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EXPANSIONS OF THE JACOBI THETA FUNCTIONS AND COMPLETE MONOTONICITY PROPERTIES

ABD RAOUF CHOUIKHA

ABSTRACT. In this paper we provide trigonometric expansions and infinite products for the theta functions $\theta_j(u, \tau)$, $j = 1, 2, 3, 4$. This allows us to highlight some complete monotonicity properties for $\log \frac{\theta_j(u, i\pi t)}{\theta_j(0, i\pi t)}$, $j = 2, 3, 4$ and $\log \frac{\theta_1(u, i\pi t)}{\theta_1'(0, i\pi t)}$ as functions with respect to $t > 0$. We also interested in the quotients $S_j(u, v, t) = \frac{\theta_j(\frac{u}{2}, i\pi t)}{\theta_j(\frac{v}{2}, i\pi t)}$. Some others functions related to θ_j will be considered.

1. INTRODUCTION

Consider the following boundary conditions of the heat equation

$$\frac{\partial^2 f}{\partial u^2} = \frac{\partial f}{\partial t}, \quad f(0, t) = f(1, t), \quad f(u, 0) = \pi \partial(u - \frac{1}{2}), \quad 0 < u < 1 \quad (1)$$

where $\partial(u)$ is the Dirac delta function. Then the general solution of the boundary problem is

$$\theta(u, t) = 2 \sum_{n \geq 0} (-1)^n e^{-\frac{(2n+1)^2 \pi t}{4}} \sin((2n+1)\pi u) \quad (2)$$

When we write $q = e^{\pi i \tau}$ with $t = Im \tau > 0$, this solution takes the form

$$\theta_1(u, \tau) = \sum_{-\infty}^{+\infty} (-1)^{\frac{n-1}{2}} q^{(n+\frac{1}{2})^2} e^{i\pi(2n+1)u} = 2 \sum_{n \geq 0} (-1)^n q^{(n+\frac{1}{2})^2} \sin((2n+1)\pi u)$$

which is the first of the four Jacobi theta functions. It is known it suffices to change the boundary conditions to obtain other theta functions. The corresponding solution of the boundary problem

$$\frac{\partial f(0, t)}{\partial u} = \frac{\partial f(1, t)}{\partial u} = 0, \quad f(u, 0) = \pi \partial(u - \frac{1}{2}), \quad 0 < u < 1$$

is given by

$$\theta_4(u, \tau) = \sum_{-\infty}^{+\infty} (-1)^n q^{n^2} e^{i\pi 2nu} = 1 + 2 \sum_{n \geq 1} (-1)^n q^{n^2} \cos(2n\pi u)$$

The function $\theta_1(u, \tau)$ is periodic of period 2. From θ_1 we obtain the second theta if we increment u by $1/2$

$$\theta_2(u, \tau) = \sum_{-\infty}^{+\infty} q^{(n+\frac{1}{2})^2} e^{i\pi(2n+1)u} = 2 \sum_{n \geq 0} q^{(n+\frac{1}{2})^2} \cos((2n+1)\pi u)$$

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The increment u by $1/2$ in θ_4 yields the third theta function

$$\theta_3(u, \tau) = 1 + 2 \sum_{n \geq 1} q^{n^2} \cos(2n\pi u)$$

These four theta functions can be extended to complex values for u and q such that $|q| < 1$.

All four theta functions are entire functions of u . All are periodic, the period of θ_1 and θ_2 is 2, and that of θ_3 and θ_4 is 1. See [1] and [11] for more details

Because of this remarkable property, the theory of theta functions has found important applications in many problems on the heat flow in planar domains. Since their invention in 19th century, the concept of theta functions becomes a source of inspiration for many mathematicians. It is common knowledge that these functions found applications in physics, theoretical chemistry and engineering sciences, and they play a central role in number theory.

Thus, any new property or characterization of theta functions may have consistent implications at least in fields that use these functions.

Indeed, thanks to trigonometric expansions and infinite products we will highlight properties of complete monotonicity (CM) for these functions as well as for their quotients. These features of theta functions seems to be new in the literature.

An interesting problem related to heat distributions was posed by A. Gonchar. For steady-state distributions of heat, we have $\frac{\partial f(u,t)}{\partial u} = 0$ Therefore (1) becomes the Laplace equation whose solutions are harmonic functions. A. Solynin [10], reduced the original problem to the question about monotonicity of the module $m(r, \omega)$ of a circular sector which is related to the quotient $\frac{\theta_2(u,t)}{\theta_2(v,t)}$.

Moreover, it is interesting to notice that the monotonicity properties of these quotients are related to certain inequalities of the sum of divisors function, which are important in number theory.

This paper improves and corrects an old version (ArXiv:1409.1498).

1.1. Selected facts about completely monotonic functions. Recall in this part some useful notions for the sequel on particular properties of C^∞ functions. A real function f is said to be *completely monotonic (CM)* on $[0, \infty[$ if f is C^∞ and all the derivatives verify

$$(-1)^k f^{(k)}(t) \geq 0, \quad k = 0, 1; 2... \quad \text{for all } t > 0.$$

This definition was introduced in 1921 by F. Hausdorff who initially called 'total monoton'. The following yields a characterization of such functions following S.N. Bernstein and D. Widder [3, p.95]

A function $f : [0, \infty[\rightarrow [0, \infty[$ is CM if and only if there exists a non-decreasing bounded function h such that $f(t) = \int_0^\infty e^{-ut} dh(u)$ and this integral converges for $0 < t < \infty$.

h may be considered as a non-negative measure on $[0, \infty[$ such that the integral converges for all $t > 0$.

Hence, a CM function cannot vanish for any $t > 0$. Notice obviously that the sum and product of CM functions are CM. Observe that if $f(t)$ is CM then $f^{2m}(t)$ and $-f^{2m+1}(t)$ are also CM. It is known that that if $f(t)$ is CM and $h(t)$ be with a CM derivative then $f(h(t))$ also is CM.

CM functions have a lot of applications in various fields. These functions have remarkable applications in different branches. In particular, they play a role in potential theory, probability theory, physics, numerical and asymptotic analysis, and combinatorics, [2], [4].

There is another class of functions very closed to such functions : the class of logarithmically completely monotonic (LCM).

A function $f :]0, \infty[\rightarrow]0, \infty[$ is logarithmically completely monotonic if

$$f \text{ is } C^\infty \text{ and } (-1)^k (\log f(t))^{(k)} \geq 0, \quad k = 1, 2, \dots$$

Remark for $k = 0$ we may only require that $f(t) > 0$.

Notice that every LCM function is CM. If this inequality is strict for all $t > 0$ and $k = 1, 2, \dots$ then f is said to be *strictly logarithmically completely monotonic*.

A function f on $]0, \infty[$ is called a *Stieltjes transform* if it can be written in the form

$$f(t) = a + \int_0^\infty \frac{d\mu(s)}{s+t}$$

where a is a non-negative number and μ a non-negative measure on $]0, \infty[$ such that $\int_0^\infty \frac{d\mu(s)}{1+t} < \infty$.

It is proved that every Stieltjes transform is a LCM function.

1.2. Statements of results. This paper is organized as follows. In the first part we describe trigonometric expansions of the four Jacobi theta functions $\theta_j(u, \tau)$, $j = 1, 2, 3, 4$. We provide the following expansions:

$$\begin{aligned} \log \left(\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} \right) &= \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{p} \left(\frac{\sin \pi v}{(\sin(k + \frac{1}{2})\pi\tau)} \right)^{2p} \right] \\ \log \left(\frac{\theta_3(v, \tau)}{\theta_4(0, \tau)} \right) &= \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{p} \left(\frac{\sin \pi v}{(\cos(k + \frac{1}{2})\pi\tau)} \right)^{2p} \right] \\ \log \left(\frac{\theta_2(v, \tau)}{(\cos \pi v)\theta_2(0, \tau)} \right) &= \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{p} \left(\frac{\sin \pi v}{\cos k\pi\tau} \right)^{2p} \right] \\ \log \left(\frac{\theta_1(v, \tau)}{\pi(\sin \pi v)\theta_1'(0, \tau)} \right) &= \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{p} \left(\frac{\sin \pi v}{\sin k\pi\tau} \right)^{2p} \right]. \end{aligned}$$

This permits us to deduce in particular infinite product expansions for $\theta_j(v, \tau)$.

In the second part we are interested in the complete monotonicity properties of these theta functions. We prove that for any fixed u real such that $0 < u < 1$ and $\tau = i\pi t$, the following functions

$$\frac{\theta_4(u, t)}{\theta_4(0, t)}, \quad \frac{\theta_3(u, t)}{\theta_4(0, t)}, \quad \frac{\theta_2(u, t)}{(\cos \pi u)\theta_1'(0, t)}, \quad \frac{\theta_1(u, t)}{(\sin \pi u)\theta_1'(0, t)}$$

are logarithmically completely monotonic (LCM) with respect to $t > 0$.

In the third part, we are interested in the complete monotonicity properties of their quotients. More precisely, for $u, v \in C$ and $\tau = i\pi t$ with $Re t > 0$, define the quotient of theta functions:

$$S_j := S_j(u, v, t) = \frac{\theta_j(\frac{u}{2}, i\pi t)}{\theta_j(\frac{v}{2}, i\pi t)}.$$

Some authors recently interested in monotonicity and convexity of these quotients [6],[7],[10]. In [7, Conj 1.1], one conjectured for u, v such that $0 \leq u < v < 1$ the functions $\frac{\partial}{\partial t} S_1(u, v; t), S_2(u, v; t), S_3(u, v; t)$ and $\frac{\partial}{\partial t} S_4(u, v; t)$ are CM for $0 < t < \infty$.

We will prove that the quotients $S_j, j = 1, 2, 3, 4$ are LCM under the condition on $t : \sinh \pi^2 t > 1$. More precisely, we prove in particular for fixed u, v such that $0 \leq u < v < 1$ the functions $\frac{1}{S_j}(\frac{\partial}{\partial t} S_j)$ for $j = 1, 4$ as well as the functions $-\frac{(\frac{\partial}{\partial t} S_j)}{S_j}$ for $j = 2, 3$ are completely monotonic for $t \in]0, \infty[$. Most of the proofs will be given in the last section.

2. FOURIER EXPANSIONS OF $\log \theta_j(u, i\pi\tau), j = 1, 2, 3, 4$

From the knowledge of the zeros of these functions it is possible to obtain infinite products representing the theta functions. For example the function $\theta_4(v, \tau)$ satisfies the triple product

$$\theta_4(v, \tau) = \prod_{n \geq 1} (1 - q^{2n})(1 - q^{2n+1} e^{2i\pi v})(1 - q^{2n-1} e^{-2i\pi v})$$

where $q = e^{i\pi\tau}, v \in C, \tau \in H_+$. From these products the Fourier expansions of $\log(\theta_j(u, i\pi t))$ and $\frac{\theta'_j}{\theta_j}$ follow.

