad (7)$$

PROOF. Based on Assumption (A1), consider the sets $\mathcal{I} := \{1, \dots, q\}$, $\mathcal{V} := \{u_i\}_{i \in \mathcal{I}} \subset \mathcal{U}$ such that $q \leq p$ and $0 \in \text{int}(\text{conv}(\mathcal{V}))$. Let $\mathcal{S} = \mathcal{V} \cup \{0\}$. From the definition of Voronoi cells, it readily follows that $V_{\mathcal{U}}(0) \subseteq V_{\mathcal{S}}(0)$, and therefore, it suffices to show that $V_{\mathcal{S}}(0) \subset \mathbb{B}_{\delta}$. Toward that end, we first observe that the Voronoi cell $V_{\mathcal{S}}(0)$ can be described as

$$V_{\mathcal{S}}(0) := \mathcal{P}_{\text{H}} \left(\begin{bmatrix} v_1 & \dots & v_q \end{bmatrix}^{\top}, \frac{1}{2} \begin{bmatrix} \|v_1\|^2 & \dots & \|v_q\|^2 \end{bmatrix}^{\top} \right). \quad (8)$$

Thus, from (8), we know that $V_{\mathcal{S}}(0)$ is a closed convex polyhedron. It remains to show that $V_{\mathcal{S}}(0)$ is bounded. Indeed, boundedness implies that we can choose $\delta = \max_{\tilde{v} \in V_{\mathcal{S}}(0)} (\|\tilde{v}\|)$, such that \mathbb{B}_{δ} is the smallest ball containing the set $V_{\mathcal{S}}(0)$, which by definition of Voronoi cell is equivalent to (7).

To show that $V_{\mathcal{S}}(0)$ is bounded, we observe that, under (A1), there exists $\mu > 0$ such that $\mathbb{B}_{\mu} \subset \text{conv}(\mathcal{V})$. Thus, for every $\tilde{v} \in V_{\mathcal{S}}(0)$, $\mu \frac{\tilde{v}}{\|\tilde{v}\|} \in \text{conv}(\mathcal{V})$. Hence, there exist $\lambda_i \geq 0$ such that $\sum_{i=1}^q \lambda_i = 1$ and $\mu \frac{\tilde{v}}{\|\tilde{v}\|} = \sum_{i=1}^q \lambda_i v_i$. Consequently, from (8), it follows that

$$\mu \frac{\tilde{v}^{\top} \tilde{v}}{\|\tilde{v}\|} = \sum_{i=1}^q \lambda_i v_i^{\top} \tilde{v} \leq \frac{1}{2} \sum_{i=1}^q \lambda_i \|v_i\|^2$$

and hence $\|\tilde{v}\| \leq \frac{1}{2\mu} \sum_{i=1}^q \lambda_i \|v_i\|^2$. \square

Example 2 A simple example of \mathcal{U} in \mathbb{R}^2 , satisfying (A1) is as follows:

$$\begin{aligned} \mathcal{U}_{\text{ex}} &:= \alpha \left\{ 0, \begin{bmatrix} \sin(\theta_{\text{ex}}) \\ \cos(\theta_{\text{ex}}) \end{bmatrix}, \begin{bmatrix} \sin(\theta_{\text{ex}} + \frac{2\pi}{3}) \\ \cos(\theta_{\text{ex}} + \frac{2\pi}{3}) \end{bmatrix}, \begin{bmatrix} \sin(\theta_{\text{ex}} + \frac{4\pi}{3}) \\ \cos(\theta_{\text{ex}} + \frac{4\pi}{3}) \end{bmatrix} \right\} \\ &=: \{0, u_{\text{ex},1}, u_{\text{ex},2}, u_{\text{ex},3}\} \end{aligned} \quad (9)$$

with some $\theta_{\text{ex}} \in \mathbb{R}$ and $\alpha \in (0, \infty)$. For this example, (A1) holds by taking $\mathcal{V} := \mathcal{U} \setminus \{0\}$. Following the proof of Lemma 5, we have $V_{\mathcal{U}}(0) := \text{conv}(\tilde{\mathcal{V}}_0)$ where

$$\tilde{\mathcal{V}}_0 := \alpha \left\{ \begin{bmatrix} \sin(\theta_{\text{ex}} + \frac{\pi}{3}) \\ \cos(\theta_{\text{ex}} + \frac{\pi}{3}) \end{bmatrix}, \begin{bmatrix} \sin(\theta_{\text{ex}} + \frac{3\pi}{3}) \\ \cos(\theta_{\text{ex}} + \frac{3\pi}{3}) \end{bmatrix}, \begin{bmatrix} \sin(\theta_{\text{ex}} + \frac{5\pi}{3}) \\ \cos(\theta_{\text{ex}} + \frac{5\pi}{3}) \end{bmatrix} \right\}.$$

Here, $\tilde{\mathcal{V}}_0$ contains all vertices of the Voronoi cell $V_{\mathcal{U}}(0)$. Then, then the smallest δ that satisfies (6) in Lemma 5 is given by $\delta = \alpha$. See Figure 1 for an illustration.

3.1 Unity output feedback

Using the result of Lemma 5 and the assumptions introduced thus far, we can define a feedback mapping ϕ which maps the measured outputs to the discrete set \mathcal{U} to achieve practical stabilization. In this regard, we first consider the mapping $\phi : \mathbb{R}^m \rightrightarrows \mathcal{U}$, defined as

$$\phi(y) := \arg \min_{v \in \mathcal{U}} \{\|v + y\|\}. \quad (10)$$

The feedback control $u = \phi(y)$, with ϕ given in (10), can be seen as a quantized version of the unity output feedback $y \mapsto -y$. This quantization rule maps $-y$ to the nearest element in the set \mathcal{U} with respect to the Euclidean distance. The partitions in the output space induced by such quantization rule indeed result in Voronoi

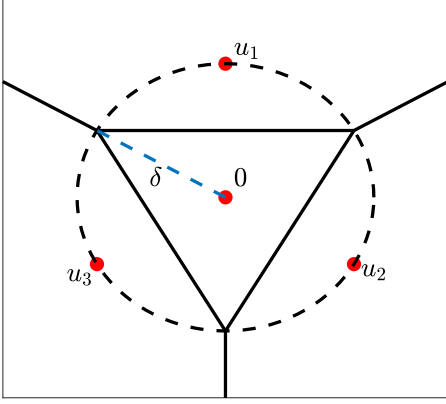


Fig. 1. Illustration of the nearest neighbor regions, or Voronoi cells, for discrete control actions set $\mathcal{U}_{ex} = \{0, u_{ex,1}, u_{ex,2}, u_{ex,3}\}$ given in Example 2. The triangular region around the origin is $V_{\mathcal{U}}(0)$ and it is contained in \mathbb{B}_{δ} , for some $\delta > 0$.

cells, and the resulting control law is hence discontinuous taking constant value in each of the Voronoi cells, see Fig. 1. By choosing $u = \phi(y)$, the closed system is thus given by

$$\begin{aligned} \dot{x} &= f(x) + g(x)\phi(y) \\ y &= h(x). \end{aligned} \quad (11)$$

As $\phi(y)$ is a non-smooth operator, we consider instead the following regularized differential inclusion

$$\begin{aligned} \dot{x} &\in \mathcal{K}(f(x) + g(x)\phi(y)) = f(x) + g(x)\mathcal{K}(\phi(y)) \\ y &= h(x). \end{aligned} \quad (12)$$

We note that the solution of (11) is basically interpreted in the sense of (12). In the following result, we analyze the asymptotic behavior of the solutions of (12) and show that they converge to \mathbb{B}_{ϵ} , for a given $\epsilon > 0$, if the elements of set \mathcal{U} satisfy certain conditions.

