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Damien Calaque, Martin Gonzalez

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## Ellipsitomic associators

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## Ellipsitomic associators

Damien Calaque, Martin Gonzalez

**Abstract.** — We develop a notion of ellipsitomic associators by means of operad theory. We take this opportunity to review the operadic point-of-view on Drinfeld associators and to provide such an operadic approach for elliptic associators too. We then show that ellipsitomic associators do exist, using the monodromy of the universal ellipsitomic KZB connection, that we introduced in a previous work. We finally relate the KZB ellipsitomic associators to certain Eisenstein series associated with congruence subgroups of  $SL_2(\mathbb{Z})$ , and to twisted elliptic multiple zeta values.

**Résumé (Associateurs ellipsitomiques).** — Nous développons la notion d'associateur ellipsitomique au moyen de la théorie des opérades. Nous saisissons cette opportunité pour revoir le point de vue opéradique sur les associateurs de Drinfeld, et pour fournir également une telle approche opéradique pour les associateurs elliptiques. Nous montrons ensuite que les associateurs ellipsitomiques existent, en utilisant la monodromie de la connexion KZB ellipsitomique universelle, que nous avons introduite dans un travail précédent. Nous relierons pour finir les associateurs ellipsitomiques KZB à certaines séries d'Eisenstein associées aux sous-groupes de congruence de  $SL_2(\mathbb{Z})$ , et aux valeurs zêta multiples elliptiques tordues.



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## Introduction

The torsor of associators was introduced by Drinfeld [17] in the early nineties, in the context of quantum groups and prounipotent Grothendieck–Teichmüller theory. Since then, it has proven to have deep connections with several areas of mathematics (and physics): number theory [34], deformation quantization [22, 33, 38], Chern–Simons theory and low-dimensional topology [32], algebraic topology and the little disks operad [39], Lie theory and the Kashiwara–Vergne conjecture [1, 2] etc... In this paper we are mostly interested in the operadic and also number theoretic aspects. For instance,

- (a) The torsor of associators can be seen as the torsor of isomorphisms between two operads in (prounipotent) groupoids related to the little disks operad, denoted **PaB** and **PaCD** (for *parenthesized braids* and *parenthesized chord diagrams*). These can be understood as the Betti and de Rham fundamental groupoids of an operad of suitably compactified configuration spaces of points in the plane. See chapter 2 for more details, and accurate references.
- (b) It is expected that associators can be seen as generating series for (variations on motivic) multiple zeta values (MZVs), as was observed for the KZ associator [34] and the Deligne associator [11].

The first example of an associator was produced by Drinfeld as the renormalized holonomy of a universal version of the so-called Knizhnik–Zamolodchikov (KZ) connection [17], which is defined on a trivial principal bundle over the configuration space of points in the plane. The defining equations of an associator can be deduced from intuitive geometric reasonings about paths on configuration spaces, and they lead to representations of braid groups.

Enriquez, Etingof and the first author [12] introduced a universal version of an elliptic variation on the KZ connection (known as Knizhnik–Zamolodchikov–Bernard, or KZB, connection, as the extension to higher genus is due to Bernard [5, 6]). It is a holomorphic connection defined on a non trivial principal bundle over configuration spaces of points on an elliptic curve. They showed that

- The holonomy of the universal KZB connection along fundamental cycles of an elliptic curve satisfy relations which lead to representations of braid groups on the (2-)torus.



- They also satisfy a modularity property, that is a consequence of the fact that the (universal) KZB connection extends from configuration spaces of points on an elliptic curve to moduli spaces of marked elliptic curves (see also [35] for when there are at most 2 marked points).

Enriquez later introduced the notion of an elliptic associator [19], and proved that the holonomy of the universal elliptic KZB connection does produce, for every elliptic curve, an example of elliptic associator. The class of elliptic associators that are obtained *via* this procedure are called *KZB associators*. In another work [20], Enriquez defined and studied an elliptic version of MZVs; he showed that KZB associators are generating series for elliptic MZVs (eMZVs).

In a recent paper [13] we introduced a generalization of the universal elliptic KZB connection: the universal *ellipsitomic* KZB connection. It is defined over twisted configuration spaces, where the twisting is by a finite quotient  $\Gamma$  of the fundamental group of the elliptic curve. When  $\Gamma = 1$  is trivial, one recovers the universal elliptic KZB connection.

The aim of the present paper is two-fold.

- First we provide an operadic interpretation of elliptic associators. We extend this approach to the ellipsitomic case, use the language of operads to define ellipsitomic associators, and sketch the rudiments of an ellipsitomic Grothendieck–Teichmüller theory.
- Then we show that holonomies of the universal ellipsitomic KZB connection along suitable paths produce examples of ellipsitomic associators, and are generating series for elliptic multiple polylogarithms at  $\Gamma$ -torsion points, that are similar to the twisted elliptic MZVs (teMZVs) studied in [10] by Broedel–Matthes–Richter–Schlotterer.

Our work fits in a more general program that aims at studying associators for an oriented surface together with a finite group acting on it. We summarize in the following table the contributions to this program that we are aware of:

gen.	group	associators	operadic approach	Universal connection / existence proof	coefficients
0	trivial	[17]	[4, 23]	rational KZ [17] / <i>ibid.</i>	MZVs [34]
0	$\mathbb{Z}/N\mathbb{Z}$	cyclotomic associators [18]	[14]	trigonometric KZ [18] / <i>ibid.</i>	colored MZVs [18]
0	fin. $c$ $PSU_2(\mathbb{C})$	unknown	unknown	[36] / unknown	unknown
1	trivial	elliptic associators [19]	this paper (Sec. 3)	elliptic KZB [12] / [19]	eMZVs [20]
1	$\mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$	ellipsitomic associators (this paper)	this paper (Sec. 4 & 5)	ellipsitomic KZB [13] / this paper (Sec. 6)	this paper (Sec. 7)
> 1	trivial	[27]	[27]	KZB [21] / conj. in [27]	maybe [25]

## Description of the paper

The first chapter is devoted to some recollection on operads and operadic modules, with some emphasis on specific features when the underlying category is the one of groupoids. Chapter 2 also recollects known results, about the operadic approach to (genuine) associators and to various Grothendieck–Teichmüller groups. The main results we state are taken from the recent book [23].

The main goal of chapter 3 is to provide a similar treatment of elliptic associators, using operadic modules in place of sole operads. We show in particular that (a variant of) the universal elliptic structure  $\mathbf{PaB}_{ell}$  (resp. its graded/de Rham counterpart  $G\mathbf{PaCD}_{ell}$ ) from [19] carries the structure of an operadic module in groupoids over the operad in groupoid  $\mathbf{PaB}$  (resp.  $G\mathbf{PaCD}$ ). We provide a generators and relations presentation for  $\mathbf{PaB}_{ell}$  (Theorem 3.3), and deduce from it the following

**Theorem (Theorem 3.15).** — *The torsor of elliptic associators from [19] coincides with the torsor of isomorphisms from (a variant of)  $\mathbf{PaB}_{ell}$  to  $G\mathbf{PaCD}_{ell}$  that are the identity on objects. Similarly, the elliptic Grothendieck–Teichmüller group (resp. its graded version) is isomorphic to the group of automorphisms of  $\mathbf{PaB}_{ell}$  (resp. of  $G\mathbf{PaCD}_{ell}$ ) that are the identity on objects.*

The fourth chapter introduces a generalization of  $\mathbf{PaB}_{ell}$ , with an additional labelling/twisting by elements of  $\Gamma$  (recall that  $\Gamma$  is the group of deck transformations of a finite cover of the torus by another torus). We give a geometric definition of the operadic module  $\mathbf{PaB}_{ell}^\Gamma$  of parenthesized ellipsitomic braids, and then provide a presentation by generators and relations for it (Theorem 4.5). In the fifth chapter we define an operadic module of ellipsitomic chord diagrams, that mixes features of  $\mathbf{PaCD}_{ell}$  from chapter 3, and of the moperad of cyclotomic chord diagrams from [14]. This allows us to identify ellipsitomic associators, which we define in purely operadic terms, with series satisfying certain algebraic equations (Theorem 5.9).

Chapter 6 is devoted to the proof of the following

**Theorem (Theorem 6.1).** — *The set of ellipsitomic associators over  $\mathbb{C}$  is non-empty.*

The proof makes crucial use of the ellipsitomic KZB connection, introduced in our previous work [13], and relies on a careful analysis of its monodromy. We actually prove that one can associate an ellipsitomic associator with every element of the upper half-plane (Theorem 6.1). In the last chapter we quickly explore some number theoretic and modular aspects of the coefficients of the “KZB produced” ellipsitomic associators from the previous chapter.

Finally, in an appendix we provide an alternative presentation for  $\mathbf{PaB}_{ell}^\Gamma$ .

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# CHAPTER 1

## Background material on operads and groupoids

In this chapter we fix a symmetric monoidal category  $(\mathcal{C}, \otimes, \mathbf{1})$  having small colimits. Let us assume for simplicity of exposition that  $\otimes$  commutes with these<sup>(1)</sup>.

### 1.1. $\mathfrak{S}$ -modules

An  $\mathfrak{S}$ -module (in  $\mathcal{C}$ ) is a functor  $S : \mathbf{Bij} \rightarrow \mathcal{C}$ , where  $\mathbf{Bij}$  denotes the category of finite sets with bijections as morphisms. It can also be defined as a collection  $(S(n))_{n \geq 0}$  of objects of  $\mathcal{C}$  such that  $S(n)$  is endowed with a right action of the symmetric group  $\mathfrak{S}_n$  for every  $n$ ; one has  $S(n) := S(\{1, \dots, n\})$ . A *morphism* of  $\mathfrak{S}$ -modules  $\varphi : S \rightarrow T$  is a natural transformation. It is determined by the data of a collection  $\varphi(n) : S(n) \rightarrow T(n)$  of  $\mathfrak{S}_n$ -equivariant morphisms in  $\mathcal{C}$ .

The category  $\mathfrak{S}\text{-mod}$  of  $\mathfrak{S}$ -modules is naturally endowed with a symmetric monoidal product  $\otimes$  defined as follows:

$$(S \otimes T)(n) := \coprod_{p+q=n} (S(p) \otimes T(q))_{\mathfrak{S}_p \times \mathfrak{S}_q}^{\mathfrak{S}_n}.$$

Here, if  $H \subset G$  is a group inclusion, then  $(-)_H^G$  is left adjoint to the restriction functor from the category of objects carrying a  $G$ -action to the category of objects carrying an  $H$ -action.

The symmetric sequence  $\mathbf{1}_\otimes$  defined by

$$\mathbf{1}_\otimes(n) := \begin{cases} \mathbf{1} & \text{if } n = 0 \\ \emptyset & \text{else} \end{cases}$$

is a monoidal unit for  $\otimes$ .

There is another (non-symmetric) monoidal product  $\circ$  on  $\mathfrak{S}\text{-mod}$ , defined as follows:

$$(S \circ T)(n) := \coprod_{k \geq 0} T(k) \otimes_{\mathfrak{S}_k} (S^{\otimes k}(n)).$$

---

<sup>(1)</sup>This latter assumption is not necessary (and we will have to get rid of it when considering the monoidal structure given by the direct sum of Lie algebras): if the monoidal product does not commute with colimits, the category of  $\mathfrak{S}$ -module still has enough structure so that one can define monoids and modules in it. Characterizations in terms of partial compositions remain unchanged. We refer to [15] for more details.

Here, if  $H$  is a group and  $X, Y$  are objects carrying an  $H$ -action, then

$$X \otimes_H Y := \operatorname{coeq} \left( \coprod_{h \in H} X \otimes Y \begin{array}{c} \xrightarrow{h \otimes \operatorname{id}} \\ \xrightarrow{\operatorname{id} \otimes h} \end{array} X \otimes Y \right).$$

The symmetric sequence  $\mathbf{1}_\circ$  defined by

$$\mathbf{1}_\circ(n) := \begin{cases} \mathbf{1} & \text{if } n = 1 \\ \emptyset & \text{else} \end{cases}$$

is a monoidal unit for  $\circ$ .

## 1.2. Operads

An *operad* (in  $\mathcal{C}$ ) is a unital monoid in  $(\mathfrak{S}\text{-mod}, \circ, \mathbf{1}_\circ)$ . The category of operads in  $\mathcal{C}$  will be denoted  $\operatorname{Op}\mathcal{C}$ .

More explicitly, an operad structure on a  $\mathfrak{S}$ -module  $\mathcal{O}$  is the data:

- of a unit map  $e : \mathbf{1} \rightarrow \mathcal{O}(1)$ .
- for every sets  $I, J$  and any element  $i \in I$ , of a *partial composition*

$$\circ_i : \mathcal{O}(I) \otimes \mathcal{O}(J) \longrightarrow \mathcal{O}(J \sqcup I - \{i\})$$

satisfying the following constraints:

- for every sets  $I, J, K$ , with elements  $i \in I, j \in J$ , the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(I) \otimes \mathcal{O}(J) \otimes \mathcal{O}(K) & \xrightarrow{\circ_i \otimes \operatorname{id}} & \mathcal{O}(J \sqcup I - \{i\}) \otimes \mathcal{O}(K) \\ \downarrow \operatorname{id} \otimes \circ_j & & \downarrow \circ_j \\ \mathcal{O}(I) \otimes \mathcal{O}(K \sqcup J - \{j\}) & \xrightarrow{\circ_i} & \mathcal{O}(K \sqcup J \sqcup I - \{i, j\}) \end{array}$$

- for every sets  $I, J_1, J_2$ , with elements  $i_1, i_2 \in I$ , the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(I) \otimes \mathcal{O}(J_1) \otimes \mathcal{O}(J_2) & \xrightarrow{\circ_{i_1} \otimes \operatorname{id}} & \mathcal{O}(J_1 \sqcup I - \{i_1\}) \otimes \mathcal{O}(J_2) \\ \downarrow (\circ_{i_2} \otimes \operatorname{id})(23) & & \downarrow \circ_{i_2} \\ \mathcal{O}(J_2 \sqcup I - \{i_2\}) \otimes \mathcal{O}(J_1) & \xrightarrow{\circ_{i_1}} & \mathcal{O}(J_2 \sqcup J_1 \sqcup I - \{i_1, i_2\}) \end{array}$$

- for every sets  $I, I', J, i \in I$ , with a bijection  $\sigma : I \rightarrow I'$ , the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(I) \otimes \mathcal{O}(J) & \xrightarrow{\mathcal{O}(\sigma)} & \mathcal{O}(I') \otimes \mathcal{O}(J) \\ \downarrow \circ_i & & \downarrow \circ_{\sigma(i)} \\ \mathcal{O}(J \sqcup I - \{i\}) & \xrightarrow{\mathcal{O}(\operatorname{id} \sqcup \sigma|_{I - \{i\}})} & \mathcal{O}(J \sqcup I' - \{\sigma(i)\}) \end{array}$$

– for every set  $I$ , with  $i \in I$ , the following diagrams commute:

$$\begin{array}{ccc}
 \mathbf{1} \otimes \mathcal{O}(I) & \xrightarrow{e \otimes \text{id}} & \mathcal{O}(\{1\}) \otimes \mathcal{O}(I) & \mathcal{O}(I) \otimes \mathbf{1} & \xrightarrow{\text{id} \otimes e} & \mathcal{O}(I) \otimes \mathcal{O}(\{1\}) \\
 & \searrow \simeq & \downarrow \circ_1 & \simeq \downarrow & & \downarrow \circ_i \\
 & & \mathcal{O}(I) & \mathcal{O}(I) & \xrightarrow[\text{i} \mapsto 1]{\simeq} & \mathcal{O}(I \sqcup \{1\} - \{i\})
 \end{array}$$

**Example 1.1.** — Let  $X$  be an object of  $\mathcal{C}$ . Then we define, for any finite set  $I$ , the set  $\underline{\text{End}}(X)(I) := \text{Hom}_{\mathcal{C}}(X^{\otimes I}, X)$ . Composition of tensor products of maps provide  $\underline{\text{End}}(X)$  with the structure of an operad in sets.

Given an operad in sets  $\mathcal{O}$ , an  $\mathcal{O}$ -algebra in  $\mathcal{C}$  is an object  $X$  of  $\mathcal{C}$  together with a morphism of operads  $\mathcal{O} \rightarrow \underline{\text{End}}(X)$ .

### 1.3. Example of an operad: Stasheff polytopes

To any finite set  $I$  we associate the configuration space  $\text{Conf}(\mathbb{R}, I) = \{\mathbf{x} = (x_i)_{i \in I} \in \mathbb{R}^I \mid x_i \neq x_j \text{ if } i \neq j\}$  and its reduced version

$$C(\mathbb{R}, I) := \text{Conf}(\mathbb{R}, I) / \mathbb{R} \rtimes \mathbb{R}_{>0}.$$

The Axelrod–Singer–Fulton–MacPherson compactification<sup>(2)</sup>  $\overline{C}(\mathbb{R}, I)$  of  $C(\mathbb{R}, I)$  is a disjoint union of  $|I|$ -th Stasheff polytopes [37], indexed by  $\mathfrak{S}_I$ . The boundary  $\partial \overline{C}(\mathbb{R}, I) := \overline{C}(\mathbb{R}, I) - C(\mathbb{R}, I)$  is the union, over all partitions  $I = J_1 \amalg \dots \amalg J_k$ , of

$$\partial_{J_1, \dots, J_k} \overline{C}(\mathbb{R}, I) := \prod_{i=1}^k \overline{C}(\mathbb{R}, J_i) \times \overline{C}(\mathbb{R}, k).$$

The inclusion of boundary components provides  $\overline{C}(\mathbb{R}, -)$  with the structure of an operad in topological spaces (where the monoidal structure is given by the cartesian product).

One can see that  $\overline{C}(\mathbb{R}, I)$  is actually a manifold with corners, and that, considering only zero-dimensional strata of our configuration spaces, we get a suboperad  $\mathbf{Pa} \subset \overline{C}(\mathbb{R}, -)$  that can be shortly described as follows:

- $\mathbf{Pa}(I)$  is the set of pairs  $(\sigma, p)$  with  $\sigma$  is a linear order on  $I$  and  $p$  a maximal parenthesization of  $\underbrace{\bullet \dots \bullet}_{|I| \text{ times}}$ ,
- the operad structure is given by substitution.

Notice that  $\mathbf{Pa}$  is actually an operad in sets, and that  $\mathbf{Pa}$ -algebras are nothing else than *magmas*.

<sup>(2)</sup>We are using the differential geometric compactification from [3], which is an analog of the algebro-geometric one from [24].

#### 1.4. Modules over an operad: Bott-Taubes polytopes

A *module* over an operad  $\mathcal{O}$  (in  $\mathcal{C}$ ) is a right  $\mathcal{O}$ -module in  $(\mathfrak{S}\text{-mod}, \circ, \mathbf{1}_\circ)$ . Notice that any operad is a module over itself. We let the reader find the very explicit description of a module in terms of partial compositions, as for operads.

To any finite set  $I$  we associate the configuration space  $\text{Conf}(\mathbb{S}^1, I) = \{\mathbf{x} = (x_i)_{i \in I} \in (\mathbb{S}^1)^I \mid x_i \neq x_j \text{ if } i \neq j\}$  and its reduced version

$$C(\mathbb{S}^1, I) := \text{Conf}(\mathbb{S}^1, I) / \mathbb{S}^1.$$

The Axelrod–Singer–Fulton–MacPherson compactification  $\overline{C}(\mathbb{S}^1, I)$  of  $C(\mathbb{S}^1, I)$  is a disjoint union of  $|I|$ -th Bott–Taubes polytopes [8], indexed by  $\mathfrak{S}_I$ . The boundary  $\partial \overline{C}(\mathbb{S}^1, I) := \overline{C}(\mathbb{S}^1, I) - C(\mathbb{S}^1, I)$  is the union, over all partitions  $I = J_1 \sqcup \cdots \sqcup J_k$ , of

$$\partial_{J_1, \dots, J_k} \overline{C}(\mathbb{S}^1, I) := \prod_{i=1}^k \overline{C}(\mathbb{R}, J_i) \times \overline{C}(\mathbb{S}^1, k).$$

The inclusion of boundary components provides  $\overline{C}(\mathbb{S}^1, -)$  with the structure of a module over the operad  $\overline{C}(\mathbb{R}, -)$  in topological spaces.

One can see that  $\overline{C}(\mathbb{S}^1, I)$  is actually a manifold with corners, and that, considering only zero-dimensional strata of our configuration spaces, we get  $\mathbf{Pa} \subset \overline{C}(\mathbb{S}^1, -)$ , which is a module over  $\mathbf{Pa} \subset \overline{C}(\mathbb{R}, -)$ .

#### 1.5. Convention: pointed versions

Observe that there is an operad *Unit* defined by

$$\text{Unit}(n) = \begin{cases} \mathbf{1} & \text{if } n = 0, 1 \\ \emptyset & \text{else} \end{cases}$$

By convention, all our operads  $\mathcal{O}$  will be *Unit*-pointed and reduced, in the sense that they will come equipped with a specific operad morphism  $\text{Unit} \rightarrow \mathcal{O}$  that is an isomorphism in arity  $\leq 1$ :  $\mathcal{O}(n) \simeq \mathbf{1}$  if  $n = 0, 1$ . Morphisms of operads are required to be compatible with this pointing.

Now, if  $\mathcal{P}$  is an  $\mathcal{O}$ -module, then it naturally becomes a *Unit*-module as well, by restriction. By convention, all our modules will be pointed as well, in the sense that they will come equipped with a specific *Unit*-module morphism  $\text{Unit} \rightarrow \mathcal{P}$  that is an isomorphism in arity  $\leq 1$ . Morphisms of modules are also required to be compatible with the pointing.

The main reason for this convention is that we need the following features, that we have in the case of compactified configuration spaces:

- For operads and modules, we want to have “deleting operations”  $\mathcal{O}(n) \rightarrow \mathcal{O}(n-1)$  that decrease arity.
- For modules, we want to be able to see the operad “inside” them, i.e. we want to have distinguished morphism  $\mathcal{O} \rightarrow \mathcal{P}$  of  $\mathfrak{S}$ -modules.

### 1.6. Group actions

Let  $G$  be a  $*$ -module in group, where  $*$  is the terminal operad: the partial composition  $\circ_i$  is a group morphism  $G(n) \rightarrow G(n+m-1)$ .

**Example 1.2.** — Let  $\Gamma$  be a group, we consider the  $\mathfrak{S}$ -module in groups  $\bar{\Gamma} := \{\Gamma^n/\Gamma^{diag}\}_{n \geq 0}$ , where  $\Gamma^{diag}$  denotes the normal closure of the diagonal subgroup in each  $\Gamma^n$ . It is equipped with the following  $*$ -module structure: the  $i$ -th partial composition is given by the partial diagonal morphism

$$\begin{aligned} \Gamma^n/\Gamma &\longrightarrow \Gamma^{n+m-1}/\Gamma \\ [\gamma_1, \dots, \gamma_n] &\longmapsto [\gamma_1, \dots, \gamma_{i-1}, \underbrace{\gamma_i, \dots, \gamma_i}_{m \text{ times}}, \gamma_{i+1}, \dots, \gamma_n] \end{aligned}$$

Given an operad  $\mathcal{O}$  in  $\mathcal{C}$ , we say that an  $\mathcal{O}$ -module  $\mathcal{P}$  carries a  $G$ -action if

- for every  $n \geq 0$ , there is an  $\mathfrak{S}_n$ -equivariant left action  $G(n) \times \mathcal{P}(n) \rightarrow \mathcal{P}(n)$ .
- for every  $m \geq 0$ ,  $n \geq 0$ , and  $1 \leq i \leq n$ , the partial composition

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{O}(m) \longrightarrow \mathcal{P}(n+m-1)$$

is equivariant along the above group morphism  $G(n) \rightarrow G(n+m-1)$ .

A morphism  $\mathcal{P} \rightarrow \mathcal{Q}$  of  $\mathcal{O}$ -modules with  $G$ -action is said  $G$ -equivariant if, for every  $n \geq 0$ , the map  $\mathcal{P}(n) \rightarrow \mathcal{Q}(n)$  is  $G(n)$ -equivariant.

Given a group  $\Gamma$ , we say that an  $\mathcal{O}$ -module  $\mathcal{P}$  carries a *diagonally trivial action* of  $\Gamma$  if it carries a  $\bar{\Gamma}$ -action.

The quotient  $G \backslash \mathcal{P}$  of an  $\mathcal{O}$ -module  $\mathcal{P}$  with a  $G$ -action is defined as follows:

- For every  $n \geq 0$ ,  $(G(n) \backslash \mathcal{P})(n) := G(n) \backslash \mathcal{P}(n)$ ;
- The equivariance of the partial composition  $\circ_i$  tells us that it descends to the quotient

$$(G(n) \backslash \mathcal{P}(n)) \otimes \mathcal{O}(m) \longrightarrow G(n+m-1) \backslash \mathcal{P}(n+m-1).$$

### 1.7. Semi-direct products and fake pull-backs

Let  $\mathbf{Grpd}$  be the category of groupoids. For a group  $G$ , we denote  $G - \mathbf{Grpd}$  the category of groupoids equipped with a  $G$ -action. There is a *semi-direct product* functor

$$\begin{aligned} G - \mathbf{Grpd} &\longrightarrow \mathbf{Grpd}_{/G} \\ \mathcal{P} &\longmapsto \mathcal{P} \rtimes G \end{aligned}$$

where the group  $G$  is viewed as a groupoid with a single object, and where  $\mathcal{P} \rtimes G$  is defined as follows:

- Objects of  $\mathcal{P} \rtimes G$  are just objects of  $\mathcal{P}$ ;
- In addition to the arrows of  $\mathcal{P}$ , for every  $g \in G$ , and for every object  $\mathbf{p}$  of  $\mathcal{P}$ , there is an arrow  $g \cdot \mathbf{p} \xrightarrow{g} \mathbf{p}$ ;
- These new arrows multiply together via the group multiplication of  $G$ ;
- For every morphism  $f$  in  $\mathcal{P}$ , and every  $g \in G$ , the relation  $gf g^{-1} = g \cdot f$  holds.



**Notation 1.3.** — We warn the reader that we use all along the paper the following rather unusual convention for arrows in a groupoid, and more generally in a category: we often concatenate arrows rather than composing them. In other words,  $f_1 f_2 = f_2 \circ f_1$ .

There is also a functor  $\mathcal{G}$  going in the other direction

$$\begin{aligned} \mathbf{Grpd}_{/G} &\longrightarrow G\text{-}\mathbf{Grpd} \\ (\mathcal{Q} \xrightarrow{\varphi} G) &\longmapsto \mathcal{G}(\varphi) \end{aligned}$$

that one can describe as follows:

- The  $G$ -set of objects of  $\mathcal{G}(\varphi)$  is the free  $G$ -set generated by  $\mathrm{Ob}(\mathcal{Q})$ ;
- A morphism  $(g, x) \rightarrow (h, y)$  in  $\mathcal{G}(\varphi)$  is a morphism  $x \xrightarrow{f} y$  in  $\mathcal{Q}$  such that  $g\varphi(f) = h$ .

**Example 1.4.** — The groupoid  $\mathcal{G}(\mathbf{B}_n \rightarrow \mathfrak{S}_n)$  is the colored braid groupoid  $\mathbf{CoB}(n)$  from [23, §5.2.8].

**Remark 1.5.** — Given an object  $q$  of  $\mathcal{Q}$ ,  $\mathrm{Aut}_{\mathcal{G}(\varphi)}(g, q)$  is the kernel of the morphism  $\mathrm{Aut}_{\mathcal{Q}}(q) \rightarrow G$  for every  $g \in G$ .

These constructions still make sense for modules over a given operad  $\mathcal{O}$  whenever  $G$  is an operadic  $*$ -module in groups.

Let  $\mathcal{P}, \mathcal{Q}$  be two operads (resp. modules) in groupoids. If we are given a morphism  $f: \mathrm{Ob}(\mathcal{P}) \rightarrow \mathrm{Ob}(\mathcal{Q})$  between the operads (resp. operad modules) of objects of  $\mathcal{P}$  and  $\mathcal{Q}$ , then (following [23]) we can define an operad (resp. operad module)  $f^*\mathcal{Q}$  in the following way:

- $\mathrm{Ob}(f^*\mathcal{Q}) := \mathrm{Ob}(\mathcal{P})$ ,
- $\mathrm{Hom}_{(f^*\mathcal{Q})(n)}(p, q) := \mathrm{Hom}_{\mathcal{Q}(n)}(f(p), f(q))$ .

In particular,  $f^*\mathcal{Q}$ , which we call the *fake pull-back* of  $\mathcal{Q}$  along  $f$ , inherits the operad structure of  $\mathcal{P}$  for its operad of objects and that of  $\mathcal{Q}$  for the morphisms.

**Remark 1.6.** — Notice that this is not a pull-back in the category of operads in groupoids.

### 1.8. Prounipotent completion

Let  $\mathbf{k}$  be a  $\mathbb{Q}$ -ring. We denote by  $\mathbf{CoAlg}_{\mathbf{k}}$  the symmetric monoidal category of complete filtered topological coassociative cocommutative counital  $\mathbf{k}$ -coalgebras, where the monoidal product is given by the completed tensor product  $\hat{\otimes}_{\mathbf{k}}$  over  $\mathbf{k}$ .

Let  $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})$  be the category of small  $\mathbf{CoAlg}_{\mathbf{k}}$ -enriched categories. It is symmetric monoidal as well, with monoidal product  $\otimes$  defined as follows:

- $\mathrm{Ob}(C \otimes C') := \mathrm{Ob}(C) \times \mathrm{Ob}(C')$ .
- $\mathrm{Hom}_{C \otimes C'}((c, c'), (d, d')) := \mathrm{Hom}_C(c, d) \hat{\otimes}_{\mathbf{k}} \mathrm{Hom}_{C'}(c', d')$ .

All the constructions of the previous section still make sense, at the cost of replacing the group  $G$  with its completed group algebra  $\widehat{\mathbf{k}G}$  (which is a Hopf algebra) in the semi-direct product construction.

Considering the cartesian symmetric monoidal structure on  $\mathbf{Grpd}$ , there is a symmetric monoidal functor

$$\begin{aligned} \mathbf{Grpd} &\longrightarrow \mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}}) \\ \mathcal{G} &\longmapsto \mathcal{G}(\mathbf{k}) \end{aligned}$$

defined as follows:

- Objects of  $\mathcal{P}(\mathbf{k})$  are objects of  $\mathcal{P}$ .
- For  $a, b \in \mathit{Ob}(\mathcal{P})$ ,

$$\mathit{Hom}_{\mathcal{P}(\mathbf{k})}(a, b) = \mathbf{k} \cdot \widehat{\mathit{Hom}_{\mathcal{P}}(a, b)}.$$

Here  $\mathbf{k} \cdot \mathit{Hom}_{\mathcal{P}}(a, b)$  is equipped with the unique coalgebra structure such that the elements of  $\mathit{Hom}_{\mathcal{P}}(a, b)$  are grouplike (meaning that they are diagonal for the coproduct and that their counit is 1), and the “ $\widehat{\phantom{x}}$ ” refers to the completion with respect to the topology defined by the sequence  $(\mathit{Hom}_{\mathcal{I}^k}(a, b))_{k \geq 0}$ , where  $\mathcal{I}^k$  is the category having the same objects as  $\mathcal{P}$  and morphisms lying in the  $k$ -th power (for the composition of morphisms) of kernels of the counits of  $\mathbf{k} \cdot \mathit{Hom}_{\mathcal{P}}(a, b)$ 's.

- For a functor  $F : \mathcal{P} \rightarrow \mathcal{Q}$ ,  $F(\mathbf{k}) : \mathcal{P}(\mathbf{k}) \rightarrow \mathcal{Q}(\mathbf{k})$  is the functor given by  $F$  on objects and by  $\mathbf{k}$ -linearly extending  $F$  on morphisms.

Being symmetric monoidal, this functor sends operads in groupoids to operads in  $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})$ .

**Example 1.7.** — For instance, viewing  $\mathbf{Pa}$  as an operad in groupoid (with only identities as morphisms), then  $\mathbf{Pa}(\mathbf{k})$  is the operad in  $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})$  with same objects as  $\mathbf{Pa}$ , and whose morphisms are

$$\mathit{Hom}_{\mathbf{Pa}(\mathbf{k})(n)}(a, b) = \begin{cases} \mathbf{k} & \text{if } a = b \\ 0 & \text{else} \end{cases}$$

with  $\mathbf{k}$  being equipped with the coproduct  $\Delta(1) = 1 \otimes 1$  and counit  $\epsilon(1) = 1$ .

The functor we have just defined has a right adjoint

$$G : \mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}}) \longrightarrow \mathbf{Grpd},$$

that we can describe as follows:

- For  $C$  in  $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})$ , objects of  $G(C)$  are objects of  $C$ .
- For  $a, b \in \mathit{Ob}(\mathcal{G})$ ,  $\mathit{Hom}_{G(C)}(a, b)$  is the subset of grouplike elements in  $\mathit{Hom}_C(a, b)$ .

Being right adjoint to a symmetric monoidal functor, it is lax symmetric monoidal, and thus it sends operads (resp. modules) to operads (resp. modules).

We thus get a  $\mathbf{k}$ -prounipotent completion functor  $\mathcal{G} \mapsto \hat{\mathcal{G}}(\mathbf{k}) := G(\mathcal{G}(\mathbf{k}))$  for (operads and modules in) groupoids.

**Remark 1.8.** — Let  $\varphi : G \rightarrow S$  be a surjective group morphism, and assume that  $S$  is finite. One can prove that the prounipotent completion  $\hat{\mathcal{G}}(\varphi)(\mathbf{k})$  of the construction from the previous section is isomorphic to  $\mathcal{G}(\varphi(\mathbf{k}))$ , where  $\varphi(\mathbf{k}) : G(\varphi, \mathbf{k}) \rightarrow S$  is Hain's relative completion [28]. This essentially follows from that, when  $S$  is finite, the kernel of the relative completion is the completion of the kernel.