Trigonometric expansions of theta functions can be proved by the methods of residue calculus as described by [8, p.358] and [11, p.489].

Thus, one obtains the following series expansions for $\log \theta_j$:

$$\log \left[\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} \right] = 4 \sum_{n \geq 1} \frac{q^n}{1 - q^{2n}} \frac{(\sin n\pi v)^2}{n} \quad (3)$$

$$\log \left[\frac{\theta_3(v, \tau)}{\theta_3(0, \tau)} \right] = 4 \sum_{n \geq 1} (-1)^n \frac{q^n}{1 - q^{2n}} \frac{(\sin n\pi v)^2}{n} \quad (4)$$

$$\log \left[\frac{\theta_2(v, \tau)}{\theta_2(0, \tau)} \right] = \log(\cos \pi v) + 4 \sum_{n \geq 1} (-1)^n \frac{q^{2n}}{1 - q^{2n}} \frac{(\sin n\pi v)^2}{n} \quad (5)$$

$$\log \left[\frac{\theta_1(v, \tau)}{\pi \theta'_1(0, \tau)} \right] = \log(\sin \pi v) + 4 \sum_{n \geq 1} \frac{q^{2n}}{1 - q^{2n}} \frac{(\sin n\pi v)^2}{n}. \quad (6)$$

Expressions of θ_3 and θ_4 are valid in the "strip" $|Imv| < \frac{1}{2}Im\tau$, those relating to θ_1 and θ_2 are valid in the "strip" $|Imv| < Im\tau$.

Now we propose the following

Theorem 2-1 Let $q = e^{i\pi\tau}$, $|q| < 1$. The functions θ_3 and θ_4 have the following expansions

$$\theta_4(v, \tau) = \theta_4(0, \tau) \exp \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{p} \left(\frac{\sin \pi v}{(\sin(k + \frac{1}{2})\pi\tau)} \right)^{2p} \right], \quad (7)$$

$$\theta_3(v, \tau) = \theta_3(0, \tau) \exp \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{p} \left(\frac{\sin \pi v}{(\cos(k + \frac{1}{2})\pi\tau)} \right)^{2p} \right] \quad (8)$$

$$= \theta_4(0, \tau) \exp \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{p} \left(\frac{\cos \pi v}{(\sin(k + \frac{1}{2})\pi\tau)} \right)^{2p} \right].$$

The above expressions of θ_3 and θ_4 are valid in the "strip" $|Imv| < \frac{1}{2}Im\tau$.

Theorem 2-3 Let $q = e^{i\pi\tau}$, $|q| < 1$. The functions θ_1 and θ_2 have the following expansions

$$\theta_2(v, \tau) = \theta_2(0, \tau) \exp \left[\log(\cos \pi v) - \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{p} \left(\frac{\sin \pi v}{\cos k\pi\tau} \right)^{2p} \right] \quad (9)$$

$$= \pi\theta'_1(0, \tau) \exp \left[\log(\cos \pi v) - \sum_{p \geq 1} \sum_{k \geq 1} \frac{1}{p} \left(\frac{\cos \pi v}{\sin k\pi\tau} \right)^{2p} \right]$$

$$\theta_1(v, \tau) = \pi\theta'_1(0, \tau) \exp \left[\log(\sin \pi v) - \sum_{p \geq 1} \sum_{k \geq 1} \frac{1}{p} \left(\frac{\sin \pi v}{\sin k\pi\tau} \right)^{2p} \right] \quad (10)$$

(where $\theta'_1(0, t) = \frac{\partial \theta_1}{\partial u}(0, t)$).

The above expressions of θ_1 and θ_2 are valid in the "strip" $|Imv| < Im\tau$.

Remark 2-6 These coefficients $c_{2p,4}(\tau) = \frac{1}{p} \sum_{k \geq 0} \frac{1}{((\sin(k + \frac{1}{2})\pi\tau))^{2p}}$ have interesting arithmetic properties. Indeed, under the action of modular group $\Gamma(1)$; $\tau, \tau + 1, \frac{-1}{\tau}$ we find relations between coefficients $c_{2p,3}, c_{2p,2}, c_{2p,1}$. The transformation theory of order n ; $n = 1, 2, \dots$ deals with the relations between coefficients. For example when we examine the theta relations $\theta_4(v + \frac{1}{2}, \tau + 1) = \theta_4(v, \tau)$, or $\theta_4(\frac{v}{\tau}, \frac{-1}{\tau}) = (-i\tau)^{1/2} \exp(i\pi v^2/\tau) \theta_2(v, \tau)$ we find

$$c_{2p,4}(\tau + 1) = -\frac{1}{p} \sum_{k \geq 0} \left[\frac{4q^{2k+1}}{(1 + q^{2k+1})^2} \right]^p = (-1)^p \sum_{k \geq p} \binom{k}{p} c_{2k}(\tau)$$

$$-\frac{2}{\tau^2} c_{2p,4}\left(\frac{-1}{\tau}\right) = 1 + (-1)^p \frac{2^p}{p} \sum_{k \geq 1} \left[\frac{q^{2k}}{1 + q^{2k}} \right]^p.$$

See [4-5] for more details.

Starting from Theorems 2-1 and 2-3 we deduce other infinite products expansions

Corollary 2-7 Let $q = e^{i\pi\tau}$, $|q| < 1$. The functions θ_j , $j = 1, 2, 3, 4$ may also be expressed as infinite products

$$\theta_4(v, \tau) = \theta_4(0, \tau) \prod_{k \geq 0} \left[1 - \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi\tau} \right)^2 \right] \quad (11)$$

$$\theta_3(v, \tau) = \theta_3(0, \tau) \prod_{k \geq 0} \left[1 - \left(\frac{\sin \pi v}{\cos(k + \frac{1}{2})\pi\tau} \right)^2 \right] \quad (12)$$

$$\theta_1(v, \tau) = (\pi \sin \pi v) \theta_1'(0, \tau) \prod_{k \geq 1} \left[1 - \left(\frac{\sin \pi v}{\sin k\pi\tau} \right)^2 \right] \quad (13)$$

$$\theta_2(v, \tau) = (\cos \pi v) \theta_2(0, \tau) \prod_{k \geq 1} \left[1 - \left(\frac{\sin \pi v}{\cos k\pi\tau} \right)^2 \right] \quad (14)$$

θ_3 and θ_4 is defined in the "strip" $|Imv| < \frac{1}{2}Im\tau$, while θ_1 and θ_2 is defined in the "strip" $|Imv| < Im\tau$.

Proof Indeed, we will prove only for $\theta_4(v, \tau)$ the others will be deduced by the same way. Starting from (7) and notice that

$$\log \left(\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} \right) = \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{p} \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi\tau} \right)^{2p} \right] = -2 \sum_{k \geq 0} \sum_{p \geq 1} \frac{1}{2p} (X_k)^{2p}$$

where

$$X_k = \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi\tau} \right)$$

On the other hand we may calculate

$$-2 \sum_{p \geq 1} \frac{1}{2p} (X_k)^{2p} = \log(1 - (X_k)^2).$$

it follows that

$$\log \left(\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} \right) = \sum_{k \geq 0} \log(1 - (X_k)^2) = \sum_{k \geq 0} \log \left(1 - \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi\tau} \right)^2 \right)$$

Therefore we obtain the infinite product of θ_4

$$\theta_4(v, \tau) = \theta_4(0, \tau) \prod_{k \geq 0} \left[1 - \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi\tau} \right)^2 \right]$$

From Corollary 2-7 we deduce the following

Corollary 2-8 Let $q = e^{i\pi\tau}$, $|q| < 1$. The logarithmic derivative of functions θ_j , $j = 1, 2, 3, 4$ with respect to v may be expressed

$$\frac{1}{\theta_4} \frac{\partial \theta_4}{\partial v}(v, \tau) = -\pi \sum_{k \geq 0} \frac{\sin 2\pi v}{(\sin(k + \frac{1}{2})\pi\tau)^2 - (\sin \pi v)^2}$$

$$\frac{1}{\theta_3} \frac{\partial \theta_3}{\partial v}(v, \tau) = \pi \sum_{k \geq 0} \frac{\sin 2\pi v}{(\cos(k + \frac{1}{2})\pi\tau)^2 - (\sin \pi v)^2}$$

$$\begin{aligned}\frac{1}{\theta_2} \frac{\partial \theta_2}{\partial v}(v, \tau) &= -(\tan \pi v) + \pi \sum_{k \geq 1} \frac{\sin 2\pi v}{(\cos k\pi\tau)^2 - (\sin \pi v)^2} \\ \frac{1}{\theta_1} \frac{\partial \theta_1}{\partial v}(v, \tau) &= (\cot \pi v) - \pi \sum_{k \geq 1} \frac{\sin 2\pi v}{(\sin k\pi\tau)^2 - (\sin \pi v)^2}\end{aligned}$$

θ_3 and θ_4 is defined in the "strip" $|Imv| < \frac{1}{2}Im\tau$, while θ_1 and θ_2 is defined in the "strip" $|Imv| < Im\tau$.

Proof Indeed, by logarithmic differentiation (11) and (13) with respect to v we obtain $\frac{1}{\theta_4} \frac{\partial \theta_4}{\partial v}$ and $\frac{1}{\theta_1} \frac{\partial \theta_1}{\partial v}$. The others are deduced in incrementing v by $\frac{1}{2}$

$$\frac{1}{\theta_1} \frac{\partial \theta_4}{\partial v}(v + \frac{1}{2}, \tau) = \frac{1}{\theta_3} \frac{\partial \theta_3}{\partial v}(v, \tau), \quad \frac{1}{\theta_1} \frac{\partial \theta_1}{\partial v}(v + \frac{1}{2}, \tau) = \frac{1}{\theta_2} \frac{\partial \theta_2}{\partial v}(v, \tau).$$

Remarks 2-9 1 - Starting from these expansions we may easily derive infinite products for the Jacobi elliptic functions. For example, the function

$$dn(z) = \frac{\theta_4(0, \tau) \theta_4(v, \tau)}{\theta_4(v, \tau) \theta_3(0, \tau)}$$

for $z = v(\theta_3(0, \tau))^2$ may be expressed as

$$dn(z) = \prod_{k \geq 0} \frac{(\cos(k + \frac{1}{2})\pi\tau)^2 - (\sin \pi v)^2}{(\sin(k + \frac{1}{2})\pi\tau)^2 - (\sin \pi v)^2}.$$

Recall the classical infinite product of that function, [8], [11]:

$$dn(z) = \frac{\theta_2(0, \tau)}{\theta_3(0, \tau)} \prod_{n \geq 1} \frac{1 + 2q^{2n-1}(\cos 2z) + q^{4n-2}}{1 - 2q^{2n-1}(\cos 2z) + q^{4n-2}}.$$