Proposition 6 *Consider a nonlinear system Σ described by (1) that satisfies (A0), and a discrete set $\mathcal{U} \subset \mathbb{R}^m$ satisfying (A1) so that (6) holds for some $\delta > 0$. For a given $\epsilon > 0$, assume that*

$$\gamma(\delta) \leq \epsilon. \quad (13)$$

Then the control law $u = \phi(y)$, with ϕ given in (10), globally practically stabilizes Σ with respect to \mathbb{B}_{ϵ} , that is, $\limsup_{t \rightarrow \infty} |x(t)| \leq \epsilon$.

PROOF. Without loss of generality, let us suppose that the set $\mathcal{V} \subset \mathcal{U}$, with the property that $0 \in \text{int}(\text{conv}(\mathcal{V}))$, admits the representation $\mathcal{V} =$

$\{u_1, \dots, u_p\}$. For a fixed $y \in \mathbb{R}^m$, suppose that $\phi(y) = \{u_i\}_{i \in J_y}$ for some $J_y \subset \{1, \dots, p\}$. It follows from (10) that $\{u_i\}_{i \in J_y}$ are the closest points to $-y$. Now, for each $i \in J_y$, we have that (see (8) also),

$$\langle u_i, -y \rangle \geq \frac{1}{2} \langle u_i, u_i \rangle \implies \langle u_i, y \rangle \leq -\frac{1}{2} \langle u_i, u_i \rangle \leq 0.$$

Therefore, for each $y \in \mathbb{R}^m$, and $u_i \in \phi(y)$, $i \in J_y$, we get

$$-\|u_i\| \cdot \|y\| \leq \langle u_i, y \rangle \leq -\frac{1}{2} \|u_i\|^2 \quad (14)$$

Based on this property of $\langle \phi(y), y \rangle$, we can now analyze the behavior of the closed-loop system given by (12).

Using the storage function H of the original system, for all solutions of (12), we have that either

(i): $0 \notin \phi(y) = \{u_i\}_{i \in J_y}$ for $J_y \subset \{1, \dots, p\}$. Let $\mathcal{W}_y := \phi(y)$, then

$$\begin{aligned} \dot{H}(x) &= \langle \nabla H(x), \dot{x} \rangle \in \langle \nabla H(x), f(x) + g(x)\mathcal{K}(\phi(y)) \rangle \\ &= \langle \nabla H(x), f(x) \rangle + \langle y, \text{conv}(\mathcal{W}_y) \rangle. \end{aligned}$$

Based on the computation of $\langle \phi(y), y \rangle$, with non-zero $\phi(y)$, it follows that

$$\langle y, \text{conv}(\mathcal{W}_y) \rangle \subset [-\|u_{y,\max}\| \|y\|, -0.5 \|u_{y,\min}\|^2],$$

where we let $\|u_{y,\max}\| := \max_{w \in \mathcal{W}_y} \|w\|$, and $\|u_{y,\min}\| := \min_{w \in \mathcal{W}_y} \|w\|$. Therefore,

$$\dot{H}(x) \leq -0.5 \|u_{y,\min}\|^2;$$

when $0 \notin \phi(y)$, or the other possibility is that, (ii): $0 \in \phi(y) = \{u_i\}_{i \in J_y} = \mathcal{W}_y$ with $J_y \subset \{1, \dots, p\}$. In this case, following the same arguments as in case (i)

$$\dot{H}(x) \in \langle \nabla H(x), f(x) \rangle + \langle y, \text{conv}(\mathcal{W}_y) \rangle.$$

Since $\{0\}$ is an element of \mathcal{W}_y ,

$$\langle y, \text{conv}(\mathcal{W}_y) \rangle \subset [-\|u_{y,\max}\| \|y\|, 0],$$

where $\|u_{y,\max}\|$ is defined as in case (i). This implies that, for the case when $0 \in \phi(y)$, we have

$$\dot{H}(x) \leq 0.$$

As $H(x)$ is non-increasing along system trajectories in both the cases (i) and (ii), and since H is proper, all system trajectories are bounded and contained in the compact set $\Omega_0 := \{z \in \mathbb{R}^n \mid H(z) \leq H(x(0))\}$. Let $\mathcal{Z}_x := \{z \in \mathbb{R}^n \mid \phi(h(z)) = 0\}$ and let M be the largest invariant

set (with respect to system (12)) contained in Z_x . By the LaSalle invariance principle, all trajectories belonging to the compact set Ω_0 converge to the set M , see for example [2, Theorem 6.5].

We next show that, because of the large-time norm observability and Lemma 5, it holds that $M \subset \mathbb{B}_\epsilon$. To see this, take an arbitrary point $z \in M$, and consider a solution of system (12) over an interval $[s, s + \tau]$ starting from z ; that is, consider $x : [s, s + \tau] \rightarrow \mathbb{R}^n$ which solves (12) and $x(s) = z \in M$. Due to the forward invariance of set M , the corresponding solution $x(t) \in M$, for each $t \in [s, s + \tau]$. Consequently, $\phi(h(x(t))) = 0$, and because of Lemma 5, $|h(x(t))| \leq \delta$ for each $t \in [s, s + \tau]$. Invoking the large-time initial state norm-observability assumption, it holds that $\|x(s)\| = \|z\| \leq \gamma(\delta) \leq \epsilon$, where the last inequality is a consequence of (13). Since $z \in M$ is arbitrary, it holds that $M \subset \mathbb{B}_\epsilon$.

In summary, we have shown that

$$x(t) \rightarrow M \subset \mathbb{B}_\epsilon \quad \text{as } t \rightarrow \infty,$$

for all initial conditions $x(0) \in \mathbb{R}^n$, and hence the desired assertion holds. \square

As first application of Proposition 6, we are interested in specifying the invariant set when the set of control actions is described by a set of equidistant points along each axis of the output space.