## CHAPTER 2

### Operads associated with configuration spaces (associators)

#### 2.1. Compactified configuration space of the plane

To any finite set  $I$  we associate a configuration space

$$\text{Conf}(\mathbb{C}, I) = \{ \mathbf{z} = (z_i)_{i \in I} \in \mathbb{C}^I \mid z_i \neq z_j \text{ if } i \neq j \}.$$

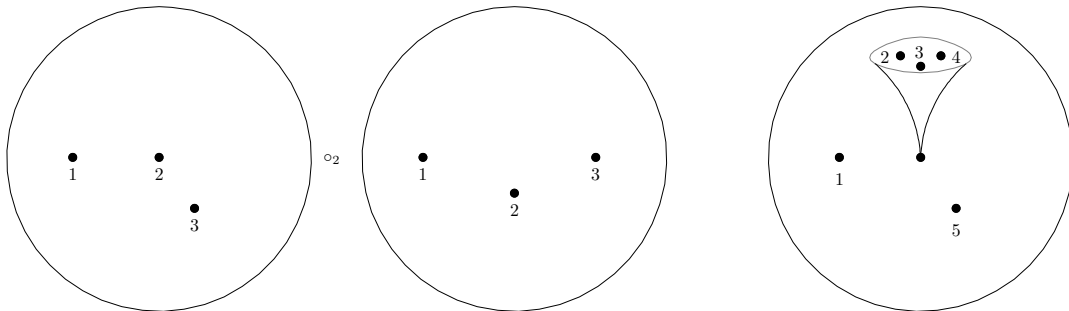
We also consider its reduced version

$$\mathbb{C}(\mathbb{C}, I) := \text{Conf}(\mathbb{C}, I) / \mathbb{C} \rtimes \mathbb{R}_{>0}.$$

We then consider the Axelrod–Singer–Fulton–MacPherson compactification  $\overline{\mathbb{C}}(\mathbb{C}, I)$  of  $\mathbb{C}(\mathbb{C}, I)$ . The boundary  $\partial \overline{\mathbb{C}}(\mathbb{C}, I) = \overline{\mathbb{C}}(\mathbb{C}, I) - \mathbb{C}(\mathbb{C}, I)$  is made of the following irreducible components: for any partition  $I = J_1 \sqcup \dots \sqcup J_k$  there is a component

$$\partial_{J_1, \dots, J_k} \overline{\mathbb{C}}(\mathbb{C}, I) \cong \overline{\mathbb{C}}(\mathbb{C}, k) \times \prod_{i=1}^k \overline{\mathbb{C}}(\mathbb{C}, J_i).$$

The inclusion of boundary components provides  $\overline{\mathbb{C}}(\mathbb{C}, -)$  with the structure of an operad in topological spaces. One can picture the partial operadic composition morphisms as follows:



## 2.2. A presentation for the pure braid group

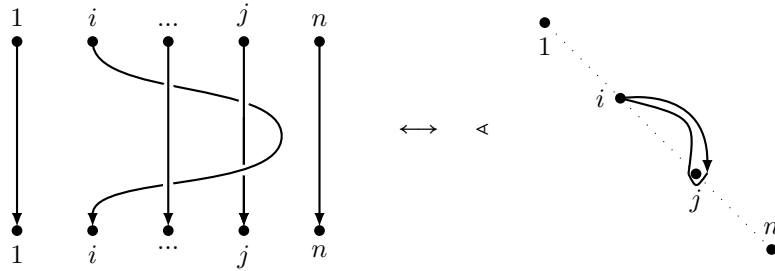
The pure braid group  $\mathbf{PB}_n$  is generated by elementary pure braids  $P_{ij}$ ,  $1 \leq i < j \leq n$ , which satisfy the following relations:

$$(PB1) \quad (P_{ij}, P_{kl}) = 1 \quad \text{if } \{i, j\} \text{ and } \{k, l\} \text{ are non crossing,}$$

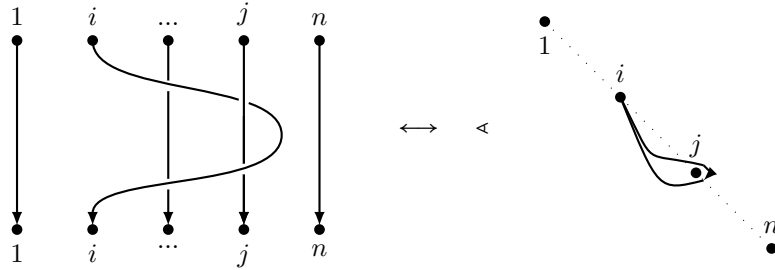
$$(PB2) \quad (P_{kj}P_{ij}P_{kj}^{-1}, P_{kl}) = 1 \quad \text{if } i < k < j < l,$$

$$(PB3) \quad (P_{ij}, P_{ik}P_{jk}) = (P_{jk}, P_{ij}P_{ik}) = (P_{ik}, P_{jk}P_{ij}) = 1 \quad \text{if } i < j < k.$$

In this article we will represent the generator  $P_{ij}$  in the following two equivalent ways:



There is another elementary braid  $\mathfrak{Q}_{i,j}$  conjugated to  $P_{i,j}$ . We can represent two incarnations of the generator  $\mathfrak{Q}_{i,j}$  in the following way



Indeed, one can define  $O_{ij} := P_{ij} \cdots P_{i(i+2)} P_{i(i+1)}$ . In other words,  $P_{ij} = O_{ij} O_{i(j-1)}^{-1}$ . And we define  $\mathfrak{Q}_{ij} := O_{i(j-1)}^{-1} O_{ij} = O_{i(j-1)}^{-1} P_{ij} O_{i(j-1)}$ .

## 2.3. The operad of parenthesized braids

There are inclusions of topological operads

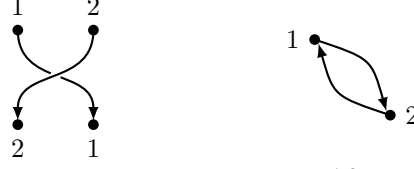
$$\mathbf{Pa} \subset \overline{\mathbf{C}}(\mathbb{R}, -) \subset \overline{\mathbf{C}}(\mathbb{C}, -).$$

Then it makes sense to define

$$\mathbf{PaB} := \pi_1(\overline{\mathbf{C}}(\mathbb{C}, -), \mathbf{Pa}),$$

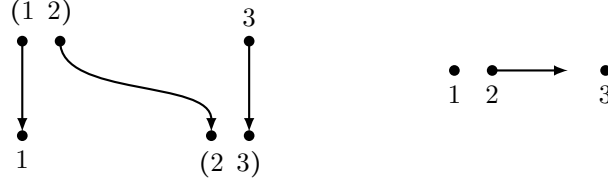
which is an operad in groupoids.

**Example 2.1 (Description of  $\mathbf{PaB}(2)$ ).** — Let us first recall that  $\mathbf{Pa}(2) = \mathfrak{S}_2$ , and that  $\overline{\mathbf{C}}(\mathbb{C}, 2) \simeq \mathbb{S}^1$ . Besides the identity morphism in  $\mathbf{PaB}(2)$  going from  $(12)$  to  $(12)$ , there is an arrow  $R^{1,2}$  in  $\mathbf{PaB}(2)$  going from  $(12)$  to  $(21)$  which can be depicted as follows<sup>(1)</sup>:

Two incarnations of  $R^{1,2}$ 

We will denote  $\tilde{R}^{1,2} := (R^{2,1})^{-1}$ .

**Example 2.2 (Notable arrows in  $\mathbf{PaB}(3)$ ).** — Let us first recall that  $\mathbf{Pa}(3) = \mathfrak{S}_3 \times \{(\bullet\bullet)\bullet, \bullet(\bullet\bullet)\}$  and that  $\overline{\mathbf{C}}(\mathbb{R}, 3) \cong \mathfrak{S}_3 \times [0, 1]$ . Therefore, there is an arrow  $\Phi^{1,2,3}$  (the identity path in  $[0, 1]$ ) from  $(12)3$  to  $1(23)$  in  $\mathbf{PaB}(3)$ . It can be depicted as follows:

Two incarnations of  $\Phi^{1,2,3}$ 

The following result is borrowed from [23, Theorem 6.2.4], even though it perhaps already appeared in [4] in a different form.

**Theorem 2.3.** — *As an operad in groupoids having  $\mathbf{Pa}$  as operad of objects,  $\mathbf{PaB}$  is freely generated by  $R := R^{1,2}$  and  $\Phi := \Phi^{1,2,3}$  together with the following relations:*

- (U1)  $\Phi^{\emptyset,1,2} = \Phi^{1,\emptyset,2} = \Phi^{1,2,\emptyset} = \text{Id}_{1,2}$  (in  $\text{Hom}_{\mathbf{PaB}(2)}((12), (12))$ ),
- (H1)  $R^{1,2}\Phi^{2,1,3}R^{1,3} = \Phi^{1,2,3}R^{1,23}\Phi^{2,3,1}$  (in  $\text{Hom}_{\mathbf{PaB}(3)}((12)3, 2(31))$ ),
- (H2)  $\tilde{R}^{1,2}\Phi^{2,1,3}\tilde{R}^{1,3} = \Phi^{1,2,3}\tilde{R}^{1,23}\Phi^{2,3,1}$  (in  $\text{Hom}_{\mathbf{PaB}(3)}((12)3, 2(31))$ ),
- (P)  $\Phi^{12,3,4}\Phi^{1,2,34} = \Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4}$  (in  $\text{Hom}_{\mathbf{PaB}(4)}(((12)3)4, 1(2(34)))$ ).

We now briefly explain the notation we have been using in the above statement, which is quite standard.

**Notation 2.4.** — *In this article, we write the composition of paths from left to right (and we draw the braids from top to bottom). If  $X$  is an arrow from  $\mathbf{p}$  to  $\mathbf{q}$  in  $\mathbf{PaB}(n)$ , then*

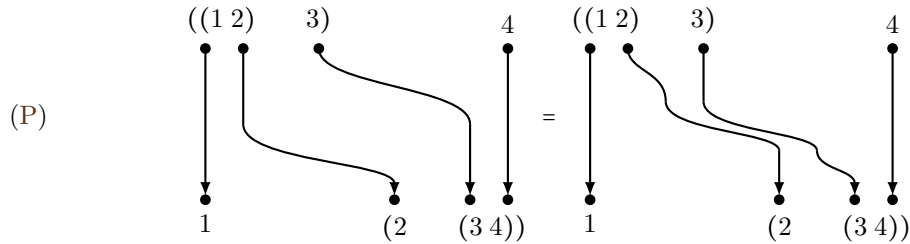
- for any  $\mathbf{r} \in \mathbf{Pa}(k)$ , the identity of  $\mathbf{r}$  in  $\mathbf{PaB}(k)$  is also denoted  $\mathbf{r}$ ,
- for any  $\mathbf{r} \in \mathbf{Pa}(k)$ , we write  $X^{1,\dots,n}$  for  $\mathbf{r} \circ_1 X \in \mathbf{PaB}(n+k-1)$ ,
- we write  $X^{\emptyset,2,\dots,n} \in \mathbf{PaB}(n+k-2)$  for the image of  $X^{1,\dots,n}$  by the first braid deleting operation,

<sup>(1)</sup>We actually have another arrow, that can be obtained from the first one as  $(R^{2,1})^{-1}$  according to the notation that is explained after Theorem 2.3, and which can be depicted as an undercrossing braid.

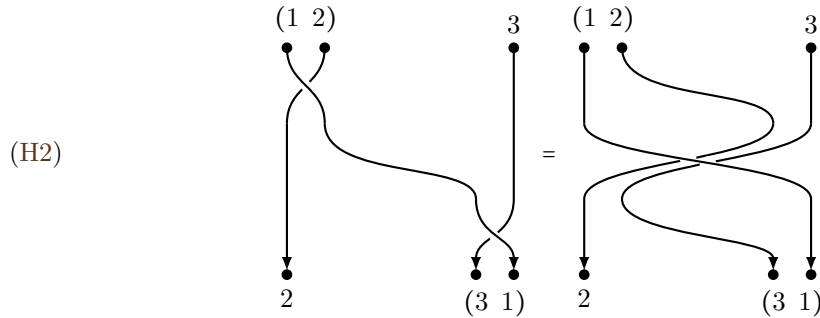
- for any  $\sigma \in \mathfrak{S}_{n+k-1}$  we define  $X^{\sigma_1, \dots, \sigma_n} := (X^{1, \dots, n}) \cdot \sigma$ ,
- for any  $\mathbf{r} \in \mathbf{Pa}(k)$ ,  $X^{\mathbf{r}, k+1, \dots, k+n-1} := X \circ_1 \mathbf{r} \in \mathbf{PaB}(n+k-1)$ ,
- we allow ourselves to combine these in an obvious way.

This notation is unambiguous as soon as we specify the starting object of our arrows.

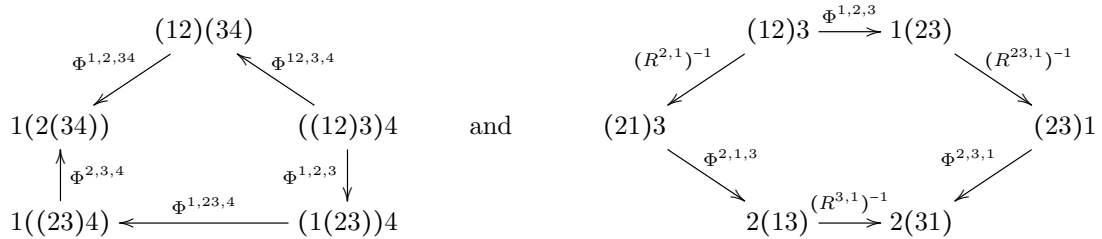
For example, the pentagon (P) and the first hexagon (H1) relations can be respectively depicted as



and



or, as commuting diagrams (giving the name of the relations)



### 2.4. The operad of chord diagrams

The holonomy Lie algebra of the configuration space

$$\text{Conf}(\mathbb{C}, n) := \{ \mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j \}$$

of  $n$  points on the complex line is isomorphic to the graded Lie  $\mathbb{C}$ -algebra  $\mathfrak{t}_n(\mathbf{k})$  generated by  $t_{ij}$ ,  $1 \leq i \neq j \leq n$ , with relations

- (S)  $t_{ij} = t_{ji}$ ,
- (L)  $[t_{ij}, t_{kl}] = 0$  if  $\#\{i, j, k, l\} = 4$ ,
- (4T)  $[t_{ij}, t_{ik} + t_{jk}] = 0$  if  $\#\{i, j, k\} = 3$ .

It is known as the Kohno–Drinfeld Lie algebra.

In [4, 23] it is shown that the collection of Lie  $\mathbf{k}$ -algebras  $\mathfrak{t}_n(\mathbf{k})$  is provided with the structure of an operad in the category  $grLie_{\mathbf{k}}$  of positively graded finite dimensional Lie algebras over  $\mathbf{k}$ , with symmetric monoidal structure given by the direct sum  $\oplus$ . Partial compositions are described as follows:

$$\begin{aligned} \circ_k : \mathfrak{t}_I(\mathbf{k}) \oplus \mathfrak{t}_J(\mathbf{k}) &\longrightarrow \mathfrak{t}_{J \sqcup I - \{k\}}(\mathbf{k}) \\ (0, t_{\alpha\beta}) &\longmapsto t_{\alpha\beta} \\ (t_{ij}, 0) &\longmapsto \begin{cases} t_{ij} & \text{if } k \notin \{i, j\} \\ \sum_{p \in J} t_{pj} & \text{if } k = i \\ \sum_{p \in J} t_{ip} & \text{if } k = j \end{cases} \end{aligned}$$

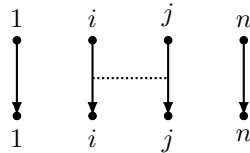
Observe that there is a lax symmetric monoidal functor

$$\hat{\mathcal{U}} : grLie_{\mathbf{k}} \longrightarrow \mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})$$

sending a positively graded Lie algebra to the degree completion of its universal enveloping algebra, which is a complete filtered cocommutative Hopf algebra, viewed as a  $\mathbf{CoAlg}_{\mathbf{k}}$ -enriched category with only one object.

We then consider the operad of *chord diagrams*  $\mathbf{CD}(\mathbf{k}) := \hat{\mathcal{U}}(\mathfrak{t}(\mathbf{k}))$  in  $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})$ .

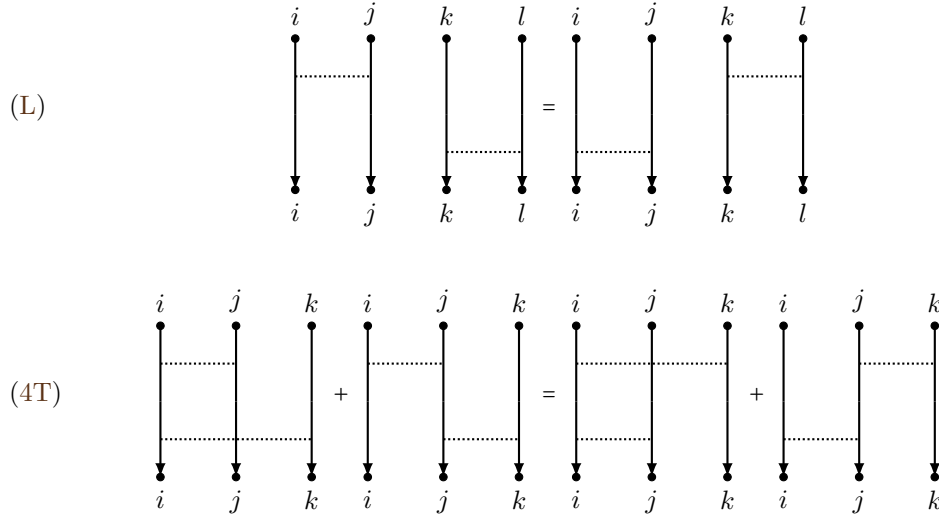
**Remark 2.5.** — This denomination comes from the fact that morphisms in  $\mathbf{CD}(\mathbf{k})(n)$  can be represented as linear combinations of diagrams of chords on  $n$  vertical strands, where the chord diagram corresponding to  $t_{ij}$  can be represented as



and the composition is given by vertical concatenation of diagrams. Partial compositions can easily be understood as “cabling and removal operations” on strands (see [4, 23]). Relations



(L) and (4T) defining each  $t_n(\mathbf{k})$  can be represented as follows:

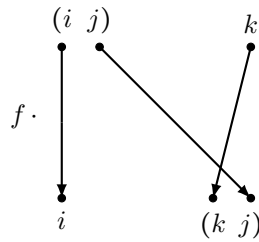


### 2.5. The operad of parenthesized chord diagrams

Recall that the operad  $\mathbf{CD}(\mathbf{k})$  has only one object in each arity. Hence we can define the operad

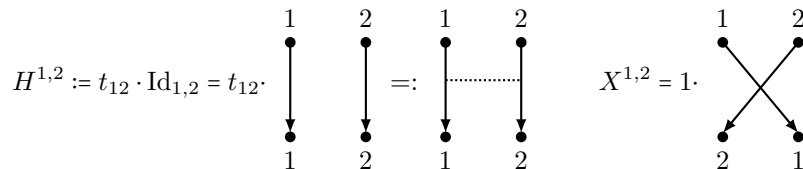
$$\mathbf{PaCD}(\mathbf{k}) := \omega_1^* \mathbf{CD}(\mathbf{k})$$

of *parenthesized chord diagrams*, where  $\omega_1 : \mathbf{Pa} = \text{Ob}(\mathbf{Pa}(\mathbf{k})) \rightarrow \text{Ob}(\mathbf{CD}(\mathbf{k}))$  is the terminal morphism. Here is a self-explanatory example of how to depict a morphism in  $\mathbf{PaCD}(\mathbf{k})(3)$ :

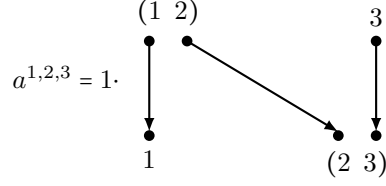


where  $f \in \mathbf{CD}(\mathbf{k})(3)$ .

**Example 2.6 (Notable arrows of  $\mathbf{PaCD}(\mathbf{k})$ ).** — There are the following arrows in  $\mathbf{PaCD}(\mathbf{k})(2)$ :



We also have the following arrow in  $\mathbf{PaCD}(\mathbf{k})(3)$ :



**Remark 2.7.** — The elements  $H^{1,2}$ ,  $X^{1,2}$  and  $a^{1,2,3}$  are generators of the operad  $\mathbf{PaCD}(\mathbf{k})$  and satisfy the following relations:

- $X^{2,1} = (X^{1,2})^{-1}$ ,
- $a^{12,3,4}a^{1,2,34} = a^{1,2,3}a^{1,23,4}a^{2,3,4}$ ,
- $X^{12,3} = a^{1,2,3}X^{2,3}(a^{1,3,2})^{-1}X^{1,3}a^{3,1,2}$ ,
- $H^{1,2} = X^{1,2}H^{2,1}(X^{1,2})^{-1}$ ,
- $H^{12,3} = a^{1,2,3}(H^{2,3} + X^{2,3}(a^{1,3,2})^{-1}H^{1,3}a^{1,3,2}X^{3,2})(a^{1,2,3})^{-1}$ .

In particular, even if  $\mathbf{PaCD}(\mathbf{k})$  does not have a presentation in terms of generators and relations (as is the case for  $\mathbf{PaB}$ ), one can show that  $\mathbf{PaCD}(\mathbf{k})$  has a universal property with respect to the generators  $H^{1,2}$ ,  $X^{1,2}$  and  $a^{1,2,3}$  and the above relations (see [23, Theorem 10.3.4] for details).

## 2.6. Drinfeld associators

Let us first introduce some terminology that we use in this paragraph, as well as later in the paper:

- Let  $\mathbf{Grpd}_{\mathbf{k}}$  denote the (symmetric monoidal) category of  $\mathbf{k}$ -prounipotent groupoids (which is the image of the completion functor  $\mathcal{G} \mapsto \hat{\mathcal{G}}(\mathbf{k})$ );
- For  $\mathcal{C}$  being  $\mathbf{Grpd}$ ,  $\mathbf{Grpd}_{\mathbf{k}}$ , or  $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}})$ , the notation

$$\mathrm{Aut}_{\mathrm{Op}\mathcal{C}}^+ \quad (\text{resp. } \mathrm{Iso}_{\mathrm{Op}\mathcal{C}}^+)$$

refers to those automorphisms (resp. isomorphisms) which are the identity on objects.

In the remainder of this section we recall some well known results on the operadic point of view on associators and Grothendieck-Teichmüller groups, which will be useful later on. Even though the statements and proofs of all the results in this section can be found in [23], it is worth mentioning that a “pre-operadic” approach was initiated by Bar-Natan in [4].

**Definition 2.8.** — A *Drinfeld  $\mathbf{k}$ -associator* is an isomorphism between the operads  $\widehat{\mathbf{PaB}}(\mathbf{k})$  and  $G\mathbf{PaCD}(\mathbf{k})$  in  $\mathbf{Grpd}_{\mathbf{k}}$ , which is the identity on objects. We denote by

$$\mathbf{Assoc}(\mathbf{k}) := \mathrm{Iso}_{\mathrm{Op}\mathbf{Grpd}_{\mathbf{k}}}^+(\widehat{\mathbf{PaB}}(\mathbf{k}), G\mathbf{PaCD}(\mathbf{k}))$$

the set of  $\mathbf{k}$ -associators.

**Theorem 2.9 (Drinfeld [17], Bar-Natan [4], Fresse [23]).** — *There is a one-to-one correspondence between the set of Drinfeld  $\mathbf{k}$ -associators and the set  $\mathbf{Ass}(\mathbf{k})$  of couples  $(\mu, \varphi) \in \mathbf{k}^\times \times \exp(\hat{\mathfrak{t}}_2(\mathbf{k}))$ , such that*

- $\varphi^{3,2,1} = (\varphi^{1,2,3})^{-1}$  in  $\exp(\hat{\mathfrak{t}}_3(\mathbf{k}))$ ,
- $\varphi^{1,2,3}e^{\mu t_{23}/2}\varphi^{2,3,1}e^{\mu t_{31}/2}\varphi^{3,1,2}e^{\mu t_{12}/2} = e^{\mu(t_{12}+t_{13}+t_{23})/2}$  in  $\exp(\hat{\mathfrak{t}}_3(\mathbf{k}))$ ,

$$- \varphi^{1,2,3} \varphi^{1,23,4} \varphi^{2,3,4} = \varphi^{12,3,4} \varphi^{1,2,34} \quad \text{in } \exp(\hat{\mathfrak{t}}_4(\mathbf{k})),$$

where  $\varphi^{1,2,3} = \varphi(t_{12}, t_{23})$  is viewed as an element of  $\exp(\hat{\mathfrak{t}}_3(\mathbf{k}))$  via the inclusion  $\hat{\mathfrak{f}}_2(\mathbf{k}) \subset \hat{\mathfrak{t}}_3(\mathbf{k})$  sending  $x$  to  $t_{12}$  and  $y$  to  $t_{23}$ .

Three observations are in order:

- The free Lie  $\mathbf{k}$ -algebra  $\mathfrak{f}_2(\mathbf{k})$  in two generators  $x, y$  is graded, with generators having degree 1, and its degree completion is denoted by  $\hat{\mathfrak{f}}_2(\mathbf{k})$ ;
- The  $\mathbf{k}$ -prounipotent group  $\exp(\hat{\mathfrak{f}}_2(\mathbf{k}))$  is thus isomorphic to the  $\mathbf{k}$ -prounipotent completion  $\widehat{F}_2(\mathbf{k})$  of the free group  $F_2$  on two generators;
- The quotient  $\hat{\mathfrak{t}}_3(\mathbf{k})$  of the Lie algebra  $\mathfrak{t}_3(\mathbf{k})$  by its center, generated by  $t_{12} + t_{13} + t_{23}$ , is isomorphic to  $\hat{\mathfrak{f}}_2(\mathbf{k})$ . Thus, the second relation in the above theorem is equivalent to

$$\varphi^{1,2,3} e^{\mu y/2} \varphi^{2,3,1} e^{\mu z/2} \varphi^{3,1,2} e^{\mu x/2} = 1$$

in  $\exp(\hat{\mathfrak{f}}_2(\mathbf{k}))$ , where  $x, y, z$  are variables subject to relation  $x + y + z = 0$ .

This Theorem was originally proven by Drinfeld in [17], though it was phrased without the operadic language. As stated, it can be found in [23, Theorem 10.2.9], and its proof relies on the universal property of  $\mathbf{PaB}$  from Theorem 2.3. In particular, a morphism  $F : \overline{\mathbf{PaB}}(\mathbf{k}) \rightarrow \mathbf{GPaCD}(\mathbf{k})$  is uniquely determined by a scalar parameter  $\mu \in \mathbf{k}$  and  $\varphi \in \exp(\hat{\mathfrak{f}}_2(\mathbf{k}))$ :

- $F(R^{1,2}) = e^{\mu t_{12}/2} \cdot X^{1,2}$ ,
- $F(\Phi^{1,2,3}) = \varphi(t_{12}, t_{23}) \cdot a^{1,2,3}$ ,

where  $R$  and  $\Phi$  are the ones from Examples 2.1 and 2.2.

**Example 2.10 (The KZ Associator).** — The first associator was constructed by Drinfeld using the KZ connection and is known as the KZ associator  $\Phi_{\text{KZ}}$ . It is defined as the renormalized holonomy from 0 to 1 of  $G'(z) = (\frac{t_{12}}{z} + \frac{t_{23}}{z-1})G(z)$ , i.e.,  $\Phi_{\text{KZ}} := G_{0^+}^{-1} G_{1^-} \in \exp(\hat{\mathfrak{t}}_3(\mathbb{C}))$ , where  $G_{0^+}, G_{1^-}$  are the solutions such that  $G_{0^+}(z) \sim z^{t_{12}}$  when  $z \rightarrow 0^+$  and  $G_{1^-}(z) \sim (1-z)^{t_{23}}$  when  $z \rightarrow 1^-$ . We have

$$\Phi_{\text{KZ}}(V, U) = \Phi_{\text{KZ}}(U, V)^{-1}, \quad \Phi_{\text{KZ}}(U, V) e^{\pi i V} \Phi_{\text{KZ}}(V, W) e^{\pi i W} \Phi_{\text{KZ}}(W, U) e^{\pi i U} = 1,$$

where  $U = \bar{t}_{12} \in \mathfrak{f}_2(\mathbb{C}) \simeq \mathfrak{t}_3(\mathbb{C}) := \mathfrak{t}_3(\mathbb{C})/(t_{12} + t_{13} + t_{23})$ ,  $V = \bar{t}_{23} \in \mathfrak{t}_3(\mathbb{C})$  and  $U + V + W = 0$ , and

$$\Phi_{\text{KZ}}^{12,3,4} \Phi_{\text{KZ}}^{1,2,34} = \Phi_{\text{KZ}}^{1,2,3} \Phi_{\text{KZ}}^{1,23,4} \Phi_{\text{KZ}}^{2,3,4},$$

hence  $(2\pi i, \Phi_{\text{KZ}})$  is an element of  $\text{Ass}(\mathbb{C})$ .

## 2.7. Grothendieck–Teichmüller group

**Definition 2.11.** — The *Grothendieck–Teichmüller group* is defined as the group

$$\mathbf{GT} := \text{Aut}_{\text{OpGrpd}}^+(\mathbf{PaB})$$

of automorphisms of the operad in groupoids  $\mathbf{PaB}$  which are the identity of objects. One defines similarly its  $\mathbf{k}$ -pro-unipotent version

$$\widehat{\mathbf{GT}}(\mathbf{k}) := \text{Aut}_{\text{OpGrpd}_{\mathbf{k}}}^+(\widehat{\mathbf{PaB}}(\mathbf{k})).$$

There are also pro- $\ell$  and profinite versions, denoted  $\mathbf{GT}_{\ell}$  and  $\widehat{\mathbf{GT}}$ , that we do not consider in this paper.

We can also characterize elements of  $\mathbf{GT}$  and  $\widehat{\mathbf{GT}}(\mathbf{k})$  as solutions of certain explicit algebraic equations. This characterization proves that the above operadic definition of  $\mathbf{GT}$  coincides with the one given by Drinfeld in his original paper [17]. In this article we will focus on the  $\mathbf{k}$ -pro-unipotent version of this group in genus 0 and 1, and twisted situations.

**Definition 2.12.** — Drinfeld’s Grothendieck–Teichmüller group  $\widehat{\mathbf{GT}}(\mathbf{k})$  consists of pairs

$$(\lambda, f) \in \mathbf{k}^\times \times \widehat{\mathbf{F}}_2(\mathbf{k})$$

which satisfy the following equations:

- $f(x, y) = f(y, x)^{-1}$  in  $\widehat{\mathbf{F}}_2(\mathbf{k})$ ,
- $x_1^\nu f(x_1, x_2) x_2^\nu f(x_2, x_3) x_3^\nu f(x_3, x_1) = 1$  in  $\widehat{\mathbf{F}}_2(\mathbf{k})$ ,
- $f(x_{13}x_{23}, x_{34}) f(x_{12}, x_{23}x_{24}) = f(x_{12}, x_{23}) f(x_{12}x_{13}, x_{24}x_{34}) f(x_{23}, x_{34})$  in  $\widehat{\mathbf{PB}}_4(\mathbf{k})$ ,

where  $x_1, x_2, x_3$  are 3 variables subject only to  $x_1x_2x_3 = 1$ ,  $\nu = \frac{\lambda-1}{2}$ , and  $x_{ij}$  is the elementary pure braid  $\mathfrak{Q}_{ij}$  from §2.2. The multiplication law is given by

$$(\lambda_1, f_1)(\lambda_2, f_2) = (\lambda_1\lambda_2, f_1(x^{\lambda_2}, f_2(x, y)y^{\lambda_2}f_2(x, y)^{-1})f_2(x, y)).$$

**Theorem 2.13.** — *There is an isomorphism between the groups  $\widehat{\mathbf{GT}}(\mathbf{k})$  and  $\widehat{\mathbf{GT}}(\mathbf{k})$ .*

This was first implicitly shown by Drinfeld in [17]. An explicit proof of this theorem can be found for example in [23, Theorem 11.1.7]. In particular, one obtains the couple  $(\lambda, f)$  from an automorphism  $F \in \widehat{\mathbf{GT}}(\mathbf{k})$  as follows. We have

$$(2.1) \quad F \left( \begin{array}{c} 1 \quad 2 \\ \bullet \quad \bullet \\ \text{---} \\ \bullet \quad \bullet \\ 2 \quad 1 \end{array} \right) = \left( \begin{array}{c} 1 \quad 2 \\ \bullet \quad \bullet \\ \text{---} \\ \bullet \quad \bullet \\ 1 \quad 2 \end{array} \right)^\nu \cdot \begin{array}{c} 1 \quad 2 \\ \bullet \quad \bullet \\ \text{---} \\ \bullet \quad \bullet \\ 2 \quad 1 \end{array} = \left( \begin{array}{c} 1 \quad 2 \\ \bullet \quad \bullet \\ \text{---} \\ \bullet \quad \bullet \\ 2 \quad 1 \end{array} \right)^{2\nu+1}$$

$$(2.2) \quad F \left( \begin{array}{c} (1 \ 2) \quad 3 \\ \bullet \quad \bullet \quad \bullet \\ \text{---} \\ \bullet \quad \bullet \quad \bullet \\ 1 \quad (2 \ 3) \end{array} \right) = f \left( \begin{array}{c} (1 \ 2) \quad 3 \quad (1 \ 2) \quad 3 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \text{---} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ (1 \ 2) \quad 3 \quad (1 \ 2) \quad 3 \end{array} \right) \cdot \begin{array}{c} (1 \ 2) \quad 3 \\ \bullet \quad \bullet \quad \bullet \\ \text{---} \\ \bullet \quad \bullet \quad \bullet \\ 1 \quad (2 \ 3) \end{array}$$

In other words, if we set  $\lambda = 2\nu + 1$ , we get the assignment

- $F(R^{1,2}) = (R^{1,2})^\lambda$ ,
- $F(\Phi^{1,2,3}) = f(x_{12}, x_{23}) \cdot \Phi^{1,2,3}$ .