2 - Since we have $\theta_4(v + \frac{1}{2}, \tau + 1) = \theta_4(v, \tau)$ then we deduce

$$\begin{aligned}\theta_4(0, \tau) \prod_{k \geq 0} \left[1 - \left(\frac{\sin \pi v}{\sin(k + \frac{1}{2})\pi\tau} \right)^2 \right] &= \theta_4(0, \tau + 1) \prod_{k \geq 0} \left[1 - \left(\frac{\cos \pi v}{\cos(k + \frac{1}{2})\pi\tau} \right)^2 \right] \\ &= \theta_3(0, \tau) \prod_{k \geq 0} \left[1 - \left(\frac{\cos \pi v}{\cos(k + \frac{1}{2})\pi\tau} \right)^2 \right].\end{aligned}$$

However we know [8] that

$$\frac{\theta_3(0, \tau)}{\theta_4(0, \tau)} = \prod_{n \geq 0} \left(\frac{1 + q^{2n+1}}{1 - q^{2n+1}} \right)^2 = \prod_{n \geq 0} (\coth \pi(n + \frac{1}{2})\tau)^2.$$

We thus obtain the equality

$$\prod_{k \geq 0} [(\cos(k + \frac{1}{2})\pi\tau)^2 - (\cos \pi v)^2] = \prod_{k \geq 0} [(\sin(k + \frac{1}{2})\pi\tau)^2 - (\sin \pi v)^2] \quad (15)$$

valid for $|Imv| < \frac{1}{2}Im\tau$. Since

$$\frac{\theta_2(0, \tau)}{\theta_1(0, \tau)} = \prod_{n \geq 1} \left(\frac{1 + q^{2n}}{1 - q^{2n}} \right)^2 = \prod_{n \geq 1} (\coth \pi n\tau)^2.$$

it seems we may also establish

$$\prod_{k \geq 1} [(\cos k\pi\tau)^2 - (\cos \pi v)^2] = \prod_{k \geq 1} [(\sin k\pi\tau)^2 - (\sin \pi v)^2] \quad (16)$$

valid for $|\operatorname{Im} v| < \operatorname{Im} \tau$.

3. COMPLETE MONOTONICITY OF $\log \frac{\theta_j(v, \tau)}{\theta_j(0, t)}$

From the knowledge of the zeros of theta functions it is possible to obtain infinite products representing these functions, see [8, p.357] or [11, p.470]. For example the function $\theta_4(v, \tau)$ satisfies the triple product

$$\theta_4(v, t) = \prod_{n \geq 1} (1 - q^{2n})(1 - q^{2n+1}e^{2i\pi v})(1 - q^{2n-1}e^{-2i\pi v})$$

where $q = e^{i\pi\tau}$, $v \in C$, $\tau \in H_+$.

From these products the Fourier expansions of $\log(\theta_j(v, i\pi\tau))$ and $\frac{\theta'_j}{\theta_j}$ follow. Trigonometric expansions of theta functions can be proved by the methods of residue calculus as described by [8, p.358] and [11].

More precisely, one has the following series expansions for $\log \theta_j$:

$$\log \left[\frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} \right] = 4 \sum_{n \geq 1} \frac{q^n}{1 - q^{2n}} \frac{(\sin n\pi v)^2}{n} \quad (17)$$

$$\log \left[\frac{\theta_3(v, \tau)}{\theta_3(0, \tau)} \right] = 4 \sum_{n \geq 1} (-1)^n \frac{q^n}{1 - q^{2n}} \frac{(\sin n\pi v)^2}{n} \quad (18)$$

$$\log \left[\frac{\theta_2(v, \tau)}{\theta_2(0, \tau)} \right] = \log(\cos \pi v) + 4 \sum_{n \geq 1} (-1)^n \frac{q^{2n}}{1 - q^{2n}} \frac{(\sin n\pi v)^2}{n} \quad (19)$$

$$\log \left[\frac{\theta_1(v, \tau)}{\pi \theta'_1(0, \tau)} \right] = \log(\sin \pi v) + 4 \sum_{n \geq 1} \frac{q^{2n}}{1 - q^{2n}} \frac{(\sin n\pi v)^2}{n}. \quad (20)$$

Expressions of θ_3 and θ_4 are valid in the "strip" $|\operatorname{Im} v| < \frac{1}{2} \operatorname{Im} \tau$, those relating to θ_1 and θ_2 are valid in the "strip" $|\operatorname{Im} v| < \operatorname{Im} \tau$.

In the sequel we will consider the case $\tau = i\pi t$ and will denote

$$\theta_j(v, \tau) = \theta_j(v, i\pi t) = \theta_j(v, t).$$

Our main result is the following

Theorem 3-1 *For any fixed u real and $\tau = i\pi t$, the following quotients of Jacobi theta functions :*

$$\frac{\theta_4(u, t)}{\theta_4(0, t)}, \quad \frac{\theta_1(u, t)}{\theta'_1(0, t)}, \quad \frac{\theta_3(u, t)}{\theta_4(0, t)}, \quad \frac{\theta_2(u, t)}{\theta'_1(0, t)},$$

(where $\theta'_1(0, t) = \frac{\partial \theta_1}{\partial u}(0, t)$) are logarithmically completely monotonic (LCM) with respect to $t > 0$. That means for $k = 0, 1, 2, \dots$

$$(-1)^k \frac{\partial^k}{\partial t^k} \left(\log \frac{\theta_4(u, t)}{\theta_4(0, t)} \right) > 0, \quad (-1)^k \frac{\partial^k}{\partial t^k} \left(\log \frac{\theta_3(u, t)}{\theta_3(0, t)} \right) > 0,$$

$$(-1)^k \frac{\partial^k}{\partial t^k} \left(\log \frac{\theta_2(u, t)}{\theta_2(0, t)} \right) > 0, \quad (-1)^k \frac{\partial^k}{\partial t^k} \left(\log \frac{\theta_1(u, t)}{\theta_1'(0, t)} \right) > 0.$$

Proof of Theorem 3-1

From Corollary 2-7, (12) may be rewritten with $q = e^{i\pi\tau} = e^{-\pi^2 t}$

$$\frac{\theta_4(u, t)}{\theta_4(0, t)} = \prod_{k \geq 0} \left[1 + \left(\frac{\sin \pi u}{\sinh(k + \frac{1}{2})\pi^2 t} \right)^2 \right].$$

Thus

$$\log \left[\frac{\theta_4(u, t)}{\theta_4(0, t)} \right] = \sum_{k \geq 0} \log \left[1 + \left(\frac{\sin \pi u}{\sinh(k + \frac{1}{2})\pi^2 t} \right)^2 \right].$$

By the same way

$$\begin{aligned} \frac{\theta_1(u, t)}{(\sin \pi u)\theta_1'(0, t)} &= \prod_{k \geq 1} \left[1 + \left(\frac{\sin \pi u}{\sinh k\pi^2 t} \right)^2 \right] \\ \log \left[\frac{\theta_1(u, t)}{(\sin \pi u)\theta_1'(0, t)} \right] &= \sum_{k \geq 1} \log \left[1 + \left(\frac{\sin \pi u}{\sinh k\pi^2 t} \right)^2 \right] \end{aligned}$$

Remark that since $\theta_4(u + \frac{1}{2}, t) = \theta_3(u, t)$ then (12) may be expressed

$$\frac{\theta_3(u, t)}{\theta_3(0, t)} = \prod_{k \geq 0} \left[1 - \left(\frac{\sin \pi u}{\cosh(k + \frac{1}{2})\pi^2 t} \right)^2 \right]$$

and

$$\begin{aligned} \log \left[\frac{\theta_3(u, t)}{\theta_3(0, t)} \right] &= \sum_{k \geq 0} \log \left[1 - \left(\frac{\sin \pi u}{\cosh(k + \frac{1}{2})\pi^2 t} \right)^2 \right] \\ \log \left[\frac{\theta_3(u, t)}{\theta_4(0, t)} \right] &= \sum_{k \geq 0} \log \left[1 + \left(\frac{\cos \pi u}{\sinh(k + \frac{1}{2})\pi^2 t} \right)^2 \right] \end{aligned}$$

Since $\theta_1(u + \frac{1}{2}, t) = \theta_2(u, t)$ then (15) may be expressed

$$\begin{aligned} \theta_2(v, \tau) &= (\cos \pi v) \theta_2(0, \tau) \prod_{k \geq 1} \left[1 - \left(\frac{\sin \pi v}{\cos k\pi\tau} \right)^2 \right] \\ \log \left[\frac{\theta_2(u, t)}{(\cos \pi u)\theta_1'(0, t)} \right] &= \sum_{k \geq 1} \log \left[1 + \left(\frac{\cos \pi u}{\sinh k\pi^2 t} \right)^2 \right] \end{aligned}$$

To prove this theorem we need the following

Lemma 3-2 *The following functions are LCM (logarithmically completely monotonic) for $t > 0$:*

$$\frac{1}{e^t - 1}, \quad \frac{1}{1 - e^{-t}}, \quad \coth(t), \quad \frac{1}{\sinh(t)}$$

Proof of Lemma 3-2 Recall that a constant function $\phi(t) = \alpha$, $\alpha \geq 0$ is CM as well as for $\phi(t) = e^{-\alpha x}$, $\alpha \geq 0$.

To prove that these functions are LCM we follow Chun-Fu Weil, Bai-Ni Guo [9] who considered the functions

$$\frac{1}{e^{\mp t} - 1}$$

and their i th derivative

$$F_i(t) = (-1)^i \frac{d^i}{dt^i} \left(\frac{1}{e^t - 1} \right) \quad \text{and} \quad G_i(t) = (-1)^i \frac{d^i}{dt^i} \left(\frac{1}{1 - e^{-t}} \right)$$

where $i = 0, 1, 2, \dots$. They proved that F_i, G_i are CM for all $t > 0$. But

$$F_0(t) = \frac{1}{e^t - 1} \quad \text{and} \quad G_0(t) = \frac{1}{1 - e^{-t}}$$

are LCM for all $t > 0$.

In fact, to see that, it suffices to prove that $G_0(t)$ is LCM for all $t > 0$ then $F_0(t)$ follows too, because we have

$$G_0(t) = 1 + F_0(t) = e^t F_0(t)$$

and the sum and the product of two LCM functions are also LCM. Consequently, we have

$$1 + 2F_0(t) = \frac{e^t + 1}{e^t - 1} = \coth \frac{t}{2}$$

is LCM for all $t > 0$ as well as for

$$\frac{1}{\sinh t} = 2e^t F_0(2t) = 2e^{-t} G_0(2t) = 2e^{-t} (1 + F_0(2t)).$$

By this lemma we may assert that the following functions with respect to $t > 0$

$$\frac{1}{\sinh(n\pi^2 t)} = 2e^{n\pi^2 t} F_0(2n\pi^2 t) = 2e^{-n\pi^2 t} G_0(2n\pi^2 t)$$

are LCM functions for $n \in \mathbb{N}$. Since the product of LCM functions is also LCM, it follows that the functions

$$1 + \left(\frac{\sin \pi u}{\sinh(k + \frac{1}{2})\pi^2 t} \right)^2, \quad 1 + \left(\frac{\cos \pi u}{\sinh(k + \frac{1}{2})\pi^2 t} \right)^2,$$

as well as

$$1 + \left(\frac{\sin \pi u}{\sinh k\pi^2 t} \right)^2, \quad 1 + \left(\frac{\cos \pi u}{\sinh k\pi^2 t} \right)^2$$

are LCM. Thus, we prove by recurrence on k that their products are LCM. Therefore the 4 quotients $\frac{\theta_4(u,t)}{\theta_3(0,t)}$, $\frac{\theta_4(u,t)}{\theta_4(0,t)}$, $\frac{\theta_2(u,t)}{\theta_1'(0,t)}$, $\frac{\theta_1(u,t)}{\theta_1'(0,t)}$ are CM.

