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Late time cosmological evolution in DHOST models

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We study the late cosmological evolution, from the nonrelativistic matter dominated era to the dark energy era, in modified gravity models described by Degenerate Higher-Order Scalar-Tensor (DHOST) theories. They represent the most general scalar-tensor theories propagating a single scalar degree of freedom and include Horndeski and Beyond Horndeski theories. We provide the homogeneous evolution equations for any quadratic DHOST theory, without restricting ourselves to theories where the speed of gravitational waves coincides with that of light since the present constraints apply to wavelengths much smaller than cosmological scales. To illustrate the potential richness of the cosmological background evolution in these theories, we consider a simple family of shift-symmetric models, characterized by three parameters and compute the evolution of dark energy and of its equation of state. We also identify the regions in parameter space where the models are perturbatively stable.

I. INTRODUCTION

One possible explanation for the observed acceleration of the cosmological expansion is that gravity is modified on cosmological scales. Concrete realisations of this idea often rely on scalar-tensor theories, which represent the simplest extension of general relativity since a scalar degree of freedom is added to the usual tensor modes of general relativity. The most general family of scalar-tensor theories that has been developed so far is that of Degenerate Higher-Order Scalar-Tensor (DHOST) theories [1], which encompass Horndeski theories [2], Beyond Horndeski (or GLPV) theories [3] which are earlier extensions of Horndeski, as well as disformal transformations of the Einstein-Hilbert action [4]. In the present work, we consider the whole family of quadratic DHOST theories, introduced in [1] (see also [5, 6] for further details and [7] for a review), but for simplicity, we do not include DHOST theories with cubic terms (in second derivatives of the scalar field) which have been fully classified in [8].

Most of the literature has recently concentrated on DHOST theories where the speed of gravitational waves coincides with that of light, following the observation of a neutron star binary merger that has set an impressively stringent constraint on the difference between these two velocities [9]. Moreover, it has been pointed out that subsets of DHOST theories can lead to the decay of gravitational waves, yielding a further tight constraint on DHOST theories [10, 11]. The cosmology of DHOST theories satisfying either the first or both of the above constraints has been studied in [12–17].

However, it should be stressed that the LIGO-Virgo measurements probe wavelengths of order 10^3 km, which are many orders of magnitude smaller than cosmological scales, and an effective theory describing cosmological scales might not be adequate to describe physics on much smaller length scales, as those probed by LIGO-Virgo (see [18] for a discussion on this point). In the present work, we adopt the point of view that DHOST theories apply only to cosmological scales

and cannot be extrapolated down to astrophysical scales within the same framework¹. In this perspective, all the constraints derived from GW170817 mentioned above are not directly relevant and it thus makes sense to study models that can lead to distinct propagation velocities for light and gravitational waves on cosmological scales.

The outline of the paper is the following. In section II, starting from the most general action for quadratic DHOST theories, we derive the Friedmann equations and the scalar field equation. These results extend those obtained recently in [12] and [13]. As in [13] we introduce an auxiliary scale factor that makes the equations manifestly second-order. The second part of the paper is devoted to the study of a simple subfamily of quadratic DHOST theories characterized by a few parameters. In section III, we write the equations of motion in the form of a dynamical system and identify the fixed points and their nature. In section IV, we turn to the linear perturbations in order to study the perturbative stability of the model. We conclude in the last section.

II. GENERAL COSMOLOGICAL EQUATIONS

In this section, we briefly recall the basic properties of DHOST theories and introduce the notations used throughout this paper. Then, we focus on the case of a homogeneous and isotropic universe and provide the cosmological evolution equations for the whole family of quadratic DHOST theories.

A. Quadratic DHOST theories

The most general theory of quadratic DHOST theory is described by the action

$$S = \int d^4x \sqrt{-g} \left(P(X, \varphi) + Q(X, \varphi) \square\varphi + F(X, \varphi) R + \sum_{i=1}^5 A_i(X, \varphi) L_i \right) \quad (2.1)$$

where the functions A_i , F , Q and P depend on the scalar field φ and its kinetic term $X \equiv \nabla^\mu\varphi\nabla_\mu\varphi$, R is the Ricci scalar. The five elementary Lagrangians L_i quadratic in second derivatives of φ are defined by

$$\begin{aligned} L_1 &\equiv \varphi_{\mu\nu}\varphi^{\mu\nu}, & L_2 &\equiv (\square\varphi)^2, & L_3 &\equiv \varphi^\mu\varphi_{\mu\nu}\varphi^\nu\square\varphi, \\ L_4 &\equiv \varphi^\mu\varphi_{\mu\nu}\varphi^{\nu\rho}\varphi_\rho, & L_5 &\equiv (\varphi^\mu\varphi_{\mu\nu}\varphi^\nu)^2, \end{aligned} \quad (2.2)$$

where we are using the standard notations $\varphi_\mu \equiv \nabla_\mu\varphi$ and $\varphi_{\mu\nu} \equiv \nabla_\nu\nabla_\mu\varphi$ for the first and second (covariant) derivatives of φ . For the theory to be degenerate and thus propagate only one extra scalar degree of freedom in addition to the usual tensor modes of gravity, the functions F and A_i have to satisfy some conditions [1, 6] whereas P and Q are totally free.

It has been established in [6] that these DHOST theories can be classified into three classes which are stable under general disformal transformations, i.e. transformations of the metric of the form

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = C(X, \varphi)g_{\mu\nu} + D(X, \varphi)\varphi_\mu\varphi_\nu, \quad (2.3)$$

where C and D are arbitrary functions (provided that the metric $\tilde{g}_{\mu\nu}$ remains regular).

¹Our motivation here is that dark energy can be described by a DHOST model. DHOST theories with a very different set of parameters could still be used to describe modified gravity in astrophysical systems, but without being able to account for dark energy because of the LIGO-Virgo constraints.

The theories belonging to the first class, named class Ia in [6], can be mapped into a Horndeski form by applying a disformal transformation. The other two classes are not physically viable [19] and will not be considered in the present work. Theories in class Ia are labelled by the three free functions F, A_1 and A_3 (in addition to P and Q) and the three remaining functions are given by the relations [1]

$$A_2 = -A_1, \quad (2.4)$$

$$A_4 = \frac{1}{8(F + XA_2)^2} \left(A_2 A_3 (16X^2 F_X - 12XF) + 4A_2^2 (16XF_X + 3F) + 16A_2 (4XF_X + 3F) F_X \right. \\ \left. + 16XA_2^3 + 8A_3 F (XF_X - F) - X^2 A_3^2 F + 48FF_X^2 \right), \quad (2.5)$$

$$A_5 = \frac{1}{8(F + XA_2)^2} \left(2A_2 + XA_3 - 4F_X \right) \left(3XA_2 A_3 - 4A_2 F_X - 2A_2^2 + 4A_2^3 F \right), \quad (2.6)$$

where F_X denotes the derivative of $F(X, \varphi)$ with respect to X . Similarly F_φ will denote the partial derivative of F with respect to φ and the same notations will be used for all functions.

The above relations (2.4-2.6) are a direct consequence of the three degenerate conditions that guarantee only one scalar degree of freedom is present [1, 20]. In conclusion, this means that all the DHOST theories we study here are characterized by five free functions of X and φ , which are P, Q, F, A_1 and A_3 . Notice that we have implicitly supposed the condition $F + XA_2 \neq 0$. Theories where $F + XA_2 = 0$ belong to the sub-class Ib which is not physically relevant [6].

B. Homogeneous and isotropic cosmology

We now wish to study the behaviour of these theories in a homogeneous and isotropic spacetime, endowed with the metric

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^j dx^i, \quad (2.7)$$

where the lapse function $N(t)$ and the scale factor $a(t)$ depend on time only. As a consequence of the spacetime symmetries, the scalar field must also be homogeneous and therefore depends only on time.