Next, one obtains the composition law of  $\widehat{\mathbf{GT}}(\mathbf{k})$  from the composition of automorphisms  $F_1$  and  $F_2$  in  $\text{Aut}_{\text{Op Grpd}_{\mathbf{k}}}^+(\widehat{\mathbf{PaB}}(\mathbf{k}))$  as follows: the associated couples  $(\lambda_1, f_1)$  and  $(\lambda_2, f_2)$  in

$$\begin{aligned}
\mathbf{k}^\times \times \hat{F}_2(\mathbf{k}) \text{ satisfy } (F_1 F_2)(R^{1,2}) &= (F_2 \circ F_1)(R^{1,2}) = (R^{1,2})^{\lambda_1 \lambda_2}, \text{ and} \\
(F_1 F_2)(\Phi^{1,2,3}) &= (F_2 \circ F_1)(\Phi^{1,2,3}) = F_2(f_1(x_{12}, x_{23})\Phi^{1,2,3}) \\
&= F_2(f_1(x_{12}, x_{23}))F_2(\Phi^{1,2,3}) \\
&= f_1(F_2(x_{12}), F_2(x_{23}))f_2(x_{12}, x_{23})\Phi^{1,2,3} \\
&= f_1(x_{12}^{\lambda_2}, f_2(x_{12}, x_{23})x_{23}^{\lambda_2}f_2(x_{12}, x_{23})^{-1})f_2(x_{12}, x_{23})\Phi^{1,2,3}.
\end{aligned}$$

**Remark 2.14.** — There are also profinite and pro- $\ell$  versions of the Grothendieck–Teichmüller group, denoted  $\widehat{\mathbf{GT}}$  and  $\mathbf{GT}_\ell$ , respectively. There are morphisms

$$\mathbf{GT} \longrightarrow \widehat{\mathbf{GT}} \twoheadrightarrow \mathbf{GT}_\ell \hookrightarrow \widehat{\mathbf{GT}}(\mathbb{Q}_\ell) \quad \text{and} \quad \mathbf{GT} \longrightarrow \widehat{\mathbf{GT}}(\mathbf{k}).$$

It is important to keep in mind that the profinite, pro- $\ell$ ,  $\mathbf{k}$ -pro-unipotent versions of the Grothendieck–Teichmüller group do not coincide with the profinite, pro- $\ell$ ,  $\mathbf{k}$ -pro-unipotent completions of the “thin” Grothendieck–Teichmüller group  $\mathbf{GT}$  which only consists of the pairs  $(1, 1)$  and  $(-1, 1)$ .

## 2.8. Graded Grothendieck–Teichmüller group

**Definition 2.15.** — The graded Grothendieck–Teichmüller group is the group

$$\mathbf{GRT}(\mathbf{k}) := \text{Aut}_{\text{OpGrpd}_{\mathbf{k}}}^+(\mathbf{GPaCD}(\mathbf{k}))$$

of automorphisms of  $\mathbf{GPaCD}(\mathbf{k})$  that are the identity on objects.

**Remark 2.16.** — When restricted to the full subcategory  $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}}^{\text{conn}})$  of  $\mathbf{CoAlg}_{\mathbf{k}}$ -enriched categories for which the hom-coalgebras are connected, the functor  $G$  leads to an equivalence between  $\mathbf{Cat}(\mathbf{CoAlg}_{\mathbf{k}}^{\text{conn}})$  and  $\mathbf{Grpd}_{\mathbf{k}}$ . Hence there is an isomorphism

$$\mathbf{GRT}(\mathbf{k}) \simeq \text{Aut}_{\text{OpCat}(\mathbf{CoAlg}_{\mathbf{k}})}^+(\mathbf{PaCD}(\mathbf{k})).$$

Again, the operadic definition of  $\mathbf{GRT}(\mathbf{k})$  happens to coincide with the one originally given by Drinfeld.

**Definition 2.17.** — Let  $\text{GRT}_1$  be the set of elements in  $g \in \exp(\hat{\mathfrak{t}}_2(\mathbf{k})) \subset \exp(\hat{\mathfrak{t}}_3(\mathbf{k}))$  such that

$$\begin{aligned}
- g^{3,2,1} &= g^{-1} \text{ and } g^{1,2,3}g^{2,3,1}g^{3,1,2} = 1, \text{ in } \exp(\hat{\mathfrak{t}}_3(\mathbf{k})), \\
- g^{1,2,3}g^{1,2,3,4}g^{2,3,4} &= g^{12,3,4}g^{1,2,3,4}, \text{ in } \exp(\hat{\mathfrak{t}}_4(\mathbf{k})),
\end{aligned}$$

One has the following multiplication law on  $\text{GRT}_1$ :

$$(g_1 * g_2)(t_{12}, t_{23}) := g_1(t_{12}, \text{Ad}(g_2(t_{12}, t_{23}))(t_{23}))g_2(t_{12}, t_{23}).$$

Drinfeld showed in [17] that  $\text{GRT}_1$  is stable under  $*$ , which defines a group structure on it, and that rescaling transformations  $g(x, y) \mapsto \lambda \cdot g(x, y) = g(\lambda x, \lambda y)$  define an action of  $\mathbf{k}^\times$  of  $\text{GRT}_1$  by automorphisms.

**Theorem 2.18.** — *There is a group isomorphism  $\mathbf{GRT}(\mathbf{k}) \cong \mathbf{k}^\times \rtimes \text{GRT}_1 =: \text{GRT}(\mathbf{k})$ .*

This was first implicitly shown by Drinfeld in [17]. An explicit proof of this theorem can be found for example in [23, Theorem 10.3.10]. In particular, we obtain the couple  $(\lambda, g)$  from an automorphism  $G \in \mathbf{GRT}(\mathbf{k})$  by the assignment

- $G(X^{1,2}) = X^{1,2}$ ,
- $G(H^{1,2}) = e^{\lambda t_{12}} H^{1,2}$ ,
- $G(a^{1,2,3}) = g(t_{12}, t_{23}) \cdot a^{1,2,3}$ .

The composition of automorphisms  $G_1$  and  $G_2$  in  $\text{Aut}_{\text{Op } \hat{\mathcal{G}}}^+(G\text{PaCD}(\mathbf{k}))$  is given as follows: the associated couples  $(\lambda, g_1)$  and  $(\mu, g_2)$  in  $\mathbf{k}^\times \times \exp(\hat{\mathfrak{t}}_3(\mathbf{k}))$  satisfy

$$(G_1 G_2)(H^{1,2}) = (G_2 \circ G_1)(H^{1,2}) = \lambda \mu H^{1,2},$$

$$(G_1 G_2)(a^{1,2,3}) = (G_2 \circ G_1)(a^{1,2,3}) = g_1(\mu t_{12}, g_2(t_{12}, t_{23})(\mu t_{23})g_2(t_{12}, t_{23})^{-1})g_2(t_{12}, t_{23}) \cdot a^{1,2,3}.$$

## 2.9. Bitorsors

Recall first that there is a free and transitive left action of  $\widehat{\text{GT}}(\mathbf{k})$  on  $\text{Ass}(\mathbf{k})$ , defined, for  $(\lambda, f) \in \widehat{\text{GT}}(\mathbf{k})$  and  $(\mu, \varphi) \in \text{Ass}(\mathbf{k})$ , by

$$((\lambda, f) * (\mu, \varphi))(t_{12}, t_{23}) := (\lambda \mu, f(e^{\mu t_{12}}, \text{Ad}(\varphi(t_{12}, t_{23}))(e^{\mu t_{23}}))\varphi(t_{12}, t_{23})),$$

where  $\text{Ad}(f)(g) := fgf^{-1}$ , for any symbols  $f, g$ .

Recall that there is also a free and transitive right action of  $\text{GRT}(\mathbf{k})$  on  $\text{Ass}(\mathbf{k})$  defined as follows: for  $(\lambda, g) \in \text{GRT}(\mathbf{k})$  and  $(\mu, \varphi) \in \text{Ass}(\mathbf{k})$ , given by

$$((\mu, \varphi) * (\lambda, g))(t_{12}, t_{23}) := (\lambda \mu, \varphi(\lambda t_{12}, \text{Ad}(g)(\lambda t_{23}))g(t_{12}, t_{23})).$$

The two actions commute making  $(\widehat{\text{GT}}(\mathbf{k}), \text{Ass}(\mathbf{k}), \text{GRT}(\mathbf{k}))$  into a bitorsor.

**Theorem 2.19.** — *There is a torsor isomorphism*

$$(2.3) \quad (\widehat{\text{GT}}(\mathbf{k}), \text{Assoc}(\mathbf{k}), \text{GRT}(\mathbf{k})) \longrightarrow (\widehat{\text{GT}}(\mathbf{k}), \text{Ass}(\mathbf{k}), \text{GRT}(\mathbf{k}))$$

*Proof.* — On the one hand, in [23, Theorem 10.3.13] it is shown that the natural free and transitive left action of  $\widehat{\text{GT}}(\mathbf{k})$  on  $\text{Assoc}(\mathbf{k})$  coincides with the action of  $\text{GT}(\mathbf{k})$  on  $\text{Ass}(\mathbf{k})$  via the correspondence of Theorem 2.13. On the other hand, in [23, Theorem 11.2.1], it is shown that the natural free and transitive right action of  $\text{GRT}(\mathbf{k})$  on  $\text{Assoc}(\mathbf{k})$  coincides with the action of  $\text{GRT}(\mathbf{k})$  over  $\text{Ass}(\mathbf{k})$  via the correspondence of Theorem 2.18.  $\square$



## CHAPTER 3

### Modules associated with configuration spaces (elliptic associators)

#### 3.1. Compactified configuration space of the torus

Let  $\mathbb{T}$  be the topological (2-)torus. To any finite set  $I$  we associate a configuration space

$$\text{Conf}(\mathbb{T}, I) = \{ \mathbf{z} = (z_i)_{i \in I} \in \mathbb{T}^I \mid z_i \neq z_j \text{ if } i \neq j \}.$$

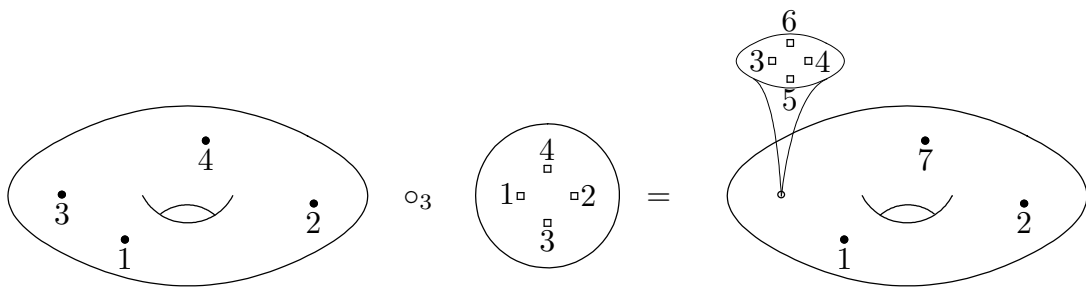
We also consider its reduced version

$$\mathbb{C}(\mathbb{T}, I) := \text{Conf}(\mathbb{T}, I) / \mathbb{T}.$$

We then consider the Axelrod–Singer–Fulton–MacPherson compactification  $\overline{\mathbb{C}}(\mathbb{T}, I)$  of  $\mathbb{C}(\mathbb{T}, I)$ . The boundary  $\partial \overline{\mathbb{C}}(\mathbb{T}, I) = \overline{\mathbb{C}}(\mathbb{T}, I) - \mathbb{C}(\mathbb{T}, I)$  is made up of the following irreducible components: for any partition  $I = J_1 \amalg \cdots \amalg J_k$  there is a component

$$\partial_{J_1, \dots, J_k} \overline{\mathbb{C}}(\mathbb{T}, I) \cong \overline{\mathbb{C}}(\mathbb{T}, k) \times \prod_{i=1}^k \overline{\mathbb{C}}(\mathbb{C}, J_i).$$

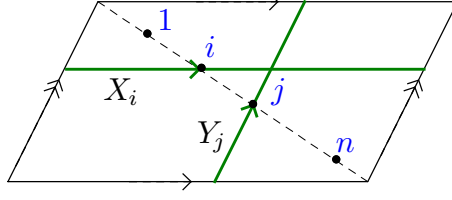
The inclusion of boundary components provide  $\overline{\mathbb{C}}(\mathbb{T}, -)$  with the structure of a module over the operad  $\overline{\mathbb{C}}(\mathbb{C}, -)$  in topological spaces.



#### 3.2. The pure braid group on the torus

The reduced pure braid group  $\overline{\text{PB}}_{1,n}$  with  $n$  strands on the torus (that is the fundamental group of  $\mathbb{C}(\mathbb{T}, n)$ ) is generated by paths  $X_i$ 's and  $Y_i$ 's ( $i = 1, \dots, n$ ), which can be represented as follows





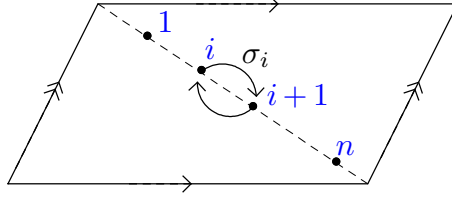
Moreover, the following relations are satisfied in  $\overline{\mathbf{PB}}_{1,n}$ :

- (T1)  $(X_i, X_j) = 1 = (Y_i, Y_j)$ , for  $i < j$ ,
- (T2)  $(X_i, Y_j) = P_{ij}$  and  $(X_j^{-1}, Y_i^{-1}) = \mathfrak{Q}_{ij}$ , for  $i < j$ ,
- (T3)  $(X_1, Y_1^{-1}) = P_{1n} \cdots P_{12}$ ,
- (T4)  $(X_i, P_{jk}) = 1 = (Y_i, P_{jk})$ , for all  $i, j < k$ ,
- (T5)  $(X_i X_j, P_{ij}) = 1 = (Y_i Y_j, P_{ij})$ , for  $i < j$ ,
- (TR)  $X_1 \cdots X_n = 1 = Y_1 \cdots Y_n$ ,

There are also the following relations, satisfied in the fundamental group  $\overline{\mathbf{B}}_{1,n}$  of  $\mathbf{C}(\mathbb{T}, n)/\mathfrak{S}_n$ :

- (N)  $X_{i+1} = \sigma_i^{-1} X_i \sigma_i^{-1}$ ,  $Y_{i+1} = \sigma_i^{-1} Y_i \sigma_i^{-1}$ ,

where  $\sigma_i$  are the generators of the braid group  $\mathbf{B}_n$  with geometric convention as follows:



### 3.3. The $\mathbf{PaB}$ -module $\mathbf{PaB}_{ell}$ of parenthesized elliptic (or beak) braids

In a similar manner as in §2.3, there are inclusions of topological modules<sup>(1)</sup>

$$\mathbf{Pa} \subset \overline{\mathbf{C}}(\mathbb{S}^1, -) \subset \overline{\mathbf{C}}(\mathbb{T}, -).$$

Then it makes sense to define

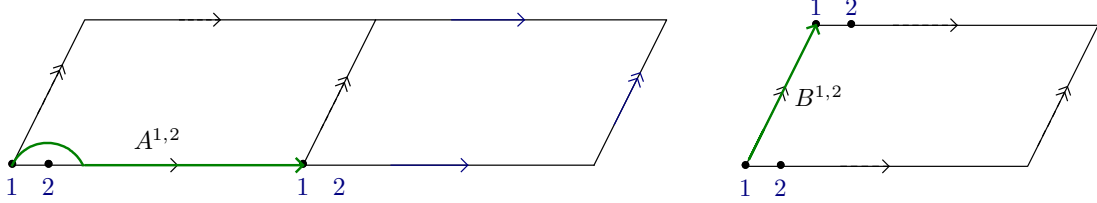
$$\mathbf{PaB}_{ell} := \pi_1(\overline{\mathbf{C}}(\mathbb{T}, -), \mathbf{Pa}),$$

which is a  $\mathbf{PaB}$ -module in groupoids.

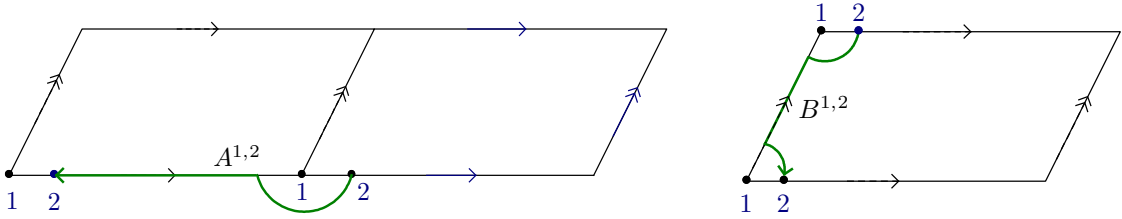
As said in section 1.5, there is a map of  $\mathfrak{S}$ -modules  $\mathbf{PaB} \rightarrow \mathbf{PaB}_{ell}$  and we abusively denote  $R^{1,2}$  and  $\Phi^{1,2,3}$  the images in  $\mathbf{PaB}_{ell}$  of the corresponding arrows in  $\mathbf{PaB}$ .

<sup>(1)</sup>The second one depends on the choice of an embedding  $\mathbb{S}^1 \hookrightarrow \mathbb{T}$ : we choose by convention the “horizontal embedding”, which corresponds to  $\mathbb{S}^1 \times \{*\}$ .

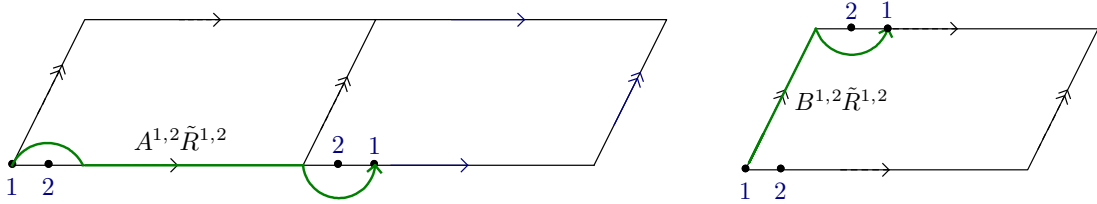
**Example 3.1 (Structure of  $\mathbf{PaB}_{ell}(2)$ ).** — As in Example 2.1 there is an arrow  $R^{1,2}$  going from (12) to (21). Additionally, we also have two automorphisms of (12), denoted  $A^{1,2}$  and  $B^{1,2}$ , corresponding to the following loops on  $\overline{\mathbb{C}}(\mathbb{T}, 2)$ :



By global translation of the torus, these are the same loops as the following



In particular,  $A^{1,2}\tilde{R}^{1,2}$  and  $B^{1,2}\tilde{R}^{1,2}$ , which are morphisms from (12) to (21), correspond to the following paths  $\overline{\mathbb{C}}(\mathbb{T}, 2)$ :



**Remark 3.2.** — The arrows  $A^{1,2}$  and  $B^{1,2}$  correspond to  $A_{1,2}^\pm$  in [19, §1.3]. Thus we will respectively depict  $A^{1,2}$  and  $B^{1,2}$  as

$$\begin{array}{c} 1 \\ \bullet \\ \downarrow \\ \bullet \\ 1 \end{array} + \begin{array}{c} 2 \\ \bullet \\ \downarrow \\ \bullet \\ 2 \end{array} \quad \text{and} \quad \begin{array}{c} 1 \\ \bullet \\ \downarrow \\ \bullet \\ 1 \end{array} - \begin{array}{c} 2 \\ \bullet \\ \downarrow \\ \bullet \\ 2 \end{array}$$

The images of  $R^{1,2}$  and  $\Phi^{1,2,3}$  by the  $\mathfrak{S}$ -module morphism  $\mathbf{PaB} \rightarrow \mathbf{PaB}_{ell}$  will still be denoted the same way. One can rephrase [19, Proposition 1.3] in the following way:

**Theorem 3.3.** — *As a  $\mathbf{PaB}$ -module in groupoids having  $\mathbf{Pa}$  as  $\mathbf{Pa}$ -module of objects,  $\mathbf{PaB}_{ell}$  is freely generated by  $A := A^{1,2}$  and  $B := B^{1,2}$ , together with the following relations:*

$$\begin{aligned} \text{(N1)} \quad & \Phi^{1,2,3} A^{1,23} \tilde{R}^{1,23} \Phi^{2,3,1} A^{2,31} \tilde{R}^{2,31} \Phi^{3,1,2} A^{3,12} \tilde{R}^{3,12} = \text{Id}_{(12)3}, \\ \text{(N2)} \quad & \Phi^{1,2,3} B^{1,23} \tilde{R}^{1,23} \Phi^{2,3,1} B^{2,31} \tilde{R}^{2,31} \Phi^{3,1,2} B^{3,12} \tilde{R}^{3,12} = \text{Id}_{(12)3}, \\ \text{(E)} \quad & R^{1,2} R^{2,1} = (\Phi^{1,2,3} A^{1,23} (\Phi^{1,2,3})^{-1}, \tilde{R}^{1,2} \Phi^{2,1,3} B^{2,13} (\Phi^{2,1,3})^{-1} \tilde{R}^{2,1}). \end{aligned}$$

All these relations hold in the automorphism group of  $(12)3$  in  $\mathbf{PaB}_{ell}(3)$ .

*Proof.* — Let  $\mathcal{Q}_{ell}$  be the  $\mathbf{PaB}$ -module with the above presentation. We first show that there is a morphism of  $\mathbf{PaB}$ -modules  $\mathcal{Q}_{ell} \rightarrow \mathbf{PaB}_{ell}$ . We have already seen that there are two automorphisms  $A, B$  of  $(12)$  in  $\mathbf{PaB}_{ell}(2)$  (see Example 3.1). We have to prove that they indeed satisfy the relations (N1), (N2) and (E).

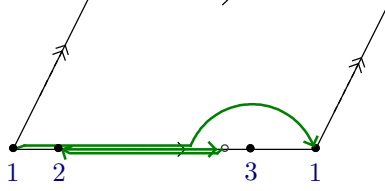
Relations (N1) and (N2) are satisfied: the two *nonagon relations* (N1) and (N2) can be depicted as

$$\text{(N1,N2)} \quad \begin{array}{ccc} \begin{array}{c} (1 \ 2) \\ \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \\ (1 \ 2) \end{array} & \begin{array}{c} 3 \\ \bullet \\ \downarrow \\ \bullet \\ 3 \end{array} & = \end{array} \quad \begin{array}{c} \begin{array}{c} (1 \ 2) \quad 3 \\ \bullet \quad \bullet \quad \bullet \\ \downarrow \quad \downarrow \quad \downarrow \\ \pm \quad \quad \quad \pm \\ \downarrow \quad \downarrow \quad \downarrow \\ \pm \quad \quad \quad \pm \\ \downarrow \quad \downarrow \quad \downarrow \\ \pm \quad \quad \quad \pm \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \quad \bullet \quad \bullet \\ (1 \ 2) \quad 3 \end{array} \end{array}$$

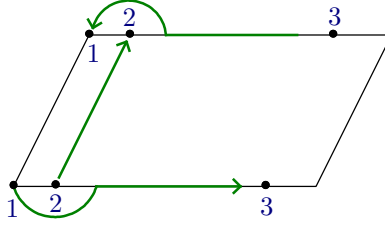
It is satisfied in  $\mathbf{PaB}_{ell}$ , expressing the fact that when all (here, three) points move in the same direction on the torus, this corresponds to a constant path in the reduced configuration space of points on the torus. The same is true with the second nonagon relation (N2).

Relation (E) is satisfied: below one sees the path that is obtained from the right-hand-side of the *mixed relation* (E):

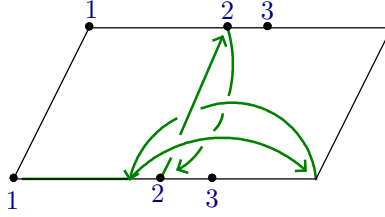
–  $\Phi^{1,2,3}A^{1,23}(\Phi^{1,2,3})^{-1}$  is the path



–  $\tilde{R}^{1,2}\Phi^{2,1,3}B^{2,13}(\Phi^{2,1,3})^{-1}\tilde{R}^{2,1}$  is the path



One can see that the commutator of these loops is homotopic to the pure braiding of the first two points in the clockwise direction, that is  $R^{1,2}R^{2,1}$ , by means of the following picture:



Thus, by the universal property of  $\mathcal{Q}_{ell}$ , there is a morphism of  $\mathbf{PaB}$ -modules  $\mathcal{Q}_{ell} \rightarrow \mathbf{PaB}_{ell}$ , which is the identity on objects. To show that this map is in fact an isomorphism, it suffices to show that it is an isomorphism at the level of automorphism groups of objects arity-wise, as all groupoids are connected. Let  $n \geq 0$ , and  $p$  be the object  $(\dots((12)3)\dots)n$  of  $\mathcal{Q}_{ell}(n)$  and  $\mathbf{PaB}_{ell}(n)$ . We want to show that the induced morphism

$$\text{Aut}_{\mathcal{Q}_{ell}(n)}(p) \longrightarrow \text{Aut}_{\mathbf{PaB}_{ell}(n)}(p) = \pi_1(\bar{C}(\mathbb{T}, n), p)$$

is an isomorphism.

On the one hand, as  $\bar{C}(\mathbb{T}, n)$  is a manifold with corners, we are allowed to move the basepoint  $p$  to a point  $p_{reg}$  which is included in the simply connected subset obtained as the image of<sup>(2)</sup>

$$D_{n,\tau} := \{z \in \mathbb{C}^n \mid z_j = a_j + b_j\tau, a_j, b_j \in \mathbb{R}, 0 < a_1 < a_2 < \dots < a_n < a_1 + 1, 0 < b_n < \dots < b_1 < b_n + 1\}$$

in  $C(\mathbb{T}, n)$ , where  $\mathbb{T} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ . We then have an isomorphism of fundamental groups  $\pi_1(\bar{C}(\mathbb{T}, n), p) \simeq \pi_1(C(\mathbb{T}, n), p_{reg})$ .

<sup>(2)</sup>We have already done so for the proof of relation (E).

On the other hand, in [19, Proposition 1.4], Enriquez constructs a universal elliptic structure  $\mathbf{PaB}_{ell}^{En}$ , that by definition carries an action of the (algebraic version of the) reduced braid group on the torus  $\overline{\mathbf{B}}_{1,n}$  in the following sense:

- $\mathbf{PaB}_{ell}^{En}$  is a category;
- for every object  $p$  of  $\mathbf{Pa}(n)$ , there is a corresponding object  $[p]$  in  $\mathbf{PaB}_{ell}^{En}$ , and  $[p] = [q]$  if  $p$  and  $q$  only differ by a permutation (but have the same underlying parenthesization);
- there are group morphisms  $\overline{\mathbf{B}}_{1,n} \xrightarrow{\sim} \text{Aut}_{\mathbf{PaB}_{ell}^{En}}(p) \rightarrow \mathfrak{S}_n$ .

Moreover, one has by construction of  $\mathbf{PaB}_{ell}^{En}$  that  $\text{Aut}_{\mathcal{Q}_{ell}(n)}(p)$  is the kernel of the map  $\text{Aut}_{\mathbf{PaB}_{ell}^{En}}([p]) \rightarrow \mathfrak{S}_n$ . One can actually show that there is a commuting diagram

$$\begin{array}{ccccccc}
\overline{\mathbf{PB}}_{1,n} & \xrightarrow{\cong} & \text{Aut}_{\mathcal{Q}_{ell}(n)}(p) & \longrightarrow & \pi_1(\overline{\mathbf{C}}(\mathbb{T}, n), p) & \xleftarrow{\cong} & \pi_1(\mathbf{C}(\mathbb{T}, n), p_{reg}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{\mathbf{B}}_{1,n} & \xrightarrow{\cong} & \text{Aut}_{\mathbf{PaB}_{ell}^{En}}(p) & \longrightarrow & \pi_1(\overline{\mathbf{C}}(\mathbb{T}, n)/\mathfrak{S}_n, [p]) & \xleftarrow{\cong} & \pi_1(\mathbf{C}(\mathbb{T}, n)/\mathfrak{S}_n, [p_{reg}]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{S}_n & \xlongequal{\quad} & \mathfrak{S}_n & \xlongequal{\quad} & \mathfrak{S}_n & \xlongequal{\quad} & \mathfrak{S}_n
\end{array}$$

where all vertical sequences are short exact sequences. Thus, in order to show that the map  $\text{Aut}_{\mathcal{Q}_{ell}(n)}(p) \rightarrow \pi_1(\overline{\mathbf{C}}(\mathbb{T}, n), p)$  is an isomorphism, we are left to show that

$$\overline{\mathbf{B}}_{1,n} \longrightarrow \pi_1(\mathbf{C}(\mathbb{T}, n), p_{reg})$$

is indeed an isomorphism. But this map is nothing else than a conjugate of the map constructed in [7, Theorem 5], identifying the algebraic and topological versions of the braid group on the torus.  $\square$

### 3.4. The $\mathbf{CD}(\mathbf{k})$ -module of elliptic chord diagrams

For any  $n \geq 0$ , recall that  $\mathfrak{t}_{1,n}(\mathbf{k})$  is defined as the bigraded Lie  $\mathbf{k}$ -algebra freely generated by  $x_1, \dots, x_n$  in degree  $(1, 0)$ ,  $y_1, \dots, y_n$  in degree  $(0, 1)$  (for  $i = 1, \dots, n$ ), and  $t_{ij}$  in degree  $(1, 1)$  (for  $1 \leq i \neq j \leq n$ ), together with the relations (S), (L), (4T), and the following additional elliptic relations as well:

- (S $_{ell}$ )  $[x_i, y_j] = t_{ij}$  for  $i \neq j$ ,
- (N $_{ell}$ )  $[x_i, x_j] = [y_i, y_j] = 0$  for  $i \neq j$ ,
- (T $_{ell}$ )  $[x_i, y_i] = -\sum_{j|j \neq i} t_{ij}$ ,
- (L $_{ell}$ )  $[x_i, t_{jk}] = [y_i, t_{jk}] = 0$  if  $\#\{i, j, k\} = 3$ ,
- (4T $_{ell}$ )  $[x_i + x_j, t_{ij}] = [y_i + y_j, t_{ij}] = 0$  for  $i \neq j$ .

The  $\sum_i x_i$  and  $\sum_i y_i$  are central in  $\mathfrak{t}_{1,n}(\mathbf{k})$ , and we also consider the quotient

$$\overline{\mathfrak{t}}_{1,n}(\mathbf{k}) := \mathfrak{t}_{1,n}(\mathbf{k}) / \left( \sum_i x_i, \sum_i y_i \right).$$

**Example 3.4.** —  $\bar{\mathfrak{t}}_{1,2}(\mathbf{k})$  is equal to the free Lie  $\mathbf{k}$ -algebra  $\mathfrak{f}_2(\mathbf{k})$  on two generators  $x = x_1$  and  $y = y_2$ .

Both  $\mathfrak{t}_{1,n}$  and  $\bar{\mathfrak{t}}_{1,n}$  are acted on by the symmetric group  $\mathfrak{S}_n$ , and one can show that the  $\mathfrak{S}$ -modules in  $grLie_{\mathbf{k}}$

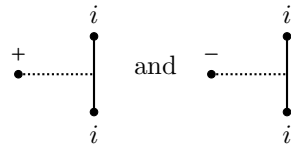
$$\mathfrak{t}_{ell}(\mathbf{k}) := \{\mathfrak{t}_{1,n}(\mathbf{k})\}_{n \geq 0} \quad \text{and} \quad \bar{\mathfrak{t}}_{ell}(\mathbf{k}) := \{\bar{\mathfrak{t}}_{1,n}(\mathbf{k})\}_{n \geq 0}$$

actually are  $\mathfrak{t}(\mathbf{k})$ -modules in  $grLie_{\mathbf{k}}$ . Partial compositions are defined as follows: for  $I$  a finite set and  $i \in I$ ,

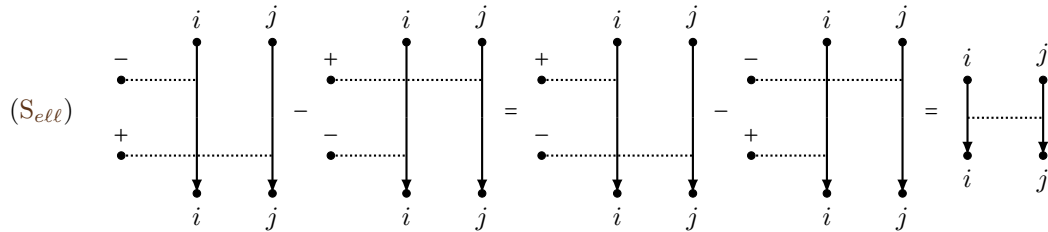
$$\begin{aligned} \circ_k : \mathfrak{t}_{1,I}(\mathbf{k}) \oplus \mathfrak{t}_J(\mathbf{k}) &\longrightarrow \mathfrak{t}_{1, J \sqcup I - \{i\}}(\mathbf{k}) \\ (0, t_{\alpha\beta}) &\longmapsto t_{\alpha\beta} \\ (t_{ij}, 0) &\longmapsto \begin{cases} t_{ij} & \text{if } k \notin \{i, j\} \\ \sum_{p \in J} t_{pj} & \text{if } k = i \\ \sum_{p \in J} t_{ip} & \text{if } j = k \end{cases} \\ (x_i, 0) &\longmapsto \begin{cases} x_i & \text{if } k \neq i \\ \sum_{p \in J} x_p & \text{if } k = i \end{cases} \\ (y_i, 0) &\longmapsto \begin{cases} y_i & \text{if } k \neq i \\ \sum_{p \in J} y_p & \text{if } k = i \end{cases} \end{aligned}$$

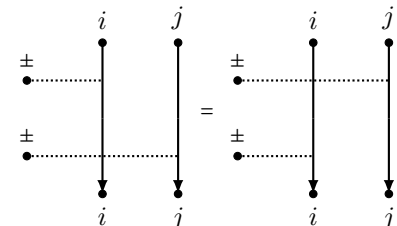
We call  $\mathfrak{t}_{ell}(\mathbf{k})$ , resp.  $\bar{\mathfrak{t}}_{ell}(\mathbf{k})$ , the module of *infinitesimal elliptic braids*, resp. of *infinitesimal reduced elliptic braids*.