The following gives an alternative proof that $\frac{\theta_4(u,t)}{\theta_4(0,t)}$ and $\frac{\theta_1(u,t)}{(\sin \pi u)\theta_1'(0,t)}$ are LCM. The method that we use seems not run for the 2 other theta functions

Corollary 3-4 *For any fixed u real such that $0 < u$ from expressions (15) and (18) we may deduce that*

$$\frac{\theta_4(u,t)}{\theta_4(0,t)}, \quad \text{and} \quad \frac{\theta_1(u,t)}{(\sin \pi u)\theta_1'(0,t)}$$

are logarithmically completely monotonic with respect to $t > 0$.

Proof of Corollary 3-4 The above expansion (16) may be rewritten with $q = e^{i\pi\tau} = e^{-\pi^2 t}$

$$\log \left[\frac{\theta_4(u, t)}{\theta_4(0, t)} \right] = 4 \sum_{n \geq 1} \frac{q^n}{1 - q^{2n}} \frac{(\sin n\pi u)^2}{n} = 4 \sum_{n \geq 1} \frac{e^{-n\pi^2 t}}{1 - e^{-2n\pi^2 t}} \frac{(\sin n\pi u)^2}{n}$$

$$\log \left[\frac{\theta_4(u, t)}{\theta_4(0, t)} \right] = 2 \sum_{n \geq 1} \frac{1}{\sinh(n\pi^2 t)} \frac{(\sin n\pi u)^2}{n}.$$

By the same way (19) may be rewritten

$$\log \left[\frac{\theta_1(u, t)}{\pi\theta'_1(0, t)} \right] = \log(\sin \pi u) + 4 \sum_{n \geq 1} \frac{q^{2n}}{1 - q^{2n}} \frac{(\sin n\pi u)^2}{n}.$$

Moreover,

$$\sum_{n \geq 1} \frac{e^{-2n\pi^2 t}}{1 - e^{-2n\pi^2 t}} \frac{(\sin n\pi u)^2}{n} = \sum_{n \geq 1} \frac{1}{e^{2n\pi^2 t} - 1} \frac{(\sin n\pi u)^2}{n}.$$

Then

$$\log \left[\frac{\theta_1(u, t)}{\pi(\sin \pi u)\theta'_1(0, t)} \right] = 4 \sum_{n \geq 1} \frac{1}{e^{2n\pi^2 t} - 1} \frac{(\sin n\pi u)^2}{n}$$

By Lemma 3-2 the following functions with respect to $t > 0$

$$\frac{1}{\sinh(n\pi^2 t)} = 2e^{n\pi^2 t} F_0(2n\pi^2 t) = 2e^{-n\pi^2 t} G_0(2n\pi^2 t)$$

$$\frac{1}{e^{2n\pi^2 t} - 1} = F_0(2n\pi^2 t) = G_0(2n\pi^2 t) - 1$$

are LCM functions for $n \in \mathbb{N}$. On the other hand it is known that the set of LCM functions is a convex cone. That means any combination with positive coefficients of LCM functions is also an LCM function.

Therefore

$$\sum_{n \geq 1} \frac{1}{\sinh(n\pi^2 t)} \frac{(\sin n\pi u)^2}{n}, \text{ and } \sum_{n \geq 1} \frac{1}{e^{2n\pi^2 t} - 1} \frac{(\sin n\pi u)^2}{n}$$

are LCM functions.

Remark 3-5 1 - Notice that the quotients $\frac{\theta_3(u, t)}{\theta_3(0, t)}$ and $\frac{\theta_2(u, t)}{\theta_2(0, t)}$ seem not to be complete monotonic. Indeed, write

$$\frac{\theta_3(u, t)}{\theta_3(0, t)} = \frac{\theta_3(u, t)}{\theta_4(0, t)} \frac{\theta_4(0, t)}{\theta_3(0, t)}, \quad \frac{\theta_2(u, t)}{\theta_2(0, t)} = \frac{\theta_2(u, t)}{\theta'_1(0, t)} \frac{\theta'_1(0, t)}{\theta_2(0, t)}$$

It is known [8] that

$$\frac{\theta_3(0, t)}{\theta_4(0, t)} = \prod_{n \geq 0} \left(\frac{1 + q^{2n+1}}{1 - q^{2n+1}} \right)^2 = \prod_{n \geq 0} (\coth \pi^2 (n + \frac{1}{2})t)^2$$

$$\frac{\theta_2(0, t)}{\theta'_1(0, t)} = \prod_{n \geq 1} \left(\frac{1 + q^{2n}}{1 - q^{2n}} \right)^2 = \prod_{n \geq 1} (\coth \pi^2 nt)^2.$$

By Lemma 3-2 the function $u \rightarrow \coth(u)$ is complete monotonic for $u > 0$. Then $\frac{\theta_3(0,t)}{\theta_4(0,t)}$ and $\frac{\theta_2(0,t)}{\theta_1(0,t)}$ are also CM, but not their inverse $\frac{\theta_4(0,t)}{\theta_3(0,t)}$ and $\frac{\theta_1(0,t)}{\theta_2(0,t)}$.

Taking the logarithmic derivatives of the infinite products (11-14) one obtains the following after replacing $\tau = i\pi t$

Corollary 3-6 *The logarithmic derivatives with respect to $t > 0$ of the Jacobi theta functions may be expressed*

$$\frac{\theta'_4(u,t)}{\theta_4(u,t)} - \frac{\theta'_4(0,t)}{\theta_4(0,t)} = -\pi^2 \sum_{k \geq 0} \frac{(2k+1)[\coth(k + \frac{1}{2})\pi^2 t][\sin \pi u]^2}{[\sinh(k + \frac{1}{2})\pi^2 t]^2 + [\sin \pi u]^2},$$

$$\frac{\theta'_3(u,t)}{\theta_3(u,t)} - \frac{\theta'_3(0,t)}{\theta_3(0,t)} = -\pi^2 \sum_{k \geq 0} \frac{(2k+1)[\tanh(k + \frac{1}{2})\pi^2 t][\sin \pi u]^2}{[\cosh(k + \frac{1}{2})\pi^2 t]^2 - [\sin \pi u]^2}$$

where $|Imu| < \frac{1}{2}\pi t$,

$$\frac{\theta'_1(u,t)}{\theta_1(u,t)} - \frac{\theta''_1(0,t)}{\theta_1(0,t)} = -\pi^2 \sum_{k \geq 1} \frac{2k[\coth k\pi^2 t][\sin \pi u]^2}{[\sinh k\pi^2 t]^2 + [\sin \pi u]^2}$$

$$\frac{\theta'_2(u,t)}{\theta_2(u,t)} - \frac{\theta'_2(0,t)}{\theta_2(0,t)} = -\pi^2 \sum_{k \geq 1} \frac{2k[\tanh k\pi^2 t][\sin \pi u]^2}{[\cosh k\pi^2 t]^2 - [\sin \pi u]^2}$$

where $|Imu| < \pi t$.

4. SOME APPLICATIONS ON THE QUOTIENTS OF THETA FUNCTIONS

For $u, v \in C$ and $\tau = i\pi t$ with $Re t > 0$, consider the quotient of theta functions as follows

$$S_j := S_j(u, v, t) = \frac{\theta_j(\frac{u}{2}, i\pi t)}{\theta_j(\frac{v}{2}, i\pi t)}.$$

In [6],[7],[10] one interested in monotonicity and convexity of these quotients. This is related to the problem of completely monotonic functions.

A. Solynin and A. Dixit, [7],[10] proved the monotonicity of $S_j(u, v, t) = \frac{\theta_j(\frac{u}{2}, i\pi t)}{\theta_j(\frac{v}{2}, i\pi t)}$. More precisely they stated for fixed u, v such that $0 \leq u < v < 1$, the functions $S_1(u, v; t)$ and $S_4(u, v; t)$ are positive and strictly increasing for $t \in]0, \infty[$ while $S_2(u, v; t)$ and $S_3(u, v; t)$ are positive decreasing for $t \in]0, \infty[$.

A. Dixit, A. Roy and A. Zaharescu [7, Th 1.2] proved for u, v such that $0 \leq u < v < 1$ the functions $S_2(u, v; t)$ and $S_3(u, v; t)$ are strictly convex for $t \in]0, \infty[$.

However, one conjectured [7, Conj 1.1] for u, v such that $0 \leq u < v < 1$ the functions $\frac{\partial}{\partial t} S_1(u, v; t)$, $S_2(u, v; t)$, $S_3(u, v; t)$ and $\frac{\partial}{\partial t} S_4(u, v; t)$ are CM for $0 < t < \infty$.

Many numerical calculus thanks to *Maple* suggest us that the odd and even derivatives in t of $\log(S_j(u, v, t))$ (and not only $S_j(u, v, t)$) have alternating signs. Thus, one naturally may ask if quotients of theta functions are LCM (logarithmically completely monotonic).

This is precisely what we are trying to prove

Theorem 4-1 For fixed u, v such that $0 \leq u < v < 1$, the quotients of theta functions $S_j := S_j(u, v, t) = \frac{\theta_j(\frac{u}{2}, t)}{\theta_j(\frac{v}{2}, t)}$, $j = 1, 2, 3, 4$ are strictly LCM (logarithmically completely monotonic) for some $t > 0$ verifying

$$\sinh\left(\frac{\pi^2 t}{2}\right) \geq \sup\left(\sin\left(\frac{\pi v}{2}\right), \cos\left(\frac{\pi v}{2}\right)\right)$$

. More precisely we have :

- for $j = 2, 3$ inequalities $(-1)^k \frac{\partial^k}{\partial t^k} \left(\frac{1}{S_j} \frac{\partial S_j}{\partial t}\right) > 0$ hold for $k = 1, 2, 3, \dots$
- for $j = 1, 4$ inequalities $(-1)^k \frac{\partial^k}{\partial t^k} \left(\frac{1}{S_j} \frac{\partial S_j}{\partial t}\right) < 0$ hold for $k = 1, 2, 3, \dots$

Some consequences of Theorem 4-1 may be derived. At first, we find again the following which has been proved in [7], [10]

Corollary 4-3 For fixed u, v such that $0 \leq u < v < 1$, the functions $S_1(u, v; t)$ and $S_4(u, v; t)$ are positive and strictly increasing for $t > 0$ while $S_2(u, v; t)$ and $S_3(u, v; t)$ are positive decreasing for $t > 0$.