Substituting the above metric (2.7) into the action (2.1), and taking into account the degeneracy conditions (2.4-2.6), one finds that the corresponding homogeneous action can be written as a functional of $N(t), a(t)$ and of the homogeneous scalar field $\varphi(t)$. It reads

$$S_{\text{hom}}[N, a, \varphi] = \int dt a^3 N \left\{ P + Q \left(\frac{\dot{N}}{N^3} \dot{\varphi} - \frac{3\dot{a}}{aN^2} \dot{\varphi} - \frac{\ddot{\varphi}}{N^2} \right) - F_\varphi \frac{6\dot{a}}{aN^2} \dot{\varphi} \right. \\ \left. - \frac{6f_1}{N^2} \left(\frac{\dot{a}}{a} + \frac{f_2}{4f_1} \left(\frac{\dot{N}\dot{\varphi}^2}{N^3} - \frac{\ddot{\varphi}\dot{\varphi}}{N^2} \right) \right)^2 \right\}, \quad (2.8)$$

where we have introduced the new functions,

$$f_1 \equiv F - XA_1, \quad f_2 \equiv 4F_X - 2A_1 + XA_3, \quad (2.9)$$

and, everywhere, the expression of X is explicitly given by

$$X = -\frac{\dot{\varphi}^2}{N^2}. \quad (2.10)$$

The Euler-Lagrange equations derived from the above action (2.8) lead to equations of motion that appear higher than second order. However, due to the degeneracy of the theory, these equations can be recast into a second order system. As done in [13], this can be demonstrated explicitly by introducing an auxiliary scale factor b , defined by the relation

$$a \equiv \Lambda(X, \varphi) b \equiv e^{\lambda(X, \varphi)} b, \quad (2.11)$$

where λ satisfies the condition

$$\lambda_X = -\frac{f_2}{8f_1}, \quad (2.12)$$

so that the terms quadratic in $\ddot{\varphi}$ in the action (2.8) are reabsorbed in the derivatives of the new scale factor.

It is also convenient to use a Hubble parameter associated with this auxiliary scale factor, defined by

$$H_b \equiv \frac{\dot{b}}{Nb} = H - \lambda_X \frac{\dot{X}}{N} - \lambda_\varphi \frac{\dot{\varphi}}{N}, \quad \dot{X} = \frac{2}{N^2} \left(\frac{\dot{N}\dot{\varphi}^2}{N} - \dot{\varphi}\ddot{\varphi} \right). \quad (2.13)$$

In fact, the auxiliary variable b corresponds to the scale factor of the disformally transformed metric $\tilde{g}_{\mu\nu}$ in (2.3) when the DHOST theory coincides with a Horndeski theory. The drawback of using this ‘‘Horndeski frame’’ is that matter is no longer minimally coupled, as it was assumed in the initial frame, which we will call here the ‘‘DHOST frame’’. The Horndeski and DHOST frames are, respectively, the analogs of the Einstein and Jordan frames for traditional scalar-tensor theories.

When expressed in terms of b instead of a , the Lagrangian L_{hom} in the action (2.8) becomes

$$\begin{aligned} L_{\text{hom}} = \Lambda^3 b^3 N \left\{ -3\lambda_\varphi \frac{\dot{\varphi}^2}{N^2} (2f_1\lambda_\varphi + Q + 2F_\varphi) + P - 6f_1 H_b^2 - \frac{3\dot{\varphi}}{N} (4\lambda_\varphi f_1 + 2F_\varphi + Q) H_b \right. \\ \left. + \left(Q - 6\lambda_X (2F_\varphi + Q) \frac{\dot{\varphi}^2}{N^2} \right) \left(\frac{\dot{N}\dot{\varphi}}{N^3} - \frac{\ddot{\varphi}}{N^2} \right) \right\}. \end{aligned} \quad (2.14)$$

The coupling to matter is described by adding to L_{hom} a matter Lagrangian L_m , and the total Lagrangian is denoted $L \equiv L_{\text{hom}} + L_m$.

We get the equations of motion by writing the Euler-Lagrange equations for N , b and φ . The first two equations provide the generalizations of the Friedmann equations. The last equation, corresponding to the scalar field equation of motion, is obtained from an Euler-Lagrange equation of the form

$$-\frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{\varphi}} + \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0. \quad (2.15)$$

Once we have derived the equations of motion, we fix the time coordinate such that $N = 1$ (and thus $\dot{N} = 0$) in order to simplify the equations.

As for matter, we assume that it is described by a perfect fluid whose equation of state is $P = w\rho$, where w is constant. The variation of the matter action S_m gives the energy-momentum tensor of the fluid, defined as usual by

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}, \quad S_m = \int d^4x \sqrt{-g} L_m. \quad (2.16)$$

As a consequence, in the DHOST frame, where matter is minimally coupled, the variation of the matter Lagrangian is immediately given by

$$\delta L_m = -a^3 \rho_m \delta N + 3Na^2 P_m \delta a, \quad (2.17)$$

where ρ_m and P_m are the fluid energy density and pressure, respectively. Using

$$\frac{\delta a}{a} = -2X\lambda_X \frac{\delta N}{N} + \frac{\delta b}{b} \quad (2.18)$$

which follows from the definition (2.11) of b , one finds that the variation of the matter Lagrangian in the Horndeski frame is given by

$$\delta L_m = Nb^3 \Lambda^3 \left[-(\rho_m + 6X\lambda_X P_m) \frac{\delta N}{N} + 3P_m \frac{\delta b}{b} \right]. \quad (2.19)$$

Inserting the above variation of the matter Lagrangian into the Euler-Lagrange equations, we obtain the set of equations of motion for the cosmological dynamics. The analogs of the two Friedmann equations (with $N = 1$) take the form

$$g_0 + g_1 H_b \dot{\varphi} + g_2 H_b^2 = (1 - 6w\lambda_X \dot{\varphi}^2) \rho_m, \quad (2.20)$$

$$g_3 + g_4(2\dot{H}_b + 3H_b^2) + g_5 H_b \dot{\varphi} + g_6 \ddot{\varphi} + g_7 H_b \dot{\varphi} \ddot{\varphi} = -w\rho_m. \quad (2.21)$$

where all the coefficients g_i can be written explicitly in terms of the functions that appear in the Lagrangian and of λ . They are given in Appendix A.

Finally the scalar field equation can be written as

$$\frac{d}{dt} (b^3 \Lambda^3 J) + b^3 \Lambda^3 U = 0, \quad (2.22)$$

where we have defined

$$J = \frac{1}{b^3 \Lambda^3} \left[\frac{d}{dt} \left(\frac{\partial L_{\text{hom}}}{\partial \ddot{\varphi}} \right) - \frac{\partial L_{\text{hom}}}{\partial \dot{\varphi}} \right], \quad U = \frac{1}{b^3 \Lambda^3} \frac{\partial L_{\text{hom}}}{\partial \varphi}. \quad (2.23)$$

The two functions U and J are of the form

$$U = g_8 + g_9 H_b \dot{\varphi} + g_{10} H_b^2 + g_{11} \ddot{\varphi}, \quad J = g_{12} \dot{\varphi} + g_{13} H_b + g_{14} H_b^2 \dot{\varphi}, \quad (2.24)$$

where the corresponding coefficients g_i are also given explicitly in Appendix A. As usual, the equation for the scalar field is not independent and can be obtained from the two Friedmann equations.

One recovers the equations of motion given in [13] when the theory is shift symmetric, i.e. when the functions in the Lagrangian are invariant under the transformation $\varphi \rightarrow \varphi + c$ and thus depend only on X . In particular, the quantity U defined above vanishes and J is conserved.