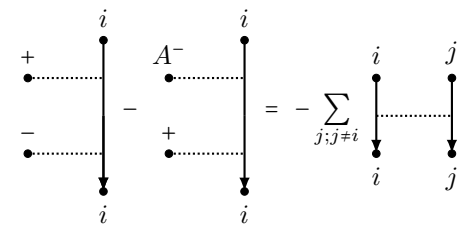
We finally define the  $\mathbf{CD}(\mathbf{k})$ -module  $\mathbf{CD}_{ell}(\mathbf{k}) := \hat{\mathcal{U}}(\bar{\mathfrak{t}}_{ell}(\mathbf{k}))$  of *elliptic chord diagrams*. Similarly to the genus 0 situation, morphisms in  $\mathbf{CD}_{ell}(\mathbf{k})(n)$  can be represented as chords on  $n$  vertical strands, with extra chords corresponding to the generators  $x_i$  and  $y_i$  as in the following picture:

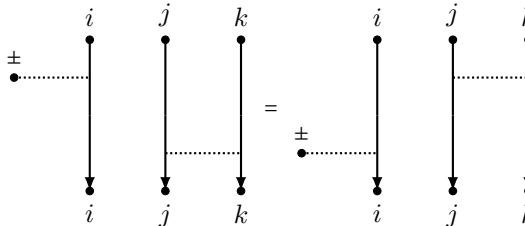


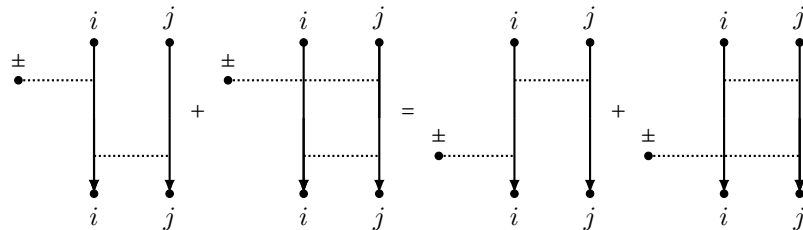
The elliptic relations introduced above can be represented as follows, analogously as for the genus 0 case:



(N<sub>ell</sub>) 

(T<sub>ell</sub>) 

(L<sub>ell</sub>) 

(4T<sub>ell</sub>) 

**Remark 3.5.** — The relation between (a closely related version of)  $\mathbf{CD}_{ell}(\mathbf{k})$  and the elliptic Kontsevich integral was studied in Philippe Humbert’s thesis [29].

### 3.5. The PaCD(k)-module of parenthesized elliptic chord diagrams

As in the genus zero case, the module of objects  $\text{Ob}(\mathbf{CD}_{ell}(\mathbf{k}))$  of  $\mathbf{CD}_{ell}(\mathbf{k})$  is terminal. Hence there is a morphism of modules  $\omega_2 : \mathbf{Pa} = \text{Ob}(\mathbf{Pa}(\mathbf{k})) \rightarrow \text{Ob}(\mathbf{CD}_{ell}(\mathbf{k}))$  over the morphism of operads  $\omega_1$  from §2.5, and thus we can define the  $\mathbf{PaCD}(\mathbf{k})$ -module<sup>(3)</sup>

$$\mathbf{PaCD}_{ell}(\mathbf{k}) := \omega_2^* \mathbf{CD}_{ell}(\mathbf{k}),$$

in  $\mathbf{Cat}(\mathbf{CoAss}_{\mathbf{k}})$ , of so-called *parenthesized elliptic chord diagrams*.

<sup>(3)</sup>Recall that  $\mathbf{PaCD}(\mathbf{k})$  is defined as  $\omega_1^* \mathbf{CD}(\mathbf{k})$ .

**Example 3.6** (Notable arrows in  $\mathbf{PaCD}_{ell}(\mathbf{k})(2)$ ). — There are the following arrows  $X_{ell}^{1,2}, Y_{ell}^{1,2}$  in  $\mathbf{PaCD}_{ell}(\mathbf{k})(2)$ :

$$X_{ell}^{1,2} = x_1 \cdot \begin{array}{c} 1 \\ \downarrow \\ 1 \end{array} \quad \begin{array}{c} 2 \\ \downarrow \\ 2 \end{array} = + \cdot \begin{array}{c} 1 \\ \cdots \downarrow \\ 1 \end{array} \quad \begin{array}{c} 2 \\ \downarrow \\ 2 \end{array} \quad Y_{ell}^{1,2} = y_1 \cdot \begin{array}{c} 1 \\ \downarrow \\ 1 \end{array} \quad \begin{array}{c} 2 \\ \downarrow \\ 2 \end{array} = - \cdot \begin{array}{c} 1 \\ \cdots \downarrow \\ 1 \end{array} \quad \begin{array}{c} 2 \\ \downarrow \\ 2 \end{array}$$

**Remark 3.7.** — As said in section 1.5, there is a map of  $\mathfrak{S}$ -modules  $\mathbf{PaCD}(\mathbf{k}) \rightarrow \mathbf{PaCD}_{ell}(\mathbf{k})$  and we abusively denote  $X^{1,2}, H^{1,2}$  and  $a^{1,2,3}$  the images in  $\mathbf{PaCD}_{ell}(\mathbf{k})$  of the corresponding arrows in  $\mathbf{PaCD}(\mathbf{k})$ . The elements  $X_{ell}^{1,2}, Y_{ell}^{1,2}$  are generators of the  $\mathbf{PaCD}(\mathbf{k})$ -module  $\mathbf{PaCD}_{ell}(\mathbf{k})$  and satisfy the following relations in  $\text{End}_{\mathbf{PaCD}_{ell}(\mathbf{k})(2)}(12)$ :

$$\begin{aligned} - X_{ell}^{1,2} + X^{1,2} X_{ell}^{2,1} (X^{1,2})^{-1} &= 0, \\ - Y_{ell}^{1,2} + X^{1,2} Y_{ell}^{2,1} (X^{1,2})^{-1} &= 0. \end{aligned}$$

They also satisfy the following relations in  $\text{End}_{\mathbf{PaCD}_{ell}(\mathbf{k})(3)}((12)3)$ :

$$\begin{aligned} - X_{ell}^{12,3} + a^{1,2,3} X^{1,23} X_{ell}^{23,1} (a^{1,2,3} X^{1,23})^{-1} + X^{12,3} (a^{3,1,2})^{-1} X_{ell}^{31,2} (X^{12,3} (a^{3,1,2})^{-1})^{-1} &= 0, \\ - Y_{ell}^{12,3} + a^{1,2,3} X^{1,23} Y_{ell}^{23,1} (a^{1,2,3} X^{1,23})^{-1} + X^{12,3} (a^{3,1,2})^{-1} Y_{ell}^{31,2} (X^{12,3} (a^{3,1,2})^{-1})^{-1} &= 0, \\ - H^{1,2} = [a^{1,2,3} X_{ell}^{1,23} (a^{1,2,3})^{-1}, X^{1,2} a^{2,1,3} Y_{ell}^{2,13} (a^{2,1,3})^{-1} (X^{1,2})^{-1}] &= 0. \end{aligned}$$

### 3.6. Elliptic associators

Let us introduce some terminology, complementing the one of §2.6. Let us write  $\text{OpRC}$  for the category of pairs  $(\mathcal{P}, \mathcal{M})$ , where  $\mathcal{P}$  is an operad and  $\mathcal{M}$  is a right  $\mathcal{O}$ -module, in  $\mathcal{C}$ . A morphism  $(\mathcal{P}, \mathcal{M}) \rightarrow (\mathcal{Q}, \mathcal{N})$  is a pair  $(f, g)$ , where  $f: \mathcal{P} \rightarrow \mathcal{Q}$  is a morphism between operads and  $g: \mathcal{M} \rightarrow \mathcal{N}$  is a morphism of  $\mathcal{P}$ -modules.

The superscript “+” still indicates that we consider morphisms that are the identity on objects.

**Definition 3.8.** — An elliptic associator over  $\mathbf{k}$  is a couple  $(F, G)$  where  $F$  is a  $\mathbf{k}$ -associator and  $G$  is an isomorphism between the  $\widehat{\mathbf{PaB}}(\mathbf{k})$ -module  $\widehat{\mathbf{PaB}}_{ell}(\mathbf{k})$  and the  $G\mathbf{PaCD}(\mathbf{k})$ -module  $G\mathbf{PaCD}_{ell}(\mathbf{k})$  which is the identity on objects and which is compatible with  $F$ :

$$\mathbf{Ell}(\mathbf{k}) := \text{Iso}_{\text{OpR Grpd}_{\mathbf{k}}}^+ \left( \left( \widehat{\mathbf{PaB}}(\mathbf{k}), \widehat{\mathbf{PaB}}_{ell}(\mathbf{k}) \right), \left( G\mathbf{PaCD}(\mathbf{k}), G\mathbf{PaCD}_{ell}(\mathbf{k}) \right) \right).$$

The following theorem identifies our definition of elliptic associators with the original one introduced by Enriquez in [19].

**Theorem 3.9.** — There is a one-to-one correspondence between the set  $\mathbf{Ell}(\mathbf{k})$  and the set  $\mathbf{Ell}(\mathbf{k})$  of quadruples  $(\mu, \varphi, A_+, A_-)$ , where  $(\mu, \varphi) \in \text{Ass}(\mathbf{k})$  and  $A_{\pm} \in \text{exp}(\hat{\mathfrak{t}}_{1,2}(\mathbf{k}))$ , such that:

$$(3.1) \quad \alpha_{\pm}^{1,2,3} \alpha_{\pm}^{2,3,1} \alpha_{\pm}^{3,1,2} = 1, \quad \text{where } \alpha_{\pm} = \varphi^{1,2,3} A_{\pm}^{1,23} e^{-\mu(t_{12}+t_{13})/2},$$

$$(3.2) \quad e^{\mu t_{12}} = \left( \varphi A_+^{1,23} \varphi^{-1}, e^{-\mu t_{12}/2} \varphi^{2,1,3} A_-^{2,13} (\varphi^{2,1,3})^{-1} e^{-\mu t_{12}/2} \right).$$

All these relations hold in the group  $\text{exp}(\hat{\mathfrak{t}}_{1,3}(\mathbf{k}))$ .



*Proof.* — An associator  $F$  corresponds uniquely to a couple  $(\mu, \varphi) \in \text{Ass}(\mathbf{k})$  and an isomorphism  $G$  between  $\widehat{\text{PaB}}_{ell}(\mathbf{k})$  and  $G\widehat{\text{PaCD}}_{ell}(\mathbf{k})$  sends the arrows  $A^{1,2}$  and  $B^{1,2}$  of  $\text{End}_{\widehat{\text{PaB}}_{ell}(\mathbf{k})(2)}(12)$  to  $A_+ \cdot X_{ell}^{1,2}$  and  $A_- \cdot Y_{ell}^{1,2}$  with  $A_{\pm} \in \exp(\hat{\mathfrak{t}}_{1,2}(\mathbf{k}))$  (recall that  $\hat{\mathfrak{t}}_{1,2}(\mathbf{k})$  is the completed free Lie algebra over  $\mathbf{k}$  in two generators). The image of relations (N1), (N2) and (E) are precisely the relations (3.1) and (3.2).  $\square$

**Example 3.10 (Elliptic KZB Associators).** — Let us fix  $\tau \in \mathfrak{h}$ . Recall that the Lie algebra  $\hat{\mathfrak{t}}_{1,2}(\mathbb{C})$  is isomorphic to the free Lie algebra  $\mathfrak{f}_2(\mathbb{C})$  generated by two elements  $x := x_1$  and  $y := y_1$ . We define the elliptic KZB associators  $e(\tau) := (A(\tau), B(\tau))$  as the renormalized holonomies from 0 to 1 and 0 to  $\tau$  of the differential equation

$$(3.3) \quad G'(z) = -\frac{\theta_{\tau}(z + \text{ad } x) \text{ad } x}{\theta_{\tau}(z)\theta_{\tau}(\text{ad } x)}(y) \cdot G(z),$$

with values in the group  $\exp(\hat{\mathfrak{t}}_{1,2}(\mathbb{C}))$ . More precisely, this equation has a unique solution  $G(z)$  defined over  $\{a + b\tau \mid a, b \in (0, 1)\}$  such that  $G(z) \simeq (-2\pi i z)^{-[x,y]}$  at  $z \rightarrow 0$ . In this case,

$$A(\tau) := G(z)G(z+1)^{-1}, \quad B(\tau) := G(z)G(z+\tau)^{-1}e^{-2\pi i x}.$$

These are elements of the group  $\exp(\hat{\mathfrak{t}}_{1,2}(\mathbb{C}))$ . More precisely, Enriquez showed in [19] that the element  $(2\pi i, \Phi_{KZ}, A(\tau), B(\tau))$  is in  $\text{Ell}(\mathbb{C})$ .

### 3.7. Elliptic Grothendieck–Teichmüller group

**Definition 3.11.** — The ( $\mathbf{k}$ -prounipotent version of the) *elliptic Grothendieck–Teichmüller group* is defined as the group

$$\widehat{\text{GT}}_{ell}(\mathbf{k}) := \text{Aut}_{\text{OpR Grpd}_{\mathbf{k}}}^+(\widehat{\text{PaB}}(\mathbf{k}), \widehat{\text{PaB}}_{ell}(\mathbf{k})).$$

Again, we now show that our definition coincides with the original one defined by Enriquez in [19]. Recall that the set  $\widehat{\text{GT}}_{ell}(\mathbf{k})$  is the set of tuples  $(\lambda, f, g_{\pm})$ , where  $(\lambda, f) \in \widehat{\text{GT}}(\mathbf{k})$ ,  $g_{\pm} \in \widehat{\text{F}}_2(\mathbf{k})$  such that, in  $\widehat{\text{B}}_{1,3}(\mathbf{k})$ ,

$$(3.4) \quad (f(\sigma_1^2, \sigma_2^2)g_{\pm}(A, B)(\sigma_1\sigma_2^2\sigma_1)^{-\frac{\lambda-1}{2}}\sigma_1^{-1}\sigma_2^{-1})^3 = 1,$$

$$(3.5) \quad u^2 = (g_+, u^{-1}g_-u^{-1}),$$

where  $u = f(\sigma_1^2, \sigma_2^2)^{-1}\sigma_1^{\lambda}f(\sigma_1^2, \sigma_2^2)$  and  $g_{\pm} = g_{\pm}(A, B)$ .

For  $(\lambda, f, g_{\pm}), (\lambda', f', g'_{\pm}) \in \widehat{\text{GT}}_{ell}(\mathbf{k})$ , we set

$$(\lambda, f, g_{\pm}) * (\lambda', f', g'_{\pm}) := (\lambda'', f'', g''_{\pm}),$$

where  $g''_{\pm}(A, B) = g_{\pm}(g'_{\pm}(A, B), g'_{\pm}(A, B))$ . This gives  $\widehat{\text{GT}}_{ell}(\mathbf{k})$  a group structure. Moreover, for  $(\lambda, f, g_+, g_-) \in \widehat{\text{GT}}_{ell}(\mathbf{k})$  and  $(\mu, \varphi, A_+, A_-) \in \text{Ell}(\mathbf{k})$ , we set

$$(\lambda, f, g_+, g_-) * (\mu, \varphi, A_+, A_-) := (\mu', \varphi', A'_+, A'_-),$$

where  $A'_{\pm} := g_{\pm}(A_+, A_-)$ . In [19], it is shown that this defines a free and transitive left action of  $\widehat{\text{GT}}_{ell}(\mathbf{k})$  on  $\text{Ell}(\mathbf{k})$ .

**Proposition 3.12.** — *There is an isomorphism  $\widehat{\text{GT}}_{ell}(\mathbf{k}) \rightarrow \widehat{\text{GT}}_{ell}(\mathbf{k})$  such that the bijection  $\text{Ell}(\mathbf{k}) \xrightarrow{\sim} \text{Ell}(\mathbf{k})$  becomes a torsor isomorphism.*

*Proof.* — Suppose that we are given an automorphism  $(F, G)$  of  $(\widehat{\mathbf{PaB}}(\mathbf{k}), \widehat{\mathbf{PaB}}_{ell}(\mathbf{k}))$  which is the identity on objects. We already know (see §2.7) that  $F$  is determined by a pair  $(\lambda, f) \in \widehat{\mathbf{GT}}(\mathbf{k})$ , and that any such pair determines an  $F$ . Moreover, the images of the two generators  $A^{1,2}, B^{1,2} \in \text{Aut}_{\widehat{\mathbf{PaB}}_{ell}(\mathbf{k})(2)}(12) = \widehat{\mathbf{PB}}_{1,2}(\mathbf{k})$  are

$$G(A^{1,2}) = g_+(A^{1,2}, B^{1,2}) \quad \text{and} \quad G(B^{1,2}) = g_-(A^{1,2}, B^{1,2}),$$

with  $g_{\pm} \in \widehat{\mathbf{F}}_2(\mathbf{k}) \simeq \widehat{\mathbf{PB}}_{1,2}(\mathbf{k})$ . It therefore follows from Theorem 3.3 that  $(\lambda, f, g_{\pm})$  satisfies relations (3.4) and (3.5) if and only if it determines an automorphism  $(F, G)$ .

Let us then prove that the bijective assignment  $(F, G) \mapsto (\lambda, f, g_{\pm})$  that we just described is a group morphism. For this we show that the composition of automorphisms corresponds to the group law of  $\mathbf{GT}_{ell}(\mathbf{k})$ . We already know (see §2.7) that the composition of automorphisms of  $\widehat{\mathbf{PaB}}(\mathbf{k})$  corresponds to the group law in  $\mathbf{GT}(\mathbf{k})$ . Now, given automorphisms  $(F_1, G)$  and  $(F_2, H)$ , and their respective images  $(\lambda_1, f_1, g_{\pm})$  and  $(\lambda_2, f_2, h_{\pm})$ , we have that

$$(H \circ G)(A) = H(g_+(A, B)) = g_+(H(A), H(B)) = g_+(h_+(A, B), h_-(A, B)),$$

and, likewise,  $(H \circ G)(B) = g_-(h_+(A, B), h_-(A, B))$ .

We finally prove the equivariance statement. Let  $(F, G) \in \mathbf{GT}_{ell}(\mathbf{k})$ , with image  $(\lambda, f, g_{\pm}) \in \mathbf{GT}_{ell}(\mathbf{k})$ , and let  $(K, H) \in \mathbf{Ell}_{ell}(\mathbf{k})$ , with image  $(\mu, \varphi, A_{\pm})$ . It is known (see §2.9) that  $K \circ F$  is sent to  $(\mu, \varpi) * (\lambda, f)$ . It remains to compute:

$$(H \circ G)(A) = H(g_+(A, B)) = g_+(H(A), H(B)) = g_+(A_+, A_-),$$

and, similarly,  $(H \circ G)(B) = g_-(A_+, A_-)$ . □

### 3.8. Graded elliptic Grothendieck–Teichmüller group

**Definition 3.13.** — The graded elliptic Grothendieck–Teichmüller group is the group

$$\mathbf{GRT}_{ell}(\mathbf{k}) := \text{Aut}_{\text{OpR Cat}(\text{CoAlg}_{\mathbf{k}})}^+(\mathbf{PaCD}(\mathbf{k}), \mathbf{PaCD}_{ell}(\mathbf{k})).$$

Notice that there is an isomorphism

$$\text{Aut}_{\text{OpR Cat}(\text{CoAlg}_{\mathbf{k}})}^+(\mathbf{PaCD}(\mathbf{k}), \mathbf{PaCD}_{ell}(\mathbf{k})) \simeq \text{Aut}_{\text{OpR Grpd}_{\mathbf{k}}}^+(\mathbf{GPaCD}(\mathbf{k}), \mathbf{GPaCD}_{ell}(\mathbf{k})).$$

As before, our goal in this paragraph is to show that our definition coincides with the one of Enriquez [19]. Recall that he defines  $\mathbf{GRT}_1^{ell}(\mathbf{k})$  as the set of triples  $(g, u_+, u_-) \in \mathbf{GRT}_1(\mathbf{k}) \times (\widehat{\mathfrak{t}}_{1,2}(\mathbf{k}))^{\times 2}$ , satisfying

$$(3.6) \quad \text{Ad}(g^{1,2,3})(u_{\pm}^{1,23}) + \text{Ad}(g^{2,1,3})(u_{\pm}^{2,13}) + u_{\pm}^{3,12} = 0,$$

$$(3.7) \quad [\text{Ad}(g^{1,2,3})(u_{\pm}^{1,23}), u_{\pm}^{3,12}] = 0,$$

$$(3.8) \quad [\text{Ad}(g^{1,2,3})(u_+^{1,23}), \text{Ad}(g^{2,1,3})(u_-^{2,13})] = t_{12},$$

as relations in  $\widehat{\mathfrak{t}}_{1,3}(\mathbf{k})$ . He defines a group structure as follows:

$$(g, u_+, u_-) * (h, v_+, v_-) := (g * h, w_+, w_-), \quad \text{where} \quad w_{\pm}(x_1, y_1) := u_{\pm}(v_+(x_1, y_1), v_-(x_1, y_1)).$$

The group  $\mathbf{k}^{\times}$  acts on  $\mathbf{GRT}_1^{ell}(\mathbf{k})$  by rescaling:  $c \cdot (g, u_{\pm}) := (c \cdot g, c \cdot u_{\pm})$ , where  $c \cdot g$  is as before, and

- $(c \cdot u_+)(x_1, y_1) := u_+(x_1, c^{-1}y_1)$ ,
- $(c \cdot u_-)(x_1, y_1) := cu_-(x_1, c^{-1}y_1)$ .

We then set  $\mathbf{GRT}_{ell}(\mathbf{k}) := \mathbf{GRT}_1^{ell}(\mathbf{k}) \rtimes \mathbf{k}^\times$ .

Moreover, there is a right action of  $\mathbf{GRT}_1^{ell}(\mathbf{k})$  on  $\mathbf{Ell}(\mathbf{k})$ : for  $(g, u_\pm) \in \mathbf{GRT}_1^{ell}(\mathbf{k})$  and  $(\mu, \varphi, A_\pm) \in \mathbf{Ell}(\mathbf{k})$ , we set  $(\mu, \varphi, A_\pm) * (g, u_\pm) := (\mu, \tilde{\varphi}, \tilde{A}_\pm)$ , where

$$\tilde{A}_\pm(x_1, y_1) := A_\pm(u_+(x_1, y_1), u_-(x_1, y_1))$$

and, for  $c \in \mathbf{k}^\times$ , we set  $(\mu, \varphi, A_\pm) * c := (\mu, c * \varphi, c \# A_\pm)$ , where  $(c \# A_\pm)(x_1, y_1) := A_\pm(x_1, cy_1)$ . In [19] this action is shown to be free and transitive. Notice that  $\tilde{A}_\pm = \theta(A_\pm)$ , where  $\theta \in \text{Aut}(\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}})$  is defined by  $x_1 \mapsto u_+(x_1, y_1)$  and  $y_1 \mapsto u_-(x_1, y_1)$ .

**Proposition 3.14.** — *There is an injective group morphism  $\mathbf{GRT}_{ell}(\mathbf{k}) \rightarrow \mathbf{GRT}_{ell}(\mathbf{k})$ . Moreover, the bijection  $\mathbf{Ell}(\mathbf{k}) \rightarrow \mathbf{Ell}(\mathbf{k})$  from Theorem 3.9 is equivariant along this morphism.*

*Proof.* — For every  $(G, U) \in \mathbf{GRT}_{ell}(\mathbf{k})$ , there are  $(\lambda, g) \in \mathbf{GRT}(\mathbf{k})$  and  $u_\pm \in \hat{\mathfrak{t}}_{1,2}^{\mathbf{k}}(\mathbf{k})$  such that

- $G(X^{1,2}) = X^{1,2}$ ,
- $G(H^{1,2}) = \lambda H^{1,2}$ ,
- $G(a^{1,2,3}) = g(t_{12}, t_{23})a^{1,2,3}$ ,
- $U(X_{ell}^{1,2}) = u_+(x, y) \text{Id}_{12}$ ,
- $U(Y_{ell}^{1,2}) = u_-(x, y) \text{Id}_{12}$ .

In light of relations of Remark 3.7, we obtain that  $(\lambda, g, u_\pm)$  satisfies relations (3.6), (3.7) and (3.8). The assignment  $(G, U) \mapsto (\lambda, g, u_\pm)$  defines an injective map  $\mathbf{GRT}_{ell}(\mathbf{k}) \rightarrow \mathbf{GRT}_{ell}(\mathbf{k})$ .

We now show that this map is a group morphism. The proof is the same as one of the analogous statement in Proposition 3.12: for two automorphisms  $(G_1, U)$  and  $(G_2, V)$ , we already know that the composition  $G_2 \circ G_1$  corresponds to the product in  $\mathbf{GRT}(\mathbf{k})$ , and we compute:

$$(V \circ U)(X_{ell}^{1,2}) = V(u_+(x_1, y_1) \text{Id}_{12}) = u_+(v_+(x_1, y_1), v_-(x_1, y_1)) \text{Id}_{12},$$

and, likewise,  $(V \circ U)(Y_{ell}^{1,2}) = u_-(v_+(x_1, y_1), v_-(x_1, y_1)) \text{Id}_{12}$ .

Finally, the equivariance of the bijection is proven in a similar way.  $\square$

### 3.9. Bitorsors

Summarizing the results we have proven so far, we get that the bijection  $\mathbf{Ell}(\mathbf{k}) \rightarrow \mathbf{Ell}(\mathbf{k})$  from Theorem 3.9 has been promoted to a bitorsor isomorphism. Indeed, we know (by definition) that

$$(\widehat{\mathbf{GT}}_{ell}(\mathbf{k}), \mathbf{Ell}(\mathbf{k}), \mathbf{GRT}_{ell}(\mathbf{k}))$$

is a bitorsor, and (from [19]) that

$$(\widehat{\mathbf{GT}}_{ell}(\mathbf{k}), \mathbf{Ell}(\mathbf{k}), \mathbf{GRT}_{ell}(\mathbf{k}))$$

is a bitorsor as well.

**Theorem 3.15.** — *There is a bitorsor isomorphism*

$$(3.9) \quad (\widehat{\mathbf{GT}}_{ell}(\mathbf{k}), \mathbf{Ell}(\mathbf{k}), \mathbf{GRT}_{ell}(\mathbf{k})) \xrightarrow{\sim} (\widehat{\mathbf{GT}}_{ell}(\mathbf{k}), \mathbf{Ell}(\mathbf{k}), \mathbf{GRT}_{ell}(\mathbf{k})).$$

*Proof.* — This is a summary of most of the above results:

- There is a group isomorphism between  $\widehat{\mathbf{GT}}_{ell}(\mathbf{k})$  and  $\widehat{\mathbf{GT}}_{ell}(\mathbf{k})$  that is such that the bijection from Theorem 3.9 is a torsor isomorphism (Proposition 3.12).
- There is an injective group morphism  $\mathbf{GRT}_{ell}(\mathbf{k}) \rightarrow \mathbf{GRT}_{ell}(\mathbf{k})$  such that the bijection from Theorem 3.9 is equivariant (Proposition 3.14).

Knowing from Example 3.10 that  $\text{Ell}(\mathbf{k})$  is non-empty, we obtain that  $\mathbf{GRT}_{ell}(\mathbf{k}) \rightarrow \mathbf{GRT}_{ell}(\mathbf{k})$  is an isomorphism.  $\square$



## CHAPTER 4

### The module of parenthesized elliptomic braids

In this chapter,  $\Gamma$  denotes the abelian group  $\Gamma = \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$  where  $M, N \geq 1$  are two integers. We also write  $\mathbf{0} := (\bar{0}, \bar{0})$ .

#### 4.1. Compactified twisted configuration space of the torus

Let  $\mathbb{T}$  be the topological torus, and consider the connected  $\Gamma$ -covering  $p: \tilde{\mathbb{T}} \rightarrow \mathbb{T}$  corresponding to the canonical surjective group morphism  $\rho: \pi_1(\mathbb{T}) = \mathbb{Z}^2 \rightarrow \Gamma$  sending the generators of  $\mathbb{Z}^2$  to their corresponding reduction in  $\Gamma$ . To any finite set  $I$  with cardinality  $n$  we associate the  $\Gamma$ -twisted configuration space

$$\text{Conf}(\mathbb{T}, I, \Gamma) := \{ \mathbf{z} = (z_1, \dots, z_n) \in \tilde{\mathbb{T}}^I \mid p(z_i) \neq p(z_j) \text{ if } i \neq j \},$$

and let  $\mathbf{C}(\mathbb{T}, I, \Gamma) := \text{Conf}(\mathbb{T}, I, \Gamma)/\tilde{\mathbb{T}}$  be its reduced version.

The inclusion

$$(4.1) \quad \text{Conf}(\mathbb{T}, I, \Gamma) \hookrightarrow \text{Conf}(\tilde{\mathbb{T}}, I \times \Gamma)$$

sending  $(z_i)_{i \in I}$  to  $(\gamma \cdot z_i)_{(i, \gamma) \in I \times \Gamma}$  induces an inclusion

$$\mathbf{C}(\mathbb{T}, I, \Gamma) \hookrightarrow \mathbf{C}(\tilde{\mathbb{T}}, I \times \Gamma) \hookrightarrow \overline{\mathbf{C}}(\tilde{\mathbb{T}}, I \times \Gamma),$$

which allows us to define  $\overline{\mathbf{C}}(\mathbb{T}, I, \Gamma)$  as the closure of  $\mathbf{C}(\mathbb{T}, I, \Gamma)$  inside  $\overline{\mathbf{C}}(\tilde{\mathbb{T}}, I \times \Gamma)$ . The boundary  $\partial \overline{\mathbf{C}}(\mathbb{T}, I, \Gamma) = \overline{\mathbf{C}}(\mathbb{T}, I, \Gamma) - \mathbf{C}(\mathbb{T}, I, \Gamma)$  is made up of the following irreducible components: for any partition  $J_1 \sqcup \dots \sqcup J_k$  of  $I$  there is a component

$$\partial_{J_1, \dots, J_k} \overline{\mathbf{C}}(\mathbb{T}, I, \Gamma) \cong \prod_{i=1}^k (\overline{\mathbf{C}}(\mathbb{C}, J_i)) \times \overline{\mathbf{C}}(\mathbb{T}, k, \Gamma).$$

The inclusion of boundary components provides  $\overline{\mathbf{C}}(\mathbb{T}, -, \Gamma)$  with the structure of a module over the operad  $\overline{\mathbf{C}}(\mathbb{C}, -)$  in topological spaces.

On the one hand, the left action of  $\Gamma$  on  $\tilde{\mathbb{T}}$  gives us an action of  $\Gamma^I$ , resp.  $\Gamma^I/\Gamma$ , on  $\text{Conf}(\tilde{\mathbb{T}}, I \times \Gamma)$ , resp.  $\mathbf{C}(\tilde{\mathbb{T}}, I \times \Gamma)$ . On the other hand,  $\Gamma^I$  also acts on  $\text{Conf}(\tilde{\mathbb{T}}, I \times \Gamma)$  and  $\mathbf{C}(\tilde{\mathbb{T}}, I \times \Gamma)$  in the following way:

$$(\alpha \cdot \mathbf{z})_{(i, \gamma)} := \mathbf{z}_{i, \gamma + \alpha}.$$

The inclusion (4.1) is  $\Gamma^I$ -equivariant, so that one gets a diagonally trivial  $\Gamma$ -action on  $\overline{\mathbb{C}}(\mathbb{C}, -)$ , in the sense of §1.6.

#### 4.2. The $\mathbf{Pa}$ -module of labelled parenthesized permutation

For every finite set  $I$ , there is a  $\Gamma^I/\Gamma$ -covering map

$$\phi_I : \mathbb{C}(\mathbb{T}, n, \Gamma) \longrightarrow \mathbb{C}(\mathbb{T}, n)$$

which extends to a continuous map

$$\bar{\phi}_I : \overline{\mathbb{C}}(\mathbb{T}, I, \Gamma) \longrightarrow \overline{\mathbb{C}}(\mathbb{T}, I),$$

everything being natural (with respect to bijections) in  $I$ . This defines a morphism  $\bar{\phi}$  of  $\overline{\mathbb{C}}(\mathbb{C}, -)$ -modules from  $\overline{\mathbb{C}}(\mathbb{T}, -, \Gamma)$  to  $\overline{\mathbb{C}}(\mathbb{T}, -)$ .

Recall from §3.3 that there are inclusions of topological operadic modules  $\mathbf{Pa} \subset \overline{\mathbb{C}}(\mathbb{S}^1, -) \subset \overline{\mathbb{C}}(\mathbb{T}, -)$  over  $\mathbf{Pa} \subset \overline{\mathbb{C}}(\mathbb{R}, -) \subset \overline{\mathbb{C}}(\mathbb{C}, -)$ . We define the  $\mathfrak{S}$ -module  $\mathbf{Pa}^\Gamma := \bar{\phi}^{-1}\mathbf{Pa}$ , which carries a  $\mathbf{Pa}$ -module structure. Indeed, it is a fiber product

$$\mathbf{Pa}^\Gamma := \mathbf{Pa} \times_{\overline{\mathbb{C}}(\mathbb{T}, -)} \overline{\mathbb{C}}(\mathbb{T}, -, \Gamma)$$

in the category of  $\mathbf{Pa}$ -modules in topological space.

The  $\mathbf{Pa}$ -module  $\mathbf{Pa}^\Gamma$  admits the following algebraic description. First of all, it is discrete, in the sense that spaces of operations are discrete (i.e., they are just sets). Then, an element of  $\mathbf{Pa}^\Gamma(n)$  is a parenthesized permutation of  $1 \dots n$  together with a label function  $\{1, \dots, n\} \rightarrow \Gamma$  that is defined up to a global relabelling (i.e. the labelling is an element of  $\Gamma^n/\Gamma$ ). For instance,  $2_\gamma 1_0 = 2_0 1_{-\gamma}$  belongs to  $\mathbf{Pa}^\Gamma(2)$  for every  $\gamma \in \Gamma$ . In geometric terms, having the label  $[\gamma_1, \dots, \gamma_n]$  means that, in our configuration of points, the  $(-\gamma_i) \cdot z_i$ 's are on the same parallel of the torus. Here is a self-explanatory example of partial composition:

$$(3_0 2_\gamma) 1_\delta \circ_2 (12) 3 = (3_0((2_\gamma 3_\gamma) 4_\gamma)) 1_\delta.$$

Finally,  $\mathbf{Pa}^\Gamma$  is acted on by  $\Gamma$  in the following way: for  $n \geq 0$ ,  $\Gamma^n$  only acts on the labellings, *via* the group law of  $\Gamma$ . For instance, if  $[\underline{\alpha}] \in \Gamma^n/\Gamma$  and  $\underline{\gamma} \in \Gamma^n$ , then  $\underline{\gamma} \cdot [\underline{\alpha}] := [\underline{\gamma} + \underline{\alpha}]$ .