Corollary 7 *Consider the system Σ as in (1) satisfying (A0), and $\mathcal{U} = \lambda\{-N, -N + 1, \dots, N - 1, N\}^m$, with $\lambda > 0$ being the step size and N a positive integer. Then the control law $u = \phi(y)$, where ϕ is as in (10), globally practically stabilizes Σ with respect to \mathbb{B}_ϵ where $\epsilon > 0$ satisfies $\gamma(\lambda\sqrt{m}) \leq \epsilon$.*

PROOF. The proof follows *mutatis mutandis* the proof of Proposition 6. The set \mathcal{U} satisfies (A1) by taking $\mathcal{V} = \lambda\{-1, 1\}^m \setminus \{0\}$. It is also seen that $\delta = \lambda\sqrt{m}$, and by requiring $\gamma(\lambda\sqrt{m}) \leq \epsilon$, all the hypotheses of Proposition 6 hold. \square

Remark 8 In contrast to the choice of \mathcal{U} in Example 2 where we used (9) to construct the discrete set \mathcal{U} in \mathbb{R}^2 , the constant δ in Corollary 7 is less than $\max_{\tilde{v} \in \tilde{\mathcal{V}}} \|\tilde{v}\|$. This is due to the choice of the set \mathcal{V} in the proof of Corollary 7 that is dense enough such that $\{y \mid \phi(y) = 0\} \subset \text{conv}(\mathcal{V})$. From this corollary, one can conclude that two-level quantization with $N = 1$ suffices to get a global practical stabilization property for passive nonlinear systems. This binary control law restricts however the convergence rate of the closed-loop system. It converges to the desired compact ball in a linear fashion and may not be desirable when the initial condition is

very far from the origin. The use of higher quantization level (e.g., $N \gg 1$) can provide a better convergence rate when it is initialized within the quantization range.

3.2 Sector bounded feedback

We next present a generalization of the result in Proposition 6 on how the nearest neighbor rule can be used to quantize more generic nonlinear feedback laws. In Proposition 6, when \mathcal{U} is the continuum space of \mathbb{R}^m , the resulting control law is simply given by $u = -y$, i.e., it is a unity output feedback law. Using standard result in passive systems theory, the closed-loop system will satisfy $\dot{H} \leq -\|y\|^2$ and the application of La-Salle invariance principle with zero-state detectability allows us to conclude that $x(t) \rightarrow 0$ asymptotically. As the underlying system is passive, we can in fact stabilize it with any sector-bounded nonlinear feedback of the form $y \mapsto -F(y)$, where $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ satisfies

$$k_1\|y\|^2 \leq \langle F(y), y \rangle \leq k_2\|y\|^2, \quad 0 < k_1 \leq k_2 \quad (15a)$$

$$\|F(y)\| \leq k_3\|y\|, \quad k_3 \geq k_1, \quad (15b)$$

for all $y \in \mathbb{R}^m$. There are a number of reasons for considering such feedback laws rather than the unity output feedback law. For instance, we can attain a prescribed L_2 -gain disturbance attenuation level or we can shape the transient behavior by adjusting the gains on different domain of y . In the following proposition, we consider such sector-bounded output feedback law $F(y)$, and how the nearest neighbor rule can be used to map such feedbacks in the limited control input set \mathcal{U} to guarantee practical stabilization.

Proposition 9 *Consider a nonlinear system Σ described by (1) that satisfies (A0), and a discrete set $\mathcal{U} \subset \mathbb{R}^m$ satisfying (A1) so that (6) holds for some $\delta > 0$. For the mapping ϕ given in (10), let $\mu_{\min,1} \in (0, 1]$ be such that², for all $z \in \mathbb{R}^m$,*

$$\phi(z) \neq 0 \Rightarrow \langle \phi(z), -z \rangle \geq \|\phi(z)\| \|z\| \mu_{\min,1}. \quad (16)$$

Assume that the constants k_1, k_2, k_3 describing the function F , as in (15), satisfy

$$\frac{k_1^2}{k_3^2} + \mu_{\min,1}^2 > 1 \quad (17a)$$

$$\gamma(\delta/k_1) \leq \epsilon \quad (17b)$$

for a given $\epsilon > 0$. Then the control law $u = \phi(F(y))$ globally practically stabilizes Σ with respect to \mathbb{B}_ϵ .

² The existence of such $\mu_{\min,1}$ is guaranteed by the assumption (A1) on \mathcal{U} .

PROOF. We basically show that, for any $y \in \mathbb{R}^m$, we have

$$\langle \phi(F(y)), y \rangle \in \{-\kappa_{i,y} \|u_i\| \|y\| \mid i \in J_y\} \quad (18)$$

for some $J_y \subset \{1, \dots, p\}$ such that $\phi(F(y)) = \{u_i\}_{i \in J_y}$ and $\kappa_{i,y} > 0$. The rest of the proof follows a pattern similar to that of Proposition 6.

First, suppose that $\phi(F(y)) = \{u_i\}_{i \in J_y}$ for some J_y and $0 \notin \phi(F(y))$. It follows from (10) that $\{u_i\}_{i \in J_y}$ are the closest points to $-F(y)$ which implies that

$$\langle \phi(F(y)), -F(y) \rangle \in \{\|u_i\| \|F(y)\| \mu_{i,1} \mid i \in J_y\}, \quad (19)$$

where $\mu_{i,1} > 0$ is such that $\langle u_i, -F(y) \rangle = \|u_i\| \|F(y)\| \mu_{i,1}$. By the hypothesis of the proposition, $\mu_{\min,1} \leq \mu_{i,1}$ for any i . On the other hand,

$$\langle -F(y), -y \rangle = \|F(y)\| \|y\| \mu_2. \quad (20)$$

Since $k_1 \|y\|^2 \leq \langle F(y), y \rangle$ and $\|F(y)\| \leq k_3 \|y\|$, the minimum value of μ_2 (with respect to all choices of F that satisfy (15)) is given by $\mu_{\min,2} = k_1/k_3$.

Now, note that, in general, $\kappa_{i,y} \in [-1, 1]$. It can be shown that if (16), (17a), and (20) hold with $\mu_2 \in [\mu_{\min,2}, 1]$, then there exist $\kappa_{\min} > 0$ such that $\kappa_{i,y} \in [\kappa_{\min}, 1]$. For each $y \in \mathbb{R}^m$ and $i \in J_y$, we introduce the Gram matrix $G_{i,y}$ as

$$G_{i,y} = \begin{bmatrix} \langle -y, -y \rangle & \langle -y, -F(y) \rangle & \langle -y, u_i \rangle \\ \langle -y, -F(y) \rangle & \langle -F(y), -F(y) \rangle & \langle -F(y), u_i \rangle \\ \langle -y, u_i \rangle & \langle -F(y), u_i \rangle & \langle u_i, u_i \rangle \end{bmatrix},$$

having the property that (see also [4]) $G_{i,y} \succcurlyeq 0$ and thus $\det(G_{i,y}) \geq 0$. This implies that

$$\begin{aligned} 0 &\leq \|y\|^2 \|F(y)\|^2 \|u_i\|^2 \\ &\quad + 2 \langle -y, -F(y) \rangle \langle -F(y), u_i \rangle \langle -y, u_i \rangle \\ &\quad - \|y\|^2 \langle -F(y), u_i \rangle^2 - \|F(y)\|^2 \langle -y, u_i \rangle^2 \\ &\quad - \|u_i\|^2 \langle -y, -F(y) \rangle^2. \end{aligned}$$