III. AN ILLUSTRATIVE TOY MODEL

We now restrict our study to a class of models described by Lagrangians that depend on three constant parameters only. These Lagrangians are shift symmetric and characterized by the simple polynomial functions

$$P = \alpha X, \quad Q = 0, \quad F = \frac{1}{2}, \quad A_1 = -A_2 = -\beta X, \quad \lambda = \frac{1}{2} \mu X^2, \quad (3.1)$$

where α , β and μ are arbitrary constants. Hence, from (2.12) and (2.9), we deduce

$$A_3 = -2(\beta + 2\mu) - 8\beta\mu X^2. \quad (3.2)$$

The expressions of A_4 and A_5 are easily obtained from the degeneracy conditions (2.5) and (2.6). One thus gets

$$A_4 = 2(\beta + 2\mu - 2\mu^2 X^2), \quad A_5 = 8\mu X(\beta + 2\mu + 3\beta\mu X^2). \quad (3.3)$$

Note that the particular choice $\mu = 0$ corresponds to a subset of Horndeski theories (in this case, the DHOST and Horndeski frames coincide, i.e. $\lambda = 0$ and thus $a = b$).

A. Horndeski frame: dynamical system analysis

Since we are interested in the transition between the matter and dark energy dominated eras, we also assume that matter is non-relativistic and thus take $w = 0$. The Friedmann-like equations (2.20) and (2.21) then reduce to

$$3H_b^2 (1 + 2(3\mu + 5\beta)X^2 + 12\mu\beta X^4) - \alpha X - 6\alpha\mu X^3 - \rho_m = 0, \quad (3.4)$$

$$3H_b^2 (1 + 2\beta X^2) + 2(1 + 2\beta X^2) \dot{H}_b + \alpha X - 4H_b((3\mu + 4\beta)X + 6\beta\mu X^3) \dot{\varphi} \ddot{\varphi} = 0. \quad (3.5)$$

Furthermore, the equation of motion for the scalar field becomes

$$\begin{aligned} & [\alpha(1 + 21\mu X^2 + 18\mu^2 X^4) - 9X(4\beta + 3\mu + 2\mu(11\beta + 3\mu)X^2 + 12\beta\mu^2 X^4) H_b^2] \ddot{\varphi} \\ & + 3 \left[\alpha(1 + 3\mu X^2) - X(4\beta + 3\mu + 6\beta\mu X^2)(3H_b^2 + 2\dot{H}_b) \right] H_b \dot{\varphi} = 0. \end{aligned} \quad (3.6)$$

To study these cosmological equations, it is convenient to rewrite them as a dynamical system (and analyse the fixed points and their stability) with the new variables

$$x_1 \equiv \alpha \frac{X}{H_b^2}, \quad x_2 \equiv \beta X^2, \quad s \equiv \frac{\mu}{\beta}, \quad \epsilon_h \equiv \frac{\dot{H}_b}{H_b^2}, \quad \epsilon_\varphi \equiv \frac{\ddot{\varphi}}{H_b \dot{\varphi}}, \quad (3.7)$$

following similar treatments for dark energy models (see e.g. [21–23] in the context of Galileons and [24] for a recent review). These variables are not independent and one can easily see that

$$\frac{dx_1}{d \ln b} = 2(\epsilon_\varphi - \epsilon_h) x_1, \quad (3.8)$$

$$\frac{dx_2}{d \ln b} = 4\epsilon_\varphi x_2, \quad (3.9)$$

where $\ln b$ plays the role of time. The two equations above can also be formulated in terms of the time $\ln a$ instead of $\ln b$ by using the relation between the two Hubble constants,

$$H = (1 + 2s x_2 \epsilon_\varphi) H_b, \quad (3.10)$$

which follows from (2.13). The two previous relations (3.8) and (3.9) then become

$$x_1' = \frac{2(\epsilon_\varphi - \epsilon_h) x_1}{1 + 2s \epsilon_\varphi x_2}, \quad (3.11)$$

$$x_2' = \frac{4\epsilon_\varphi x_2}{1 + 2s \epsilon_\varphi x_2}, \quad (3.12)$$

where a prime denotes a derivative with respect to $N \equiv \ln a$.

So far, we have not yet used the equations of motion, namely the Friedmann-like equations, Eqs. (3.4) and (3.5), and the scalar equation (3.6). They can be reformulated, respectively, as

$$\frac{x_1}{3} - 12sx_2^2 - 6sx_2 + 2sx_1x_2 - 10x_2 + \Omega_m = 1, \quad (3.13)$$

$$2(1 + 2x_2)\epsilon_h + 4x_2(4 + 3s + 6sx_2)\epsilon_\varphi + (3 + x_1 + 6x_2) = 0, \quad (3.14)$$

$$\begin{aligned} & -\frac{2}{3}(x_1(18s^2x_2^2 + 21sx_2 + 1) - 9x_2(6s^2x_2(2x_2 + 1) + s(22x_2 + 3) + 4))\epsilon_\varphi \\ & + 4x_2\epsilon_h(s(6x_2 + 3) + 4) + 36sx_2^2 + 18sx_2 - 6sx_1x_2 + 24x_2 - 2x_1 = 0. \end{aligned} \quad (3.15)$$

These equations can be seen as constraints for the dynamical system (3.11-3.12). The first constraint, Eq. (3.13), involves the matter density, whereas the last two equations (3.14) and (3.15) can be used to determine ϵ_φ and ϵ_h in terms of x_1 and x_2 . After a straightforward calculation, one gets

$$\begin{aligned} \epsilon_h = & [3x_1(72s^2x_2^3 + 12s(3s + 8)x_2^2 + 26x_2 - 1) - x_1^2(18s^2x_2^2 + 21sx_2 + 1) \\ & - 9x_2(18s^2x_2(2x_2 + 1)^2 + 3s(20x_2^2 + 4x_2 - 3) + 40x_2 - 12)] / \Delta, \end{aligned} \quad (3.16)$$

$$\epsilon_\varphi = -6x_1(12sx_2^2 + 6(s + 1)x_2 + 1) / \Delta, \quad (3.17)$$

where the common denominator Δ is given by

$$\begin{aligned} \Delta \equiv & 6x_2(18x_2(2sx_2 + s)^2 + 3s(20x_2^2 + 4x_2 - 3) + 40x_2 - 12) \\ & + 2x_1(2x_2 + 1)(18s^2x_2^2 + 21sx_2 + 1). \end{aligned} \quad (3.18)$$

Hence, the equations of motion are now given in the form (3.11) and (3.12) with ϵ_h and ϵ_φ given by the equations (3.16) and (3.17).

The critical points are found by solving the equations (3.11) and (3.12) for $x_1' = 0$ and $x_2' = 0$. The number and stability properties of these fixed points are summarized in Table I. We see that there are at most two stable fixed points corresponding to a de Sitter solution. To find the conditions on the parameters of the theory for these fixed points to exist, we have to study the signs of x_1 and x_2 at the fixed points. It is immediate to show that

- at the point C : $x_1 < 0$ and $x_2 < 0$ for all values of s ;
- at the point D : $x_1 > 0$ and $x_2 < 0$ for $s > 0$ whereas $x_1 < 0$ and $x_2 > 0$ for $s < 0$.

From the definition of x_1 and x_2 (3.7), we see immediately that $\alpha x_1 < 0$ (because $X < 0$) and $\beta x_2 > 0$. As a consequence, we deduce that the fixed point C exists only if $\alpha > 0$ and $\beta < 0$ whereas the fixed point D exists only if $\mu < 0$.

Notice that, in the limit $s = 0$, i.e. $\mu = 0$, corresponding to a DHOST theory that belongs to the Horndeski subclass, the dynamical system admits a single fixed point given by the limit $s \rightarrow 0$ of C ,

$$x_1 = -2, \quad x_2 = -\frac{1}{6}, \quad (3.19)$$

whereas the limit of the fixed point D is ill-defined.

points	x_1	x_2	Eigenvalues
A	0	x_2	(0, 3) (unstable)
B	3	0	(-12, 3) (Saddle)
C	$(3 - 3s - \sqrt{3(3s^2 + 2s + 3)})/2s$	$(-3 - 3s + \sqrt{3(3s^2 + 2s + 3)})/12s$	(-3, -3) (Stable)
D	$(3 - 3s + \sqrt{3(3s^2 + 2s + 3)})/2s$	$(-3 + 3s + \sqrt{3(3s^2 + 2s + 3)})/12s$	(-3, -3) (Stable)

TABLE I: Fixed points of the dynamical system with the eigenvalues of the corresponding Hessian matrix. Only the last two points are stable.