Proof Notice that

$$\frac{\partial}{\partial t} \left(\frac{1}{S_j} \frac{\partial S_j}{\partial t}\right)(u, v, t) = \frac{\theta'_j}{\theta_j} \left(\frac{u}{2}, t\right) - \frac{\theta'_j}{\theta_j} \left(\frac{v}{2}, t\right), \quad j = 1, 2, 3, 4$$

By Corollary 3-6 we derive

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{S_4} \frac{\partial S_4}{\partial t}\right)(u, v, t) &= \frac{\theta'_4}{\theta_4} \left(\frac{u}{2}, t\right) - \frac{\theta'_4}{\theta_4} \left(\frac{v}{2}, t\right) \\ &= \sum_{k \geq 0} \frac{(2k+1) \coth\left(\left(k + \frac{1}{2}\right) \pi^2 t\right) \left(\sin\left(\pi \frac{u}{2}\right)\right)^2}{\left(\sinh\left(\left(k + \frac{1}{2}\right) \pi^2 t\right)\right)^2 + \left(\sin\left(\pi \frac{u}{2}\right)\right)^2} - \frac{(2k+1) \coth\left(\left(k + \frac{1}{2}\right) \pi^2 t\right) \left(\sin\left(\pi \frac{v}{2}\right)\right)^2}{\left(\sinh\left(\left(k + \frac{1}{2}\right) \pi^2 t\right)\right)^2 + \left(\sin\left(\pi \frac{v}{2}\right)\right)^2} \\ &= \sum_{k \geq 0} \frac{(2k+1) \cosh\left(\frac{1}{2} (2k+1) \pi^2 t\right) \sinh\left(\frac{1}{2} (2k+1) \pi^2 t\right) \left(\left(\cos\left(\pi \frac{u}{2}\right)\right)^2 - \left(\cos\left(\pi \frac{v}{2}\right)\right)^2\right)}{\left[\left(\sinh\left(\left(k + \frac{1}{2}\right) \pi^2 t\right)\right)^2 + \left(\sin\left(\pi \frac{u}{2}\right)\right)^2\right] \left[\left(\sinh\left(\left(k + \frac{1}{2}\right) \pi^2 t\right)\right)^2 + \left(\sin\left(\pi \frac{v}{2}\right)\right)^2\right]} > 0. \end{aligned}$$

Then $\frac{\partial}{\partial t} \left(\frac{\partial S_4}{\partial t}\right)(u, v, t) > 0$. By the same way using again Corollary 3-6, we prove $\frac{\partial}{\partial t} \left(\frac{\partial S_1}{\partial t}\right)(u, v, t) > 0$, $\frac{\partial}{\partial t} \left(\frac{\partial S_2}{\partial t}\right)(u, v, t) < 0$, and $\frac{\partial}{\partial t} \left(\frac{\partial S_3}{\partial t}\right)(u, v, t) < 0$.

The following extends Theorem 1.2 of [10]

Corollary 4-4 For fixed u, v such that $0 \leq u < v < 1$, the functions S_j are such that

- For any $t > 0$ the second derivatives $\frac{\partial^2}{\partial t^2} S_j(u, v, t) < 0$ for $j = 1, 4$.
- For any $t > 0$ the second derivatives $\frac{\partial^2}{\partial t^2} S_j(u, v, t) > 0$ for $j = 2, 3$.

Proof We will show the result only for S_4 . We will operate in the same way for others S_j , $j = 1, 2, 3$.

Derive another time the function $\frac{\partial}{\partial t} \left(\frac{1}{S_4} \frac{\partial S_4}{\partial t}\right)(u, v, t)$. We obtain thank to Maple

$$\frac{\partial^2}{\partial t^2} \left(\frac{1}{S_4} \frac{\partial S_4}{\partial t}\right)(u, v, t) = - \sum_{k \geq 0} \left[\left(\cos\left(\pi \frac{u}{2}\right)\right)^2 - \left(\cos\left(\pi \frac{v}{2}\right)\right)^2 \right] (2k+1)^2 \pi^2 \frac{A_k}{B_k}$$

where

$$\begin{aligned}
A_k &= -2 \left(\cos \left(\pi \frac{u}{2} \right) \right)^2 \left(\cosh \left(\frac{1}{2} (2k+1) \pi^2 t \right) \right)^2 \left(\cos \left(\pi \frac{v}{2} \right) \right)^2 \\
&+ \left(\cos \left(\pi \frac{u}{2} \right) \right)^2 \left(\cosh \left(\frac{1}{2} (2k+1) \pi^2 t \right) \right)^2 + \left(\cos \left(\pi \frac{u}{2} \right) \right)^2 \left(\cos \left(\pi \frac{v}{2} \right) \right)^2 \\
&+ 2 \left(\cosh \left(\frac{1}{2} (2k+1) \pi^2 t \right) \right)^6 - 3 \left(\cosh \left(\frac{1}{2} (2k+1) \pi^2 t \right) \right)^4 \\
&+ \left(\cosh \left(\frac{1}{2} (2k+1) \pi^2 t \right) \right)^2 \left(\cos \left(\pi \frac{v}{2} \right) \right)^2
\end{aligned}$$

$$\begin{aligned}
B_k &= 2 \left(\cosh \left(\frac{1}{2} (2k+1) \pi^2 t \right) - \cos \left(\pi \frac{v}{2} \right) \right)^2 \left(\cosh \left(\frac{1}{2} (2k+1) \pi^2 t \right) + \cos \left(\pi \frac{v}{2} \right) \right)^2 \\
&\times \left(\cosh \left(\frac{1}{2} (2k+1) \pi^2 t \right) - \cos \left(\pi \frac{u}{2} \right) \right)^2 \left(\cosh \left(\frac{1}{2} (2k+1) \pi^2 t \right) + \cos \left(\pi \frac{u}{2} \right) \right)^2
\end{aligned}$$

Here $X > 1$, $(\cos \pi v)^2 < (\cos \pi u)^2 < 1$.

B is a square and *Maple* shows that A is non negative.

Then $\frac{\partial^2}{\partial t^2} \left(\frac{1}{S_4} \frac{\partial S_4}{\partial t} \right) (u, v, t) = \frac{\partial^2 S_4}{S_4} - \left(\frac{\partial S_4}{S_4} \right)^2 < 0$. As well as $\frac{\partial^2}{\partial t^2} S_4(u, v, t) < 0$.

5. CONCLUDING REMARKS 4-5

1 - It seems that the hypothesis $\sinh \left(\frac{\pi^2 t}{2} \right) \geq \sup \left(\sin \left(\frac{\pi v}{2} \right), \cos \left(\frac{\pi v}{2} \right) \right)$ is superfluous and that we can extend Theorem 4-1 for $t > 0$ as confirmed by calculations. It is clear we should use another approach to get around this obstacle. Recall that Corollaries 4-3 and 4-4 confirm this fact.

2 - As we have seen before by the result of S.N. Bernstein and D. Widder we may deduce that :

- there exists a non-decreasing function ω_j such that $S_j(u, v; t) = \int_0^\infty e^{-\nu t} d\omega_j(u) d\nu$ for $j = 2, 3$ and $\frac{\partial}{\partial t} S_j(u, v; t) = \int_0^\infty e^{-\nu t} d\omega_j(u) d\nu$ for $j = 3, 4$.

- there exists a non-decreasing function ω_j such that $\frac{\theta_j(u, t)}{\theta_j(0, t)} = \int_0^\infty e^{-\nu t} d\omega_j(u) d\nu$ for $j = 1, 2, 3, 4$.

3 - Since every Stieltjes transform f is a LCM function, in the sense that

$$f(t) = a + \int_0^\infty \frac{d\mu(s)}{s+t}$$

where a is a non-negative number and μ a non-negative measure on $]0, \infty[$.

It is natural to ask if the quotients $\frac{\theta_j(u, t)}{\theta_j(0, t)}$ and $S_j(u, v; t)$ share this property.

4 - Among the interesting questions that may arise; what possible applications of this new property of theta functions which yet are very well known? are there other functions of the same type sharing this property of monotonicity? Concerning the second question we may have an answer. For example the sigma functions of Weierstrass

$$\sigma_0(z) = e^{\frac{\eta z^2}{2\omega}} \frac{\theta_1(v)}{\theta_1'(0)} \quad \sigma_\alpha(z) = e^{\frac{\eta z^2}{2\omega}} \frac{\theta_{\alpha+1}(v)}{\theta_{\alpha+1}'(0)}, \quad \alpha = 1, 2, 3$$

where (see [11] p.473) $z = 2\omega v$, $\tau = \frac{\omega'}{\omega}$, and

$$\eta = -\frac{\theta'''(0, \tau)}{12\omega\theta'(0, \tau)} = 24 \sum_{n \geq 1} \frac{q^{2n}}{(1 - q^{2n})^2} - 1 = 24 \sum_{n \geq 1} \frac{1}{(\sinh n\pi^2 t)^2} - 1$$

When $\omega = 1$ and $\tau = \omega' = i\pi t$ we then deduce from Theorems 3-1 or Corollary 3-5

Corollary 4-6 *The sigma functions of Weierstrass $\sigma_\alpha(z)$, $\alpha = 0, 1, 2, 3$ such that $\omega = 1$ and $\eta = -\frac{\theta_\alpha'''(0, i\pi t)}{12\theta_\alpha'(0, i\pi t)}$ are logarithmically completely monotonic as functions with respect to $t > 0$.*

6. PROOFS

6.1. Proof of Theorem 2-1. We shall compare different trigonometric expansions for the theta functions. This will allow us to establish the expression of the coefficients.