In other words, according to the terminology of §1.6 and §1.7,  $\mathbf{Pa}^\Gamma$  is identified with  $\mathcal{G}(\mathbf{Pa} \rightarrow \overline{\Gamma})$ .

#### 4.3. The $\mathbf{PaB}$ -module of parenthesized ellipsitomic braids

We define

$$\mathbf{PaB}_{ell}^\Gamma := \pi_1(\overline{\mathbb{C}}(\mathbb{T}, -, \Gamma), \mathbf{Pa}^\Gamma),$$

which is a  $\mathbf{PaB}$ -module (in groupoids), that also carries a diagonally trivial action of  $\Gamma$ . The morphism  $\bar{\phi}$  induces a  $\mathbf{PaB}$ -module morphism  $\mathbf{PaB}_{ell}^\Gamma \rightarrow \mathbf{PaB}_{ell}$ .

**Example 4.1 (Notable arrows in  $\mathbf{PaB}_{ell}^\Gamma$ ).** — Recall the following notable arrows in  $\mathbf{PaB}_{ell}$ :

- $A^{1,2}$  and  $B^{1,2}$  are automorphisms of 12 in  $\mathbf{PaB}_{ell}(2)$ .

- $R^{1,2}$  goes from 12 to 21 in  $\mathbf{PaB}_{ell}(2)$ .
- $\Phi^{1,2,3}$  goes from (12)3 to 1(23) in  $\mathbf{PaB}_{ell}(2)$ .

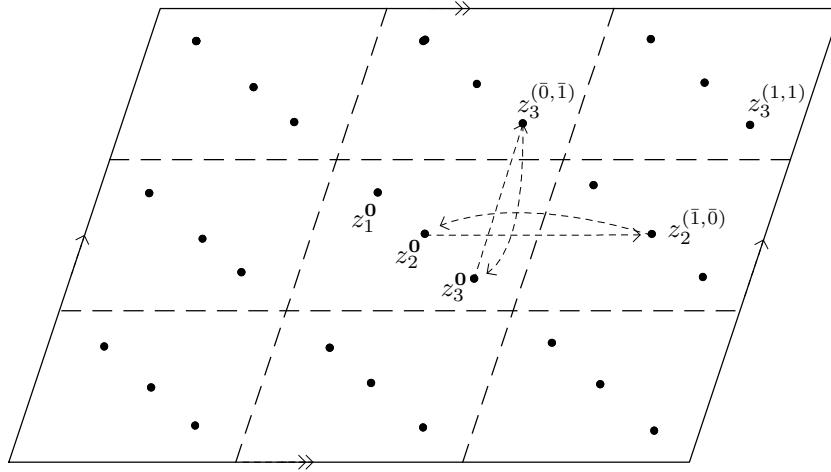
All are represented by paths which, apart from the endpoints that are in  $\mathbf{Pa}$ , remain in the open part  $C(\mathbb{T}, n)$  of the configuration spaces ( $n = 2, 3$ ). Let us set  $\alpha := (\bar{1}, \bar{0})$  and  $\beta := (\bar{0}, \bar{1})$ . Since there are covering maps

$$C(\mathbb{T}, n, \Gamma) \longrightarrow C(\mathbb{T}, n),$$

then these paths admits unique lifts, with starting point being the same parenthesized permutation with the trivial labelling (the one being constantly equal to  $\mathbf{0}$ ). These lifts are denoted the same way:

- The lift  $A^{1,2}$  goes from  $1_0 2_0$  to  $1_\alpha 2_0 = 1_0 2_{-\alpha}$  in  $\mathbf{PaB}_{ell}^\Gamma(2)$ .
- The lift  $B^{1,2}$  goes from  $1_0 2_0$  to  $1_\beta 2_0 = 1_0 2_{-\beta}$  in  $\mathbf{PaB}_{ell}^\Gamma(2)$ .
- etc...

Here is a drawing of paths representing  $A^{1,2}$  and  $B^{1,2}$ :



We may chose to alternatively depict them as diagrams representing elliptic pure braids (i.e. loops in the base configuration space) together with appropriate base points (which uniquely determines the lift in the covering twisted configuration space):

$$A^{1,2} = + \begin{array}{c} 1_0 \quad 2_0 \\ \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \\ 1_\alpha \quad 2_0 \end{array} \quad \text{and} \quad B^{1,2} = - \begin{array}{c} 1_0 \quad 2_0 \\ \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \\ 1_\beta \quad 2_0 \end{array}$$

**Remark 4.2.** — It is important to observe that, the action of  $\Gamma$  being diagonally trivial, one can shift the global labelling of the indexed points, and thus  $A^{1,2}$  and  $B^{1,2}$  can also be represented as follows:



$$A^{1,2} = + \begin{array}{c} 1_0 \quad 2_0 \\ \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ 1_0 \quad 2_{-\alpha} \end{array} \quad \text{and} \quad B^{1,2} = - \begin{array}{c} 1_0 \quad 2_0 \\ \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ 1_0 \quad 2_{-\beta} \end{array}$$

As for  $R^{1,2}$  and  $\Phi^{1,2,3}$ , they are depicted in the usual way:

$$R^{1,2} = \begin{array}{c} 1_0 \quad 2_0 \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ 2_0 \quad 1_0 \end{array} \quad \text{and} \quad \Phi^{1,2,3} = \begin{array}{c} (1_0 2_0) \quad 3_0 \\ \bullet \quad \bullet \quad \bullet \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \quad \bullet \quad \bullet \\ \downarrow \quad \downarrow \quad \downarrow \\ 1_0 \quad (2_0 3_0) \end{array}$$

Actually, every morphism in  $\mathbf{PaB}_{ell}$  can be uniquely lifted to  $\mathbf{PaB}_{ell}^\Gamma$ , once the lift of the source object has been fixed; all other lifts are obtained by the  $\bar{\Gamma}$ -action. Moreover, all morphisms can be obtained like this. This shows that the  $\mathbf{PaB}$ -module  $\mathbf{PaB}_{ell}^\Gamma$  has an alternative simple algebraic description that we explain now. First observe that the  $\mathbf{PaB}$ -module  $\mathbf{PaB}_{ell}$  comes with a morphism  $\pi$  to the  $\ast$ -module  $\bar{\Gamma}$ , which is the composition of the abelianization morphism to  $\bar{\mathbb{Z}^2}$  with the projection  $\bar{\mathbb{Z}^2} \rightarrow \bar{\Gamma}$ .

In terms of the presentation from Theorem 3.3,

$$\pi(A) = \alpha_1 = [(\bar{1}, 0), \mathbf{0}] \quad \text{and} \quad \pi(B) = \beta_1 = [(0, \bar{1}), \mathbf{0}],$$

where we adopt the following notation:

**Notation 4.3.** — For  $\gamma \in \Gamma$  and  $1 \leq i \leq n$ , then we write

$$\gamma_i := [\mathbf{0}, \dots, \mathbf{0}, \gamma, \mathbf{0}, \dots, \mathbf{0}] \in \Gamma^n / \Gamma.$$

**Proposition 4.4.** — There is an isomorphism

$$\mathcal{G}(\mathbf{PaB}_{ell} \rightarrow \bar{\Gamma}) \xrightarrow{\sim} \mathbf{PaB}_{ell}^\Gamma$$

of  $\mathbf{PaB}$ -modules with a  $\bar{\Gamma}$ -action, which is the identity on objects.

*Proof.* — We first describe the morphism:

- It is the identity on objects;
- Given two labelled parenthesized permutations  $(\mathbf{p}, \gamma)$  and  $(\mathbf{q}, \delta)$ , it sends the class in  $\mathbf{PaB}_{ell}$  of a path  $f : \mathbf{p} \rightarrow \mathbf{q}$  such that  $\gamma + \pi(f) = \delta$  to the class of the unique lift of  $f$  that starts at the base point determined by  $(\mathbf{p}, \gamma)$ .

As we have already seen, to show that this morphism is in fact an isomorphism, it suffices to show that it is an isomorphism at the level of automorphism groups of objects arity-wise. This is indeed the case, as on both sides, in arity  $n$ , the automorphism group of an object is the kernel of the morphism  $\bar{\mathbf{PB}}_{1,n} \rightarrow \Gamma^n / \Gamma$  sending  $X_i$  to  $(\bar{1}, \bar{0})_i$  and  $Y_j$  to  $(\bar{0}, \bar{1})_j$ .  $\square$

#### 4.4. The universal property of $\mathbf{PaB}_{ell}^\Gamma$

We are now ready to provide an explicit presentation for the  $\mathbf{PaB}$ -module  $\mathbf{PaB}_{ell}^\Gamma$ . As before, we keep the convention that  $\alpha = (\bar{1}, \bar{0})$  and  $\beta = (\bar{0}, \bar{1})$ .

**Theorem 4.5.** — *As a  $\mathbf{PaB}$ -module in groupoids with a diagonally trivial  $\Gamma$ -action and having  $\mathbf{Pa}^\Gamma$  as  $\mathbf{Pa}$ -module of objects,  $\mathbf{PaB}_{ell}^\Gamma$  is freely generated by  $A : 1_0 2_0 \rightarrow 1_\alpha 2_0$  and  $B : 1_0 2_0 \rightarrow 1_\beta 2_0$ , together with the following relations, satisfied in  $\text{Aut}_{\mathbf{PaB}_{ell}^\Gamma(3) \times (\Gamma^3/\Gamma)}((1_0 2_0) 3_0)$ :*

$$(tN1) \quad \Phi^{1,2,3} \underline{A}^{1,23} \tilde{R}^{1,23} \Phi^{2,3,1} \underline{A}^{2,31} \tilde{R}^{2,31} \Phi^{3,1,2} \underline{A}^{3,12} \tilde{R}^{3,12} = \text{Id}_{(1_0 2_0) 3_0}$$

$$(tN2) \quad \Phi^{1,2,3} \underline{B}^{1,23} \tilde{R}^{1,23} \Phi^{2,3,1} \underline{B}^{2,31} \tilde{R}^{2,31} \Phi^{3,1,2} \underline{B}^{3,12} \tilde{R}^{3,12} = \text{Id}_{(1_0 2_0) 3_0}$$

$$(tE) \quad R^{1,2} R^{2,1} = (\Phi^{1,2,3} \underline{A}^{1,23} (\Phi^{1,2,3})^{-1}, \tilde{R}^{1,2} \Phi^{2,1,3} \underline{B}^{2,13} (\Phi^{2,1,3})^{-1} \tilde{R}^{2,1})$$

where  $\underline{A} := A\alpha_1$  and  $\underline{B} := B\beta_1$ .

**Remark 4.6.** — The above relations are clearer when stated within the semidirect product, even though they can be written within  $\mathbf{PaB}_{ell}^\Gamma$  itself. For instance, (tN1) can be written as

$$\Phi^{1,2,3} A^{1,23} \alpha_1 \cdot (\tilde{R}^{1,23} \Phi^{2,3,1} A^{2,31} \alpha_2 \cdot (\tilde{R}^{2,31} \Phi^{3,1,2} A^{3,12})) \tilde{R}^{3,12} = \text{Id}_{(1_0 2_0) 3_0}.$$

The expression for (tE) becomes unpleasant to write.

*Proof of the Theorem.* — Let  $\mathcal{Q}_{ell}^\Gamma$  be the  $\mathbf{PaB}$ -module with the above presentation, and let  $\mathcal{Q}_{ell}$  be the  $\mathbf{PaB}$ -module with the presentation in Theorem 3.3. Our goal is to prove that there is an isomorphism

$$\mathcal{G}(\mathcal{Q}_{ell} \rightarrow \bar{\Gamma}) \xrightarrow{\sim} \mathcal{Q}_{ell}^\Gamma$$

of  $\mathbf{PaB}$ -modules with a  $\bar{\Gamma}$ -action, which is the identity on objects. The result will then follow from Proposition 4.4.

By definition there is a morphism  $\mathcal{Q}_{ell} \rightarrow \mathcal{Q}_{ell}^\Gamma \rtimes \bar{\Gamma}$ , which sends  $A$  to  $\underline{A}$ , and  $B$  to  $\underline{B}$ . Moreover, when we compose this morphism with the projection  $\mathcal{Q}_{ell}^\Gamma \rtimes \bar{\Gamma} \rightarrow \bar{\Gamma}$ , we get back the morphism  $\pi : \mathcal{Q}_{ell} \rightarrow \bar{\Gamma}$  from the previous chapter, that sends  $A$  to  $\alpha_1$  and  $B$  to  $\beta_1$ .

By the adjunction from §1.7, we therefore get a morphism

$$\mathcal{G}(\mathcal{Q}_{ell} \rightarrow \bar{\Gamma}) \rightarrow \mathcal{Q}_{ell}^\Gamma.$$

of  $\mathbf{PaB}$ -modules with a  $\bar{\Gamma}$ -action. It is surjective on morphisms, because both generators of  $\mathcal{Q}_{ell}^\Gamma$  have preimages. Finally, as we have already seen, to show that this is in fact an isomorphism, it suffices to show that it is an isomorphism at the level of automorphism groups of objects arity-wise, and it is sufficient to do it for a single object in every arity.

Let  $n \geq 1$  and let  $\tilde{p}$  be the object  $(\dots((1_0 2_0) 3_0) \dots) n_0$  of  $\mathcal{Q}_{ell}^\Gamma(n)$  and  $\mathcal{G}(\mathcal{Q}_{ell}(n) \rightarrow \Gamma^n/\Gamma)$ , which lifts  $p = (\dots((12)3) \dots) n$  in  $\mathcal{Q}_{ell}(n)$ . There is a commuting diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \text{Aut}_{\mathcal{G}(\mathcal{Q}_{ell}(n) \rightarrow \Gamma^n/\Gamma)}(\tilde{p}) & \longrightarrow & \text{Aut}_{\mathcal{Q}_{ell}^\Gamma(n)}(p) & \longrightarrow & \Gamma^n/\Gamma \longrightarrow 1 \\ & & \downarrow & \nearrow & & & \\ & & \text{Aut}_{\mathcal{Q}_{ell}^\Gamma(n)}(\tilde{p}) & & & & \end{array}$$

where the horizontal sequence is exact. Therefore the vertical morphism is injective, and we are done.  $\square$

#### 4.5. Elliptic Grothendieck–Teichmüller groups

**Definition 4.7.** — The ( $\mathbf{k}$ -pro-unipotent version of the) *elliptic Grothendieck–Teichmüller group* is defined as

$$\widehat{\mathbf{GT}}_{ell}^{\Gamma}(\mathbf{k}) := \text{Aut}_{\text{OpR Grpd}_{\mathbf{k}}}^+(\widehat{\mathbf{PaB}}(\mathbf{k}), \widehat{\mathbf{PaB}}_{ell}^{\Gamma}(\mathbf{k}))^{\Gamma},$$

where, as usual, the superscript  $\Gamma$  means that we are considering the subgroup of  $\Gamma$ -equivariant automorphisms.

Let  $M', N' \geq 1$ , and assume we are given a surjective group morphism

$$\rho : \Gamma \twoheadrightarrow \Gamma' := \mathbb{Z}/M'\mathbb{Z} \times \mathbb{Z}/N'\mathbb{Z}.$$

This gives a (surjective) map between the corresponding covering spaces of the torus, which can be used to construct a morphism of  $\overline{\mathcal{C}}(\mathbb{C}, -)$ -modules

$$\overline{\mathcal{C}}(\mathbb{T}, -, \Gamma) \longrightarrow \overline{\mathcal{C}}(\mathbb{T}, -, \Gamma').$$

Following the construction of sections 4.2 and 4.3, we get a morphism of  $\mathbf{PaB}$ -modules

$$\mathbf{PaB}_{ell}^{\rho} : \mathbf{PaB}_{ell}^{\Gamma} \longrightarrow \mathbf{PaB}_{ell}^{\Gamma'}.$$

The morphism  $\mathbf{PaB}_{ell}^{\rho}$  is  $\Gamma$ -equivariant, and has a straightforward algebraic description:

- On objects, it consists in applying  $\rho$  to the labelling, keeping the underlying parenthesized permutation unchanged;
- It sends the generating morphisms  $A^{1\circ 2\circ}$  and  $B^{1\circ 2\circ}$  in  $\mathbf{PaB}_{ell}^{\Gamma}$  to their counterparts (which are denoted the same way) in  $\mathbf{PaB}_{ell}^{\Gamma'}$ .

As a consequence, the  $\mathbf{PaB}$ -module  $\mathbf{PaB}_{ell}^{\Gamma'}$  can be obtained as the quotient of  $\mathbf{PaB}_{ell}^{\Gamma}$  by  $\ker \rho$ . We therefore obtain a group morphism  $\widehat{\mathbf{GT}}_{ell}^{\Gamma}(\mathbf{k}) \longrightarrow \widehat{\mathbf{GT}}_{ell}^{\Gamma'}(\mathbf{k})$ .

## CHAPTER 5

### Ellipsitomic chord diagrams and ellipsitomic associators

#### 5.1. Infinitesimal ellipsitomic braids

In this paragraph and the next one,  $(\Gamma, \mathbf{0}, +)$  can be any finite abelian group.

For any  $n \geq 0$  we define  $\mathfrak{t}_{1,n}^\Gamma(\mathbf{k})$  to be the bigraded  $\mathbf{k}$ -Lie algebra with generators  $x_i$  ( $1 \leq i \leq n$ ) in degree  $(1, 0)$ ,  $y_i$  ( $1 \leq i \leq n$ ) in degree  $(0, 1)$ , and  $t_{ij}^\gamma$  ( $\gamma \in \Gamma$ ,  $i \neq j$ ) in degree  $(1, 1)$ , and relations

$$\begin{aligned}
 (\text{tS}_{ell1}) \quad & t_{ij}^\gamma = t_{ji}^{-\gamma}, \\
 (\text{tS}_{ell2}) \quad & [x_i, y_j] = [x_j, y_i] = \sum_{\gamma \in \Gamma} t_{ij}^\gamma, \\
 (\text{tN}_{ell}) \quad & [x_i, x_j] = [y_i, y_j] = 0, \\
 (\text{tT}_{ell}) \quad & [x_i, y_i] = - \sum_{j:j \neq i} \sum_{\gamma \in \Gamma} t_{ij}^\gamma, \\
 (\text{tL}_{ell1}) \quad & [t_{ij}^\gamma, t_{kl}^\delta] = 0, \\
 (\text{tL}_{ell2}) \quad & [x_i, t_{jk}^\gamma] = [y_i, t_{jk}^\gamma] = 0, \\
 (\text{t4T}_{ell1}) \quad & [t_{ij}^\gamma, t_{ik}^{\gamma+\delta} + t_{jk}^\delta] = 0, \\
 (\text{t4T}_{ell2}) \quad & [x_i + x_j, t_{ij}^\gamma] = [y_i + y_j, t_{ij}^\gamma] = 0,
 \end{aligned}$$

where  $1 \leq i, j, k, l \leq n$  are pairwise distinct and  $\gamma, \delta \in \Gamma$ . We will call  $\mathfrak{t}_{1,n}^\Gamma(\mathbf{k})$  the  $\mathbf{k}$ -Lie algebra of *infinitesimal ellipsitomic braids*. Observe that  $\sum_i x_i$  and  $\sum_i y_i$  are central in  $\mathfrak{t}_{1,n}^\Gamma$ . Then we denote by  $\bar{\mathfrak{t}}_{1,n}^\Gamma(\mathbf{k})$  the quotient of  $\mathfrak{t}_{1,n}^\Gamma(\mathbf{k})$  by  $\sum_i x_i$  and  $\sum_i y_i$ , and the natural morphism  $\mathfrak{t}_{1,n}^\Gamma(\mathbf{k}) \rightarrow \bar{\mathfrak{t}}_{1,n}^\Gamma(\mathbf{k}); u \mapsto \bar{u}$ .

There is an alternative presentation of  $\mathfrak{t}_{1,n}^\Gamma(\mathbf{k})$  and  $\bar{\mathfrak{t}}_{1,n}^\Gamma(\mathbf{k})$ :

**Lemma 5.1.** — *The Lie  $\mathbf{k}$ -algebra  $\mathfrak{t}_{1,n}^\Gamma(\mathbf{k})$  (resp.  $\bar{\mathfrak{t}}_{1,n}^\Gamma(\mathbf{k})$ ) can equivalently be presented with the same generators, and the following relations:  $(\text{tS}_{ell1})$ ,  $(\text{tS}_{ell2})$ ,  $(\text{tN}_{ell})$ ,  $(\text{tL}_{ell1})$ ,  $(\text{tL}_{ell2})$ ,  $(\text{t4T}_{ell1})$ , and, for every  $i \in I$ ,*

$$(\text{tC}_{ell}) \quad \left[ \sum_j x_j, y_i \right] = \left[ \sum_j y_j, x_i \right] = 0$$

(resp.  $\sum_j x_j = \sum_j y_j = 0$ ).

*Proof.* — If  $x_i, y_i$  and  $t_{ij}^\alpha$  satisfy the initial relations, then

$$[\sum_j x_j, y_i] = [x_i, y_i] + [\sum_{j \neq i} x_j, y_i] = - \sum_{j: j \neq i} \sum_{\gamma \in \Gamma} t_{ij}^\gamma + \sum_{j: j \neq i} \sum_{\gamma \in \Gamma} t_{ij}^\gamma = 0.$$

Now, if  $x_i, y_i$  and  $t_{ij}^\alpha$  satisfy the above relations, then relations  $[\sum_j x_j, y_i] = 0$  and  $[x_j, y_i] = \sum_{\gamma \in \Gamma} t_{ij}^\gamma$ , for  $i \neq j$ , imply that  $[x_i, y_i] = - \sum_{j: j \neq i} \sum_{\gamma \in \Gamma} t_{ij}^\gamma$ . Now, relations  $[\sum_k x_k, y_j] = 0$  and  $[\sum_k x_k, x_i] = 0$  imply that  $[\sum_k x_k, \sum_{\gamma \in \Gamma} t_{ij}^\gamma] = 0$ . Thus, as  $[x_i, t_{jk}^\gamma] = 0$  if  $\text{card}\{i, j, k\} = 3$ , we obtain relation  $[x_i + x_j, t_{ij}^\gamma] = 0$ , for  $i \neq j$ . In the same way we obtain  $[y_i + y_j, t_{ij}^\gamma] = 0$ , for  $i \neq j$ .  $\square$

Both  $\mathfrak{t}_{1,n}^\Gamma(\mathbf{k})$  and  $\bar{\mathfrak{t}}_{1,n}^\Gamma(\mathbf{k})$  are acted on by the symmetric group  $\mathfrak{S}_n$ , we get that

$$\mathfrak{t}_{ell}^\Gamma(\mathbf{k}) := \{\mathfrak{t}_{1,n}^\Gamma(\mathbf{k})\}_{n \geq 0} \quad \text{and} \quad \bar{\mathfrak{t}}_{ell}^\Gamma(\mathbf{k}) := \{\bar{\mathfrak{t}}_{1,n}^\Gamma(\mathbf{k})\}_{n \geq 0}$$

define  $\mathfrak{S}$ -modules in  $grLie_{\mathbf{k}}$ . They are actually  $\mathfrak{t}(\mathbf{k})$ -module in  $grLie_{\mathbf{k}}$ , where partial compositions are defined as follows<sup>(1)</sup>: for  $I$  a finite set and  $k \in I$ ,

$$\begin{aligned} \circ_k : \mathfrak{t}_{1,I}^\Gamma(\mathbf{k}) \oplus \mathfrak{t}_J(\mathbf{k}) &\longrightarrow \mathfrak{t}_{1, J \sqcup I - \{k\}}^\Gamma(\mathbf{k}) \\ (0, t_{uv}) &\longmapsto t_{uv}^{\mathbf{0}} \\ (t_{ij}^\gamma, 0) &\longmapsto \begin{cases} t_{ij}^\gamma & \text{if } k \notin \{i, j\} \\ \sum_{p \in J} t_{pj}^\gamma & \text{if } k = i \\ \sum_{p \in J} t_{ip}^\gamma & \text{if } j = k \end{cases} \\ (x_i, 0) &\longmapsto \begin{cases} x_i & \text{if } k \neq i \\ \sum_{p \in J} x_p & \text{if } k = i \end{cases} \\ (y_i, 0) &\longmapsto \begin{cases} y_i & \text{if } k \neq i \\ \sum_{p \in J} y_p & \text{if } k = i \end{cases} \end{aligned}$$

We call  $\mathfrak{t}_{ell}^\Gamma(\mathbf{k})$ , resp.  $\bar{\mathfrak{t}}_{ell}^\Gamma(\mathbf{k})$ , the module of *infinitesimal ellipsitomic braids*, resp. of *infinitesimal reduced ellipsitomic braids*. When  $\mathbf{k} = \mathbb{C}$  we write  $\mathfrak{t}_{1,n}^\Gamma := \mathfrak{t}_{1,n}^\Gamma(\mathbb{C})$  and  $\bar{\mathfrak{t}}_{1,n}^\Gamma := \bar{\mathfrak{t}}_{1,n}^\Gamma(\mathbb{C})$ .

Both  $\mathfrak{t}(\mathbf{k})$ -modules are acted on by  $\Gamma$ : any element  $\underline{\gamma} = (\gamma_i)_{i \in I} \in \Gamma^I$  acts as

$$\begin{aligned} \underline{\gamma} \cdot x_i &= x_i & (i \in I), \\ \underline{\gamma} \cdot y_i &= y_i & (i \in I), \\ \underline{\gamma} \cdot t_{ij}^\delta &= t_{ij}^{\delta + \gamma_i - \gamma_j} & (\delta \in \Gamma \text{ and } i \neq j). \end{aligned}$$

We also have the following functoriality in  $\Gamma$ , with respect to surjections:

**Proposition 5.2.** — *Let  $\rho : \Gamma_1 \twoheadrightarrow \Gamma_2$  be a surjective group morphism, and let  $a, b, c, d \in \mathbf{k}$  such that  $ad - bc = |\ker \rho|$ . There are  $\Gamma_1^I$ -equivariant surjective comparison morphisms  $\mathfrak{t}_{1,I}^{\Gamma_1}(\mathbf{k}) \rightarrow \mathfrak{t}_{1,I}^{\Gamma_2}(\mathbf{k})$  and  $\bar{\mathfrak{t}}_{1,I}^{\Gamma_1}(\mathbf{k}) \rightarrow \bar{\mathfrak{t}}_{1,I}^{\Gamma_2}(\mathbf{k})$ , defined by*

$$x_i \mapsto ax_i + by_i, \quad y_i \mapsto cx_i + dy_i, \quad t_{ij}^\gamma \mapsto t_{ij}^{\rho(\gamma)}.$$

*These are morphisms of  $\mathfrak{t}(\mathbf{k})$ -modules in  $grLie_{\mathbf{k}}$ .*

<sup>(1)</sup>We give the formulæ for  $\mathfrak{t}_{ell}^\Gamma(\mathbf{k})$ . Formulæ for  $\bar{\mathfrak{t}}_{ell}^\Gamma(\mathbf{k})$  are the exact same.

*Proof.* — This follows from direct computations.  $\square$

Actually, these morphisms exhibit  $\mathfrak{t}_{1,I}^{\Gamma_2}(\mathbf{k})$ , resp.  $\bar{\mathfrak{t}}_{1,I}^{\Gamma_2}(\mathbf{k})$ , as the quotient of  $\mathfrak{t}_{1,I}^{\Gamma_1}(\mathbf{k})$ , resp.  $\bar{\mathfrak{t}}_{1,I}^{\Gamma_1}(\mathbf{k})$ , by  $(\ker \rho)^I$ .

**Remark 5.3.** — Whenever  $\Gamma_i = \mathbb{Z}/M_i\mathbb{Z} \times \mathbb{Z}/N_i\mathbb{Z}$ , there is a natural choice for the scalars  $a, b, c, d$ . Indeed, if  $\rho : \Gamma_1 \rightarrow \Gamma_2$  is surjective, then there exists elements  $(a, b)$  and  $(c, d)$  in the lattice  $M_2\mathbb{Z} \times N_2\mathbb{Z}$  that generate the sublattice  $M_1\mathbb{Z} \times N_1\mathbb{Z}$ . Hence, in particular, the determinant  $ad - bc$  equals  $\frac{M_1N_1}{M_2N_2} = |\ker \rho|$ .

### 5.2. Horizontal ellipsitomic chord diagrams

In this paragraph we define the  $\mathbf{CD}(\mathbf{k})$ -module  $\mathbf{CD}_{ell}^{\Gamma}(\mathbf{k})$  of *ellipsitomic chord diagrams*.

We first consider the  $\mathbf{CD}(\mathbf{k})$ -module  $\hat{\mathcal{U}}(\bar{\mathfrak{t}}_{ell}^{\Gamma}(\mathbf{k}))$ . Morphisms in  $\hat{\mathcal{U}}(\bar{\mathfrak{t}}_{ell}^{\Gamma}(\mathbf{k}))$  can be given a pictorial description, which mixes the features of the horizontal  $N$ -chord diagrams from [9] (see also [14]) together with the elliptic chord diagrams from §3.4. Diagrams corresponding to  $x_i$  and  $y_j$  are, respectively,

$$\begin{array}{c} + \\ \bullet \cdots \vdots \\ i \end{array} = \begin{array}{c} + \\ \bullet \cdots \vdots \\ \gamma \\ \bullet \\ -\gamma \\ \bullet \\ i \end{array} \quad \text{and} \quad \begin{array}{c} - \\ \bullet \cdots \vdots \\ j \end{array} = \begin{array}{c} - \\ \bullet \cdots \vdots \\ \gamma \\ \bullet \\ -\gamma \\ \bullet \\ j \end{array}$$

and the one corresponding to  $t_{ij}^{\gamma} = t_{ji}^{-\gamma}$  is

$$\begin{array}{c} i \\ \bullet \\ \gamma \\ \bullet \\ -\gamma \\ \bullet \\ i \end{array} \cdots \begin{array}{c} j \\ \bullet \\ \bullet \\ j \end{array} = \begin{array}{c} i \\ \bullet \\ \bullet \\ i \end{array} \cdots \begin{array}{c} j \\ \bullet \\ \gamma \\ \bullet \\ -\gamma \\ \bullet \\ j \end{array}$$

Relations can be depicted as follows:

$$(\text{tS}_{ell2}) \quad \begin{array}{c} \mp \\ \bullet \cdots \vdots \\ i \end{array} \cdots \begin{array}{c} j \\ \bullet \\ \bullet \\ j \end{array} - \begin{array}{c} \pm \\ \bullet \cdots \vdots \\ i \end{array} \cdots \begin{array}{c} j \\ \bullet \\ \bullet \\ j \end{array} = \sum_{\gamma \in \Gamma} -\gamma \begin{array}{c} \bullet \\ \gamma \\ \bullet \\ -\gamma \\ \bullet \\ i \end{array} \cdots \begin{array}{c} j \\ \bullet \\ \bullet \\ j \end{array}$$

$$(\text{tN}_{ell}) \quad \begin{array}{c} \pm \\ \bullet \cdots \vdots \\ i \end{array} \cdots \begin{array}{c} j \\ \bullet \\ \bullet \\ j \end{array} = \begin{array}{c} \pm \\ \bullet \cdots \vdots \\ i \end{array} \cdots \begin{array}{c} j \\ \bullet \\ \bullet \\ j \end{array}$$

(tT<sub>ell</sub>)

Diagrammatic equation (tT<sub>ell</sub>) showing two vertical lines with dots and signs, equated to a sum over indices  $j$  and  $\gamma$ .

(tL<sub>ell1</sub>)

Diagrammatic equation (tL<sub>ell1</sub>) involving four vertical lines with various signs and parameters  $\gamma$  and  $\delta$ .

(tL<sub>ell2</sub>)

Diagrammatic equation (tL<sub>ell2</sub>) involving three vertical lines with signs and parameters  $\gamma$ .

(t4T<sub>ell1</sub>)

Diagrammatic equation (t4T<sub>ell1</sub>) involving three vertical lines with multiple dots and signs, equated to a sum of two terms.

(t4T<sub>ell2</sub>)

Diagrammatic equation (t4T<sub>ell2</sub>) involving two vertical lines with multiple dots and signs, equated to a sum of two terms.

$$(tC_{ell}) \quad \sum_i \begin{array}{c} i \\ \bullet \cdots \bullet \\ \vdots \\ \bullet \\ i \end{array} = 0$$

One can notice that labels sum to  $\mathbf{0}$  on each strand of all the above diagrams.

We are now ready to define the  $\mathbf{CD}(\mathbf{k})$ -module  $\mathbf{CD}_{ell}^\Gamma(\mathbf{k})$ .

- In arity  $n$ , objects of  $\mathbf{CD}_{ell}^\Gamma(\mathbf{k})$  are just labellings  $\{1, \dots, n\} \rightarrow \Gamma$  up to a global shift:  $\text{Ob}(\mathbf{CD}_{ell}^\Gamma(\mathbf{k}))(n) = \Gamma^n / \Gamma$ ;
- The  $\ast$ -module structure is given as follows on objects: for every  $i$ ,  $\circ_i : \Gamma^n \rightarrow \Gamma^{n+m-1}$  is the partial diagonal

$$(\alpha_1, \dots, \alpha_n) \mapsto (\alpha_1, \dots, \alpha_{i-1}, \underbrace{\alpha_i, \dots, \alpha_i}_{m \text{ times}}, \alpha_{i+1}, \dots, \alpha_n);$$

- Given two objects  $[\underline{\alpha}] = [\alpha_1, \dots, \alpha_n]$  and  $[\underline{\beta}] = [\beta_1, \dots, \beta_n]$  in arity  $n$ , the  $\mathbf{k}$ -vector space of morphisms from  $[\underline{\alpha}]$  to  $[\underline{\beta}]$  in  $\mathbf{CD}_{ell}^\Gamma(\mathbf{k})$  is the vector space of horizontal  $\Gamma$ -chord diagrams such that, for every  $i$ , the sum of labels on the  $i$ -th strand is  $\beta_i - \alpha_i$ ;
- The  $\mathbf{CD}(\mathbf{k})$ -module structure on morphisms is the exact same as the one for  $\hat{\mathcal{U}}(\hat{t}_{ell}^\Gamma(\mathbf{k}))$ .