B. DHOST frame: Effective Friedmann equations

In the frame where matter is minimally coupled, it is always possible to write effectively the Friedmann equations in the usual form,

$$3H^2 = \rho_m + \rho_{\text{DE}}, \quad 2\dot{H} + 3H^2 = P_m + P_{\text{DE}}, \quad (3.20)$$

where all new terms are “hidden” in the effective dark energy density and pressure, denoted ρ_{DE} and P_{DE} respectively. Hence, one can also define an equation of state parameter w_{DE} for dark energy as usual by the ratio

$$w_{\text{DE}} \equiv \frac{P_{\text{DE}}}{\rho_{\text{DE}}}. \quad (3.21)$$

Moreover, one can define a global effective equation of state parameter as

$$w_{\text{eff}} \equiv \frac{P_m + P_{\text{DE}}}{\rho_m + \rho_{\text{DE}}} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2}. \quad (3.22)$$

For the models we are considering here, this parameter can be expressed in terms of the variables introduced earlier and reads

$$w_{\text{eff}} = -1 - \frac{2}{3} \frac{H'}{H} = -1 - \frac{2\epsilon_h + 4s(\epsilon_\varphi x_2' + \epsilon_\varphi' x_2)}{3(1 + 2sx_2\epsilon_\varphi)}. \quad (3.23)$$

Using the fact that $P_m = 0$ for non-relativistic matter, we can write, from (3.20) and (3.22), a relation between w_{DE} and w_{eff} given by

$$w_{\text{DE}} = \frac{w_{\text{eff}}}{\Omega_{\text{DE}}}, \quad \Omega_{\text{DE}} \equiv \frac{\rho_{\text{DE}}}{3H^2}. \quad (3.24)$$

The dynamical equations (3.8) and (3.9) can be solved numerically and the right amount of nonrelativistic matter today, i.e. $\Omega_m = \rho_m/(3H^2) \approx 0.3$, can be reached by tuning the initial conditions for x_1 and x_2 . We choose our initial conditions deep in the matter dominated era, i.e. when $\Omega_m \simeq 1$. According to the constraint (3.13), taking $|x_1| \ll 1$ and $|x_2| \ll 1$ initially guarantees that we are deep in the matter dominated era. Moreover, in order to observe a relatively rapid transition from the matter era to de Sitter era, we take initial conditions such that

$$|x_1| \ll |x_2| \ll 1. \quad (3.25)$$

Indeed, in this regime, the dynamical system reduces to

$$x_1' \approx 3x_1, \quad x_2' \approx 4x_1/(9s + 12), \quad (3.26)$$

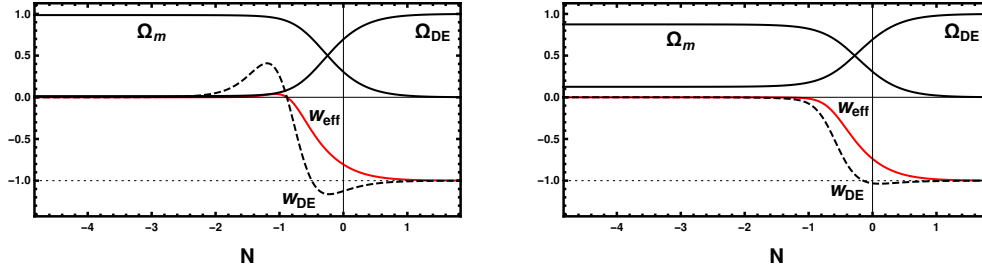


FIG. 1: Evolution of the parameters w_{DE} , w_{eff} , Ω_{DE} , and Ω_m as functions of $N = \ln(a)$ for various choices for the parameter s and the initial conditions. Left: $s = -4$ and initial conditions $x_1^{(i)} = -3.50 \times 10^{-7}$ and $x_2^{(i)} = 10^{-3}$. Right: $s = -10$ and initial conditions $x_1^{(i)} = -7.00 \times 10^{-7}$ and $x_2^{(i)} = 2.50 \times 10^{-3}$.

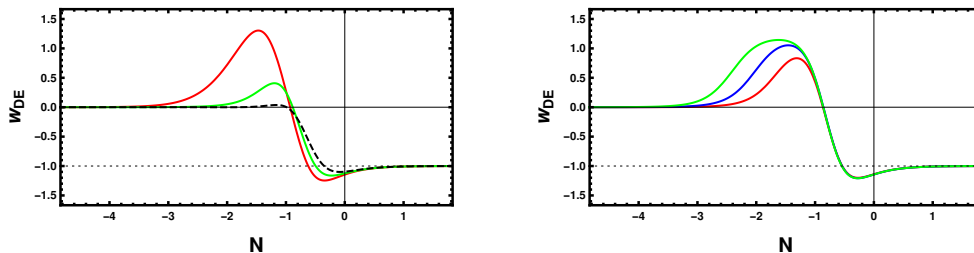


FIG. 2: Evolution of the dark energy ratio $w_{\text{DE}} = P_{\text{DE}}/\rho_{\text{DE}}$ as a function of $N = \ln(a)$ for various choices of s and initial conditions (set at $N = -5$). Left: $x_2^{(i)} = 10^{-3}$; $(x_1^{(i)}, s) = (-2.50 \times 10^{-7}, -2)$ for the green curve, $(x_1^{(i)}, s) = (-3.50 \times 10^{-7}, -4)$ for the red one and $(x_1^{(i)}, s) = (-4.85 \times 10^{-7}, -10)$ for the black dashed one. Right: $s = -4$; $(x_1^{(i)}, x_2^{(i)}) = (-1.15 \times 10^{-7}, 10^{-4})$ for the red curve, $(x_1^{(i)}, x_2^{(i)}) = (-3.60 \times 10^{-8}, 10^{-5})$ for the blue one and $(x_1^{(i)}, x_2^{(i)}) = (-1.15 \times 10^{-8}, 10^{-6})$ for the green one.

which shows that the system moves quickly away from the region where x_1 and x_2 are very small (this is not the case if we take $|x_2| \ll |x_1| \ll 1$ instead). The initial dark energy parameter Ω_{DE} is then approximated, according to (3.13), by

$$\Omega_{\text{DE}} \approx -2(3s + 5)x_2. \quad (3.27)$$

If we choose $x_2 > 0$ initially, in order to get $c_T < 1$ (as we will see in Eq. (4.15) of the next section), then the parameter s must satisfy $s < -5/3$.

We have plotted some illustrative examples of numerical results in (Fig. 1) and (Fig. 2). The first figure shows the evolution of the cosmological parameters Ω_m , Ω_{DE} , w_{DE} and w_{eff} . We observe a cosmological transition from the matter era to the dark energy era. We also observe that the dark energy behaves like pressureless matter deep in the matter dominated era and like a cosmological constant with $w_{\text{DE}} \approx -1$ at very late times, with a transition going through an intermediate regime where w_{DE} can even reach some significant positive values.