1- Expansion of $\theta_4(v, \tau)$

By (3) we have

$$\theta_4(v, \tau) = \theta_4(0, \tau) \exp \left[4 \sum_{n \geq 1} \frac{q^n}{1 - q^{2n}} \frac{(\sin n\pi v)^2}{n} \right].$$

Comparing the latter with the expansion

$$\theta_4(v, \tau) = \theta_4(0, \tau) \exp \left[4 \sum_{p \geq 1} c_{2p}(\tau) (\sin \pi v)^{2p} \right]$$

one derives the following

Lemma 2-2 *The coefficients c_{2p} defined above may be written as*

$$c_{2p}(\tau) = -\frac{1}{p} \sum_{k \geq 0} \frac{1}{(\sin(k + \frac{1}{2})\pi\tau)^{2p}} = -\frac{1}{p} \sum_{k \geq 0} \left[\frac{(-4)q^{2k+1}}{(1 - q^{2k+1})^2} \right]^p.$$

In particular, we have

$$c_2(\tau) = 4 \sum_{n \geq 1} \frac{nq^n}{(1 - q^{2n})} = \frac{1}{2\pi^2} \frac{\theta_4''(0, \tau)}{\theta_4(0, \tau)}$$

Proof of Lemma 2-2 It is enough to express $\cos(2n\alpha)$ in terms of $\sin \alpha$ as

$$\begin{aligned} \cos(2n\alpha) &= \frac{(-1)^n}{2} (2 \sin \alpha)^{2n} + \sum_{k=0}^{n-1} (-1)^{n+k+1} \frac{2n}{k+1} \binom{2n-k-1}{k} (2 \sin \alpha)^{2n-2k-2} \\ &= n \sum_{0 \leq p \leq n} \frac{(-1)^p (n+p-1)!}{(2p)!(n-p)!} (2 \sin \alpha)^{2p} = \sum_{p=0}^n b_{2p,n} (\sin \alpha)^{2p}, \end{aligned}$$

where

$$b_{2n,n} = (-1)^n 2^{2n-1}; \quad b_{0,n} = 1; \quad \text{and} \quad b_{2p,n} = (-1)^p 2^{2p} \frac{n}{n-p} \binom{n+p-1}{n-p-1} \text{ for } n \neq p.$$

It can also be rewritten

$$b_{2p,n} = n \frac{(-1)^p (n+p-1)!}{(2p)!(n-p)!} 2^{2p} \quad \text{for } 0 \leq p \leq n.$$

Furthermore, we may use

$$\log \frac{\theta_4(v, \tau)}{\theta_4(0, \tau)} = 2 \sum_{n \geq 1} \gamma_n - 2 \sum_{n \geq 1} \gamma_n \cos(2nv) = \sum_{p \geq 1} c_{2p}(\tau) (\sin \pi v)^{2p},$$

where $\gamma_n = \frac{q^n}{n(1-q^{2n})}$.

So, we obtain the relations

$$c_{2p} = -2 \sum_{n \geq p} \gamma_n b_{2p,n} = -(-1)^p 2^{2p} \left[\gamma_p b_{2p,p} + \sum_{n \geq p} \frac{q^n}{(n-p)(1-q^{2n})} \binom{n+p-1}{n-p-1} \right].$$

Therefore

$$c_{2p} = -2(-1)^p 2^{2p-1} \frac{q^p}{1-q^{2p}} + (-1)^{p+1} 2^{2p+1} \sum_{m > p} \binom{m+p-1}{m-p-1} \frac{q^m}{(m-p)(1-q^{2m})}.$$

i.e.,

$$c_{2p}(\tau) = (-1)^{p+1} \frac{2^{2p+1}}{(2p)!} \sum_{n \geq p} \frac{(n+p-1)!}{(n-p)!} \frac{q^n}{(1-q^{2n})}.$$

In particular, for $p = 1$,

$$c_2(\tau) = 2^2 \frac{q}{1-q^2} + 8 \sum_{m > 1} \binom{m}{m-2} \frac{q^m}{(m-1)(1-q^{2m})},$$

and

$$c_2(\tau) = 2^2 \frac{q}{1-q^2} + 4 \sum_{m > 1} \frac{mq^m}{(1-q^{2m})} = 4 \sum_{n \geq 1} \frac{nq^n}{(1-q^{2n})}.$$

Thus,

$$c_2 = 4 \sum_{n \geq 1} \frac{nq^n}{(1-q^{2n})} = \frac{1}{2\pi^2} \frac{\theta_4''(0, \tau)}{\theta_4(0, \tau)}$$

We have also

$$c_4 = \frac{1}{3} \theta_2^4(0, \tau) \theta_3^4(0, \tau) + \frac{1}{3} c_2.$$

These series $c_{2p}(\tau)$ can be expressed as

$$\begin{aligned} (-1)^{p+1} \frac{(2p)!}{2^{2p+1}} c_{2p}(\tau) &= \sum_{n \geq p} \frac{(n+p-1)!}{(n-p)!} \frac{q^n}{(1-q^{2n})} = \sum_{n \geq p} \frac{q^n (n+p-1)!}{(n-p)!} \sum_{k \geq 0} q^{2kn} \\ &= \sum_{k \geq 0, n \geq p} \frac{(n+p-1)!}{(n-p)!} q^{2kn+n} = \sum_{k \geq 0} q^{(2k+1)p} \sum_{n \geq p} \frac{(n+p-1)!}{(n-p)!} q^{(2k+1)(n-p)} \\ &= \sum_{k \geq 0} g_p(q^{2k+1}), \end{aligned}$$

where

$$g_p(z) = \sum_{n \geq p} \frac{(n+p-1)!}{(n-p)!} z^n = z^p \sum_{m \geq 0} \frac{(m+2p-1)!}{(m)!} z^m.$$

Now using the notation

$$(2p)_n = 2p(2p+1)\dots(2p+n-1), \quad \text{then} \quad (m+2p-1)! = (2p)_m(2p-1)!,$$

and one gets

$$\frac{1}{(2p-1)!} g_p(z) = z^p \sum_{m \geq 0} \frac{(2p)_m}{(m)!} z^m.$$

However, we know that

$$z^p \sum_{m \geq 0} \frac{(2p)_m}{(m)!} z^m = z^p {}_2F_1(2p, \alpha, \alpha, z) = \frac{z^p}{(1-z)^{2p}}.$$

$$g_p(\tau) = (2p-1)! \frac{z^p}{(1-z)^{2p}}.$$

As a result, after replacing we derive the expression

$$c_{2p}(\tau) = (-1)^{p+1} \frac{2^{2p+1}}{(2p)!} \sum_{k \geq 0} g_p(q^{2k+1}) = (-1)^{p+1} \frac{2^{2p+1}}{2p} \sum_{k \geq 0} \frac{q^{(2k+1)p}}{(1-q^{2k+1})^{2p}}.$$

Finally,

$$c_{2p}(\tau) = (-1)^{p+1} \frac{2^{2p+1}}{2p} \sum_{k \geq 0} \left(\frac{q^{2k+1}}{(1-q^{2k+1})^2} \right)^p = -\frac{1}{p} \sum_{k \geq 0} \frac{1}{(\sin(k + \frac{1}{2})\pi\tau)^{2p}}.$$

2- Expansion of $\theta_3(v, \tau)$

This function is related with the preceding by two manners, [2], [11]. Under the action of the modular group $\Gamma(1)$ in changing v by $v+1$ or τ by $\tau+1$ we find the relations

$$\theta_3(v, \tau) = \theta_4(v, \tau+1) = \theta_4(v+1, \tau)$$

We then deduce expressions for θ_3

$$\begin{aligned} \theta_3(v, \tau) &= \theta_4(v+1, \tau) = \theta_4(0, \tau) \exp \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{p} \left(\frac{\cos \pi v}{(\sin(k + \frac{1}{2})\pi\tau)} \right)^{2p} \right] \\ \theta_3(v, \tau) &= \theta_4(v, \tau+1) = \theta_4(0, \tau+1) \exp \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{p} \left(\frac{\sin \pi v}{(\cos(k + \frac{1}{2})\pi\tau)} \right)^{2p} \right] \\ &= \theta_3(0, \tau) \exp \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{p} \left(\frac{\sin \pi v}{(\cos(k + \frac{1}{2})\pi\tau)} \right)^{2p} \right]. \end{aligned}$$

On the other hand $|Imv| < \frac{1}{2} Im\tau$ implies $\left| \frac{\sin \pi v}{(\sin(k + \frac{1}{2})\pi\tau)} \right| < 1$
and $\left| \frac{\sin \pi v}{(\cos(k + \frac{1}{2})\pi\tau)} \right| < 1$.

Moreover we may prove

$$c_2(\tau+1) = -4 \sum_{k \geq 0} \frac{q^{2k+1}}{(1+q^{2k+1})^2} = \frac{1}{2\pi^2} \frac{\theta_3''(0)}{\theta_3(0)}.$$

The proof of Theorem 2-1 is then achieved.

6.2. Proof of Theorem 2-3. 1- Expansion of $\theta_2(v, \tau)$

We will proceed as before. Consider (3)

$$\theta_2(v, \tau) = \theta_2(0, \tau) \exp \left[\log(\cos \pi v) + 4 \sum_{n \geq 1} \frac{(-1)^n q^{2n}}{1 - q^{2n}} \frac{(\sin n\pi v)^2}{n} \right].$$

Comparing the latter with the expansion

$$\theta_2(v, \tau) = \theta_2(0, \tau) \exp \left[\sum_{p \geq 1} d_{2p}(\tau) (\sin \pi v)^{2p} \right]$$

one derives the following

Lemma 2-4 *The coefficients d_{2p} defined above may be written as*

$$d_{2p}(\tau) = -\frac{1}{p} \sum_{k \geq 1} \frac{1}{(\cos k\pi\tau)^{2p}}.$$

In particular, we have

$$d_2(\tau) = 4 \sum_{n \geq 1} \frac{n(-1)^n q^{2n}}{(1 - q^{2n})} = \frac{1}{2\pi^2} \frac{\theta_2''(0, \tau)}{\theta_2(0, \tau)}$$

Proof of Lemma 2-4 Take again the expression of $\cos(2n\alpha)$ in terms of $\sin \alpha$ as

$$\begin{aligned} \cos(2n\alpha) &= \frac{(-1)^n}{2} (2 \sin \alpha)^{2n} + \sum_{k=0}^{n-1} (-1)^{n+k+1} \frac{2n}{k+1} \binom{2n-k-1}{k} (2 \sin \alpha)^{2n-2k-2} \\ &= n \sum_{0 \leq p \leq n} \frac{(-1)^p (n+p-1)!}{(2p)!(n-p)!} (2 \sin \alpha)^{2p} = \sum_{p=0}^n b_{2p,n} (\sin \alpha)^{2p}, \end{aligned}$$

where

$$\begin{aligned} b_{2n,n} &= (-1)^n 2^{2n-1}; b_{0,n} = 1; \quad \text{and} \quad b_{2p,n} = (-1)^p 2^{2p} \frac{n}{n-p} \binom{n+p-1}{n-p-1} \text{ for } n \neq p. \\ &= n \frac{(-1)^p (n+p-1)!}{(2p)!(n-p)!} 2^{2p} \quad \text{for } 0 \leq p \leq n. \end{aligned}$$

Consider the expansion

$$\begin{aligned} \log \frac{\theta_2(v, \tau)}{\theta_2(0, \tau)} &= \log(\cos(\pi v)) + 4 \sum_{n \geq 1} (-1)^n \frac{q^{2n}}{1 - q^{2n}} \frac{(\sin(n\pi v))^2}{n} \\ &= 2 \sum_{n \geq 1} \gamma_n - 2 \sum_{n \geq 1} \gamma_n \cos(2nv) = \sum_{p \geq 1} d_{2p}(\tau) (\sin \pi v)^{2p}, \end{aligned}$$

where $\gamma_n = \frac{(-1)^n q^{2n}}{n(1 - q^{2n})}$.