By rewriting above inequality in terms of their respective norms and constants $\mu_{i,1}, \mu_2$, and $\kappa_{i,y}$, we have that, for each $u_i, i \in J_y$

$$\begin{aligned} \kappa_{i,y}^2 - 2 \mu_{i,1} \mu_2 \kappa_{i,y} &\leq 1 - (\mu_{i,1}^2 + \mu_2^2) \\ \Rightarrow (\kappa_{i,y} - \mu_{i,1} \mu_2)^2 &\leq 1 - (\mu_{i,1}^2 + \mu_2^2) + \mu_{i,1}^2 \mu_2^2 \\ \Leftrightarrow |\kappa_{i,y} - \mu_{i,1} \mu_2| &\leq \sqrt{1 - (\mu_{i,1}^2 + \mu_2^2) + \mu_{i,1}^2 \mu_2^2} \end{aligned}$$

where the last inequality always hold when $\kappa_{i,y} \geq \mu_{i,1} \mu_2 \geq 0$. On the other hand, when $\kappa_{i,y} \leq \mu_{i,1} \mu_2$,

the condition (17a) gives the following result

$$\begin{aligned} \kappa_{i,y} &\geq \mu_{i,1} \mu_2 - \sqrt{1 - (\mu_{i,1}^2 + \mu_2^2) + \mu_{i,1}^2 \mu_2^2} \\ &= \mu_{i,1} \mu_2 - \sqrt{(1 - \mu_{i,1}^2)(1 - \mu_2^2)} \\ &\geq \mu_{\min,1} \frac{k_1}{k_3} - \sqrt{(1 - \mu_{\min,1}^2) \left(1 - \frac{k_1^2}{k_3^2}\right)} \\ &= \mu_{\min,1} \frac{k_1}{k_3} - \sqrt{1 - \left(\mu_{\min,1}^2 + \frac{k_1^2}{k_3^2}\right) + \mu_{\min,1}^2 \frac{k_1^2}{k_3^2}} \\ &> \mu_{\min,1} \frac{k_1}{k_3} - \sqrt{\mu_{\min,1}^2 \frac{k_1^2}{k_3^2}} = 0. \end{aligned}$$

Note that above arguments hold for all $i \in J_y$ and (18) holds for some $\kappa_{i,y} > 0$.

Following the same line of arguments as in the proof of Proposition 6, (18) implies that the storage function is nondecreasing along the solutions of the closed-loop system and converge to a set M , where M is the largest invariant set contained in $\mathcal{X}_x := \{z \in \mathbb{R}^n \mid \phi(F(h(z))) = 0\}$. Hence for any trajectory starting with initial condition $x(s) = z \in M$, it holds that the corresponding output satisfies $\|F(y(t))\| \leq \delta$ for all $t \geq s$. Since $k_1 \|v\|^2 \leq \langle F(v), v \rangle \leq \|F(v)\| \|v\|$ holds for all $v \in \mathbb{R}^m$, it follows that $\|y(t)\| \leq \frac{\delta}{k_1}$ for all $t \geq s$. By the property of large-time initial-state norm-observability of (2), it holds that,

$$\|z\| = \|x(s)\| \leq \gamma(\delta/k_1) \leq \epsilon \quad \forall t \geq s$$

and this holds for each $z \in M$. Hence, $M \subseteq \mathbb{B}_\epsilon$ and in particular, each trajectory converges to \mathbb{B}_ϵ as $t \rightarrow \infty$. \square

Remark 10 The condition (17a) requires that the nonlinearity should lie in a relatively thin sector bound. When $F(y) = ky$, i.e., it is a proportional controller with a scalar gain $k > 0$, then the condition (17a) holds trivially, since $\mu_{\min,1} > 0$ and $\frac{k_1}{k_3} = \frac{k}{k} = 1$. Consequently, it follows from this proposition that we can make the practical stabilization ball arbitrary small by assigning a large gain k .

3.3 An illustrative example

Example 3 Consider the following nonlinear system

$$\Sigma_{\text{ex}} : \begin{cases} \dot{x} = \begin{bmatrix} -x_2 + x_3^3 \\ x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \\ y = \begin{bmatrix} x_1 \\ x_3^3 \end{bmatrix} \end{cases} \quad (21)$$

where $x := [x_1 \ x_2 \ x_3]^\top \in \mathbb{R}^3$ and $y := [y_1 \ y_2]^\top$, $u := [u_1 \ u_2]^\top \in \mathbb{R}^2$. It can be checked that by using the proper storage function $H(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}x_3^4$, the system Σ_{ex} is passive. Indeed, a straightforward computation gives us $\dot{H} = \langle y, u \rangle$.

We will now show that Σ_{ex} satisfies the large-time initial-state norm observability condition. As the bound on x_3 for the large-time norm observability can directly be obtained from the output y , we need to compute the bound on $[x_1 \ x_2]^\top$. If we consider the sub-system of $[x_1 \ x_2]^\top$ with x_1 as its output (and is equal to y_1), it is a linear system with $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and its input is $x_3^3 = y_2$. Thus as (A, C) is observable, the observability Gramian is given by

$$W_\pi(t) = \int_t^{t+\pi} e^{A^\top(s-t)} C^\top C e^{A(s-t)} ds = \frac{\pi}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

whose inverse is simply given by $W_\pi^{-1} = \frac{2}{\pi} I_2$ and $\|W_\pi^{-1}\| = \frac{2}{\pi}$. Then for any $t > 0$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = W_\pi^{-1} \int_t^{t+\pi} e^{A^\top(s-t)} C^\top \left(x_1(s) - \left(H * \begin{bmatrix} x_3^3 \\ 0 \end{bmatrix} \right)(s) \right) ds,$$

where $*$ denotes the convolution operation and H is the convolution matrix kernel given by $H(t) = C e^{At}$. Since $\|e^{At}\| = 1$ for all t , it follows then that

$$\begin{aligned} & \left\| \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \right\| \\ & \leq \|W_\pi^{-1}\| \int_t^{t+\pi} \left\| e^{A^\top(s-t)} C^\top \left\| x_1(s) - \left(H * \begin{bmatrix} x_3^3 \\ 0 \end{bmatrix} \right)(s) \right\| ds \\ & \leq \frac{2}{\pi} \pi (\|y_1\|_{[t, t+\pi]} + \|y_2\|_{[t, t+\pi]}) \\ & \leq 4\|y\|_{[t, t+\pi]}. \end{aligned}$$

Since by the definition of y , $\|x_3\|_{[t, t+\pi]} = \|y_2\|_{[t, t+\pi]}^{\frac{1}{3}} \leq \|y\|_{[t, t+\pi]}^{\frac{1}{3}}$, it follows from the inequality above that

$$\|x(t)\| \leq 4\|y\|_{[t, t+\pi]} + \|y\|_{[t, t+\pi]}^{\frac{1}{3}}.$$

In other words, the function γ in (3) is given by $\gamma(s) = 4s + s^{\frac{1}{3}}$.