IV. PERTURBATIVE LINEAR STABILITY

In this section, we study the linear stability of the models studied in the previous section. For that purpose, we work in the framework of the Effective Theory of Dark Energy developed in [25–27] and extended to DHOST theories in [19]. This effective approach relies on the ADM formulation where the metric is parametrized by the lapse function N , the shift vector N^i and the

spatial metric h_{ij} as follows,

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (4.1)$$

In the ADM framework, the ‘‘velocity’’ of the spatial metric is encoded in the extrinsic curvature tensor K_{ij} defined by

$$K_{ij} \equiv \frac{1}{2N} \left(\dot{h}_{ij} - D_i N_j - D_j N_i \right), \quad (4.2)$$

where D_i denotes the spatial covariant derivative associated to h_{ij} . The DHOST action can be reformulated in terms of the ADM variables and the dynamics of the linear perturbations about an FLRW background is governed by the expansion of this action at quadratic order in the variables δN , δK_{ij} and δh_{ij} . After a long but straightforward calculation, one finds that the quadratic action for the perturbations is given by [19]

$$S^{\text{quad}} = \int d^3x dt a^3 \frac{M^2}{2} \left\{ \delta K_{ij} \delta K^{ij} - \left(1 + \frac{2}{3} \alpha_L \right) \delta K^2 + (1 + \alpha_T) \left(R \frac{\delta \sqrt{h}}{a^3} + \delta_2 R \right) \right. \\ \left. + H^2 \alpha_K \delta N^2 + 4H \alpha_B \delta K \delta N + (1 + \alpha_H) R \delta N + 4\beta_1 \delta K \delta \dot{N} + \beta_2 \delta \dot{N}^2 + \frac{\beta_3}{a^2} (\partial_i \delta N)^2 \right\}, \quad (4.3)$$

where $\delta^2 R$ stands for the second order term in the perturbative expansion of the Ricci scalar R and h is the determinant of the spatial metric. The coefficients M , α_L , α_T , α_K , α_B , α_H , β_1 , β_2 and β_3 , which fully characterize the quadratic action, are functions of time as they depend on the background. They can be expressed explicitly in terms of the functions entering the DHOST action (2.1), as recalled in the Appendix B.

After integrating out the gauge degrees of freedom, and ignoring the coupling to matter for the moment, it has been shown in [19] that the quadratic action reduces to the sum of an action for the curvature perturbation ζ , representing the scalar mode,

$$S_{\text{quad}}[\zeta] = \int d^3x dt a^3 \frac{M^2}{2} \left[A_\zeta \dot{\zeta}^2 - B_\zeta \frac{(\partial_i \zeta)^2}{a^2} \right], \quad (4.4)$$

and an action for the tensor modes γ_{ij} ,

$$S_{\text{quad}}[\gamma_{ij}] = \int d^3x dt a^3 \frac{M^2}{8} \left[\dot{\gamma}_{ij}^2 - \frac{c_T^2}{a^2} (\partial_k \gamma_{ij})^2 \right]. \quad (4.5)$$

The coefficients A_ζ and B_ζ that appear in the scalar action are given by

$$A_\zeta = \frac{1}{(1 + \alpha_B - \dot{\beta}_1/H)^2} \left[\alpha_K + 6\alpha_B^2 - \frac{6}{a^3 H^2 M^2} \frac{d}{dt} (a^3 H M^2 \alpha_B \beta_1) \right], \quad (4.6)$$

$$B_\zeta = -2(1 + \alpha_T) + \frac{2}{aM^2} \frac{d}{dt} \left[\frac{aM^2 (1 + \alpha_H + \beta_1(1 + \alpha_T))}{H(1 + \alpha_B) - \dot{\beta}_1} \right], \quad (4.7)$$

while the speed of gravitational waves c_T , which appears in the tensor action, is given by $c_T^2 = 1 + \alpha_T$. Therefore the stability conditions for the linear perturbations are simply given by

$$M^2 > 0, \quad A_\zeta > 0, \quad B_\zeta > 0, \quad c_T^2 > 0. \quad (4.8)$$

The expressions of these coefficients in terms of the dynamical variables (3.7) are given in Appendix B. As we will see, they will be useful for the numerical analysis of the linear stability of the model.

In the presence of matter, these stability conditions (for the scalar mode) are modified. They have been derived explicitly in [19] for the simple case where matter is described by a scalar field σ whose dynamics is governed by a k-essence type action,

$$S_m = \int d^4x \sqrt{-g} K(Y), \quad Y \equiv g^{\mu\nu} \sigma_\mu \sigma_\nu, \quad (4.9)$$

which is added to the DHOST action. The link with a perfect fluid description of matter, with energy density ρ_m , pressure P_m and sound speed c_m is given by the expressions

$$\rho_m = 2YK_Y - K, \quad P_m = K, \quad c_m^2 = \frac{K_Y}{K_Y + 2YK_{YY}}, \quad (4.10)$$

where all terms are evaluated on a background solution.

It has been shown in [19] that the conditions for the stability of scalar linear perturbations are modified and more involved than the case without matter. Indeed, in addition to ζ , there is an extra scalar degree of freedom that we denote $\delta\sigma$, and the dynamics of the two modes are entangled. The quadratic action for these two scalar perturbations takes the form [13]

$$S_{\text{quad}} = \int d^3x dt a^3 \frac{M^2}{2} \left[\dot{V}^T \mathbf{K} \dot{V} - \frac{1}{a^2} \partial_i V^T \mathbf{G} \partial^i V + \dots \right], \quad (4.11)$$

where the vector $V^T = (\zeta, H \frac{\delta\sigma}{\sigma})$ contains the two scalar degrees of freedom and the dots stand for the terms with fewer than two (space or time) derivatives, which are not relevant for the stability discussion. The kinetic and gradient matrices read (see [13] for details)

$$\mathbf{K} = \begin{pmatrix} A_\zeta + \frac{\rho_m(1+w_m)}{M^2 c_m^2 (H(1+\alpha_B) - \dot{\beta}_1)^2} & \frac{\rho_m(1+w_m)(3c_m^2 \beta_1 - 1)}{M^2 c_m^2 H (H(1+\alpha_B) - \dot{\beta}_1)} \\ \frac{\rho_m(1+w_m)(3c_m^2 \beta_1 - 1)}{M^2 c_m^2 H (H(1+\alpha_B) - \dot{\beta}_1)} & \frac{\rho_m(1+w_m)}{M^2 c_m^2 H^2} \end{pmatrix}, \quad (4.12)$$

$$\mathbf{G} = \begin{pmatrix} B_\zeta & -\frac{\rho_m(1+w_m)(1+\alpha_H + (1+\alpha_T)\beta_1)}{M^2 H (H(1+\alpha_B) - \dot{\beta}_1)} \\ -\frac{\rho_m(1+w_m)(1+\alpha_H + (1+\alpha_T)\beta_1)}{M^2 H (H(1+\alpha_B) - \dot{\beta}_1)} & \frac{\rho_m(1+w_m)}{M^2 H^2} \end{pmatrix}. \quad (4.13)$$

In order to avoid ghost and gradient instabilities, both matrices \mathbf{K} and \mathbf{G} must be positive definite. When matter satisfies $c_m \ll 1$ and $w_m \ll 1$, one can expand the expressions of the eigenvalues of \mathbf{K} and \mathbf{G} with respect to c_m and w_m and one obtains, at leading order,

$$\lambda_{K_1} = \frac{A_\zeta M^2 H^2 (1 + \alpha_B - \beta_1')^2 + 6\rho_m \beta_1}{M^2 H^2 (1 + (1 + \alpha_B - \beta_1')^2)}, \quad \lambda_{K_2} = \frac{\rho_m}{c_m^2 H^2 M^2} \left[\frac{1}{(1 + \alpha_B - \beta_1')^2} + 1 \right], \quad (4.14)$$

$$\lambda_{G_\pm} = \frac{B_\zeta}{2} + \frac{1}{2M^2} \left[\frac{\rho_m}{H^2} \pm \sqrt{\frac{4\rho_m^2 (1 + \alpha_H + (1 + \alpha_T)\beta_1)^2}{H^4 (1 + \alpha_B - \beta_1')^2} + \left(\frac{\rho_m}{H^2} - M^2 B_\zeta\right)^2} \right],$$

where $\lambda_{K_{1,2}}$ and λ_{G_\pm} are the eigenvalues of \mathbf{K} and \mathbf{G} respectively. One thus finds that λ_{K_2} is always positive while the sign of the three other eigenvalues depends on the specific background solution.