Moreover,  $\mathbf{CD}_{ell}^\Gamma(\mathbf{k})$  carries an action of  $\bar{\Gamma}$ , by translation on the labelling of objects.

For every surjective group morphism  $\rho : \Gamma \twoheadrightarrow \Gamma'$ , the morphism of Proposition 5.2 gives rise to a  $\Gamma$ -equivariant surjective morphism

$$\mathbf{CD}_{ell}^\Gamma(\mathbf{k}) \rightarrow \mathbf{CD}_{ell}^{\Gamma'}(\mathbf{k})$$

which exhibits  $\mathbf{CD}_{ell}^{\Gamma'}(\mathbf{k})$  as the quotient  $\overline{\ker(\rho)} \backslash \mathbf{CD}_{ell}^\Gamma(\mathbf{k})$ .

**Example 5.4 (Notable arrows in  $\mathbf{CD}_{ell}^\Gamma(\mathbf{k})(2)$ ).** — Assume that  $\Gamma = \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ . In addition to the arrows of  $\hat{\mathcal{U}}(\hat{t}_{1,2}^\Gamma(\mathbf{k}))$ , we also have, in  $\mathbf{CD}_{ell}^\Gamma(\mathbf{k})(2)$ ,

$$I_{ell}^{1,2} = \begin{array}{cc} 1_0 & 2_0 \\ \bullet & \bullet \\ \downarrow \alpha & \downarrow \\ \bullet & \bullet \\ 1_\alpha & 2_0 \end{array} \quad \text{and} \quad J_{ell}^{1,2} = \begin{array}{cc} 1_0 & 2_0 \\ \bullet & \bullet \\ \downarrow \beta & \downarrow \\ \bullet & \bullet \\ 1_\beta & 2_0 \end{array}$$

recalling that  $\alpha = (\bar{1}, \bar{0})$  and  $\beta = (\bar{0}, \bar{1})$ .

Let us introduce  $\underline{I}_{ell}^{1,2} := I_{ell}^{1,2} \alpha_1$  and  $\underline{J}_{ell}^{1,2} := J_{ell}^{1,2} \beta_1$ , that are automorphisms of  $1_0 2_0$  in the semi-direct product groupoid  $\mathbf{CD}_{ell}^\Gamma(\mathbf{k})(2) \rtimes (\Gamma^2 / \Gamma)$ . Then, by definition, for every  $\gamma = (\bar{p}, \bar{q}) \in \Gamma$ ,

$$\text{Ad}((\underline{I}_{ell}^{1,2})^p (\underline{J}_{ell}^{1,2})^q)(t_{12}^0) = t_{12}^\gamma.$$

**Notation 5.5.** — For later purposes, we also introduce the notation



$$X_{ell}^{1,2} = x_1 \cdot \text{Id}_{1_0 2_0} = \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} + \begin{array}{c} 1_0 \\ \bullet \\ \vdots \\ \bullet \\ 1_0 \end{array} \quad \text{and} \quad Y_{ell}^{1,2} = y_1 \cdot \text{Id}_{1_0 2_0} = \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} - \begin{array}{c} 1_0 \\ \bullet \\ \vdots \\ \bullet \\ 1_0 \end{array}$$

### 5.3. Parenthesized ellipsitomic chord diagrams

There is a  $\Gamma$ -equivariant morphism of modules  $\omega_3 : \mathbf{Pa}^\Gamma \rightarrow \text{Ob}(\mathbf{CD}_{ell}^\Gamma(\mathbf{k}))$ , which forgets the parenthesized permutation (and only remembers the labelling), over the terminal operad morphism  $\omega_1 : \mathbf{Pa} \rightarrow * = \text{Ob}(\mathbf{CD}(\mathbf{k}))$  from §2.5. Hence we can consider the fake pull-back  $\mathbf{PaCD}(\mathbf{k})$ -module

$$\mathbf{PaCD}_{ell}^\Gamma(\mathbf{k}) := \omega_3^* \mathbf{CD}_{ell}^\Gamma(\mathbf{k})$$

of parenthesized ellipsitomic chord diagrams, which is still acted on by  $\bar{\Gamma}$ .

**Remark 5.6.** — As explained in section 1.5, there is a map of  $\mathfrak{S}$ -modules  $\mathbf{PaCD}(\mathbf{k}) \rightarrow \mathbf{PaCD}_{ell}^\Gamma(\mathbf{k})$  and we keep the same symbol for the image in  $\mathbf{PaCD}_{ell}^\Gamma(\mathbf{k})$  an arrows in  $\mathbf{PaCD}(\mathbf{k})$ .

$$X^{1,2} = 1 \cdot \begin{array}{c} 1_0 \quad 2_0 \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ 2_0 \quad 1_0 \end{array} \quad H^{1,2} = t_{12}^0 \cdot \begin{array}{c} 1_0 \quad 2_0 \\ \bullet \quad \bullet \\ \vdots \\ \bullet \quad \bullet \\ 1_0 \quad 2_0 \end{array} \quad a^{1,2,3} = 1 \cdot \begin{array}{c} (1_0 2_0) \quad 3_0 \\ \bullet \quad \bullet \\ \vdots \\ \bullet \quad \bullet \\ 1_0 \quad (2_0 3_0) \end{array}$$

Assuming again that  $\Gamma = \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ , the following relations hold in  $\text{End}_{\mathbf{PaCD}_{ell}^\Gamma(\mathbf{k})(2) \times (\Gamma^2/\Gamma)}(1_0 2_0)$ :

- $(\underline{I}_{ell}^{1,2})^M = \text{Id}_{1_0 2_0}$ ,
- $(\underline{J}_{ell}^{1,2})^N = \text{Id}_{1_0 2_0}$ ,
- $(\underline{I}_{ell}^{1,2}, \underline{J}_{ell}^{1,2}) = \text{Id}_{1_0 2_0}$ ,

These relations allow to unambiguously define, for every  $\gamma = (\bar{p}, \bar{q}) \in \Gamma$ , a morphism  $K_\gamma^{1,2} : 1_0 2_0 \rightarrow 1_\gamma 2_\gamma$  by

$$\underline{K}_\gamma^{1,2} := K_\gamma \gamma_1 = (\underline{I}_{ell}^{1,2})^p (\underline{J}_{ell}^{1,2})^q,$$

so that the assignement  $\gamma \mapsto \underline{K}_\gamma$  is multiplicative.

We also have the following relations in  $\text{End}_{\mathbf{PaCD}_{ell}^\Gamma(\mathbf{k})(3) \times (\Gamma^3/\Gamma)}((1_0 2_0) 3_0)$ :

$$\begin{aligned} 0 &= X_{ell}^{12,3} + \text{Ad}(a^{1,2,3} X_{ell}^{1,23})(X_{ell}^{23,1}) + \text{Ad}(X_{ell}^{12,3}(a^{3,1,2})^{-1})(X_{ell}^{31,2}), \\ 0 &= Y_{ell}^{12,3} + \text{Ad}(a^{1,2,3} X_{ell}^{1,23})(Y_{ell}^{23,1}) + \text{Ad}(X_{ell}^{12,3}(a^{3,1,2})^{-1})(Y_{ell}^{31,2}), \\ 0 &= \underline{I}_{ell}^{12,3} + \text{Ad}(a^{1,2,3} X_{ell}^{1,23})(\underline{I}_{ell}^{23,1}) + \text{Ad}(X_{ell}^{12,3}(a^{3,1,2})^{-1})(\underline{I}_{ell}^{31,2}), \\ 0 &= \underline{J}_{ell}^{12,3} + \text{Ad}(a^{1,2,3} X_{ell}^{1,23})(\underline{J}_{ell}^{23,1}) + \text{Ad}(X_{ell}^{12,3}(a^{3,1,2})^{-1})(\underline{J}_{ell}^{31,2}), \\ \sum_{\gamma \in \Gamma} \text{Ad}(\underline{K}_\gamma^{1,2})(H^{1,2}) &= [\text{Ad}(a^{1,2,3})(X_{ell}^{1,23}), \text{Ad}(X_{ell}^{1,2} a^{2,1,3})(Y_{ell}^{2,13})]. \end{aligned}$$

**Definition 5.7.** — The graded ellipsitomic Grothendieck-Teichmüller group is defined as

$$\mathbf{GRT}_{ell}^\Gamma(\mathbf{k}) := \text{Aut}_{\text{OpR Cat}(\text{CoAlg}_{\mathbf{k}})}^+(\mathbf{PaCD}(\mathbf{k}), \mathbf{PaCD}_{ell}^\Gamma(\mathbf{k}))^\Gamma$$

Recall that there is an isomorphism

$$\mathrm{Aut}_{\mathrm{OpR}\text{-Cat}(\mathrm{CoAlg}_{\mathbf{k}})}^+(\mathbf{PaCD}(\mathbf{k}), \mathbf{PaCD}_{\mathrm{ell}}^{\Gamma}(\mathbf{k}))^{\Gamma} \simeq \mathrm{Aut}_{\mathrm{OpR}\text{-Grpd}_{\mathbf{k}}}^+(\mathrm{GPaCD}(\mathbf{k}), \mathrm{GPaCD}_{\mathrm{ell}}^{\Gamma}(\mathbf{k}))^{\Gamma}.$$

For every group surjective morphism  $\rho : \Gamma \rightarrow \Gamma'$ , and every  $a, b, c, d \in \mathbf{k}$  such that  $ad - bc = |\ker(\rho)|$ , using the fact that  $\overline{\ker(\rho)} \backslash \mathbf{PaCD}_{\mathrm{ell}}^{\Gamma}(\mathbf{k}) \simeq \mathbf{PaCD}_{\mathrm{ell}}^{\Gamma'}(\mathbf{k})$ , we obtain a group morphism

$$\mathbf{GRT}_{\mathrm{ell}}^{\Gamma}(\mathbf{k}) \longrightarrow \mathbf{GRT}_{\mathrm{ell}}^{\Gamma'}(\mathbf{k}).$$

#### 5.4. Ellipsitomic associators

We now fix  $\Gamma := \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ .

**Definition 5.8.** — The set of *ellipsitomic  $\mathbf{k}$ -associators* is

$$\mathbf{Ell}^{\Gamma}(\mathbf{k}) := \mathrm{Iso}_{\mathrm{OpR}\text{-Grpd}_{\mathbf{k}}}^+\left(\left(\widehat{\mathbf{PaB}}(\mathbf{k}), \widehat{\mathbf{PaB}}_{\mathrm{ell}}^{\Gamma}(\mathbf{k})\right), \left(\mathrm{GPaCD}(\mathbf{k}), \mathrm{GPaCD}_{\mathrm{ell}}^{\Gamma}(\mathbf{k})\right)\right)^{\Gamma}.$$

**Theorem 5.9.** — There is a one-to-one correspondence between the set  $\mathbf{Ell}^{\Gamma}(\mathbf{k})$  and the set  $\mathbf{Ell}^{\Gamma}(\mathbf{k})$  consisting of quadruples  $(\mu, \varphi, A, B) \in \mathrm{Ass}(\mathbf{k}) \times \exp(\hat{\mathfrak{t}}_{1,2}^{\Gamma}(\mathbf{k}))$  such that, for  $\underline{A} := A\alpha_1$  and  $\underline{B} := B\beta_1$ , the following relations hold in  $\exp(\hat{\mathfrak{t}}_{1,3}^{\Gamma}(\mathbf{k})) \rtimes (\Gamma^3/\Gamma)$ :

$$(5.1) \quad 1 = \varphi^{1,2,3} \underline{A}^{1,23} e^{-\mu(\bar{t}_{12}^0 + \bar{t}_{13}^0)/2} \varphi^{2,3,1} \underline{A}^{2,31} e^{-\mu(\bar{t}_{23}^0 + \bar{t}_{12}^0)/2} \varphi^{3,1,2} \underline{A}^{3,12} e^{-\mu(\bar{t}_{31}^0 + \bar{t}_{32}^0)/2},$$

$$(5.2) \quad 1 = \varphi^{1,2,3} \underline{B}^{1,23} e^{-\mu(\bar{t}_{12}^0 + \bar{t}_{13}^0)/2} \varphi^{2,3,1} \underline{B}^{2,31} e^{-\mu(\bar{t}_{23}^0 + \bar{t}_{12}^0)/2} \varphi^{3,1,2} \underline{B}^{3,12} e^{-\mu(\bar{t}_{31}^0 + \bar{t}_{32}^0)/2},$$

$$(5.3) \quad e^{\mu \bar{t}_{12}^0} = \left( \varphi^{1,2,3} \underline{A}^{1,23} \varphi^{3,2,1}, e^{-\mu \bar{t}_{12}^0/2} \varphi^{2,1,3} \underline{B}^{2,13} \varphi^{3,2,1} e^{-\mu \bar{t}_{12}^0/2} \right).$$

*Proof.* — Let  $(F, G)$  be an ellipsitomic associator. We have already seen that the choice of the operad isomorphism  $F$  corresponds bijectively to the choice of an element  $(\mu, \varphi) \in \mathrm{Ass}(\mathbf{k})$ . From the presentation of  $\mathbf{PaB}_{\mathrm{ell}}^{\Gamma}$ , we know that  $G$  is uniquely determined by the images of  $A^{1,2} \in \mathrm{Hom}_{\mathbf{PaB}_{\mathrm{ell}}^{\Gamma}(\mathbf{k})(2)}(1_0 2_0, 1_{\alpha} 2_0)$  and  $B^{1,2} \in \mathrm{Hom}_{\mathbf{PaB}_{\mathrm{ell}}^{\Gamma}(\mathbf{k})(2)}(1_0 2_0, 1_{\beta} 2_0)$ . There are elements  $A, B \in \exp(\hat{\mathfrak{t}}_{1,2}^{\Gamma}(\mathbf{k}))$  such that

$$\begin{aligned} - G(A^{1,2}) &= A \cdot I_{\mathrm{ell}}^{1,2}, \\ - G(B^{1,2}) &= B \cdot J_{\mathrm{ell}}^{1,2}. \end{aligned}$$

These elements must satisfy relations (5.1), (5.2) and (5.3), that are images of (tN1), (tN2) and (tE). Conversely, if (5.1), (5.2) and (5.3) are satisfied, then  $G$  is well-defined.  $\square$

**Remark 5.10.** — It follows from the alternative presentation of  $\mathbf{PaB}_{\mathrm{ell}}^{\Gamma}$  (see Theorem A.3) that  $\mathbf{Ell}^{\Gamma}(\mathbf{k})$  is also in bijection with the set of  $(\mu, \varphi, A, B) \in \mathrm{Ass}(\mathbf{k}) \times \left( \exp(\hat{\mathfrak{t}}_{1,2}^{\Gamma}(\mathbf{k})) \right)^{\times 2}$  satisfying

$$(5.4) \quad \underline{A}^{12,3} = \varphi^{1,2,3} \underline{A}^{1,23} \varphi^{3,2,1} e^{-\mu \bar{t}_{12}^0/2} \varphi^{2,1,3} \underline{A}^{2,13} \varphi^{3,1,2} e^{-\mu \bar{t}_{12}^0/2}$$

$$(5.5) \quad \underline{B}^{12,3} = \varphi^{1,2,3} \underline{B}^{1,23} \varphi^{3,2,1} e^{-\mu \bar{t}_{12}^0/2} \varphi^{2,1,3} \underline{B}^{2,13} \varphi^{3,1,2} e^{-\mu \bar{t}_{12}^0/2}$$

$$(5.6) \quad \varphi^{1,2,3} e^{\mu \bar{t}_{23}^0} \varphi^{3,2,1} = \left( \underline{A}^{12,3} \varphi^{1,2,3} (\underline{A}^{1,23})^{-1} \varphi^{3,2,1}, (\underline{B}^{12,3})^{-1} \right)$$

As before, if we are given a surjective group morphism

$$\rho : \Gamma \twoheadrightarrow \Gamma' := \mathbb{Z}/M'\mathbb{Z} \times \mathbb{Z}/N'\mathbb{Z},$$

then, given  $a, b, c, d \in \mathbf{k}$  as in Remark 5.3, there is a bitorsor morphism

$$(\widehat{\mathbf{GT}}^\Gamma(\mathbf{k}), \mathbf{Ell}^\Gamma(\mathbf{k}), \mathbf{GRT}^\Gamma(\mathbf{k})) \longrightarrow (\widehat{\mathbf{GT}}^{\Gamma'}(\mathbf{k}), \mathbf{Ell}^{\Gamma'}(\mathbf{k}), \mathbf{GRT}^{\Gamma'}(\mathbf{k})).$$

In chapter 6 we prove that ellipsitomic associators (with complex coefficients) do exist.

**Remark 5.11.** — Drinfeld's argument in [17] (see also [4]) that shows how to deduce the existence of an associator over  $\mathbb{Q}$  from the existence of an associator over  $\mathbb{C}$  can be repeated *verbatim* for ellipsitomic associators. We leave the details for future work.

## CHAPTER 6

### The KZB ellipsitomic associator

In this chapter,  $\Gamma$  still denotes the abelian group  $\Gamma = \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$  where  $M, N \geq 1$  are two integers.

Recall from Theorem 5.9 that the set of ellipsitomic associators can be regarded, either as the set of  $\bar{\Gamma}$ -equivariant  $\overline{\mathbf{PaB}}(\mathbf{k})$ -module isomorphisms  $\overline{\mathbf{PaB}}_{ell}^{\Gamma}(\mathbf{k}) \rightarrow \mathbf{GPaCD}_{ell}^{\Gamma}(\mathbf{k})$  which are the identity on objects, or as quadruples  $(\lambda, \Phi, A_+, A_-)$ , where  $(\lambda, \Phi) \in \text{Ass}(\mathbf{k})$  and  $A_{\pm} \in \exp(\hat{\mathfrak{t}}_{1,2}^{\Gamma}(\mathbf{k}))$ , satisfying relations (5.1), (5.2), (5.3). The following result tells us that the set  $\text{Ell}^{\Gamma}(\mathbb{C})$  is not empty. We write  $\text{Ell}_{\text{KZB}}^{\Gamma} := \text{Ell}^{\Gamma}(\mathbb{C}) \times_{\text{Ass}(\mathbb{C})} \{2\pi i, \Phi_{\text{KZ}}\}$ .

**Theorem 6.1.** — *There is an analytic map*

$$\begin{aligned} \mathfrak{H} &\longrightarrow \text{Ell}_{\text{KZB}}^{\Gamma} \\ \tau &\longmapsto e^{\Gamma}(\tau) = (A^{\Gamma}(\tau), B^{\Gamma}(\tau)). \end{aligned}$$

*In particular, for each  $\tau \in \mathfrak{H}$ , where  $\mathfrak{H}$  is the upper half-plane, the element  $(2\pi i, \Phi_{\text{KZ}}, A^{\Gamma}(\tau), B^{\Gamma}(\tau))$  is an ellipsitomic  $\mathbb{C}$ -associator (i.e. it belongs to  $\text{Ell}^{\Gamma}(\mathbb{C})$ ).*

The rest of this chapter is devoted to the proof of the above theorem.

#### 6.1. The pair $e^{\Gamma}(\tau)$

We adopt the convention for monodromy actions of [13, Appendix A]. First of all, recall that  $\hat{\mathfrak{t}}_{1,2}^{\Gamma}$  is the Lie  $\mathbb{C}$ -algebra generated by  $x := \bar{x}_1$ ,  $y := \bar{y}_2$  and  $t^{\alpha} := \bar{t}_{12}^{\alpha}$ , for  $\alpha \in \Gamma$ , such that  $[x, y] = \sum_{\alpha \in \Gamma} t^{\alpha}$ . We define the *KZB ellipsitomic associator* as the couple

$$e^{\Gamma}(\tau) := (A^{\Gamma}(\tau), B^{\Gamma}(\tau)) \in \exp(\hat{\mathfrak{t}}_{1,2}^{\Gamma}) \times \exp(\hat{\mathfrak{t}}_{1,2}^{\Gamma})$$

consisting in the renormalized holonomies along the straight paths from 0 to  $1/M$  and from 0 to  $\tau/N$ , respectively, of the differential equation

$$(6.1) \quad J'(z) = \left( y + \sum_{\alpha \in \Gamma} \left( e^{-\frac{2\pi i v_{\alpha}}{N} \text{ad}(x)} \frac{\theta(z - \tilde{\alpha} + \text{ad}(x)|\tau)}{\theta(z - \tilde{\alpha}|\tau)\theta(\text{ad}(x)|\tau)} - \frac{1}{\text{ad}(x)} \right) (t^{\alpha}) \right) \cdot J(z),$$

with values in the group  $\exp(\hat{\mathfrak{t}}_{1,2}^{\Gamma}) \rtimes \Gamma$  (here,  $v_{\alpha}$  is any integer such that  $\alpha = (v_{\alpha}^-, v_{\alpha}^+) \in \Gamma$ ).

More precisely, equation (6.1) has a unique solution  $J(z)$  defined over  $\{\frac{s_1}{M} + \frac{s_2}{N}\tau \mid s_1, s_2 \in \mathbb{R}, s_1 \text{ or } s_2 \in (0, 1)\}$  and such that

$$J(z) \simeq (-z)^{t^0}$$

at  $z \rightarrow 0$ .

**Remark 6.2.** — We always consider a branch of log that is defined outside the half line  $\mathbb{R}_+\tau$ , and we always make sure that the domains of definition never contain this half-line. Above, we indeed have that every  $z$  in the domain of definition satisfies  $-z \notin \mathbb{R}_+\tau$ .

We define

$$\underline{A}^\Gamma(\tau) := J\left(z + \frac{1}{M}\right)^{-1}(\bar{1}, \bar{0})J(z) \in \exp(\hat{\mathfrak{t}}_{1,2}^\Gamma) \rtimes \Gamma.$$

Then the  $A$ -elliptitomic KZB associator  $A^\Gamma(\tau)$  is the  $\exp(\hat{\mathfrak{t}}_{1,2}^\Gamma)$ -component of  $\underline{A}^\Gamma(\tau)$ :

$$A^\Gamma(\tau) := \underline{A}^\Gamma(\tau)(-\bar{1}, \bar{0}) = J\left(z + \frac{1}{M}\right)^{-1}(\bar{1}, \bar{0}) \cdot J(z) \in \exp(\hat{\mathfrak{t}}_{1,2}^\Gamma).$$

In the same way, we define

$$\underline{B}^\Gamma(\tau) := J\left(z + \frac{\tau}{N}\right)^{-1}e^{-\frac{2\pi i x}{N}}(\bar{0}, \bar{1})J(z),$$

and the  $B$ -elliptitomic KZB associator  $B^\Gamma(\tau)$  is then its  $\exp(\hat{\mathfrak{t}}_{1,2}^\Gamma)$ -component:

$$B^\Gamma(\tau) := \underline{B}^\Gamma(\tau)(\bar{0}, -\bar{1}) = J\left(z + \frac{\tau}{N}\right)^{-1}e^{-\frac{2\pi i x}{N}}(\bar{0}, \bar{1}) \cdot J(z) \in \exp(\hat{\mathfrak{t}}_{1,2}^\Gamma).$$

## 6.2. The elliptitomic KZB system

Recall from [13] the elliptitomic KZB system, that is a several variables version of the differential equation from the previous subsection:

$$(6.2) \quad \begin{cases} \partial_{z_i} F(\mathbf{z}|\tau) = K_i(\mathbf{z}|\tau)F(\mathbf{z}|\tau) & (i = 1, \dots, n) \\ \partial_\tau F(\mathbf{z}|\tau) = \Delta(\mathbf{z}|\tau)F(\mathbf{z}|\tau) \end{cases}$$

Here  $F(\mathbf{z}|\tau)$  is a holomorphic function  $(\mathbb{C}^n \times \mathfrak{H}) - \text{Diag}_{n,\Gamma} \supset U \longrightarrow \mathbf{G}_n^\Gamma$ ,

$$\text{Diag}_{n,\Gamma} = \left\{ (\mathbf{z}|\tau) \in \mathbb{C}^n \times \mathfrak{H} \mid z_i - z_j \in \frac{1}{M}\mathbb{Z} + \frac{\tau}{N}\mathbb{Z} \text{ for } i \neq j \right\},$$

and the  $\mathbb{C}$ -group  $\mathbf{G}_n^\Gamma$ ,  $K_i(\mathbf{z}|\tau) \in \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \subset \mathbf{G}_n^\Gamma$ , and  $\Delta(\mathbf{z}|\tau) \in \mathbf{G}_n^\Gamma$ , are defined in [13].

If one denotes  $z_{ij} = z_i - z_j$ , then

$$\begin{aligned} K_i(\mathbf{z}|\tau) &= -y_i + \sum_{j:j \neq i \alpha \in \Gamma} \sum_{\alpha \in \Gamma} \left( e^{-2\pi i \alpha \text{ad}(x_i)} \frac{\theta(z_{ij} - \tilde{\alpha} + \text{ad}(x_i)|\tau)}{\theta(z_{ij} - \tilde{\alpha}|\tau)\theta(\text{ad}(x_i)|\tau)} - \frac{1}{\text{ad}(x_i)} \right) (t_{ij}^\alpha) \\ &= \sum_{j:j \neq i \alpha \in \Gamma} \sum_{\alpha \in \Gamma} \left( \frac{1}{\text{ad}(x_i)} + \frac{t_{ij}^\alpha}{z_{ij} - \tilde{\alpha}} - \frac{1}{\text{ad}(x_i)} \right) (t_{ij}^\alpha) + O(1) \\ &= \sum_{j:j \neq i \alpha \in \Gamma} \sum_{\alpha \in \Gamma} \frac{t_{ij}^\alpha}{z_{ij} - \tilde{\alpha}} + O(1) = \sum_{j:j \neq i \alpha \in \Gamma} \sum_{\alpha \in \Gamma} \frac{t_{ij}^\alpha}{z_{ij} - \frac{\alpha_0}{M}} + O(1), \end{aligned}$$

where  $O(1)$  stands for a holomorphic function on  $\mathbb{C}^n \times \mathfrak{H}$ . Then it follows directly from the definition of  $\Delta(\mathbf{z}|\tau)$  in [13, §3.3] that, for  $|z_{ij}| \ll 1$ ,

$$\Delta(\mathbf{z}|\tau) = -\frac{1}{2\pi i} \left( \Delta_0 + \frac{1}{2} \sum_{s \geq 0} \sum_{\gamma \in \Gamma} A_{s,\gamma}(\tau) \left( \delta_{s,\gamma} - 2 \sum_{i < j} \text{ad}(x_i)^s(t_{ij}^{-\gamma}) \right) \right) + o(1),$$

where  $o(1)$  denotes a function of the form  $\sum_{ij} z_{ij} f_{ij}(\mathbf{z}|\tau)$ , with  $f_{ij}$ 's being holomorphic on  $\mathbb{C}^n \times \mathfrak{H}$ .

**Remark 6.3.** — In chapter 7 we study the modularity properties of the coefficients  $A_{s,\gamma}(\tau)$ .

We now determine a particular solution  $F_{\tau_0,n,\Gamma}$  of the elliptic KZB system (6.2), associated with every  $\tau_0 \in \mathfrak{H}$ .

Let  $D_{n,\Gamma} \subset (\mathbb{C}^n \times \mathfrak{H}) - \text{Diag}_{n,\Gamma}$  be defined as

$$\left\{ (\mathbf{z}, \tau) \in \mathbb{C}^n \times \mathfrak{H} \mid z_i = a_i + b_i \tau, a_i, b_i \in \mathbb{R}, a_1 < a_2 < \dots < a_n < a_1 + \frac{1}{M}, b_n < \dots < b_1 < b_n + \frac{1}{N} \right\},$$

which is simply connected. A solution of the elliptic KZB system on this domain is then unique, up to right multiplication by a constant element in  $\mathbf{G}_n^\Gamma$ . Then, by applying [12, Appendix A, Proposition 85] with  $u_{n-1} = z_{n1}$ ,  $u_{n-2} = z_{(n-1)1}/z_{n1}, \dots, u_1 = z_{21}/z_{31}$ , we obtain a unique solution  $F_{\tau_0,n,\Gamma}$  with the expansion

$$F_{\tau_0,n,\Gamma}(\mathbf{z}|\tau_0) \simeq z_{21}^{t_{12}^0} z_{31}^{t_{13}^0 + t_{23}^0} \dots z_{n1}^{t_{1n}^0 + \dots + t_{n-1,n}^0}$$

in the region  $|z_{21}| \ll |z_{31}| \ll \dots \ll |z_{n1}| \ll 1$ ,  $(\mathbf{z}, \tau_0) \in D_{n,\Gamma}$ . The sign  $\simeq$  means here that any of the ratios of both sides is of the form

$$1 + \sum_{k > 0} \sum_{i, a_1, \dots, a_{n-1}} r_k^{i, a_1, \dots, a_{n-1}}(u_1, \dots, u_{n-1}|\tau_0),$$

where the second sum is finite with  $a_i \geq 0$ ,  $i \in \{1, \dots, n-1\}$ ,  $r_k^{i, a_1, \dots, a_{n-1}}(u_1, \dots, u_{n-1}|\tau_0)$  has degree  $k$ , and is  $O(u_i (\log u_1)^{a_1} \dots (\log u_{n-1})^{a_{n-1}})$ .

In the remainder of this chapter, we keep  $\tau$  fixed and consider  $F_{\tau,n}(\mathbf{z}) := F_{\tau,n,\Gamma}(\mathbf{z}|\tau)$ , which is a solution of the first line of the elliptic KZB system (6.2), defined on

$$D_{\tau,n,\Gamma} := \{ \mathbf{z} \in \mathbb{C}^n \mid (\mathbf{z}, \tau) \in D_{n,\Gamma} \},$$

and taking its values in  $\exp(\mathfrak{t}_{1,n}^\Gamma) \subset \mathbf{G}_n^\Gamma$ .

### 6.3. Generators for the group $B_{1,n}^\Gamma$

Let us define, for  $(\mathbf{z}_0, \tau) \in D_{n,\Gamma}$ , the group  $B_{1,n}^\Gamma := \pi_1(\text{Conf}(E_{\tau,\Gamma}, n, \Gamma)/\mathfrak{S}_n, [\mathbf{z}_0])$ , and recall that  $B_{1,n} = \pi_1(\text{Conf}(E_{\tau,\Gamma}, n)/\mathfrak{S}_n, [\mathbf{z}_0])$ . Now, since the canonical surjective map

$$\text{Conf}(E_{\tau,\Gamma}, n, \Gamma)/\mathfrak{S}_n \twoheadrightarrow \text{Conf}(E_{\tau,\Gamma}, n)/\mathfrak{S}_n$$

defines a  $\Gamma$ -covering, then  $B_{1,n}^\Gamma = \ker(\rho)$ , where  $\rho: B_{1,n} \rightarrow \Gamma$  sends  $\sigma_i$  to  $\mathbf{0} = (\bar{0}, \bar{0})$ ,  $X_i$  to  $(\bar{1}, \bar{0})$  and  $Y_i$  to  $(\bar{0}, \bar{1})$ . We let  $A_i$  (resp.  $B_i$ ) be the class of the path given by  $[0, 1] \ni t \mapsto \mathbf{z}_0 - \frac{t}{M} \sum_{j=i}^n \delta_j$

(resp.  $[0, 1] \ni t \mapsto \mathbf{z}_0 - \frac{t}{N} \tau \sum_{j=i}^n \delta_j$ ), so that  $X_i = A_i^{-1} A_{i+1}$  (resp.  $Y_i = B_i^{-1} B_{i+1}$ ). It then follows from the geometric description of  $B_{1,n}^\Gamma$  that  $A_i^M, B_i^N$  ( $i = 1, \dots, n$ ) and

$$\sigma_i^{(\bar{p}, \bar{q})} := X_i^p Y_i^q \sigma_i Y_{i+1}^{-q} X_{i+1}^{-p} \quad (1 \leq p \leq M, 1 \leq q \leq N)$$

are generators of  $B_{1,n}^\Gamma$ . Similarly,  $A_i^M, B_i^N$  ( $i = 1, \dots, n$ ) and

$$P_{ij}^{(\bar{p}, \bar{q})} := X_i^p Y_i^q P_{ij} Y_i^{-q} X_i^{-p} \quad (i < j, 1 \leq p \leq M, 1 \leq q \leq N)$$

generate  $PB_{1,n}^\Gamma$ .

We denote with the same symbols  $A_i^M, B_i^N, \sigma_i^\alpha$  and  $P_{ij}^\alpha$  ( $\alpha \in \Gamma, i = 1, \dots, n$ ) for the projections of these elements to  $\bar{B}_{1,n}^\Gamma := \pi_1(C(E_{\tau, \Gamma}, n, \Gamma)/\mathfrak{S}_n, [\mathbf{z}_0])$ .

#### 6.4. The monodromy morphism $\mu_n : B_{1,n} \rightarrow \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$

Recall from [13, §3.1] the moduli space

$$\mathcal{M}_{1,n}^\Gamma := (\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2^\Gamma \setminus ((\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{n, \Gamma})$$

of  $\Gamma$ -structured elliptic curves with  $n$  ordered marked points, where

$$\mathrm{SL}_2^\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv 1 \pmod{M}, d \equiv 1 \pmod{N}, b \equiv 0 \pmod{N} \text{ and } c \equiv 0 \pmod{M} \right\}.$$

The elliptitomic KZB system (6.2) can be used to define a flat  $\mathbf{G}_n^\Gamma$ -bundle  $(\mathcal{P}_{n, \Gamma}, \nabla_{n, \Gamma})$  on  $\mathcal{M}_{1,n}^\Gamma$  (see [13, Theorem 3.9 & Theorem 3.12]), that descends to a flat  $\mathbf{G}_n^\Gamma \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$ -bundle  $(\mathcal{P}_{(\Gamma), [n]}, \nabla_{(\Gamma), [n]})$  on  $(\Gamma^n \rtimes \mathfrak{S}_n) \setminus \mathcal{M}_{1,n}^\Gamma$  (see [13, §3.5]). For every  $\tau \in \mathfrak{H}$ ,  $(\mathcal{P}_{n, \Gamma}, \nabla_{n, \Gamma})$  restricts to a flat  $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)$ -bundle  $(\mathcal{P}_{\tau, n, \Gamma}, \nabla_{\tau, n, \Gamma})$  on  $\mathrm{Conf}(E_{\tau, \Gamma}, n, \Gamma)$  (see [13, Theorem 1.11]), that descends to a flat  $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$ -bundle  $(\mathcal{P}_{(\tau, \Gamma), [n]}, \nabla_{(\tau, \Gamma), [n]})$  on  $\Gamma^n \rtimes \mathfrak{S}_n \setminus \mathrm{Conf}(E_{\tau, \Gamma}, n, \Gamma) = \mathfrak{S}_n \setminus \mathrm{Conf}(E_{\tau, \Gamma}, n)$ .