We can now use the results in Proposition 6 to practically stabilize Σ_{ex} . We choose the control set to be \mathcal{U}_{ex} given in (9), and the desired stability margin to be $\epsilon = 1$. Then, based on the function γ computed for the system Σ_{ex} , we get $\gamma(\delta) < 1$ if $\delta \in (0, \frac{1}{8}]$. Using the same discrete

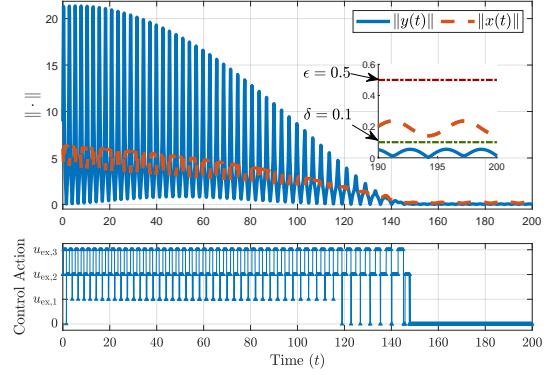


Fig. 2. Simulation results of Σ_{ex} using the control approach proposed in the Proposition 6 with discrete input set \mathcal{U}_{ex} as in (9) and fixed parameters $\theta_{\text{ex}} = 0$ and $\alpha = 0.1$. It can be seen that once both the state x and the output y enters their respective convergence ball, the control input is zero.

set as in (9) along with the function ϕ as in (10), we can fix $\theta_{\text{ex}} = 0$ and $\alpha = 0.1$ such that the system Σ_{ex} is globally practically stable with respect to \mathbb{B}_ϵ , with $\epsilon = 1$, as shown in the simulation results in Figure 2.

4 Nearest-Neighbor Control for Incrementally Passive Systems With Constant Inputs

In many cases, the desired equilibrium point of the passive nonlinear system Σ as in (1) is not equal to the minimum of the associated storage function H . Instead, it may correspond to an arbitrary constant input. For these cases, a constant input $u^* \in \mathbb{R}^m$ with its corresponding steady-state solution $x^* \in \mathbb{R}^n$ defines the steady-state relation given by the set

$$\mathcal{E} := \left\{ (x^*, u^*) \in \mathbb{R}^n \times \mathbb{R}^m \mid 0 = f(x^*) + g(x^*)u^* \right\}. \quad (22)$$

The problem of practically stabilizing the system Σ around $x^* \in \mathbb{R}^n$ is equivalent to practically stabilizing $\bar{x} = x - x^*$ around the origin, with $\bar{(\cdot)} = (\cdot) - (\cdot)^*$ denoting the incremental variable. Thus, the incremental system is given by

$$\bar{\Sigma} : \begin{cases} \dot{\bar{x}} = \bar{f}(\bar{x}) + g(\bar{x} + x^*)\bar{u}, \\ \bar{y} = h(\bar{x} + x^*) - h(x^*), \end{cases} \quad (23)$$

with

$$\bar{f}(\bar{x}) = f(\bar{x} + x^*) - f(x^*) + (g(\bar{x} + x^*) - g(x^*))u^*.$$

For this matter, the passivity of the mapping $\bar{u} \mapsto \bar{y}$ is, in the original system Σ , referred to as incremental passivity with respect to constant input; and is defined as follows [17].

Definition 11 (Constant Incremental Passivity)

Consider the nonlinear system Σ as in (1). The system Σ is said to be incrementally passive with respect to constant input if, for every $(x^*, u^*) \in \mathcal{E}$, the corresponding incremental system $\bar{\Sigma}$ in (23) with input \bar{u} and output \bar{y} , is passive; that is, there exists a storage function $H_0 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\dot{H}_0 = \langle \nabla H_0, \dot{\bar{x}} \rangle \leq \langle \bar{u}, \bar{y} \rangle. \quad (24)$$

Note that constant incremental passivity defined above is equivalent to shifted passivity as in [30,22] and equilibrium-independent passivity as in [14]. Nevertheless, the term constant incremental passivity is preferred in this paper because the pair (x^*, u^*) can be arbitrary and most importantly, the incremental function is used in the definition. In the remainder of this section, we study stabilization of incrementally passive systems with finite set of control actions.

4.1 Steady-state $u^* \in \mathcal{U}$

In the case of constant incremental passivity, the corresponding constant input u^* is often known from the knowledge of the nominal system (1). Then we can simply design the finite input set \mathcal{U} such that it contains u^* . Thus it is natural to adapt the assumption (A1) to the current setting that brings us to the following proposition.

Proposition 12 Consider the system Σ as in (1), and a finite set of control actions $\mathcal{U} = \{u_0, u_1, \dots, u_p\} \subset \mathbb{R}^m$. Assume that:

- (A2) Σ is constant-incrementally passive with the proper storage function $H_0(x, x^*)$ for all pair $(x^*, u^*) \in \mathcal{E}$;
- (A3) $u^* \in \mathcal{U}$, with $u_0 = u^*$, and there exists a subset \mathcal{V} of \mathcal{U} such that $u^* \in \text{int}(\text{conv}(\mathcal{V}))$; and
- (A4) the autonomous incremental system $\bar{\Sigma}$ with $u = u^*$ is large-time initial-state norm-observable, i.e. there exists $\tau > 0$ and $\bar{\gamma} \in \mathcal{K}_\infty$ such that the solution of the autonomous incremental system $\bar{\Sigma}_{u=u^*}$ satisfies

$$\|\bar{x}(t)\| \leq \bar{\gamma} \left(\|\bar{y}\|_{[t, t+\tau]} \right)$$

for all $\bar{x}(0) \in \mathbb{R}^n, t \geq 0$.

Furthermore, for a given $\epsilon > 0$, assume that $\bar{\gamma}(\delta) \leq \epsilon$, where $\delta > 0$ is the smallest number that satisfies

$$V_{\mathcal{U}}(u^*) \subseteq \mathbb{B}_\delta(u^*). \quad (25)$$

Then the control law $u = \phi(\bar{y} - u^*)$, with $\phi : \mathbb{R}^m \rightarrow \mathcal{U}$ defined in (10), globally practically stabilizes Σ with respect to $\mathbb{B}_\epsilon(x^*)$.

PROOF. We start by introducing the set $\bar{\mathcal{U}}$, which is a translation of the set \mathcal{U} , and is defined as

$$\bar{\mathcal{U}} := \{\bar{v} \in \mathbb{R}^m \mid \bar{v} = v - u^*; v \in \mathcal{U}\}. \quad (26)$$

Since $u = \phi(\bar{y} - u^*) \in \mathcal{U}$, the incremental input variable $\bar{u} = u - u^* \in \bar{\mathcal{U}}$. We now consider the mapping $\bar{\phi} : \mathbb{R}^m \mapsto \bar{\mathcal{U}}$ given by

$$\bar{\phi}(\eta) := \arg \min_{\bar{v} \in \bar{\mathcal{U}}} \{\|\bar{v} + \eta\|\}. \quad (27)$$

The desired statement can then be proven by noting that,

$$\bar{\phi}(\bar{y}) = \phi(\bar{y} - u^*) - u^*. \quad (28)$$

Following the same argument as in the proof of Proposition 6, as the mapping ϕ is a non-smooth operator, we consider instead the differential inclusion

$$\begin{aligned} \dot{\bar{x}} &\in \mathcal{K}(\bar{f}(\bar{x}) + g(\bar{x} + x^*)(\phi(\bar{y} - u^*) - u^*)) \\ &= \bar{f}(\bar{x}) + g(\bar{x} + x^*)\mathcal{K}(\phi(\bar{y} - u^*) - u^*) \\ &= \bar{f}(\bar{x}) + g(\bar{x} + x^*)\mathcal{K}(\bar{\phi}(\bar{y})), \end{aligned} \quad (29)$$

$$\bar{y} = h(\bar{x} + x^*) - h(x^*).$$

Then, using the storage function H_0 for system (29), introduced in (A2), we can consider the two cases:

(i): Take $y \in \mathbb{R}^m \setminus \mathbb{B}_\delta(u^*)$, and let $\bar{\phi}(\bar{y}) = \{u_i - u^*\}_{i \in J_y} =: \bar{\mathcal{W}}_y$ for $J_y \subset \{1, \dots, p\}$, so that $0 \notin \bar{\mathcal{W}}_y$, then

$$\begin{aligned} \dot{H}_0 &\in \langle \nabla H_0(\bar{x}), \bar{f}(\bar{x}) + g(\bar{x} + x^*)\mathcal{K}(\bar{\phi}(\bar{y})) \rangle \\ &= \langle \nabla H_0(\bar{x}), \bar{f}(\bar{x}) \rangle + \langle \bar{y}, \text{conv}(\bar{\mathcal{W}}_y) \rangle \\ &\subset (-\infty, 0] + \langle \bar{y}, \text{conv}(\bar{\mathcal{W}}_y) \rangle. \end{aligned}$$

Adapting the computation of $\langle \phi(y), y \rangle$ for nonzero $\phi(y)$ in Proposition 6 to the computation of $\langle \bar{\phi}(\bar{y}), \bar{y} \rangle$ for nonzero $\bar{\phi}(\bar{y})$, we can conclude that, for $y \in \mathbb{R}^m \setminus \mathbb{B}_\delta(u^*)$, we have

$$\dot{H}_0 \in (-\infty, 0),$$

(ii): similarly, for any $y \in \mathbb{B}_\delta(u^*)$, we have that

$$\dot{H}_0 \in (-\infty, 0].$$

Therefore, the proper storage function $H_0(\bar{x})$ is non-increasing in both cases of (i) and (ii), and \dot{H}_0 vanishes only when $x \in \bar{\mathcal{Z}}_x := \{z \in \mathbb{R}^n \mid \bar{\phi}(h(z) - h(x^*)) = 0\}$. By LaSalle's invariance principle, all trajectories of (29) converge to the largest invariant set $\bar{M} \subset \mathbb{R}^n$ contained in $\bar{\mathcal{Z}}_x$. Invoking (A3), and the reasoning similar to the proof of Proposition 6, we can conclude that for each $z \in \bar{M}$,

$$\|z\| \leq \bar{\gamma}(\delta) \leq \epsilon,$$

and hence, $\bar{x}(t) \rightarrow \bar{M} \subset \mathbb{B}_\epsilon$ as $t \rightarrow \infty$. Therefore, in the original coordinate of Σ , it holds that $x(t) \rightarrow \mathbb{B}_\epsilon(x^*)$ as $t \rightarrow \infty$. \square

4.2 Sector bounded feedback

Similar to the results in the previous section, sector bounded nonlinear mapping F that satisfies (15) can easily be included in the constant-incrementally passive systems case. This is due to the fact given by (28). Then the following proposition is true.

Proposition 13 *Consider a nonlinear system Σ described by (1) that satisfies (A2) and (A4); and a discrete set $\mathcal{U} \subset \mathbb{R}^m$ satisfying (A3) so that (25) holds for some $\delta > 0$. Let ϕ be as given in (10); and let $\mu_{\min,1} \in (0, 1]$ be such that (16) holds for all $z \in \mathbb{R}^m$. Assume that (17a) holds with the mapping F , along with constants k_1, k_2, k_3 , satisfying (15). For a given $\epsilon > 0$, assume that*

$$\bar{\gamma}(\delta/k_1) \leq \epsilon.$$

Then, the control law $u = \phi(F(\bar{y}) - u^)$ globally practically stabilizes Σ with respect to $\mathbb{B}_\epsilon(x^*)$.*

4.3 Revisiting an illustrative example

Example 3 (Continued) Consider the nonlinear system Σ_{ex} along with the associated storage function $H(x)$ as in Example 3. It can be shown easily (following the main results in [17]) that Σ_{ex} is constant-incrementally passive. Indeed, for any $(x^*, u^*) \in \mathcal{E}$, we can define

$$H_0(x, x^*) = \frac{1}{2}(x_1 - x_1^*)^2 + \frac{1}{2}(x_2 - x_2^*)^2 + \frac{1}{4}(x_3 - x_3^*)^4.$$

It follows immediately that $\dot{H}_0 = \langle \bar{y}, \bar{u} \rangle$.

We will now show that the autonomous incremental system of Σ_{ex} satisfies the large-time initial-state norm observability conditions. Let the function $\bar{\gamma}$ be computed by considering $u = u^*$ for all $t \geq 0$ as provided in the following. Consider the incremental system of Σ_{ex} with $u = u^*$ for all $t \geq 0$, i.e.

$$\bar{\Sigma}_{\text{ex}} \Big|_{u=u^*} : \begin{cases} \dot{\bar{x}} = \begin{bmatrix} -x_2 + x_2^* \\ x_1 - x_1^* \\ -x_1 + x_1^* \end{bmatrix} + \begin{bmatrix} \bar{y}_2 \\ 0 \\ 0 \end{bmatrix}, \\ \bar{y} = \begin{bmatrix} x_1 - x_1^* \\ x_3^2 - x_3^{*2} \end{bmatrix}. \end{cases} \quad (30)$$

Following the computation in Example 3, we first compute the bound on the subsystem $\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$ by considering \bar{y}_1 as the input and \bar{y}_2 as the output. Then we have a linear system with $\bar{A} = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\bar{B} = B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\bar{C} = C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Hence, following a similar routine computation as before, we get

$$\begin{aligned} \left\| \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} \right\| &\leq 2 (\|\bar{y}_1\|_{[t,t+\pi]} + \|\bar{y}_2\|_{[t,t+\pi]}) \\ &\leq 4\|\bar{y}\|_{[t,t+\pi]}. \end{aligned}$$

Accordingly, for \bar{x}_3 , we have that

$$\bar{x}_3 = \frac{\bar{y}_2}{x_3^2 + x_3^{*2} + x_3 x_3^*}.$$

For all $x_3^* \neq 0$, we have that $x_3^2 + x_3^{*2} + x_3 x_3^* \geq \frac{3}{4}x_3^{*2}$, for all x_3 . Hence,

$$\begin{aligned} \|\bar{x}(t)\| &\leq \left\| \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} \right\| + \|\bar{x}_3\| \\ &\leq 4\|\bar{y}\|_{[t,t+\pi]} + \frac{4}{3x_3^{*2}} \|\bar{y}_2\|_{[t,t+\pi]}^2 \\ &\leq 4\|\bar{y}\|_{[t,t+\pi]} + \frac{4}{3x_3^{*2}} \|\bar{y}\|_{[t,t+\pi]}^2. \end{aligned}$$

In other words, the large-time initial-state norm-observability function is given by $\bar{\gamma}(s, x_3^*) = 4s + \frac{4}{3x_3^{*2}}s^2$.