All eigenvalues can be expressed in terms of x_1 , x_2 and Ω_m . Moreover, the coefficients M^2 and c_T^2 which appear in the tensor action are given explicitly by

$$M^2 = 1 + 2x_2, \quad c_T^2 = \frac{1}{1 + 2x_2}. \quad (4.15)$$

Deep in the matter dominated era when $|x_1| \ll |x_2| \ll 1$, the leading order behaviour of the eigenvalues λ_{K_1} and $\lambda_{G_{\pm}}$ is given by

$$\lambda_{K_1} \approx -9(3s + 4)x_2, \quad \lambda_{G_+} \approx 6 - 17(s + 1)x_2, \quad \lambda_{G_-} \approx -5(s + 1)x_2. \quad (4.16)$$

and they are all positive when we take $0 < x_2 \ll 1$ and $s < -5/3$, as discussed below (3.27).

In Fig. (3), we plot the time evolution of the eigenvalues, as well as c_T^2 . We have chosen parameters and initial conditions such that all eigenvalues remain positive and $c_T^2 < 1$. With these choices, we see that the tensor and scalar perturbations remain stable from the matter era to the de Sitter era.

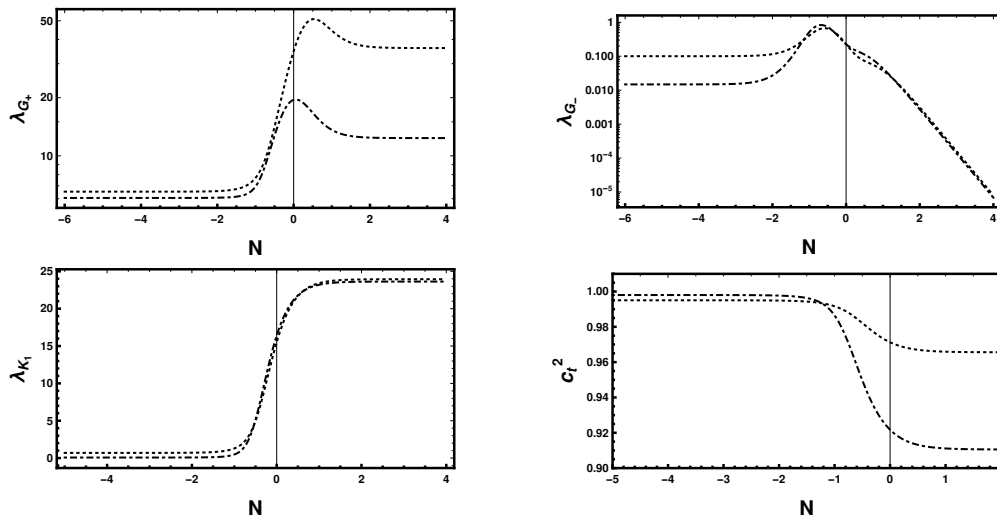


FIG. 3: Evolution of the eigenvalues $\lambda_{G_{\pm}}$ (top), λ_{K_1} (bottom left) and c_T^2 (at the bottom right) viewed as functions of $N = \ln(a)$. Dotted curves: $s = -4$ and initial conditions $x_1^{(i)} = -3.50 \times 10^{-7}$ and $x_2^{(i)} = 10^{-3}$. Dot-Dashed curves: $s = -10$ and initial conditions $x_1^{(i)} = -7.00 \times 10^{-7}$ and $x_2^{(i)} = 2.50 \times 10^{-3}$.

V. CONCLUSIONS

In this paper, we have studied the cosmology of DHOST theories. We have considered the most general action for quadratic DHOST theories and derived the general equations of motion in an isotropic and homogeneous background in the presence of a perfect fluid. We have presented these equations in full generality, without restricting ourselves to shift-symmetric Lagrangians in the first part. Then, we have considered a particular family of shift-symmetric DHOST models characterized by three parameters only. We have performed a dynamical system analysis and obtained the conditions for our models to admit self-accelerating solutions at late time. Then, we have examined the linear stability of both tensor and scalar modes, in the presence of pressureless matter, and found that the models studied here are stable in some region of the parameters space.

With the advent of stage IV cosmological probes (LSST, Euclid), the sharp increase in the amount of data will enable us to test gravitational laws on cosmological scales. In order to analyse

such a trove of data, it will be very useful to rely on a parametrized set of models that can quantify, in a flexible way, deviations from general relativity. DHOST theories, which describe the most general and simplest scalar-tensor theories (the simplest in the sense that they propagate a single additional degree of freedom), are natural candidates to serve as benchmark models for the analysis of future data.

Acknowledgments

We thank Marco Crisostomi for instructive discussions and for providing the full expressions for the parameters α_K and α_B in terms of the functions of the general Lagrangian. H. B. would like to thank APC for their hospitality during his two stays when this project was initiated and then continued.

Appendix A: Coefficients in the cosmological equations

The coefficients entering in the Friedmann equations (2.20) and (2.21) are given by

$$\begin{aligned}
g_0 &= P + X \left[-6 (F\lambda_\varphi^2 + F_\varphi\lambda_\varphi + P\lambda_X) - 2P_X + Q_\varphi \right] \\
&\quad - 6X^2 \left[\lambda_\varphi^2 (2F_X + 6F\lambda_X - 3A_1) + \lambda_\varphi (2F_{\varphi X} + 4F\lambda_{X\varphi} + Q_X) - 2F_{\varphi\varphi}\lambda_X - Q_\varphi\lambda_X \right] \\
&\quad + 12\lambda_\varphi X^3 \left[\lambda_\varphi (3A_1\lambda_X + A_{1X}) + 2A_1\lambda_{X\varphi} \right], \\
g_1 &= 6 \{ 2F\lambda_\varphi + F_\varphi + X [Q_X + 2F_{\varphi X} + 4F\lambda_{X\varphi} + \lambda_\varphi (4F_X + 12F\lambda_X - 6A_1)] \\
&\quad - 4X^2 [\lambda_\varphi (3A_1\lambda_X + A_{1X}) + A_1\lambda_{X\varphi}] \}, \\
g_2 &= 6F + 6X (6F\lambda_X + 2F_X - 3A_1) - 12X^2 (3A_1\lambda_X + A_{1X}), \\
g_3 &= P - X (4F\lambda_{\varphi\varphi} + 6F\lambda_\varphi^2 + 4F_\varphi\lambda_\varphi + 2F_{\varphi\varphi} + Q_\varphi) + X^2 [(4\lambda_{\varphi\varphi} + 6\lambda_\varphi^2)A_1 + 4\lambda_\varphi A_{1\varphi}], \\
g_4 &= 2(F - A_1X), \\
g_5 &= 4[F_\varphi - XA_{1\varphi} + 3\lambda_\varphi(F - A_1X)], \\
g_6 &= 2F_\varphi + 4\lambda_\varphi F + 2X [Q_X + 2F_{X\varphi} + 4\lambda_{X\varphi}F + \lambda_\varphi(4F_X + 12\lambda_X F - 6A_1)] \\
&\quad - 8X^2 (\lambda_\varphi(A_{1X} + 3\lambda_X A_1) + \lambda_{X\varphi}A_1), \\
g_7 &= -8 [F_X + 3\lambda_X F - A_1 - X(A_{1X} + 3\lambda_X A_1)],
\end{aligned}$$