This flat bundle determines a monodromy morphism

$$\mu_n^{\mathbf{z}_0} = \mu_{(\tau, \Gamma), [n]}^{\mathbf{z}_0} : B_{1,n} \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n).$$

whose restriction to  $PB_{1,n}^\Gamma$  takes values in  $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)$ . In other words, there is a morphism of short exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & PB_{1,n}^\Gamma & \longrightarrow & B_{1,n} & \longrightarrow & \Gamma^n \rtimes \mathfrak{S}_n \longrightarrow 1 \\ & & \downarrow & & \downarrow \mu_n^{\mathbf{z}_0} & & \parallel \\ 1 & \longrightarrow & \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) & \longrightarrow & \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n) & \longrightarrow & \Gamma^n \rtimes \mathfrak{S}_n \longrightarrow 1 \end{array}$$

where the first vertical morphism is the monodromy morphism of  $\nabla_{\tau, n, \Gamma}$ . It is important to keep in mind that the monodromy morphism depends on the base point  $[\mathbf{z}_0]$ , e.g. for  $\mathbf{z}_0 \in D_{\tau, n, \Gamma}$ .

Note that every local solution  $F$  of (the first line of) the elliptitomic KZB system (6.2) around  $\mathbf{z}_0 \in D_{\tau, n, \Gamma}$  determines a local  $\nabla_{(\tau, \Gamma), [n]}$ -flat section of  $\mathcal{P}_{(\tau, \Gamma), [n]}$ , and thus can be used to compute the monodromy in a way that we explain now (we refer to [13, Appendix A] for more details on our conventions).

For every loop  $\gamma$  based at  $[\mathbf{z}_0]$  in  $\text{Conf}(E_{\tau,\Gamma},n)/\mathfrak{S}_n$ , we consider its unique lift  $\tilde{\gamma}$  starting at  $\mathbf{z}_0 \in D_{\tau,n,\Gamma}$ , and choose a simply connected open neighbourhood  $U$  of  $\tilde{\gamma}$  that contains  $D_{\tau,n,\Gamma}$ . Then the solution  $F$  extends uniquely to  $U$ , and we define

$$\mu_n^{\mathbf{z}_0}([\gamma]) := F(\mathbf{z}_0)F(h_\gamma \cdot \mathbf{z}_0)^{-1} c_{h_\gamma},$$

where  $h_\gamma \in \Gamma^n \rtimes \mathfrak{S}_n$  is such that  $\tilde{\gamma}(1) = h_\gamma \tilde{\gamma}(0)$ , and  $c$  is the non-abelian 1-cocycle from [13] defining the underlying principal bundle of the flat connection.

Recall that for any other solution  $G$  defined on  $U$ , there exists  $g \in \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$  such that  $G(\mathbf{z}) = F(\mathbf{z})g$  for every  $\mathbf{z} \in U$ . Hence  $\mu_n^{\mathbf{z}_0}([\gamma]) := G(\mathbf{z}_0)G(h_\gamma \cdot \mathbf{z}_0)^{-1} c_{h_\gamma}$ , and the monodromy does not depend on the choice of local solution.

**Example 6.4.** — Let us consider the domains

$$H_{\tau,n,\Gamma} := \left\{ \mathbf{z} \in \mathbb{C}^n \mid z_i = a_i + b_i \tau, a_i, b_i \in \mathbb{R}, b_n < \dots < b_1 < b_n + \frac{1}{N} \right\}$$

and

$$V_{\tau,n,\Gamma} := \left\{ \mathbf{z} \in \mathbb{C}^n \mid z_i = a_i + b_i \tau, a_i, b_i \in \mathbb{R}, a_1 < a_2 < \dots < a_n < a_1 + \frac{1}{M} \right\}.$$

Both of these domains are simply connected, and contain  $D_{\tau,n,\Gamma}$ . We denote  $F_H(\mathbf{z})$ , resp.  $F_V(\mathbf{z})$ , the prolongation to  $H_{\tau,n,\Gamma}$ , resp. to  $V_{\tau,n,\Gamma}$ , of a given local solution  $F(\mathbf{z})$  defined on  $D_{\tau,n,\Gamma}$ . We then consider

$$\underline{A}_i^{\mathbf{z}_0} := \mu_n^{\mathbf{z}_0}(A_i) = F_H(\mathbf{z}_0)F_H\left(\mathbf{z}_0 - \sum_{j=i}^n \frac{\delta_j}{M}\right)^{-1} (-\bar{1}, \bar{0})_{i,\dots,n} \in \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \Gamma^n$$

and

$$\underline{B}_i^{\mathbf{z}_0} := \mu_n^{\mathbf{z}_0}(B_i) = F_V(\mathbf{z}_0)F_V\left(\mathbf{z}_0 - \tau \sum_{j=i}^n \frac{\delta_j}{N}\right)^{-1} e^{\frac{2\pi i}{N}(x_i + \dots + x_n)} (\bar{0}, -\bar{1})_{i,\dots,n} \in \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \Gamma^n.$$

We also consider the projections of these elements on the first factor  $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)$ :

$$A_i^{\mathbf{z}_0} := \underline{A}_i^{\mathbf{z}_0}(\bar{1}, \bar{0})_{i,\dots,n} = F_H(\mathbf{z}_0)F_H\left(\mathbf{z}_0 - \sum_{j=i}^n \frac{\delta_j}{M}\right)^{-1} \in \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)$$

and

$$B_i^{\mathbf{z}_0} := \underline{B}_i^{\mathbf{z}_0}(\bar{0}, \bar{1})_{i,\dots,n} = F_V(\mathbf{z}_0)F_V\left(\mathbf{z}_0 - \tau \sum_{j=i}^n \frac{\delta_j}{N}\right)^{-1} e^{\frac{2\pi i}{N}(x_i + \dots + x_n)} \in \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma).$$

We finally introduce the simply connected domain  $S_{\tau,n,\Gamma}$  consisting of  $\mathbf{z} \in \mathbb{C}^n$ , with  $z_i = a_i + b_i \tau$  ( $a_i, b_i \in \mathbb{R}$ ) satisfying the following conditions:

- for every  $i < j$ ,  $|a_i - a_j| < \frac{1}{M}$  and  $|b_i - b_j| < \frac{1}{N}$ ;
- for every  $i < j$ ,  $z_{ji} \notin \mathbb{R}_+ \tau$ .

Note that  $\mathfrak{S}_n(D_{\tau,n,\Gamma}) \subset S_{\tau,n,\Gamma}$ . We denote  $F_S(\mathbf{z})$  the prolongation to  $S_{\tau,n,\Gamma}$  of a given local solution  $F(\mathbf{z})$  defined on  $D_{\tau,n,\Gamma}$ , and then consider for every  $\sigma \in \mathfrak{S}_n$ ,

$$\underline{\sigma}^{\mathbf{z}_0} := F_S(\mathbf{z}_0)F_S(\sigma \cdot \mathbf{z}_0)^{-1} \sigma \in \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \mathfrak{S}_n.$$



Observe that the (unique) homotopy class of a path going from  $\mathbf{z}_0$  to  $\sigma \cdot \mathbf{z}_0$  represents the unique braid with underlying permutation  $\sigma$  such that for every  $i < j$ , the  $i$ -th strand passes under the  $j$ -th strand whenever they cross (this is just a translation, in terms of braids, of the condition that  $z_{ji} \notin \mathbb{R}_+\tau$ ). In other words, denoting this braid  $\tilde{\sigma}$ ,  $\underline{\sigma}^F = \mu_n^F(\tilde{\sigma})$ . As before, we also consider the projection of  $\underline{\sigma}^{\mathbf{z}_0} \in \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \mathfrak{S}_n$  on the first factor:

$$\sigma^{\mathbf{z}_0} := \underline{\sigma}^{\mathbf{z}_0} \sigma \in \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma).$$

Even though  $\mu_n^{\mathbf{z}_0}$  does not depend on the choice of local solution  $F$ , it is conjugated to a morphism that does depend on  $F$ . Indeed, one can define

$$\mu_n^F([\gamma]) := F(\mathbf{z}_0)^{-1} \mu_n^{\mathbf{z}_0}([\gamma]) F(\mathbf{z}_0) = F(h_\gamma \cdot \mathbf{z}_0)^{-1} c_{h_\gamma} F(\mathbf{z}_0).$$

The resulting monodromy morphism  $\mu_n^F$  does not depend on  $\mathbf{z}_0$  (because it is a ratio of two solutions of the elliptic KZB system), but does depend on  $F$ . Whenever  $F(\mathbf{z}_0) = 1$ , we obviously have  $\mu_n^F = \mu_n^{\mathbf{z}_0}$ .

In what follows, we consider the monodromy morphism  $\mu_n := \mu_n^F$  associated with the particular solution  $F = F_{\tau,n}$  from Section 6.2.

### 6.5. Formulæ for $\mu_n : B_{1,n} \rightarrow \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$

**Lemma 6.5.** — *If  $\sigma = (12) \in \mathfrak{S}_n$  then  $\tilde{\sigma} = \sigma_1$ , and  $\mu_n(\tilde{\sigma}) = e^{\pi i t_{12}^0} \sigma$ .*

*Proof.* — Only the last claim requires a proof. Let us consider  $\mathbf{z}$  such that  $a_1 < a_2$  (e.g.  $\mathbf{z} \in D_{\tau,n,\Gamma}$ ), guaranteeing that  $\sigma \mathbf{z} = (z_2, z_1, z_3, \dots, z_n) \in S_{\tau,n,\Gamma}$ . Recall the expansion

$$F(\mathbf{z}) \simeq z_{21}^{t_{12}^0} z_{31}^{t_{13}^0 + t_{23}^0} \dots z_{n1}^{t_{1n}^0 + \dots + t_{n-1,n}^0}$$

in the region  $|z_{21}| \ll |z_{31}| \ll \dots \ll |z_{n1}| \ll 1$ . Hence  $\sigma \cdot F(\mathbf{z})$  has a similar expansion, and

$$F(\sigma \mathbf{z}) \simeq z_{12}^{t_{12}^0} z_{32}^{t_{13}^0 + t_{23}^0} \dots z_{n2}^{t_{1n}^0 + \dots + t_{n-1,n}^0}$$

in the same region. With our choice of branch of  $\log$  (see Remark 6.2, and the definition of the domain  $S_{\tau,n,\Gamma}$ ), one gets that  $\log(z_{12}) = \log(z_{21}) - \pi i$ . Therefore

$$\mu_n(\tilde{\sigma}) = F(\sigma \mathbf{z})^{-1} \sigma \cdot F(\mathbf{z}) \simeq e^{\pi i t_{12}^0} \sigma.$$

The last equivalence is an equality, as  $\mu_n(\tilde{\sigma})$  is constant. □

Let  $\Phi = \Phi_0^{1,2,3}$  be the image in  $\exp(\hat{\mathfrak{t}}_{1,3}^\Gamma)$  of the KZ associator  $\Phi_{\text{KZ}}^{1,2,3}$  from Example 2.10 along the map  $\exp(\hat{\mathfrak{t}}_3) \rightarrow \exp(\hat{\mathfrak{t}}_{1,3}^\Gamma)$  given by  $t_{ij} \mapsto t_{ij}^0$ . Define

$$\Phi_i := \Phi_0^{1 \dots i-1, i, i+1 \dots n} \dots \Phi_0^{1 \dots n-2, n-1, n} \in \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma).$$

**Proposition 6.6.** — *For every  $n \geq 3$ , and every  $i = 2, \dots, n$ ,*

$$\mu_n(A_i) = \Phi_i \mu_2(A_2)^{1 \dots i-1, i \dots n} \Phi_i^{-1} \quad \text{and} \quad \mu_n(B_i) = \Phi_i \mu_2(B_2)^{1 \dots i-1, i \dots n} \Phi_i^{-1}.$$

*Proof.* — We first compute the monodromy  $\mu_{i,n} := \mu_n^G$  associated with another solution  $G$  of the (first line of the) elliptic KZB system: the one (for a fixed  $\tau$ ) having the expansion

$$G(\mathbf{z}) \simeq z_{21}^{\mathbf{0}} \cdots z_{i-1,1}^{\mathbf{0}} z_{i-1,1}^{t_{12}^{\mathbf{0}} + \cdots + t_{1,i-1}^{\mathbf{0}}} z_{n,i}^{t_{i,n}^{\mathbf{0}} + \cdots + t_{n-1,n}^{\mathbf{0}}} \cdots z_{n,n-1}^{\mathbf{0}}$$

in the region where  $|z_{21}| \ll \cdots \ll |z_{i-1,1}| \ll 1$  and  $|z_{n,n-1}| \ll \cdots \ll |z_{n,i}| \ll 1$ .

We claim that

$$\mu_{i,n}(A_i) = \mu_2(A_2)^{1 \dots i-1, i \dots n} \quad \text{and} \quad \mu_{i,n}(B_i) = \mu_2(B_2)^{1 \dots i-1, i \dots n},$$

and only give the proof for  $A_i$ , as the proof for  $B_i$  is the exact same. The element  $A_i$  can be represented by a path inside the region  $|z_{21}| \ll \cdots \ll |z_{i-1,1}| \ll 1$  and  $|z_{n,n-1}| \ll \cdots \ll |z_{n,i}| \ll 1$ , that keeps the coordinates  $z_1, \dots, z_{i-1}$ , as well as the differences between the remaining coordinates, fixed. Hence, computing  $\mu_n^G(A_i)$  amounts to compute the monodromy along the same path for the differential equation

$$\partial_{z_n} G(\mathbf{z}) = \sum_{j=i}^n K_j(\mathbf{z}|\tau) G(\mathbf{z}),$$

where  $\mathbf{z} = (z_1, \dots, z_{i-1}, z_n + s_i, z_n + s_{i+1}, \dots, z_n + s_{n-1}, z_n)$ . Now observe that the difference  $\sum_{j=i}^n K_j(\mathbf{z}|\tau) - K_2(z_1, z_n|\tau)^{1 \dots i-1, i \dots n}$  (where  $K_2$  is from the  $n = 2$  points system) tends to 0 whenever  $z_j \rightarrow z_n$  for  $j \geq i$  and  $z_\ell \rightarrow z_1$  for  $\ell < i$ .<sup>(1)</sup> Hence  $\mu_{i,n}(A_i) = \mu_2(A_2)^{1 \dots i-1, i \dots n}$ .

Finally, the two monodromy representations  $\mu_n = \mu_n^F$  and  $\mu_{i,n} = \mu_n^G$  are conjugated. Indeed,

$$\mu_n^F([\gamma]) = \Phi_{F,G}(h_\gamma \cdot \mathbf{z}) \mu_n^G([\gamma]) \Phi_{F,G}(\mathbf{z})^{-1},$$

with  $\Phi_{F,G}(\mathbf{z}) := F(\mathbf{z})^{-1} G(\mathbf{z})$  being constant as it is a ratio of two solutions of the elliptic KZB system. To conclude, we prove that  $\Phi_{F,G} = \Phi_i$ . For this we consider the rational universal KZ system from [17, (2.2)] (with  $\hbar = 1$ , and  $t^{ij} = t_{ij}^{\mathbf{0}}$ ), and denote by  $\tilde{F}$  (resp.  $\tilde{G}$ ) the solution of this KZ system that have the same expansion as  $F$  (resp.  $G$ ). In the whole region where  $|z_{ij}| \ll 1$  for every  $i \neq j$ , the elliptic KZB system and the rational KZ system only differ by a holomorphic part, therefore  $F \simeq \tilde{F}$  and  $G \simeq \tilde{G}$  in this region. Therefore, as they are constant,  $\Phi_{F,G} = \Phi_{\tilde{F},\tilde{G}}$ . Finally, it is a standard fact that  $\Phi_{\tilde{F},\tilde{G}} = \Phi_i$  (see again [17]).  $\square$

<sup>(1)</sup>More precisely,

$$K_j(\mathbf{z}|\tau) = -y_j + \sum_{\ell \neq j} \sum_{\alpha \in \Gamma} k_\alpha(\text{ad}(x_j), z_{j\ell}|\tau) (t_{j\ell}^\alpha),$$

where  $k_\alpha(u, v|\tau)$  are formal power series in  $u$  with coefficient being meromorphic functions in  $v$ , and satisfying the identity

$$k_\alpha(-u, -v|\tau) + k_{-\alpha}(u, v|\tau) = 0.$$

We refer to [13] for more details (see also the next chapter). In particular

$$k_\alpha(\text{ad}(x_\ell), z_{\ell j}|\tau) (t_{\ell j}^\alpha) + k_{-\alpha}(\text{ad}(x_j), z_{j\ell}|\tau) (t_{j\ell}^\alpha) = 0,$$

and thus

$$\sum_{j=i}^n K_j(\mathbf{z}|\tau) = - \sum_{j=i}^n y_j + \sum_{j=i}^n \sum_{\ell=1}^{i-1} \sum_{\alpha \in \Gamma} k_\alpha(\text{ad}(x_j), z_{j\ell}|\tau) (t_{j\ell}^\alpha).$$

On the other hand,

$$K_2(z_1, z_n|\tau) = - \sum_{j=i}^n y_j + \sum_{\ell=1}^{i-1} \sum_{j=i}^n \sum_{\alpha \in \Gamma} k_\alpha(\text{ad}(x_j), z_{n1}|\tau) (t_{j\ell}^\alpha).$$

Therefore their difference indeed tends to 0 whenever  $z_j \rightarrow z_n$  for  $j \geq i$  and  $z_\ell \rightarrow z_1$  for  $\ell < i$ .

Using similar techniques, one can actually prove that the restriction of  $\mu_n$  on  $B_n \subset \overline{B}_{1,n}$  coincides with the monodromy morphism for the rational KZ system from [17, (2.2)] associated with the solution  $\tilde{F}$  having the same expansion as  $F$ . In particular,  $\mu_3(\sigma_2) = \Phi e^{\pi i \tilde{t}_{23}^0} (23) \Phi^{-1}$ .

## 6.6. Algebraic relations for the elliptic KZB associator

We now finish the proof of Theorem 6.1.

**Remark 6.7.** — The results of Sections 6.2, 6.4 and 6.5 remain true in the reduced case, and we will make use of the same notation as in the previous sections.

Let us set  $\underline{A} := \mu_2(A_2)$  and  $\underline{B} := \mu_2(B_2)$ , both viewed in  $\exp(\hat{t}_{1,2}^\Gamma) \rtimes \Gamma^2/\Gamma$ . In other words,

$$\underline{A} = F_{\tau,2}(z, -\frac{1}{M})^{-1}(-\bar{1}, \bar{0})_2 F_{\tau,2}(z, 0) = F_{\tau,2}(z + \frac{1}{M}, 0)^{-1}(\bar{1}, \bar{0})_1 F_{\tau,2}(z, 0)$$

and

$$\underline{B} = F_{\tau,2}(z, \frac{\tau}{N})^{-1}(\bar{0}, -\bar{1})_2 F_{\tau,2}(z, 0) = F_{\tau,2}(z + \frac{\tau}{N}, 0)^{-1}(\bar{0}, \bar{1})_1 F_{\tau,2}(z, 0).$$

One can easily check that the pair  $(\underline{A}, \underline{B})$  coincides with the pair  $(\underline{A}^\Gamma(\tau), \underline{B}^\Gamma(\tau))$  from Section 6.1. Indeed, if  $F_{\tau,2}(z_1, z_2)$  is the solution of the elliptic KZB system defined on  $D_{\tau,2,\Gamma}$  with expansion  $F_{\tau,2}(z_1, z_2) \simeq z_{21}^{\tilde{t}_{12}^0}$  whenever  $|z_{21}| \ll 1$ , then  $J(z) = F_{\tau,2}(z, 0)$  is the solution of the differential equation (6.1) with expansion  $J(z) \simeq (-z)^{\tilde{t}^0}$  whenever  $z \rightarrow 0$  from Section 6.1.

The identity  $A_3^{-1}A_2 = \sigma_1 A_2^{-1} \sigma_1$  obviously holding in  $\overline{B}_{1,3}$ , is equivalent to the identity  $A_3 = A_2 \sigma_1^{-1} A_2 \sigma_1^{-1}$ . Applying the monodromy morphism  $\mu_3$  therefore yields

$$(6.3) \quad \underline{A}^{12,3} = \Phi^{1,2,3} \underline{A}^{1,23} (\Phi^{1,2,3})^{-1} e^{-\pi i \tilde{t}_{12}^0} \Phi^{2,1,3} \underline{A}^{2,13} (\Phi^{2,1,3})^{-1} e^{-\pi i \tilde{t}_{12}^0},$$

that is (5.4). Similarly, the identity  $B_3 = B_2 \sigma_1^{-1} B_2 \sigma_1^{-1}$  yields

$$(6.4) \quad \underline{B}^{12,3} = \Phi^{1,2,3} \underline{B}^{1,23} (\Phi^{1,2,3})^{-1} e^{-\pi i \tilde{t}_{12}^0} \Phi^{2,1,3} \underline{B}^{2,13} (\Phi^{2,1,3})^{-1} e^{-\pi i \tilde{t}_{12}^0},$$

that is (5.5).

In  $\overline{B}_{1,3}$ , one also has  $(X_2, Y_3) = P_{23}$ . Recalling that  $X_2 = A_3 A_2^{-1}$  and  $Y_3 = B_3^{-1}$ , one gets  $P_{23} = (A_3 A_2^{-1}, B_3^{-1})$  which, after applying the monodromy morphism  $\mu_3$ , yields

$$(6.5) \quad \Phi e^{2\pi i \tilde{t}_{23}^0} \Phi^{-1} = \underline{A}^{12,3} \Phi (\underline{A}^{1,23})^{-1} \Phi^{-1} (\underline{B}^{12,3})^{-1} \Phi \underline{A}^{1,23} \Phi^{-1} (\underline{A}^{12,3})^{-1} \underline{B}^{12,3},$$

which is (5.6)

This proves that the pair  $(\underline{A}^\Gamma(\tau), \underline{B}^\Gamma(\tau)) = (\underline{A}, \underline{B})$  satisfies (5.4), (5.5) and (5.6). Hence, according to Remark 5.10 it satisfies (5.1), (5.2) and (5.3), and thus  $e^\Gamma(\tau) = (A^\Gamma(\tau), B^\Gamma(\tau))$  defines an element in  $\text{Ell}_{\text{KZB}}^\Gamma$ .

This concludes the proof of Theorem 6.1.  $\square$

**Remark 6.8.** — If  $\Gamma$  is trivial, we retrieve relations (22), (23), (25) and (26) from [12], up to some changes of convention (for the monodromy action, and for the open subset of “base configurations” of marked points).

## CHAPTER 7

### Number theoretic aspects: Eisenstein series

In the previous chapter we studied (the first line of) the elliptic KZB system (6.2) of differential equations and deduced from it an element in the set of elliptic associators over  $\mathbb{C}$ . One of the main ingredients defining this differential system is given by

$$(7.1) \quad k_\gamma(x, z|\tau) := e^{-\frac{2\pi i v}{N}x} \frac{\theta(z - \tilde{\gamma} + x|\tau)}{\theta(z - \tilde{\gamma}|\tau)\theta(x|\tau)} - \frac{1}{x},$$

where  $\tau \in \mathfrak{h}$ ,  $\gamma = (\bar{u}, \bar{v}) \in \Gamma := \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$  and  $\tilde{\gamma} = \frac{u}{M} + \frac{v}{N}\tau \in \Lambda_{\tau, \Gamma} := \frac{1}{M}\mathbb{Z} + \frac{\tau}{N}\mathbb{Z}$  is any lift of  $\gamma$ . Here we implicitly used the canonical identification  $\Gamma \simeq \Lambda_{\tau, \Gamma}/\Lambda_\tau$ , where  $\Lambda_\tau := \mathbb{Z} + \tau\mathbb{Z}$ .

Denote by  $g_\gamma(x, z|\tau) := \partial_x k_\gamma(x, z|\tau)$  its partial derivative with respect to  $x$ . In this chapter we take a closer look at the functions  $A_{s, \gamma}(\tau)$ , defined in [13, Subsection 3.3] as the Taylor coefficients of  $g_{-\gamma}(x, 0|\tau)$ :

$$g_{-\gamma}(x, 0|\tau) = \sum_{s \geq 0} A_{s, \gamma}(\tau) x^s.$$

After a brief account on Eisenstein series for congruence subgroups, we express  $A_{s, \gamma}(\tau)$  in terms of these Eisenstein series, giving evidence that they should be quasi-modular forms for the group  $\mathrm{SL}_2^\Gamma$ .

We end the chapter with some perspectives about elliptic Grothendieck–Teichmüller theory and twisted elliptic multiple zeta values.

#### 7.1. Eisenstein series for $\mathrm{SL}_2^\Gamma$

We refer to [16, 43] for generalities about modular forms and Eisenstein series.

Recall the *Eisenstein series*  $G_s$ , defined for all integers  $s \geq 2$  by

$$G_s(\tau) := \sum_{n=-\infty}^{+\infty} \left( \sum_{\substack{m=-\infty \\ m \neq 0 \text{ if } n=0}}^{+\infty} \frac{1}{(m+n\tau)^s} \right).$$

The Eisenstein series  $G_s$  are modular forms for  $\mathrm{SL}_2(\mathbb{Z})$  of weight  $s$  for  $s \geq 3$ , and  $G_2$  is quasimodular (in the sense of [31]). One easily sees that  $G_s(\tau) = 0$  whenever  $s$  is odd, and that the value of  $G_s$  at the cusp  $i\infty$  for an even  $s = 2n$  is  $2\zeta(s) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{(2n)!}$ , where  $B_s$

are Bernoulli numbers and  $\zeta$  is the Riemann zeta function. Hence, for every integer  $s \geq 2$ , one defines the *normalized Eisenstein series*  $E_s(\tau) := \frac{G_s(\tau)}{2\zeta(s)}$ .

Let us now introduce the functions  $G_s(z|\tau)$  defined for  $(z|\tau) \in \mathbb{C} \times \mathfrak{h}$  such that  $z \notin \Lambda_\tau$ , and for every integer  $s \geq 2$  as

$$G_s(z|\tau) := \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{(m+n\tau+z)^s}.$$

For  $s \geq 3$ , the series  $G_s(z|\tau)$  is absolutely and locally uniformly convergent, and defines a holomorphic function on  $\mathfrak{h}$  for every  $z \in \mathbb{C} - \Lambda_\tau$ . For  $s = 2$ , it is still locally uniformly convergent, and thus still holomorphic, but is no longer absolutely convergent (so that we are not allowed to re-order terms in the series).

For a fixed  $\tau \in \mathfrak{h}$ , one can see that  $G_s(-|\tau)$  is  $\Lambda_\tau$ -periodic, so that  $G_s(z|\tau)$  only depends on the class  $[z] \in E_\tau^\times = (\mathbb{C} - \Lambda_\tau)/\Lambda_\tau$ . Hence, for  $\gamma = (\bar{u}, \bar{v}) \in \Gamma \simeq \Lambda_{\tau,\Gamma}/\Lambda_\tau \subset E_\tau$ , we can define

$$G_{s,\gamma}(\tau) := \begin{cases} G_s(z|\tau) & \text{with } z = \frac{u}{M} + \frac{v}{N}\tau \quad \text{if } \gamma \neq \mathbf{0}, \\ G_s(\tau) & \text{else.} \end{cases}$$

**Proposition 7.1.** — *Let  $s \geq 3$  and  $\gamma \in \Gamma$ . The function  $G_{s,\gamma}$  is a modular form of weight  $s$  with respect to  $\mathrm{SL}_2^\Gamma$ .*

*Proof.* — This is a classical fact for  $\gamma = \mathbf{0}$  (see [16, 43]). The proof is probably standard and known to experts even in the case  $\gamma \neq \mathbf{0}$ , but we provide it here as we could not find a reference for it. We will first prove weak modularity, and then holomorphy at the cusps.

**Lemma 7.2 (Weak modularity).** — *For every  $s \geq 3$ ,  $\tau \in \mathfrak{h}$ ,  $z \in \mathbb{C} - \Lambda_\tau$ , and  $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ ,*

$$G_s(\alpha \cdot (z|\tau)) = (c\tau + d)^s G_s(z|\tau).$$

*In particular, if  $s \geq 3$  and  $\gamma \in \Gamma - \{\mathbf{0}\}$ ,  $G_{s,\gamma}$  is weakly modular of weight  $s$  with respect to  $\mathrm{SL}_2^\Gamma$ .*

Recall that  $\alpha \cdot (z|\tau) := \left( \frac{z}{c\tau+d} \middle| \frac{a\tau+b}{c\tau+d} \right)$ .

*Proof.* — For  $z \notin \Lambda_\tau$ , we compute:

$$\begin{aligned} G_s(\alpha \cdot (z|\tau)) &= \sum_{(m,n) \in \mathbb{Z}^2} \frac{1}{\left( m + n \left( \frac{a\tau+b}{c\tau+d} \right) + \frac{z}{c\tau+d} \right)^s} \\ &= (c\tau + d)^s \sum_{(m,n) \in \mathbb{Z}^2} \frac{1}{\left( m(c\tau + d) + n(a\tau + b) + z \right)^s} \\ &= (c\tau + d)^s \sum_{(m,n) \in \mathbb{Z}^2} \frac{1}{(m + n\tau + z)^s} = G_s(z|\tau). \end{aligned}$$

Observe that if  $(z'|\tau') = \alpha \cdot (z|\tau)$  with  $z' = x + \tau'y$ , then  $z = (c\tau + d)x + (a\tau + b)y$ . Hence the inverse of the action of  $\alpha$  gives an isomorphism  $E_{\tau'} \rightarrow E_\tau$  that is precisely given by  $x + \tau'y \mapsto (c\tau + d)x + (a\tau + b)y$ .

Now assume that  $\alpha \in \mathrm{SL}_2^\Gamma$ , which means that  $a \equiv 1 \pmod{M}$ ,  $d \equiv 1 \pmod{N}$ ,  $b \equiv 0 \pmod{N}$  and  $c \equiv 0 \pmod{M}$ . In particular,  $z' \in \Lambda_{\tau', \Gamma}$  if and only if  $z \in \Lambda_{\tau, \Gamma}$ , and moreover the induced composed isomorphism

$$\Gamma \simeq \Lambda_{\tau', \Gamma} / \Lambda_{\tau'} \xrightarrow{\sim} \Lambda_{\tau, \Gamma} / \Lambda_\tau \simeq \Gamma$$

is the identity. Indeed,  $z' = \frac{u}{M} + \frac{v}{N}\tau' \in \Lambda_{\tau', \Gamma}$  is sent to  $\frac{u}{M}d + \frac{v}{N}b + \left(\frac{v}{N}a + \frac{u}{M}c\right)\tau \in \frac{u}{M} + \frac{v}{N}\tau + \Lambda_\tau$ . Therefore, if  $\gamma \in \Gamma - \{\mathbf{0}\}$ ,

$$G_{s, \gamma}(\alpha \cdot \tau) = G_s\left(\alpha \cdot \left(\frac{u}{M} + \frac{v}{N}\tau|\tau\right)\right) = (c\tau + d)^s G_s\left(\frac{u}{M} + \frac{v}{N}\tau|\tau\right) = G_{s, \gamma}(\tau).$$

This ends the proof of the Lemma.  $\square$

**Remark 7.3.** — The proof does not work in the case  $s = 2$  because we need to reorder the terms of the series to prove the required identity. Nevertheless, for elements of the form  $\alpha = \begin{pmatrix} 1 & H \\ 0 & 1 \end{pmatrix}$ , we can keep  $n$  fixed and apply a shift by  $nH$  in the internal series (the one running over  $m$ ). Hence, for these  $\alpha$ 's, the required identity is true even in the case  $s = 2$ .

As the function  $G_{s, \gamma}$  is holomorphic on  $\mathfrak{h}$ , it remains to show that it is also holomorphic at all cusps for  $\mathrm{SL}_2^\Gamma$ . Recall that these cusps are orbits of the action of  $\mathrm{SL}_2^\Gamma$  on  $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ .

**Lemma 7.4.** — *For every  $s \geq 3$  and  $\gamma \in \Gamma - \{\mathbf{0}\}$ , the function  $G_{s, \gamma}$  is holomorphic at all cusps for  $\mathrm{SL}_2^\Gamma$ .*

*Proof.* — Recall that for every  $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , the width of the cusp  $[\alpha(\infty)]$  is the smallest positive integer  $H$  such that  $\alpha \begin{pmatrix} 1 & H \\ 0 & 1 \end{pmatrix} \alpha^{-1} \in \mathrm{SL}_2^\Gamma$ . This condition is equivalent to the following requirements:  $M|acH$ ,  $M|c^2H$ ,  $N|a^2H$ , and  $N|acH$ . Since  $a$  and  $c$  are relatively prime, this in turn boils down to the condition that  $M|cH$  and  $N|aH$ .