We can now use the results in Proposition 12 to practically stabilize Σ_{ex} around any arbitrary steady-state relation $(x^*, u^*) \in \mathcal{E}$. Fix $x^* = [0 \ 0 \ -1]^\top$, $u^* = [1 \ 0]^\top$, and $\epsilon = 0.5$. Then, by the large-time initial-state norm-observability property of the incremental system, we can choose $\delta = 0.1$ to generate the discrete set of control actions. In this case, we can translate the previously used discrete set such that u^* is in the realizable control actions, i.e. $\bar{\mathcal{U}}_{\text{ex}} := \mathcal{U}_{\text{ex}} + u^*$ with \mathcal{U}_{ex} be as the discrete input set used in Example 3. The mapping control law with the mapping ϕ can then be demonstrated as shown in Figure 3.

5 Minimal Control Actions: Constructions and Bounds

In the earlier sections, we have shown that a nearest neighbor approach is a powerful tool for global practical stabilization of passive nonlinear systems. Indeed, when we are given a limited choice of static control inputs, assumptions (A1) and (A3) provide us a way to check whether the given set of inputs is applicable by means of nearest neighbor approach for the practical stabilization problem. If these assumptions hold for a finite set \mathcal{U} , then it is of interest to compute the smallest number $\delta > 0$ associated with Voronoi cell $V_{\mathcal{U}}(u^*)$, such that

$$V_{\mathcal{U}}(u^*) \subset \mathbb{B}_\delta.$$

Since our control design achieves convergence up to a ball of radius $\gamma(\delta)$, with $\gamma(\cdot)$ being the output-to-state gain in

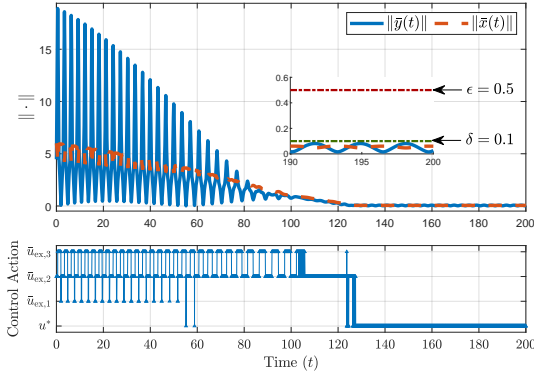


Fig. 3. Simulation results of Σ_{ex} using the control approach proposed in the Proposition 12 with discrete input set $\bar{\mathcal{U}}_{ex} := \mathcal{U}_{ex} + u^*$ with \mathcal{U}_{ex} be as the input set used in Example 3. Here we have that $u^* \in \bar{\mathcal{U}}_{ex}$. Similar to before, in this simulation, once both the state x and the output y enter their respective convergence ball, the control action is switched to u^* for the rest of the simulation.

large-time initial-state norm-observability assumption, the knowledge of δ basically determines how close the trajectories can get to the desired equilibrium with our proposed controller. Let us recall the assumption (A1) and its generalization (A3), where we assume that, for a finite set \mathcal{U} , the desired equilibrium $u^* \in \text{int}(\text{conv}(\mathcal{V}))$. To obtain \mathcal{U} of minimal cardinality, the following result, borrowed from [3], is of interest:

Lemma 14 *For a finite set $\mathcal{S} \subset \mathbb{R}^m$, the minimal cardinality of \mathcal{S} such that $\text{int}(\text{conv}(\mathcal{S})) \neq \emptyset$ is equal to $m + 1$.*

An immediate consequence is that, for practical stabilization of passive systems, it suffices to consider a control set \mathcal{U} with $m + 1$ elements, provided they satisfy a certain geometric configuration.

Corollary 15 *Let the set \mathcal{U} be such that $u^* \in \text{int}(\text{conv}(\mathcal{U}))$. If $\text{conv}(\mathcal{U} \setminus \{u^*\})$ is an m -simplex, then \mathcal{U} is a minimal set that satisfies (A3).*

In the remainder of this section, we will work with two particular choices of the set \mathcal{U} with cardinality $m + 1$ that satisfy (A1) or (A3). We give a closed-form expression of δ for these sets in terms of the elements \mathcal{U} . For the sake of simplicity, we fix $u^* = 0$ in these computations. The two cases we consider are:

- (1) The set $\mathcal{U} = \mathcal{S}_{\text{reg}} \cup \{0\}$, where

$$\mathcal{S}_{\text{reg}} = \lambda \left\{ e_1, \dots, e_m, \frac{1 - \sqrt{m+1}}{m} \mathbb{1} \right\}, \quad (31)$$

for some $\lambda > 0$, and the barycenter of $\mathcal{S}_{m,\text{reg}} =$

$\text{conv}(\mathcal{S}_{\text{reg}})$ is

$$b_{\mathcal{S}_{\text{reg}}} = \lambda \frac{\sqrt{m+1} - 1}{m\sqrt{m+1}} \mathbb{1}. \quad (32)$$

- (2) We take $\mathcal{U} = \mathcal{S}_{\text{reg}}^0 \cup \{0\}$, where $\mathcal{S}_{\text{reg}}^0 = \mathcal{S}_{\text{reg}} - b_{\mathcal{S}_{\text{reg}}}$ and $\text{conv}(\mathcal{S}_{\text{reg}}^0)$ has barycenter at the origin.

In the next two lemmas, we basically compute a bound on the sets $V_{\mathcal{S}_{\text{reg}}}(0)$ and $V_{\mathcal{S}_{\text{reg}}^0}(0)$. It is noted that the results apply to the case when $u^* \neq 0$ since the set $\mathcal{V} = \mathcal{S}_{\text{reg}} + u^*$ (or $\mathcal{V} = \mathcal{S}_{\text{reg}}^0 + u^*$) is such that $u^* \in \text{int}(\text{conv}(\mathcal{V}))$ and the relative position of u^* with respect to the vertices of \mathcal{V} is the same as the relative position of the origin with respect to the vertices of \mathcal{S}_{reg} (or $\mathcal{S}_{\text{reg}}^0$); hence it has the same bound.

Lemma 16 *For the set $V_{\mathcal{S}_{\text{reg}} \cup \{0\}}(0)$, the smallest bound $\delta > 0$ such that $V_{\mathcal{S}_{\text{reg}} \cup \{0\}}(0) \subset \mathbb{B}_\delta$ is given by*

$$\delta = \frac{\lambda}{2} \sqrt{m-1 + (2-m-\sqrt{m+1})^2}.$$

PROOF. First, we observe that the set $V_{\mathcal{S}_{\text{reg}} \cup \{0\}}(0)$ is equal to the solution set of the following inequalities.