The coefficients entering in the scalar equation (2.22) through the functions (2.24) are given by

$$\begin{aligned}
g_8 &= P_\varphi + 3\lambda_\varphi P + 3X (2F_{\varphi\varphi}\lambda_\varphi + 4F\lambda_\varphi\lambda_{\varphi\varphi} + 2F_\varphi\lambda_{\varphi\varphi} + 6F\lambda_\varphi^3 + 8F_\varphi\lambda_\varphi^2 + 3Q\lambda_\varphi^2 + \lambda_\varphi Q_\varphi + Q\lambda_{\varphi\varphi}) \\
&\quad - 6X^2 \lambda_\varphi (2A_1\lambda_{\varphi\varphi} + 3A_1\lambda_\varphi^2 + A_{1\varphi}\lambda_\varphi), \\
g_9 &= -3 (12F\lambda_\varphi^2 + 10F_\varphi\lambda_\varphi + 4F\lambda_{\varphi\varphi} + 2F_{\varphi\varphi} + Q_\varphi + 3\lambda_\varphi Q) + 12X (3A_1\lambda_\varphi^2 + A_{1\varphi}\lambda_\varphi + A_1\lambda_{\varphi\varphi}), \\
g_{10} &= -6 (F_\varphi + 3\lambda_\varphi F) + 6X (A_{1\varphi} + 3\lambda_\varphi A_1), \\
g_{11} &= -(Q_\varphi + 3\lambda_\varphi Q) - 6X (6F_\varphi\lambda_\varphi\lambda_X + 2F_{\varphi\varphi}\lambda_X + 2F_\varphi\lambda_{X\varphi} + Q_\varphi\lambda_X + 3Q\lambda_\varphi\lambda_X + Q\lambda_{X\varphi}), \\
g_{12} &= 12\lambda_\varphi (F_\varphi + \lambda_\varphi F) + 2(P_X + 3\lambda_X P) - Q_\varphi + 3\lambda_\varphi Q \\
&\quad + 6X (-4A_1\lambda_\varphi^2 + 2F_{X\varphi}\lambda_\varphi + 2F_X\lambda_\varphi^2 + 6F\lambda_\varphi^2\lambda_X - 2F_{\varphi\varphi}\lambda_X + 4F\lambda_\varphi\lambda_{X\varphi} + \lambda_\varphi Q_X - Q_\varphi\lambda_X) \\
&\quad - 12X^2 (A_{1X}\lambda_\varphi^2 + 3A_1\lambda_\varphi^2\lambda_X + 24A_1\lambda_\varphi\lambda_{X\varphi}), \\
g_{13} &= 12F\lambda_\varphi + 6F_\varphi + 6X (-6A_1\lambda_\varphi + 2F_{X\varphi} + 4F_X\lambda_\varphi + 12F\lambda_\varphi\lambda_X + 4F\lambda_{X\varphi} + Q_X) \\
&\quad - 24X^2 (A_{1X}\lambda_\varphi + 3A_1\lambda_\varphi\lambda_X + A_1\lambda_{X\varphi}), \\
g_{14} &= -12 [F_X + 3\lambda_X F - A_1 - X(A_{1X} + 3\lambda_X A_1)].
\end{aligned}$$

Appendix B: Effective parameters in the quadratic action of perturbations

In this section, we recall the expressions of the effective parameters entering in the quadratic action of the perturbations about a FLRW background,

$$S^{\text{quad}} = \int d^3x dt a^3 \frac{M^2}{2} \left\{ \delta K_{ij} \delta K^{ij} - \left(1 + \frac{2}{3} \alpha_L \right) \delta K^2 + (1 + \alpha_T) \left(R \frac{\delta \sqrt{h}}{a^3} + \delta_2 R \right) \right. \\ \left. + H^2 \alpha_K \delta N^2 + 4H \alpha_B \delta K \delta N + (1 + \alpha_H) R \delta N + 4\beta_1 \delta K \delta \dot{N} + \beta_2 \delta \dot{N}^2 + \frac{\beta_3}{a^2} (\partial_i \delta N)^2 \right\}, \quad (\text{B1})$$

in terms of the functions (evaluated in the background solution) entering in the DHOST action,

$$S = \int d^4x \sqrt{-g} \left(P(X, \varphi) + Q(X, \varphi) \square \varphi + F(X, \varphi) R + \sum_{i=1}^5 A_i(X, \varphi) L_i \right). \quad (\text{B2})$$

We restrict ourselves to shift-symmetric theories where all the functions in the action above depend on X only.

All parameters but α_K and α_B depend on F and A_i only, and they were given in [19],

$$\frac{M^2}{2} = F - A_1 X, \quad \frac{M^2}{2} (1 + \alpha_T) = F, \quad \frac{M^2}{2} (1 + \alpha_H) = F - 2X F_X, \\ \frac{M^2}{2} \left(1 + \frac{2}{3} \alpha_L \right) = F + A_2 X, \quad \frac{M^2}{2} \beta_2 = -X (A_1 + A_2 + (A_3 + A_4) X + A_5 X^2), \quad (\text{B3}) \\ 2M^2 \beta_1 = X (4F_X + 2A_2 + A_3 X), \quad \frac{M^2}{2} \beta_3 = -X (4F_X - 2A_1 - A_4 X),$$

where the right-hand side quantities are evaluated on the homogeneous and isotropic background.

The expressions of α_K and α_B are much more complicated and they involve, in addition to F and A_i , the functions P and Q . A long calculation gives,

$$2HM^2 \alpha_B = (4HX + \dot{X}) A_1 + 2(3HX + \dot{X}) A_2 + \frac{3}{2} X (-2HX + \dot{X}) A_3 - X \dot{X} A_4 - X^2 \dot{X} A_5 \\ + 4HX^2 A_{1X} + 2X(6HX + \dot{X}) A_{2X} + X^2 \dot{X} A_{3X} \\ + (-4HX + 6\dot{X}) F_X + 2\sqrt{-X} X Q_X + 4X \dot{X} P_{XX}, \quad (\text{B4}) \\ \frac{M^2}{2} H^2 \alpha_K = (-3H^2 X + 3H \dot{X} - \frac{3\dot{X}^2}{2X} + 2\ddot{X}) A_1 + (9\dot{H} X + 3H \dot{X} - \frac{3\dot{X}^2}{2X} + 2\ddot{X}) A_2 \\ + \frac{3}{4} (18H^2 X^2 + 10\dot{H} X^2 + 8HX \dot{X} - \dot{X}^2 + 4X \ddot{X}) A_3 + (6HX \dot{X} - \frac{3X^2}{4} + 3X \ddot{X}) \\ + X(9HX \dot{X} + \dot{X}^2 + 4X \ddot{X}) A_5 + (-15H^2 X^2 + 3HX \dot{X} + \frac{3\dot{X}^2}{4} + X \ddot{X}) A_{1X} \\ + (-27H^2 X^2 + 3HX \dot{X} + \frac{3\dot{X}^2}{4} + 6X^2 \dot{H} + X \ddot{X}) A_{2X} + \frac{X}{4} (12HX \dot{X} + 7\dot{X}^2 + 4X \ddot{X}) A_{4X} \\ + (9H^2 X^3 + 3\dot{H} X^3 + 3HX^2 \dot{X} + \frac{7X \dot{X}^2}{4} + X^2 \ddot{X}) A_{3X} + \frac{X^2}{4} (12HX \dot{X} + 11\dot{X}^2 + 4X \ddot{X}) A_{5X} \\ + (-6H^2 X^3 + \frac{X \dot{X}^2}{2}) A_{1XX} + (-18H^2 X^3 + \frac{X \dot{X}^2}{2}) A_{2XX} + \frac{X^2 \dot{X}^2}{2} A_{3XX} \\ + \frac{X^2 \dot{X}^2}{2} A_{4XX} + \frac{X^3 \dot{X}^2}{2} A_{5XX} + 6X(2H^2 + 3\dot{H}) F_X \\ + 12X^2(2H^2 + \dot{H}) F_{XX} + 2X^2 Q_{XX} - 6HX^2 \sqrt{-X} Q_{XX}. \quad (\text{B5})$$