So, we obtain the relations

$$d_{2p} = -2 \sum_{n \geq p} \gamma_n b_{2p,n} = -(-1)^p 2^{2p} \left[\gamma_p b_{2p,p} + \sum_{n \geq p} \frac{(-1)^n q^{2n}}{(n-p)(1 - q^{2n})} \binom{n+p-1}{n-p-1} \right].$$

Therefore

$$d_{2p} = -2(-1)^p 2^{2p-1} \frac{q^p}{1-q^{2p}} + (-1)^{p+1} 2^{2p+1} \sum_{m>p} \binom{m+p-1}{m-p-1} \frac{(-1)^m q^{2m}}{(m-p)(1-q^{2m})}.$$

i.e.,

$$d_{2p}(\tau) = (-1)^{p+1} \frac{2^{2p+1}}{(2p)!} \sum_{n \geq p} \frac{(n+p-1)!}{(n-p)!} \frac{(-1)^n q^{2n}}{(1-q^{2n})}.$$

In particular, we have

$$d_2(\tau) = 4 \sum_{n \geq 1} \frac{n(-1)^n q^{2n}}{(1-q^{2n})}$$

These series $d_{2p}(\tau)$ can be expressed as

$$\begin{aligned} (-1)^{p+1} \frac{(2p)!}{2^{2p+1}} d_{2p}(\tau) &= \sum_{n \geq p} \frac{(-1)^n (n+p-1)!}{(n-p)!} \frac{q^{2n}}{(1-q^{2n})} \\ &= \sum_{n \geq p} \frac{(-1)^n q^{2n} (n+p-1)!}{(n-p)!} \sum_{k \geq 0} q^{2kn} \\ &= \sum_{k \geq 0, n \geq p} \frac{(-1)^n (n+p-1)!}{(n-p)!} q^{2kn+2n} = \sum_{k \geq 0} q^{(2k+2)p} \sum_{n \geq p} \frac{(-1)^n (n+p-1)!}{(n-p)!} q^{(2k+2)(n-p)} \\ &= \sum_{k \geq 0} g_p(q^{2k+2}) = g(z), \end{aligned}$$

where

$$g_p(z) = \sum_{n \geq p} (-1)^n \frac{(n+p-1)!}{(n-p)!} z^n = z^p \sum_{m \geq 0} (-1)^n \frac{(m+2p-1)!}{(m)!} z^m.$$

Now using the notation

$$(2p)_n = 2p(2p+1)\dots(2p+n-1), \quad \text{then} \quad (m+2p-1)! = (2p)_m (2p-1)!,$$

and one gets

$$\frac{1}{(2p-1)!} g_p(z) = (-1)^p z^p \sum_{m \geq 0} \frac{(-1)^m (2p)_m}{(m)!} z^m.$$

However, we have

$$\sum_{m \geq 0} (-1)^m \frac{(2p)_m}{(m)!} z^m = {}_2F_1(2p, \alpha, \alpha, -z) = \frac{1}{(1+z)^{2p}}.$$

$$g_p(z) = (2p-1)! \frac{(-z)^p}{(1+z)^{2p}}.$$

We then derive the expression

$$(-1)^{p+1} \frac{(2p)!}{2^{2p+1}} d_{2p}(\tau) = \sum_{k \geq 0} g_p(q^{2k+2}) = (2p-1)! \frac{(-q^{2k+2})^p}{(1+q^{2k+2})^{2p}}.$$

$$d_{2p}(\tau) = (-1)^{p+1} \frac{2^{2p+1}}{(2p)!} \sum_{k \geq 0} g_p(q^{2k+2}) = -\frac{2^{2p}}{p} \sum_{k \geq 0} \frac{q^{(2k+2)p}}{(1+q^{2k+2})^{2p}}.$$

Finally,

$$\begin{aligned} d_{2p}(\tau) &= -\frac{2^{2p}}{p} \sum_{k \geq 0} \left(\frac{q^{2k+2}}{(1+q^{2k+2})^2} \right)^p = -\frac{2^{2p}}{p} \sum_{k \geq 0} \left(\frac{1}{(q^{-(k+1)} + q^{(k+1)})^2} \right)^p \\ &= -\frac{1}{p} \sum_{k \geq 0} \frac{1}{(\cos(k+1)\pi\tau)^{2p}}. \end{aligned}$$

2- Expansion of $\theta_1(v, \tau)$

This function is related with the preceding by changing v in $v + \frac{1}{2}$, [B], [W-W]:

$$\theta_1(v, \tau) = -\theta_2(v + \frac{1}{2}, \tau) = \exp(i\pi)\theta_2(v + \frac{1}{2}, \tau)$$

We then deduce a first expression for θ_1

$$\theta_1(v, \tau) = -\theta_2(0, \tau) \exp \left[\log(\sin \pi v) - \sum_{p \geq 1} \sum_{k \geq 1} \frac{1}{p} \left(\frac{\cos \pi v}{\cos k\pi\tau} \right)^{2p} \right]. \quad (21)$$

To obtain the other expansion we start from (6) and we proceed as above

$$\theta_1(v, \tau) = \pi\theta'_1(0, \tau) \exp \left[\log(\sin \pi v) + \sum_{n \geq 1} \frac{q^{2n}}{1 - q^{2n}} \frac{(\sin n\pi v)^2}{n} \right].$$

Comparing the latter with the expansion

$$\theta_1(v, \tau) = \pi\theta'_1(0, \tau) \exp \left[\sum_{p \geq 1} e_{2p}(\tau) (\sin \pi v)^{2p} \right]$$

one derives the following

Lemma 2-5 *The coefficients e_{2p} defined above may be written as*

$$e_{2p}(\tau) = -\frac{1}{p} \sum_{k \geq 1} \frac{1}{(\sin k\pi\tau)^{2p}}.$$

In particular, we have

$$e_2(\tau) = 4 \sum_{n \geq 1} \frac{nq^{2n}}{(1 - q^{2n})} = \frac{1}{2\pi^2} \frac{\theta'_1(0, \tau)}{\pi\theta'_1(0, \tau)}$$

We may demonstrate this lemma exactly with the same techniques as those used for the proofs of Lemma 2-2 and 2-4.

On the other hand $|Imv| < Im\tau$ implies $|\frac{\sin \pi v}{(\sin k\pi\tau)}| < 1$ and $|\frac{\sin \pi v}{(\cos k\pi\tau)}| < 1$.

The proof of Theorem 2-3 is then achieved.

6.3. Proof of Theorem 4-1. Starting from relations (7-10) we find the following expressions for $S_j(u, v, t)$.

(7) implies

$$\begin{aligned} \log S_4(u, v, t) &= \log \left[\theta_4\left(\frac{u}{2}, t\right) \right] - \log \left[\theta_4\left(\frac{v}{2}, t\right) \right] = \log \left[\frac{\theta_4\left(\frac{u}{2}, t\right)}{\theta_4(0, t)} \right] - \log \left[\frac{\theta_4\left(\frac{v}{2}, t\right)}{\theta_4(0, t)} \right] \\ &= \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{(-1)^p}{p} \left(\frac{\sin \pi \frac{u}{2}}{(\sinh(k + \frac{1}{2})\pi^2 t)} \right)^{2p} \right] - \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{(-1)^p}{p} \left(\frac{\sin \pi \frac{v}{2}}{(\sinh(k + \frac{1}{2})\pi^2 t)} \right)^{2p} \right] \end{aligned}$$

Thus

$$\log S_4(u, v, t) = \sum_{p \geq 1} \sum_{k \geq 0} \frac{(-1)^p}{p} \left(\frac{(\sin \pi \frac{v}{2})^{2p} - (\sin \pi \frac{u}{2})^{2p}}{(\sinh(k + \frac{1}{2})\pi^2 t)^{2p}} \right). \quad (22)$$

By the same way (8) implies

$$\begin{aligned} \log S_3(u, v, t) &= \log \left[\theta_3\left(\frac{u}{2}, t\right) \right] - \log \left[\theta_3\left(\frac{v}{2}, t\right) \right] \\ &= \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{(-1)^p}{p} \left(\frac{\cos \pi \frac{u}{2}}{(\sinh(k + \frac{1}{2})\pi^2 t)} \right)^{2p} \right] - \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{(-1)^p}{p} \left(\frac{\cos \pi \frac{v}{2}}{(\sinh(k + \frac{1}{2})\pi^2 t)} \right)^{2p} \right] \end{aligned}$$

Thus

$$\log S_3(u, v, t) = \sum_{p \geq 1} \sum_{k \geq 0} \frac{(-1)^p}{p} \left(\frac{(\cos \pi \frac{v}{2})^{2p} - (\cos \pi \frac{u}{2})^{2p}}{(\sinh(k + \frac{1}{2})\pi^2 t)^{2p}} \right). \quad (23)$$

(9) implies

$$\begin{aligned} \log S_1(u, v, t) &= \log \left[\frac{\theta_1\left(\frac{u}{2}, t\right)}{\pi \theta_1'(0, t)} \right] - \log \left[\frac{\theta_1\left(\frac{v}{2}, t\right)}{\pi \theta_1'(0, t)} \right] \\ &= \log\left(\sin \frac{u\pi}{2}\right) \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{(-1)^p}{p} \left(\frac{\sin \pi \frac{u}{2}}{(\sinh k\pi^2 t)} \right)^{2p} \right] \\ &\quad - \log\left(\sin \frac{v\pi}{2}\right) - \left[- \sum_{p \geq 1} \sum_{k \geq 0} \frac{(-1)^p}{p} \left(\frac{\sin \pi \frac{v}{2}}{(\sinh k\pi^2 t)} \right)^{2p} \right] \\ \log S_1(u, v, t) &= \log \left(\frac{\sin \pi \frac{u}{2}}{\sin \pi \frac{v}{2}} \right) + \sum_{p \geq 1} \sum_{k \geq 0} \frac{(-1)^p}{p} \left(\frac{(\sin \pi \frac{v}{2})^{2p} - (\sin \pi \frac{u}{2})^{2p}}{(\sinh k\pi^2 t)^{2p}} \right) \end{aligned} \quad (24)$$

By the same way (10) implies

$$\log S_2(u, v, t) = \log \left(\frac{\cos \pi \frac{u}{2}}{\cos \pi \frac{v}{2}} \right) + \sum_{p \geq 1} \sum_{k \geq 0} \frac{(-1)^p}{p} \left(\frac{(\cos \pi \frac{v}{2})^{2p} - (\cos \pi \frac{u}{2})^{2p}}{(\sinh k\pi^2 t)^{2p}} \right). \quad (25)$$

Notice that since $u < v < 1$ then we have

$$(\cos \pi \frac{v}{2})^{2p} - (\cos \pi \frac{u}{2})^{2p} < 0, \quad (\sin \pi \frac{v}{2})^{2p} - (\sin \pi \frac{u}{2})^{2p} > 0.$$

We have also $\log \left(\frac{\sin \pi \frac{u}{2}}{\sin \pi \frac{v}{2}} \right) < 0$ and $\log \left(\frac{\cos \pi \frac{u}{2}}{\cos \pi \frac{v}{2}} \right) > 0$.