Now observe that, in order to prove that  $G_{s, \gamma}$  is holomorphic at this cusp, one is reduced to prove that the function

$$G_{s, \gamma, \alpha} : \tau \mapsto (c\tau + d)^{-s} G_{s, \gamma}(\alpha \cdot \tau)$$

is holomorphic at  $\infty$ . From the modularity property of  $G_{s, \gamma}$ , we know that  $G_{s, \gamma, \alpha}$  is  $H$ -periodic, and thus descends to a holomorphic function  $\hat{G}_{s, \gamma, \alpha}(q) = G_{s, \gamma, \alpha}(\tau)$  defined on the punctured unit disk, with  $q = e^{\frac{2\pi i \tau}{H}}$ . Hence it remains to show that  $\hat{G}_{s, \gamma, \alpha}$  has a  $q$ -expansion with non-negative Fourier coefficients. Note furthermore that, according to Lemma 7.2,  $G_{s, \gamma, \alpha}(\tau) = \tilde{G}_s(z|\tau)$  with

$$z = (c\tau + d)\frac{u}{M} + (a\tau + b)\frac{v}{N} = \frac{u}{M}d + \frac{v}{N}b + \left(\frac{v}{N}a + \frac{u}{M}c\right)\tau = x + \frac{K}{H}\tau,$$

$K = u\frac{aH}{N} + v\frac{cH}{M} \in \mathbb{Z}$  (and  $x = \frac{u}{M}d + \frac{v}{N}b \in \mathbb{Q}$ ). We let  $K = QH + R$  be the euclidean division of  $K$  by  $H$ , define  $w := x + \frac{R}{H}\tau$ , and compute:

$$\begin{aligned} G_{s, \gamma, \alpha}(\tau) &= \sum_{(m, n) \in \mathbb{Z}^2} \frac{1}{(m + n\tau + z)^s} = \sum_{(m, n) \in \mathbb{Z}^2} \frac{1}{(m + n\tau + w)^s} \\ &= \sum_{m \in \mathbb{Z}} \frac{1}{(m + w)^s} + \sum_{n \geq 1} \sum_{m \in \mathbb{Z}} \frac{1}{(m + n\tau + w)^s} + \sum_{n \geq 1} \sum_{m \in \mathbb{Z}} \frac{1}{(m - n\tau + w)^s} \end{aligned}$$

Let us show that the three series in the last expression have a  $q$ -expansion with non-negative coefficients, and start with  $\sum_{m \in \mathbb{Z}} (m+w)^{-s}$ . If  $w \in \mathbb{R}$  then it is constant in  $\tau$ , and we are done. If  $w \notin \mathbb{R}$ , a standard calculation shows that

$$\sum_{m \in \mathbb{Z}} \frac{1}{(m+w)^s} = \frac{(-2\pi i)^s}{(s-1)!} \sum_{r=1}^{+\infty} r^{s-1} e^{2\pi i r w} = \frac{(-2\pi i)^s}{(s-1)!} \sum_{r=1}^{+\infty} r^{s-1} e^{2\pi i r x} q^{Rr}.$$

The proofs for both double series are identical, hence we restrict ourselves to the first one, and compute:

$$\sum_{n \geq 1} \sum_{m \in \mathbb{Z}} \frac{1}{(m+n\tau+w)^s} = \sum_{n \geq 1} \frac{(-2\pi i)^s}{(s-1)!} \sum_{r=1}^{+\infty} r^{s-1} e^{2\pi i r x} q^{(nH+R)r} = \sum_{k \geq 1} \left( \sum_{\substack{r|k \\ \frac{k}{r} \equiv R \pmod{H}}} r^{s-1} e^{2\pi i r x} \right) q^k.$$

This ends the proof of the Lemma.  $\square$

The proof of Proposition 7.1 is now completed.  $\square$

**Remark 7.5.** — It follows from the proof of Lemma 7.4 that in many cases (i.e. whenever  $w \notin \mathbb{R}$ ),  $G_{s,\gamma}$  actually vanishes at the corresponding cusp. In the case of the cusp  $[\infty]$ ,  $G_{s,\gamma}$  does not vanish at the cusp only if  $\gamma = (\bar{u}, \bar{0})$ ,  $u \in \{1, \dots, M-1\}$ . In this case, the value at the cusp is  $\zeta(s, u/M) + (-1)^s \zeta(s, -u/M) - (M/u)^s$ , where

$$\zeta(s, z) := \sum_{m \geq 0} \frac{1}{(m+z)^s}$$

is the Hurwitz zeta function.

**Remark 7.6.** — It is likely that, using a variation on Hecke's trick (see e.g. [43, Proposition 6]), one could prove that  $G_{2,\gamma}$  is quasi-modular with respect to  $\mathrm{SL}_2^\Gamma$ .

## 7.2. The coefficients $A_{s,\gamma}(\tau)$

Let us recall some standard properties of the Weierstrass function  $\wp : \mathbb{C} \times \mathfrak{h} \rightarrow \mathbb{C}$  given by

$$\wp(z|\tau) = \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z}^2 - \{(0,0)\}} \left( \frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right) = G_2(z|\tau) - G_2(\tau).$$

In the variable  $z$ , it is even, periodic with respect to  $\Lambda_\tau$ , and meromorphic with poles of order two in  $\Lambda_\tau$ . There exists a constant  $c \in \mathbb{C}$  such that

$$\wp(z|\tau) = -\partial_z^2 \log(\theta(z|\tau)) + c.$$

In a suitable punctured neighbourhood of 0 (e.g. the maximal punctured open disk centred at 0 which does not contain any lattice point), there is a Laurent expansion

$$\wp(z|\tau) = \frac{1}{z^2} + \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} z^{2k} = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)G_{2k+2}(\tau)z^{2k},$$

where  $f(z) = \wp(z|\tau) - \frac{1}{z^2}$ .

**Proposition 7.7.** — For every  $s \geq 0$ ,  $A_{s,\mathbf{0}}(\tau) = -(s+1)G_{s+2}(\tau)$ .

Therefore  $A_{s,0}(\tau)$  is a modular form of weight  $s+2$  for  $\mathrm{SL}_2^\Gamma$  whenever  $s > 0$ , while  $A_{0,0}(\tau)$  is only quasi-modular (of weight 2).

*Proof.* — This is proven in [12]. Roughly, one first sees that  $g_0(x, 0|\tau) = \partial_x^2 \log(\theta(x|\tau)) + 1/x^2$ , which proves the required equality for  $s \geq 1$ . Then a specific analysis of the constant term (see e.g. [12, §4.1]) tells us that  $g_0(0, 0|\tau) = 4\pi i \partial_\tau \log \eta(\tau) = -G_2(\tau)$ , where  $\eta$  is the Dedekind eta function.  $\square$

Let now  $\gamma = (\bar{u}, \bar{v}) \in \Gamma - \{\mathbf{0}\}$ , and  $\tilde{\gamma} = \frac{u}{M} + \frac{v}{N}\tau \in \Lambda_{\tau,\Gamma} - \Lambda_\tau$  be a lift of  $\gamma$ . Recall that we are interested in the Taylor coefficients  $A_{s,\gamma}(\tau)$  of

$$g_{-\gamma}(x, 0|\tau) = \partial_x \left( e^{\frac{2\pi i v}{N}x} \frac{\theta(\tilde{\gamma} + x|\tau)}{\theta(\tilde{\gamma}|\tau)\theta(x|\tau)} - \frac{1}{x} \right).$$

We define  $F_\gamma(x|\tau) := e^{\frac{2\pi i v}{N}x} \frac{\theta(\tilde{\gamma} + x|\tau)}{\theta(\tilde{\gamma}|\tau)\theta(x|\tau)}$ , so that  $g_{-\gamma}(x, 0|\tau) = \partial_x F_\gamma(x|\tau) + \frac{1}{x^2}$ .

**Lemma 7.8.** — *If we define*

$$a_{1,\gamma}(\tau) := \frac{2\pi i v}{N} + \frac{\partial_x \theta(\tilde{\gamma}|\tau)}{\theta(\tilde{\gamma}|\tau)} \quad \text{and} \quad a_{s,\gamma}(\tau) := (-1)^s \frac{G_s(\tau) - G_{s,\gamma}(\tau)}{s} \quad (s \geq 2),$$

then  $F_\gamma(x|\tau) = \frac{1}{x} \exp\left(\sum_{s \geq 1} a_{s,\gamma}(\tau) x^s\right)$ .

*Proof.* — We first compute:

$$\partial_x^2 \log(F_\gamma(x|\tau)) = \partial_x^2 \log(\theta(\tilde{\gamma} + x|\tau)) - \partial_x^2 \log(\theta(x|\tau)) = G_2(x|\tau) - G_2(\tilde{\gamma} + x|\tau)$$

A simple calculation shows that

$$G_2(x|\tau) = \frac{1}{x^2} + \sum_{s \geq 0} (-1)^s (s+1) G_{s+2}(\tau) x^s$$

and that

$$G_2(\tilde{\gamma} + x|\tau) = \sum_{s \geq 0} (-1)^s (s+1) G_{s+2,\gamma}(\tau) x^s.$$

Hence

$$\partial_x^2 \log(F_\gamma(x|\tau)) - \frac{1}{x^2} = \sum_{s \geq 0} (-1)^s (s+1) (G_{s+2}(\tau) - G_{s+2,\gamma}(\tau)) x^s.$$

Knowing that  $\left(\frac{1}{x} - \frac{\partial_x \theta(x|\tau)}{\theta(x|\tau)}\right)_{x=0} = 0$ , we obtain

$$\left(\partial_x \log(F_\gamma(x|\tau)) + \frac{1}{x}\right)_{x=0} = \frac{2\pi i v}{N} + \frac{\partial_x \theta(\tilde{\gamma}|\tau)}{\theta(\tilde{\gamma}|\tau)} = a_{1,\gamma}(\tau),$$

and thus

$$\partial_x \log(F_\gamma(x|\tau)) + \frac{1}{x} = a_{1,\gamma}(\tau) + \sum_{s \geq 0} (-1)^s (G_{s+2}(\tau) - G_{s+2,\gamma}(\tau)) x^{s+1}.$$

Finally, since  $\left(\log(x) - \log(\theta(x|\tau))\right)_{x=0} = 0$ , we get that  $\left(\log(F_\gamma(x|\tau)) + \log(x)\right)_{x=0} = 0$ , and thus

$$\log(F_\gamma(x|\tau)) + \log(x) = a_{1,\gamma}(\tau)x + \sum_{s \geq 0} (-1)^s \frac{G_{s+2}(\tau) - G_{s+2,\gamma}(\tau)}{s+2} x^{s+2} = \sum_{s \geq 1} a_{s,\gamma}(\tau) x^s.$$



This shows that  $F_\gamma(x|\tau) = \frac{1}{x} \exp\left(\sum_{s \geq 1} a_{s,\gamma}(\tau)x^s\right)$ . □

As a consequence, we get

$$F_\gamma(x|\tau) - \frac{1}{x} = \frac{\exp\left(\sum_{s \geq 1} a_{s,\gamma}(\tau)x^s\right) - 1}{x},$$

so that

$$\sum_{s \geq 0} A_{s,\gamma}(\tau) := g_{-\gamma}(x, 0|\tau) = \partial_x \left( \frac{\exp\left(\sum_{s \geq 1} a_{s,\gamma}(\tau)x^s\right) - 1}{x} \right).$$

Hence  $A_{s,\gamma}$  can be expressed as an explicit linear combination of products of the form  $a_{s_1,\gamma} \cdots a_{s_k,\gamma}$ , with  $s_1 + \cdots + s_k = s + 2$ , and we expect  $A_{s,\gamma}$  to be quasi-modular of weight  $s + 2$  with respect to  $\mathrm{SL}_2^\Gamma$  (as hinted from Remark 7.6 and Proposition 7.1).

### 7.3. Concluding remarks and outlook

Observe that  $\bar{\mathfrak{t}}_{1,2}^\Gamma$  is the free Lie algebra generated by  $x := \bar{x}_1$ ,  $y := \bar{y}_2$  and  $t^\alpha := \bar{t}_{12}^\alpha$ , for  $\alpha \in \Gamma - \{\mathbf{0}\}$ . Moreover, the element defining the differential equation (6.1) lies in the (degree completion of the) ideal generated by  $y$  and  $t^\alpha$ ,  $\alpha \in \Gamma - \{\mathbf{0}\}$ . By Lazard's elimination theorem (see [30, Theorem 1]), as a Lie algebra this ideal is isomorphic to the free Lie algebra generated by  $y_n$  and  $t_n^\alpha$ ,  $n \geq 0$  and  $\alpha \in \Gamma - \{\mathbf{0}\}$ ; the isomorphism sends  $y_n$  to  $\mathrm{ad}(x)^n(y)$  and  $t^\alpha$  to  $\mathrm{ad}(x)^n(t^\alpha)$ .

As a consequence,  $A^\Gamma(\tau)$  and  $B^\Gamma(\tau)$  can be seen as elements of the formal power series algebra  $\mathbb{C}\langle\langle y_n, t_n^\alpha \mid n \geq 0, \alpha \in \Gamma - \mathbf{0} \rangle\rangle$ . The coefficients of these series can be computed as iterated integrals, and are  $\Gamma$ -twisted versions of Enriquez's elliptic analogs of multiple zeta values [20].

This approach to elliptic multiple zeta values at torsion points seems different to that of the work of Broedel–Matthes–Richter–Schlotterer [10]. The relation between the twisted elliptic multiple zeta values obtained in this paper and that in [10] deserves further investigations. A comparison with the values at torsion points of Goncharov's multiple elliptic polylogarithms [25, Section 8] would also be interesting.

Finally, in addition to the algebraic properties of  $e^\Gamma(\tau)$ , that are essentially given by Theorem 6.1, it would be interesting to study its analytic and modularity properties. In the elliptic case, when  $\Gamma$  is the trivial group, this was done in [19, §5.4 & §5.5], and we expect something similar in the more general ellipsitomic case.

For the analytic properties of the ellipsitomic associator, it amounts to understanding how  $e^\Gamma(\tau)$  depends on small variations of the modulus  $\tau$ . For that, one can use the second line of the ellipsitomic KZB system (6.2), and compute  $\partial_\tau e^\Gamma(\tau)$ . Indeed, recall from [13, Subsection 2.3] that  $\delta_{s,\gamma}$  acts as a derivation on  $\bar{\mathfrak{t}}_{1,2}^\Gamma$ . We can modify it in the following way, by introducing a new derivation

$$\varepsilon_{s,\gamma} := \delta_{s,\gamma} - 2[(\mathrm{ad} x)^s t^{-\gamma}, -].$$

Then the second line of the elliptic KZB system (6.2) for  $n = 2$  reads as

$$2\pi i \partial_\tau F(z|\tau) = - \left( \Delta_0 + \frac{1}{2} \sum_{s \geq 0} \sum_{\gamma \in \Gamma} A_{s,\gamma}(\tau) \varepsilon_{s,\gamma} + O(z) \right) \cdot F(z|\tau),$$

where  $z = z_{12}$  and  $O(z)$  denotes a term of the form  $zf(z|\tau)$ , with  $f$  being holomorphic on  $\mathbb{C} \times \mathfrak{H}$ . Hence, going along the lines of [19, §5.4] one can prove that

$$2\pi i \partial_\tau A^\Gamma(\tau) = - \left( \Delta_0 + \frac{1}{2} \sum_{\gamma \in \Gamma} \sum_{s \geq 0} A_{s,\gamma}(\tau) \varepsilon_{s,\gamma} \right) \cdot A^\Gamma(\tau)$$

and

$$2\pi i \partial_\tau B^\Gamma(\tau) = - \left( \Delta_0 + \frac{1}{2} \sum_{\gamma \in \Gamma} \sum_{s \geq 0} A_{s,\gamma}(\tau) \varepsilon_{s,\gamma} \right) \cdot B^\Gamma(\tau).$$

The derivation  $\varepsilon_{s,\gamma}$  shall be relevant for the study of the elliptic Grothendieck-Teichmüller group, as well as of a yet to be defined analog of Tsunogai's special derivation algebra from [40] in the elliptic case.



## APPENDIX A

### An alternative presentation for $\mathbf{PaB}_{ell}^\Gamma$

In this appendix, we provide an alternative presentation for  $\mathbf{PaB}_{ell}^\Gamma$ , that we use in chapter 6 when proving that the monodromy of the elliptic KZB connection indeed gives rise to an elliptic associator.

#### A.1. An alternative presentation for $\mathbf{PaB}_{ell}$

We first state and prove the result when the group  $\Gamma$  is trivial.

**A.1.0.0.1.** *The relations (N1bis) and (N2bis).* — We introduce three new relations, which are satisfied in the automorphism group of the object (12)3 in  $\mathbf{PaB}_{ell}$  (this can be seen topologically):

$$(N1bis) \quad A^{12,3} = \Phi^{1,2,3} A^{1,23} (\Phi^{1,2,3})^{-1} \tilde{R}^{1,2} \Phi^{2,1,3} A^{2,13} (\Phi^{2,1,3})^{-1} \tilde{R}^{2,1},$$

$$(N2bis) \quad B^{12,3} = \Phi^{1,2,3} B^{1,23} (\Phi^{1,2,3})^{-1} \tilde{R}^{1,2} \Phi^{2,1,3} B^{2,13} (\Phi^{2,1,3})^{-1} \tilde{R}^{2,1},$$

$$(Ebis) \quad \Phi^{1,2,3} R^{2,3} R^{3,2} (\Phi^{1,2,3})^{-1} = (A^{12,3} \Phi^{1,2,3} (A^{1,23})^{-1} (\Phi^{1,2,3})^{-1}, (B^{12,3})^{-1}).$$

For instance, equations (N1bis) and (N2bis) can be depicted as

$$(N1bis, N2bis) \quad \begin{array}{c} (1 \ 2) \\ \bullet \quad \bullet \\ \pm \quad \quad \quad \downarrow \\ \bullet \quad \bullet \\ (1 \ 2) \end{array} \quad \begin{array}{c} 3 \\ \bullet \\ \downarrow \\ \bullet \\ 3 \end{array} = \begin{array}{c} (1 \ 2) \\ \bullet \quad \bullet \\ \pm \quad \quad \quad \downarrow \\ \bullet \quad \bullet \\ (1 \ 2) \end{array} \quad \begin{array}{c} 3 \\ \bullet \\ \downarrow \\ \bullet \\ 3 \end{array}$$

**A.1.0.0.2.** *The statement.* —

**Theorem A.1.** — *As a  $\mathbf{PaB}$ -module in groupoids having  $\mathbf{Pa}$  as  $\mathbf{Pa}$ -module of objects,  $\mathbf{PaB}_{e\ell\ell}$  is freely generated by  $A := A^{1,2}$  and  $B := B^{1,2}$ , together with the relations (N1bis), (N2bis), and (Ebis).*

The above theorem is a direct consequence of Theorem 3.3 together with the following

**Proposition A.2.** — *Let us consider a  $\mathbf{PaB}$ -module in groupoids  $\mathbf{PaM}$ , having  $\mathbf{Pa}$  as  $\mathbf{Pa}$ -module of objects, and let  $A, B$  be a automorphisms of 12. Then*

- (i) *Equations (N1) and (N1bis) are equivalent;*
- (ii) *Equations (N2) and (N2bis) are equivalent;*
- (iii) *If (N1) and (N2) are satisfied, then equations (E) and (Ebis) are equivalent.*

**A.1.0.0.3.** *A useful observation.* — Both (N1) and (N1bis) imply

$$A^{1,2} \tilde{R}^{1,2} A^{2,1} \tilde{R}^{2,1} = \text{Id}_{12}.$$

For both, this is obtained by applying  $(-)^{1,2,\emptyset}$ . Similarly, both (N1) and (N1bis) imply

$$B^{1,2} \tilde{R}^{1,2} B^{2,1} \tilde{R}^{2,1} = \text{Id}_{12}.$$

**A.1.0.0.4.** *Proof of (i) and (ii) in Proposition A.2.* — The following calculation takes place in  $\mathbf{PaM}(3)$ . For ease of comprehension, we put a brace under a sequence of symbols where we use a relation in order to pass to the next step.

$$\begin{aligned} & \Phi^{1,2,3} A^{1,23} \underbrace{\tilde{R}^{1,23} \Phi^{2,3,1}} A^{2,31} \tilde{R}^{2,31} \Phi^{3,1,2} A^{3,12} \tilde{R}^{3,12} \\ = & \Phi^{1,2,3} A^{1,23} (\Phi^{1,2,3})^{-1} \tilde{R}^{1,2} \Phi^{2,1,3} \underbrace{\tilde{R}^{1,3} A^{2,31} \tilde{R}^{2,31} \Phi^{3,1,2} A^{3,12} \tilde{R}^{3,12}} \\ = & \Phi^{1,2,3} A^{1,23} (\Phi^{1,2,3})^{-1} \tilde{R}^{1,2} \Phi^{2,1,3} A^{2,13} \underbrace{\tilde{R}^{1,3} \tilde{R}^{2,31} \Phi^{3,1,2} A^{3,12} \tilde{R}^{3,12}} \\ = & \Phi^{1,2,3} A^{1,23} (\Phi^{1,2,3})^{-1} \tilde{R}^{1,2} \Phi^{2,1,3} A^{2,13} (\Phi^{2,1,3})^{-1} \tilde{R}^{2,1} \underbrace{\tilde{R}^{12,3} A^{3,12} \tilde{R}^{3,12}} \\ = & \Phi^{1,2,3} A^{1,23} (\Phi^{1,2,3})^{-1} \tilde{R}^{1,2} \Phi^{2,1,3} A^{2,13} (\Phi^{2,1,3})^{-1} \tilde{R}^{2,1} (A^{12,3})^{-1} \end{aligned}$$

We repeatedly used (various forms of) the hexagon equation, and only at the last step we used the useful observation from the previous paragraph. This gives that (N1) and (N1bis) are both a consequence of each other. The proof that (N2) and (N2bis) are equivalent is the same.  $\square$

**A.1.0.0.5.** *Another useful fact.* — One can also show that (N1) and (N1bis) are equivalent to

$$(A.1) \quad A^{12,3} \Phi^{1,2,3} \tilde{R}^{1,23} A^{23,1} \Phi^{2,3,1} \tilde{R}^{2,31} A^{31,2} \Phi^{3,1,2} \tilde{R}^{3,12} = \text{Id}_{(12)3}.$$

**A.1.0.0.6.** *Proof of (iii) in Proposition A.2.* — Relation (N1bis) is equivalent to

$$\Phi^{1,2,3}(A^{1,23})^{-1}(\Phi^{1,2,3})^{-1}A^{12,3} = \tilde{R}^{1,2}\Phi^{2,1,3}A^{2,13}(\Phi^{2,1,3})^{-1}\tilde{R}^{2,1}.$$

Thus, (Ebis) is equivalent to

$$\Phi R^{2,3}R^{3,2}\Phi^{-1} = (\tilde{R}^{1,2}\Phi^{2,1,3}A^{2,13}(\Phi^{2,1,3})^{-1}\tilde{R}^{2,1}, (B^{12,3})^{-1}).$$

Using  $\tilde{R}^{2,1}B^{12,3} = B^{21,3}\tilde{R}^{2,1}$ , we deduce that (Ebis) is equivalent to

$$\Phi R^{2,3}R^{3,2}\Phi^{-1} = \tilde{R}^{1,2}\Phi^{2,1,3}A^{2,13}(\Phi^{2,1,3})^{-1}(B^{21,3})^{-1}\Phi^{2,1,3}(A^{2,13})^{-1}(\Phi^{2,1,3})^{-1}B^{21,3}(\tilde{R}^{1,2})^{-1},$$

which is equivalent to

$$(\Phi^{2,1,3})^{-1}(\tilde{R}^{1,2})^{-1}\Phi R^{2,3}R^{3,2}\Phi^{-1}\tilde{R}^{1,2}(B^{21,3})^{-1}\Phi^{2,1,3} = A^{2,13}(\Phi^{2,1,3})^{-1}B^{21,3}\Phi^{2,1,3}(A^{2,13})^{-1}.$$

Now, by using

$$\begin{aligned} & - (\Phi^{2,1,3})^{-1}(\tilde{R}^{1,2})^{-1}\Phi R^{2,3}R^{3,2}\Phi^{-1}\tilde{R}^{1,2} = \tilde{R}^{1,3}(\Phi^{2,3,1})^{-1}R^{2,3}\Phi^{3,2,1}R^{3,21} \\ & - (B^{21,3})^{-1} = (R^{3,21})^{-1}B^{3,21}(R^{21,3})^{-1} \\ & - \Phi^{2,1,3} = R^{21,3}(\Phi^{3,2,1})^{-1}\tilde{R}^{3,2}\Phi^{2,3,1}\tilde{R}^{1,3}, \\ & - (\Phi^{2,1,3})^{-1} = \tilde{R}^{1,3}(\Phi^{2,3,1})^{-1}\tilde{R}^{2,3}\Phi^{3,2,1}R^{3,21} \end{aligned}$$

we obtain

$$\begin{aligned} & \tilde{R}^{1,3}(\Phi^{2,3,1})^{-1}R^{2,3}\Phi^{3,2,1}B^{3,21}(\Phi^{3,2,1})^{-1}\tilde{R}^{3,2}\Phi^{2,3,1}\tilde{R}^{1,3} = A^{2,13}\tilde{R}^{1,3}(\Phi^{2,3,1})^{-1} \\ & \tilde{R}^{2,3}\Phi^{3,2,1}B^{3,21}(\Phi^{3,2,1})^{-1}\tilde{R}^{3,2}\Phi^{2,3,1}\tilde{R}^{1,3}(A^{2,13})^{-1}. \end{aligned}$$

After performing  $A^{2,13}\tilde{R}^{1,3} = \tilde{R}^{1,3}A^{2,31}$  in the r.h.s. of the above equation, one can cancel out the  $\tilde{R}^{1,3}$  terms in both sides of the equation. We obtain, by performing the permutation (123)  $\mapsto$  (312) that

$$\begin{aligned} & (\Phi^{1,2,3})^{-1}R^{1,2}\Phi^{2,1,3}B^{2,13}(\Phi^{2,1,3})^{-1}\tilde{R}^{2,1}\Phi^{1,2,3} \\ & = A^{1,23}(\Phi^{1,2,3})^{-1}\tilde{R}^{1,2}\Phi^{2,1,3}B^{2,13}(\Phi^{2,1,3})^{-1}\tilde{R}^{2,1}\Phi^{1,2,3}(A^{1,23})^{-1}. \end{aligned}$$

This is equivalent to

$$\begin{aligned} & \Phi^{1,2,3}A^{1,23}(\Phi^{1,2,3})^{-1}\tilde{R}^{1,2}\Phi^{2,1,3}B^{2,13}(\Phi^{2,1,3})^{-1}\tilde{R}^{2,1}\Phi^{1,2,3}(A^{1,23})^{-1}(\Phi^{1,2,3})^{-1} \\ & = R^{1,2}\Phi^{2,1,3}B^{2,13}(\Phi^{2,1,3})^{-1}\tilde{R}^{2,1}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \Phi^{1,2,3}A^{1,23}(\Phi^{1,2,3})^{-1}\tilde{R}^{1,2}\Phi^{2,1,3}B^{2,13}(\Phi^{2,1,3})^{-1}\tilde{R}^{2,1}\Phi^{1,2,3}(A^{1,23})^{-1}(\Phi^{1,2,3})^{-1} \\ & (\tilde{R}^{2,1})^{-1}\Phi^{2,1,3}(B^{2,13})^{-1}(\Phi^{2,1,3})^{-1}(R^{1,2})^{-1} = \text{Id}_{(12)3}. \end{aligned}$$

As  $(R^{1,2})^{-1}R^{1,2}R^{2,1} = R^{2,1} = (\tilde{R}^{1,2})^{-1}$ , we obtain

$$R^{1,2}R^{2,1} = (\Phi^{1,2,3}A^{1,23}(\Phi^{1,2,3})^{-1}, \tilde{R}^{1,2}\Phi^{2,1,3}B^{2,13}(\Phi^{2,1,3})^{-1}\tilde{R}^{2,1}).$$

### A.2. An alternative presentation for $\mathbf{PaB}_{e\ell\ell}^\Gamma$

Below, we borrow the notation from Theorem 4.5.

**Theorem A.3.** — *As a  $\mathbf{PaB}$ -module in groupoids with a diagonally trivial  $\Gamma$ -action and having  $\mathbf{Pa}^\Gamma$  as  $\mathbf{Pa}$ -module of objects,  $\mathbf{PaB}_{e\ell\ell}^\Gamma$  is freely generated by  $A$  and  $B$  together with the relations*

$$\begin{aligned} \text{(tN1bis)} \quad & \underline{A}^{12,3} = \Phi^{1,2,3} \underline{A}^{1,23} (\Phi^{1,2,3})^{-1} \tilde{R}^{1,2} \Phi^{2,1,3} \underline{A}^{2,13} (\Phi^{2,1,3})^{-1} \tilde{R}^{2,1}, \\ \text{(tN2bis)} \quad & \underline{B}^{12,3} = \Phi^{1,2,3} \underline{B}^{1,23} (\Phi^{1,2,3})^{-1} \tilde{R}^{1,2} \Phi^{2,1,3} \underline{B}^{2,13} (\Phi^{2,1,3})^{-1} \tilde{R}^{2,1}, \\ \text{(tEbis)} \quad & \Phi^{1,2,3} R^{2,3} R^{3,2} (\Phi^{1,2,3})^{-1} = (\underline{A}^{12,3} \Phi^{1,2,3} (\underline{A}^{1,23})^{-1} (\Phi^{1,2,3})^{-1}, (\underline{B}^{12,3})^{-1}). \end{aligned}$$

In order to prove Theorem A.3, one can

- (i) Either deduce it from Theorem 4.5 in a similar manner as we deduced Theorem A.1 from Theorem 3.3;
- (ii) Or deduce it from Theorem A.1 in a similar manner as we deduced Theorem 4.5 from Theorem 3.3.

Both strategies are straightforward to implement; this is left to the reader.

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## Glossary

### Operads. —

- PaB** : Operad of parenthesized braids. 10
- PaCD(k)** : Operad of parenthesized chord diagrams. 14
- PaB<sub>ell</sub>** : **PaB**-module of elliptic parenthesized braids. 22
- PaCD<sub>ell</sub>(k)** : **PaCD(k)**-module of elliptic parenthesized chord diagrams. 28
- PaB<sup>Γ</sup><sub>ell</sub>** : **PaB**-module of ellipsitomic parenthesized braids. 36
- PaCD<sup>Γ</sup><sub>ell</sub>(k)** : **PaCD(k)**-module of ellipsitomic parenthesized chord diagrams. 46

### Groups. —

- PB<sub>n</sub>** : Pure braid group on the complex plane. 10
- GT** : Operadic Grothendieck-Teichmüller group. 16
- GT(k)** : **k**-pro-unipotent Grothendieck-Teichmüller group. 17
- GRT(k)** : Operadic graded Grothendieck-Teichmüller group. 18
- GRT(k)** : Graded Grothendieck-Teichmüller group. 18
- PB<sub>1,n</sub>** : Reduced pure braid group on the torus. 21
- GT<sub>ell</sub>(k)** : Operadic **k**-pro-unipotent elliptic Grothendieck-Teichmüller group. 30
- GT<sub>ell</sub>(k)** : **k**-pro-unipotent elliptic Grothendieck-Teichmüller group. 30
- GRT<sub>ell</sub>(k)** : Operadic graded elliptic Grothendieck-Teichmüller group. 31
- GRT<sub>ell</sub>(k)** : Graded elliptic Grothendieck-Teichmüller group. 32
- GT<sup>Γ</sup><sub>ell</sub>(k)** : Operadic **k**-pro-unipotent ellipsitomic Grothendieck-Teichmüller group. 40
- GRT<sup>Γ</sup><sub>ell</sub>(k)** : Operadic graded ellipsitomic Grothendieck-Teichmüller group. 46
- B<sup>Γ</sup><sub>1,n</sub>** :  $\Gamma$ -decorated braid group on the torus. 51

### Spaces. —

- C(C, I)** : Reduced configuration space of  $I$ -indexed points in  $\mathbb{C}$ . 9
- C̄(C, I)** : Fulton-MacPherson compactification of **C(C, I)**. 9
- Conf(C, n)** : Configuration space of  $n$  points in  $\mathbb{C}$ . 12
- C(T, I)** : Reduced configuration space of  $I$ -indexed points in  $\mathbb{T}$ . 21
- C̄(T, I)** : Fulton-MacPherson compactification of **C(T, I)**. 21
- Conf(T, I, Γ)** :  $\Gamma$ -decorated configuration space of  $I$ -indexed points in  $\mathbb{T}$ . 35
- C(T, I, Γ)** : Reduced  $\Gamma$ -decorated configuration space of  $I$ -indexed points in  $\mathbb{T}$ . 35
- C̄(T, I, Γ)** : Fulton-MacPherson compactification of **C(T, I, Γ)**. 35

**Lie and associative algebras.** —

- $\mathfrak{t}_n(\mathbf{k})$  : Rational Kohno-Drinfeld Lie  $\mathbf{k}$ -algebra. 13
- $\mathfrak{t}_{1,n}(\mathbf{k})$  : Elliptic Kohno-Drinfeld Lie  $\mathbf{k}$ -algebra. 26
- $\mathfrak{t}_{1,n}^\Gamma(\mathbf{k})$  :  $\Gamma$ -ellipsitomic Kohno-Drinfeld Lie  $\mathbf{k}$ -algebra. 41

**Torsor sets.** —

- Assoc**( $\mathbf{k}$ ) : Operadic  $\mathbf{k}$ -associators. 15
- Ass**( $\mathbf{k}$ ) :  $\mathbf{k}$ -associators. 15
- Ell**( $\mathbf{k}$ ) : Operadic elliptic  $\mathbf{k}$ -associators. 29
- Ell**( $\mathbf{k}$ ) : Elliptic  $\mathbf{k}$ -associators. 29
- Ell** $^\Gamma$ ( $\mathbf{k}$ ) : Operadic ellipsitomic  $\mathbf{k}$ -associators. 47
- Ell** $^\Gamma$ ( $\mathbf{k}$ ) : Ellipsitomic  $\mathbf{k}$ -associators. 47

**Series.** —

- $\Phi_{\text{KZ}}$  : KZ associator. 16
- $e(\tau)$  : Elliptic KZB associator. 30
- $A^\Gamma(\tau)$  :  $A$ -ellipsitomic KZB associator. 50
- $B^\Gamma(\tau)$  :  $B$ -ellipsitomic KZB associator. 50
- $G_{s,\gamma}(\tau)$  : Eisenstein-Hurwitz series. 58