When applied to the model we are considering in the paper,

$$\begin{aligned} P &= \alpha X, \quad Q = 0, \quad F = \frac{1}{2}, \quad A_2 = \beta X, \quad A_3 = -2(\beta + 2\mu) - 8\beta\mu X^2, \\ A_4 &= 2(\beta + 2\mu - 2\mu^2 X^2), \quad A_5 = 8\mu X(\beta + 2\mu + 3\beta\mu X^2), \end{aligned} \quad (\text{B6})$$

the expressions of (B3) yield

$$\begin{aligned} M^2 &= 1 + 2\beta X, \quad \alpha_T = \alpha_H = -\frac{2\beta X^2}{1 + 2\beta X^2}, \quad \alpha_L = 0, \\ \beta_1 &= -2\mu X^2, \quad \beta_2 = -24\mu^2 X^4, \quad \beta_3 = -\frac{8\mu X^2(-1 + \mu X^2)}{1 + 2\beta X^2}, \end{aligned} \quad (\text{B7})$$

while the expressions for α_B and α_K simplify into

$$\alpha_B = \frac{2\dot{\varphi}^3 (4\beta\dot{\varphi}H + \mu (6\beta X^2 (\dot{\varphi}H - 3\dot{\varphi}) + 3\dot{\varphi}H - 5\ddot{\varphi}) - 6\mu^2 X^2 (2\beta X^2 + 1) \ddot{\varphi})}{2\beta X^2 H + H}, \quad (\text{B8})$$

$$\begin{aligned} \alpha_K &= -\frac{2X}{H^2 (2\beta X^2 + 1)} (6X(2\mu^2 X(24\dot{\varphi}H(3\beta X^2 + 1)\ddot{\varphi} + 2\dot{\varphi}\ddot{\varphi}(14\beta X^2 + 5) \\ &+ (126\beta X^2 + 25)(\ddot{\varphi})^2) + \mu(6\beta X^2(7H^2 + 3\dot{H}) + 9H^2 + 5\dot{H}) + 6\beta H^2) - \alpha), \end{aligned} \quad (\text{B9})$$

where we have used $X = -\dot{\varphi}^2$, $\dot{X} = -2\dot{\varphi}\ddot{\varphi}$ and $\ddot{X} = -2(\ddot{\varphi}^2 + \dot{\varphi}\ddot{\varphi})$.

In terms of the variables introduced in (3.7), these coefficients become

$$\begin{aligned} M^2 &= 1 + 2x_2, \quad \alpha_T = \alpha_H = -\frac{2x_2}{2x_2 + 1}, \quad \alpha_L = 0, \\ \beta_1 &= -2sx_2, \quad \beta_2 = -24s^2 x_2^2, \quad \beta_3 = -\frac{8sx_2 (sx_2 - 1)}{2x_2 + 1}, \\ \alpha_B &= -\frac{2x_2 (5(2sx_2 + s)\epsilon_\varphi - 6sx_2 - 3s - 4)}{(2x_2 + 1)(2sx_2\epsilon_\varphi + 1)}, \\ \alpha_K &= \frac{2}{(2x_2 + 1)(2sx_2\epsilon_\varphi + 1)^2} (x_1 + \epsilon_\varphi (x_2^3 (336s^2\epsilon_h - 144s^2) + 24sx_2^2 (5s\epsilon_h + 3s - 6)) \\ &- (\dot{H}/H^2) ((432s^3 x_2^4 + 120s^3 x_2^3) \epsilon_\varphi^2 + (432s^2 x_2^3 + 120s^2 x_2^2) \epsilon_\varphi + 108sx_2^2 + 30sx_2) \\ &+ \epsilon'_\varphi ((672s^3 x_2^4 + 240s^3 x_2^3) \epsilon_\varphi + 336s^2 x_2^3 + 120s^2 x_2^2) \\ &+ (720s^3 x_2^4 + 24s^2(15s + 71)x_2^3 + 420s^2 x_2^2) \epsilon_\varphi^2 - 252sx_2^2 - (54s + 36)x_2). \end{aligned} \quad (\text{B10})$$

Notice that in the last equation, we have used the relation $\ddot{\varphi} = \epsilon'_\varphi(1 + 2sx_2\epsilon_\varphi) + \epsilon_\varphi^2 + \epsilon_\varphi\epsilon_h$, which can be deduced from the relations

$$\dot{X} = -2\dot{\varphi}\ddot{\varphi} = 2\epsilon_\varphi H_b X, \quad (\text{B11})$$

which comes from the definition of ϵ_φ , and

$$\ddot{X} = -2(\ddot{\varphi}^2 + \dot{\varphi}\ddot{\varphi}) = 2XH_b^2 \left(\frac{H}{H_b} \epsilon'_\varphi + \epsilon_\varphi\epsilon_h + 2\epsilon_\varphi^2 \right). \quad (\text{B12})$$

We could also replace \dot{H}/H^2 in the expression for α_K by using the relation

$$\frac{\dot{H}}{H^2} = \frac{\epsilon_h + 2s(x'_2\epsilon_\varphi + x_2\epsilon'_\varphi)}{1 + 2sx_2\epsilon_\varphi}, \quad (\text{B13})$$

which follows from (3.10).

Finally, these results allow us to express the coefficients A_ζ in (4.6) and B_ζ in (4.7), entering in the quadratic action for the scalar perturbation, in terms of the dynamical variables in the form,

$$A_\zeta = \frac{\mathcal{A}_1(x_2) + x_1 \mathcal{A}_2(x_2)}{\mathcal{A}_3(x_2)}, \quad (\text{B14})$$

$$B_\zeta = \frac{\mathcal{B}_1(x_2) + x_1 \mathcal{B}_2(x_2) + x_1^2 \mathcal{B}_3(x_2)}{\mathcal{B}_4(x_2) + x_1 \mathcal{B}_5(x_2)}, \quad (\text{B15})$$

where the functions \mathcal{A}_i and \mathcal{B}_i are polynomials of the variable x_2 only (which depends on the parameter s) given by

$$\mathcal{A}_1(x_2) = 6x_2 (216s^2 x_2^3 + (54s^2 + 84s + 40)x_2 + 36s(6s + 5)x_2^2 - 3(s + 4)),$$

$$\mathcal{A}_2(x_2) = 2(2x_2 + 1)(-18s^2 x_2^2 + 15s x_2 + 1),$$

$$\mathcal{A}_3(x_2) = (12s x_2^2 + 2(3s + 5)x_2 + 1)^2,$$

$$\begin{aligned} \mathcal{B}_1(x_2) = & 3x_2 (38016s^4 x_2^7 + 576s^3(132s + 125)x_2^6 + 288s^2(198s^2 + 344s + 215)x_2^5 \\ & + 16s(1188s^3 + 2538s^2 + 2784s + 1775)x_2^4 - 4(279s^3 + 1176s^2 + 1154s + 500)x_2^3 \\ & + (120 + 96s - 90s^2)x_2 + 8(297s^4 + 288s^3 - 282s^2 + 600s + 500)x_2^3 + 9(3s + 4)), \end{aligned}$$

$$\begin{aligned} \mathcal{B}_2(x_2) = & -3456s^4 x_2^7 - 576s^3(9s - 20)x_2^6 - 96s^2(27s^2 - 240s - 76)x_2^5 \\ & - 16s(27s^3 - 900s^2 - 822s - 280)x_2^4 + 40(72s^3 + 96s^2 + 32s + 47)x_2^3 \\ & - 4(114s^2 + 400s + 21)x_2^2 - 2(88s + 67)x_2 - 3, \end{aligned}$$

$$\begin{aligned} \mathcal{B}_3(x_2) = & (2x_2 + 1)(432s^4 x_2^5 + 216s^3(s + 3)x_2^4 + 12s^2(15s + 1)x_2^3 \\ & - 2s(45s + 101)x_2^2 - 5(5s + 2)x_2 - 1), \end{aligned}$$

$$\mathcal{B}_4(x_2) = 3x_2(2x_2 + 1)(12s x_2^2 + 2(3s + 5)x_2 + 1)^2 \left[18s^2 x_2(2x_2 + 1)^2 + 60s x_2^2 + 4(3s + 10)x_2 - 3(3s + 4) \right],$$

$$\mathcal{B}_5(x_2) = (2x_2 + 1)^2 (12s x_2^2 + 2(3s + 5)x_2 + 1)^2 (18s^2 x_2^2 + 21s x_2 + 1).$$

These are the expressions we use to plot Fig. (3) and to express the eigenvalues (4.16) in the matter era where $|x_1| \ll |x_2| \ll 1$.

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