The vector $x = [x_1 \dots x_m]^\top \in V_{\mathcal{S}_{\text{reg}} \cup \{0\}}(0)$ if

$$x_i \leq \frac{\lambda}{2}, \quad i = 1, \dots, m, \quad (33)$$

$$\frac{1 - \sqrt{m+1}}{m} \mathbb{1}^\top x \leq \lambda \frac{(1 - \sqrt{m+1})^2}{2m} \quad (34)$$

Since $m \geq 1$, the inequality (34) can simply be rewritten as

$$\mathbb{1}^\top x \geq \frac{\lambda}{2} (1 - \sqrt{m+1}).$$

Next, we observe that each of the vertices of the Voronoi cell $V_{\mathcal{S}_{\text{reg}} \cup \{0\}}(0)$ can be obtained by solving m equations taken from (33) and/or (34). Let \mathcal{V} be the set of all vertices of $V_{\mathcal{S}_{\text{reg}} \cup \{0\}}(0)$. Then $\mathcal{V} = \{\frac{\lambda}{2} \mathbb{1}\} \cup_{i=1}^m \{\frac{\lambda}{2} \tilde{v}_i\}$ with \tilde{v}_i being a column vector where the i -th element is given by $2 - m - \sqrt{m+1}$ and the other $m - 1$ elements are 1. Therefore, the minimum value of δ for which $V_{\mathcal{S}_{\text{reg}} \cup \{0\}}(0) \subset \mathbb{B}_\delta$ is given by

$$\begin{aligned} \delta &= \max_{\tilde{v} \in \mathcal{V}} \{\|\tilde{v}\|\} = \frac{\lambda}{2} \|\tilde{v}_i\| \\ &= \frac{\lambda}{2} \sqrt{m-1 + (2-m-\sqrt{m+1})^2}. \end{aligned}$$

which is the desired expression. \square

Next, let us consider the regular m -simplex centered at the origin with vertices $\mathcal{S}_{\text{reg}}^0$.

Lemma 17 Consider the vertices of a regular m -simplex centered at the origin $\mathcal{S}_{\text{reg}}^0 = \mathcal{S}_{\text{reg}} - b_{\mathcal{S}_{\text{reg}}}$ where \mathcal{S}_{reg} and $b_{\mathcal{S}_{\text{reg}}}$ are defined in (31) and (32). Then the bound $\delta > 0$ such that $V_{\mathcal{S}_{\text{reg}}^0 \cup \{0\}}(0) \subset \mathbb{B}_\delta$ is given by

$$\delta = \lambda \frac{m}{2} \sqrt{\frac{m}{m+1}}.$$

PROOF. Let us denote the set $\mathcal{S} := \mathcal{S}_{\text{reg}}^0 \cup \{0\}$. Then, by following the same proof as before, we have that the set $V_{\mathcal{S}}(0)$ is equal to the solution set of the following system of inequalities,

$$\left(e_i - \frac{\sqrt{m+1}-1}{m\sqrt{m+1}} \mathbb{1} \right)^\top x \leq \frac{\lambda}{2} \frac{m}{m+1}, \quad i = 1, \dots, m, \quad (35)$$

$$-\frac{1}{m+1} \mathbb{1}^\top x \leq \frac{\lambda}{2} \frac{m}{m+1}. \quad (36)$$

Since all points in $\mathcal{S}_{\text{reg}}^0$ have the same distance from the origin, we can pick any set of m equations from the above inequalities in order to get one of the vertices of $V_{\mathcal{S}}(0)$. Let us now choose all m equations from (35) because they have a nice symmetric structure given by,

$$\underbrace{\begin{bmatrix} a & -1 & \dots & -1 \\ -1 & a & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & a \end{bmatrix}}_{\mathcal{A}} v = \frac{\lambda}{2} \frac{m^2}{m+1 - \sqrt{m+1}} \mathbb{1}, \quad (37)$$

where $a = m + \sqrt{m+1}$. Since \mathcal{A} is symmetric, we can find a symmetric matrix \mathcal{A}^{-1} such that $\mathcal{A}\mathcal{A}^{-1} = \mathcal{A}^{-1}\mathcal{A} = I_{m \times m}$. Via routine computation, we have that the matrix \mathcal{A}^{-1} is of the form

$$\mathcal{A}^{-1} = \begin{bmatrix} b & c & \dots & c \\ c & b & \dots & c \\ \vdots & \vdots & \ddots & \vdots \\ c & c & \dots & b \end{bmatrix}$$

whose main diagonal elements are

$$b = \frac{m-2 + \frac{2}{\sqrt{m+1}}}{m^2}$$

and its off-diagonal elements are

$$c = \frac{1}{2(1 + \sqrt{m+1}) + m(2 + \sqrt{m+1})}.$$

Thus, the point v is given by

$$v = \frac{\lambda}{2} \frac{m}{\sqrt{m+1}} \mathbb{1}.$$

Therefore, the minimum bound on the set $V_{\mathcal{S}}(0)$ is,

$$\delta = \|v\| = \frac{\lambda}{2} \frac{m}{\sqrt{m+1}} \|\mathbb{1}\| = \lambda \frac{m}{2} \sqrt{\frac{m}{m+1}}.$$

which completes the proof. \square

We have shown in Lemma 16 and Lemma 17 above that for the two types of discrete sets, whose elements form the vertices of regular m -simplices, the minimum bounds of the Voronoi cell of the origin can be computed in a closed-form manner. Now, for a given incrementally passive system Σ and admissible reference signal u^* with large-time norm-observability function $\gamma \in \mathcal{K}$ when $u = u^*$, for a given stability margin $\epsilon > 0$, the value of the bound δ can be chosen as large as possible such that $\gamma(\delta) \leq \epsilon$. Thus, for a given $\epsilon > 0$, norm-observability function γ of the system Σ , and a desired rotation matrix R , we can choose $\delta > 0$ that satisfies $\gamma(\delta) \leq \epsilon$ and construct the minimal set $\mathcal{U} \subset \mathbb{R}^m$ that satisfies (A3) as follows:

- (1) $\mathcal{U} := (R\mathcal{S}_{\text{reg}} \cup \{0\}) + u^*$ with

$$\lambda = \frac{2\delta}{m-1 + (2-m-\sqrt{m+1})^2},$$

- or;
(2) $\mathcal{U} := (R\mathcal{S}_{\text{reg}}(0) \cup \{0\}) + u^*$ with

$$\lambda = \frac{2\delta}{m} \sqrt{\frac{m+1}{m}}.$$

Example 4 Recall the discrete set \mathcal{U}_{ex} as in Example 2. The same discrete set can be constructed by using $\mathcal{U} := (R\mathcal{S}_{\text{reg}}(0) \cup \{0\}) + u^*$; by fixing $\alpha = \delta$ and

$$R = -\frac{\sqrt{2}}{2} \begin{bmatrix} \sin \theta_{\text{ex}} + \cos \theta_{\text{ex}} & \sin \theta_{\text{ex}} - \cos \theta_{\text{ex}} \\ -\sin \theta_{\text{ex}} + \cos \theta_{\text{ex}} & \sin \theta_{\text{ex}} + \cos \theta_{\text{ex}} \end{bmatrix}.$$

6 CONCLUSIONS AND FURTHER RESEARCH

We have considered practical stabilization of continuous-time (incrementally) passive nonlinear systems using output-feedback where the control inputs only take values among the available actions in a given finite discrete set. We propose simple ways to select the control actions at each time instance where we have shown that our proposed control laws are able to stabilize the systems up to some desirable distance from the equilibrium. In addition, our results provide an insight on the lower bound on the number of control elements that guarantee practical stability. We have also provided methods

to design the finite set of control actions with minimal cardinality. Questions related to improving the convergence rate with more (than necessary) control elements and/or to eliminate the chattering effects are being investigated as further directions of research.

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