Prove now that $S_4(u, v, t)$ is LCM. Rewrite

$$\begin{aligned} \log S_4(u, v, t) &= \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{2p} \left(\frac{(\sin \pi \frac{v}{2})^{4p} - (\sin \pi \frac{u}{2})^{4p}}{(\sinh(k + \frac{1}{2})\pi^2 t)^{2p}} \right) \\ &\quad - \frac{1}{2p-1} \left(\frac{(\sin \pi \frac{v}{2})^{4p-2} - (\sin \pi \frac{u}{2})^{4p-2}}{(\sinh(k + \frac{1}{2})\pi^2 t)^{2p}} \right) \\ &= \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{(\sinh(k + \frac{1}{2})\pi^2 t)^{4p-2}} \left[\frac{(\sin \pi \frac{v}{2})^{4p} - (\sin \pi \frac{u}{2})^{4p}}{2p(\sinh(k + \frac{1}{2})\pi^2 t)^2} - \frac{(\sin \pi \frac{v}{2})^{4p-2} - (\sin \pi \frac{u}{2})^{4p-2}}{2p-1} \right] \end{aligned}$$

$$= \sum_{p \geq 1} \sum_{k \geq 0} \frac{(\sin \pi \frac{u}{2})^{4p}}{(\sinh(k + \frac{1}{2})\pi^2 t)^{4p-2}} \left[\frac{(\frac{\sin \pi \frac{v}{2}}{\sin \pi \frac{u}{2}})^{4p} - 1}{2p(\sinh(k + \frac{1}{2})\pi^2 t)^2} - \frac{(\frac{\sin \pi \frac{v}{2}}{\sin \pi \frac{u}{2}})^{4p-2} - 1}{(2p-1)(\sin \pi \frac{u}{2})^2} \right].$$

We will show that the bracket

$$\left[\frac{(\frac{\sin \pi \frac{v}{2}}{\sin \pi \frac{u}{2}})^{4p} - 1}{2p(\sinh(k + \frac{1}{2})\pi^2 t)^2} - \frac{(\frac{\sin \pi \frac{v}{2}}{\sin \pi \frac{u}{2}})^{4p-2} - 1}{(2p-1)(\sin \pi \frac{u}{2})^2} \right] \quad (b_1)$$

is non-positive. Consider for that the following function

$$l(x) = \frac{(2x-1)(\sin \pi \frac{u}{2})^2}{2x} \frac{(\frac{\sin \pi \frac{v}{2}}{\sin \pi \frac{u}{2}})^{4x} - 1}{(\frac{\sin \pi \frac{v}{2}}{\sin \pi \frac{u}{2}})^{4x-2} - 1}$$

We may verify thanks to *Maple* that $l(x)$ is increasing as a function with respect to x , and then the following inequalities hold : $l(1) \leq l(x) \leq l(\infty)$. i.e. :

$$\frac{(\sin \pi \frac{u}{2})^2}{2} \frac{(\frac{\sin \pi \frac{v}{2}}{\sin \pi \frac{u}{2}})^4 - 1}{(\frac{\sin \pi \frac{v}{2}}{\sin \pi \frac{u}{2}})^2 - 1} \leq \frac{(2x-1)(\sin \pi \frac{u}{2})^2}{2x} \frac{(\frac{\sin \pi \frac{v}{2}}{\sin \pi \frac{u}{2}})^{4x} - 1}{(\frac{\sin \pi \frac{v}{2}}{\sin \pi \frac{u}{2}})^{4x-2} - 1} \leq (\sin \pi \frac{v}{2})^2.$$

Therefore we may assert when $\sinh(\frac{1}{2}\pi^2 t)^2 \geq (\sin \pi \frac{v}{2})^2 \geq (\sin \pi \frac{u}{2})^2$ then

$$\sinh(\frac{1}{2}\pi^2 t)^2 \geq \frac{2p-1}{2p} \frac{(\sin \pi \frac{v}{2})^{4p} - (\sin \pi \frac{u}{2})^{4p}}{(\sin \pi \frac{v}{2})^{4p-2} - (\sin \pi \frac{u}{2})^{4p-2}}.$$

Thus the bracket (b_1) is non-positive for all integer $p \geq 1$. This means that $\log S_4(u, v, t)$ may be viewed as a polynomial of $\frac{1}{(\sinh(k + \frac{1}{2})\pi^2 t)^2}$ with negative coefficients.

We proceed exactly with the same manner to establish that $\log S_1(u, v, t)$ may be viewed as a polynomial of $\frac{1}{(\sinh k \pi^2 t)^2}$ with negative coefficients.

Turn out to the other quotients. We rewrite

$$\begin{aligned} \log S_3(u, v, t) &= \sum_{p \geq 1} \sum_{k \geq 0} \frac{1}{2p} \left(\frac{(\cos \pi \frac{v}{2})^{4p} - (\cos \pi \frac{u}{2})^{4p}}{(\sinh(k + \frac{1}{2})\pi^2 t)^{2p}} \right) \\ &\quad - \frac{1}{2p-1} \left(\frac{(\cos \pi \frac{v}{2})^{4p-2} - (\cos \pi \frac{u}{2})^{4p-2}}{(\sinh(k + \frac{1}{2})\pi^2 t)^{2p}} \right) \\ &= \sum_{p \geq 1} \sum_{k \geq 0} \frac{(\cos \pi \frac{u}{2})^{4p}}{(\sinh(k + \frac{1}{2})\pi^2 t)^{4p-2}} \left[\frac{(\frac{\cos \pi \frac{v}{2}}{\cos \pi \frac{u}{2}})^{4p} - 1}{2p(\sinh(k + \frac{1}{2})\pi^2 t)^2} - \frac{(\frac{\cos \pi \frac{v}{2}}{\cos \pi \frac{u}{2}})^{4p-2} - 1}{(2p-1)(\cos \pi \frac{u}{2})^2} \right]. \end{aligned}$$

We will prove that the bracket

$$\left[\frac{(\frac{\cos \pi \frac{v}{2}}{\cos \pi \frac{u}{2}})^{4p} - 1}{2p(\sinh(k + \frac{1}{2})\pi^2 t)^2} - \frac{(\frac{\cos \pi \frac{v}{2}}{\cos \pi \frac{u}{2}})^{4p-2} - 1}{(2p-1)(\cos \pi \frac{u}{2})^2} \right] \quad (b_2)$$

is non-negative. Indeed, the following function

$$m(x) = \frac{(2x-1)(\cos \pi \frac{u}{2})^2}{2x} \frac{(\frac{\cos \pi \frac{v}{2}}{\cos \pi \frac{u}{2}})^{4x} - 1}{(\frac{\cos \pi \frac{v}{2}}{\cos \pi \frac{u}{2}})^{4x-2} - 1}$$

is increasing and the following inequalities hold

$$\frac{(\cos \pi \frac{u}{2})^2 \left(\frac{\cos \pi \frac{v}{2}}{\cos \pi \frac{u}{2}} \right)^4 - 1}{2 \left(\frac{\cos \pi \frac{v}{2}}{\cos \pi \frac{u}{2}} \right)^2 - 1} \leq \frac{(2x-1)(\cos \pi \frac{u}{2})^2}{2x} \frac{\left(\frac{\cos \pi \frac{v}{2}}{\cos \pi \frac{u}{2}} \right)^{4x} - 1}{\left(\frac{\cos \pi \frac{v}{2}}{\cos \pi \frac{u}{2}} \right)^{4x-2} - 1} \leq (\cos \pi \frac{v}{2})^2.$$

We may assert when $\sinh(\frac{1}{2}\pi^2 t)^2 \geq (\cos \pi \frac{v}{2})^2 \geq (\cos \pi \frac{u}{2})^2$ then

$$\sinh(\frac{1}{2}\pi^2 t)^2 \geq \frac{2p-1}{2p} \frac{(\cos \pi \frac{v}{2})^{4p} - (\cos \pi \frac{u}{2})^{4p}}{(\cos \pi \frac{v}{2})^{4p-2} - (\cos \pi \frac{u}{2})^{4p-2}}.$$

Recall that since $u < v < 1$ then $\cos(\pi \frac{u}{2}) > \cos(\pi \frac{v}{2})$.

Thus the bracket (b_2) is non-negative for all integer $p \geq 1$. This means that $\log S_3(u, v, t)$ may be viewed as a polynomial of $\frac{1}{(\sinh(k+\frac{1}{2})\pi^2 t)^2}$ with positive coefficients.

We proceed exactly with the same manner to establish that $\log S_2(u, v, t)$ may be viewed as a power series of $\frac{1}{(\sinh k\pi^2 t)^2}$ with positive coefficients.

On the other hand we refer again to Lemma 2-2. Recall that the functions $\frac{1}{\sinh(n\pi^2 t)}$ are LCM functions for $n \in \mathbb{N} - \{0\}$ with respect to $t > 0$. Moreover, any combination with positive coefficients of CM functions is also an CM function.

We have seen that

$$\sum_{p \geq 1} \frac{(-1)^p}{p} \left(\frac{(\sin \pi \frac{v}{2})^{2p} - (\sin \pi \frac{u}{2})^{2p}}{(\sinh(k+\frac{1}{2})\pi^2 t)^{2p}} \right), \quad \sum_{p \geq 1} \frac{(-1)^p}{p} \left(\frac{(\sin \pi \frac{v}{2})^{2p} - (\sin \pi \frac{u}{2})^{2p}}{(\sinh(k+\frac{1}{2})\pi^2 t)^{2p}} \right)$$

are series power of $\frac{1}{(\sinh(k+\frac{1}{2})\pi^2 t)^2}$ or $\frac{1}{(\sinh k\pi^2 t)^2}$ with negative coefficients for fixed k . Then we deduce by recurrence on k that $\log S_4(u, v, t)$ and by the same manner $\log S_1(u, v, t)$ are CM. While we have seen that

$$\sum_{p \geq 1} \frac{(-1)^p}{p} \left(\frac{(\cos \pi \frac{v}{2})^{2p} - (\cos \pi \frac{u}{2})^{2p}}{(\sinh(k+\frac{1}{2})\pi^2 t)^{2p}} \right), \quad \sum_{p \geq 1} \frac{(-1)^p}{p} \left(\frac{(\cos \pi \frac{v}{2})^{2p} - (\cos \pi \frac{u}{2})^{2p}}{(\sinh k\pi^2 t)^{2p}} \right)$$

is a power series of $\frac{1}{(\sinh(k+\frac{1}{2})\pi^2 t)^2}$ or $\frac{1}{(\sinh k\pi^2 t)^2}$ with negative coefficients for fixed k . Then we deduce by recurrence on k that $-\log S_3(u, v, t)$ and $-\log S_2(u, v, t)$ are CM.

The proof of Theorem 4-1 is then achieved.

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