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Conjecture: the set of prime numbers is supernatural

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ABSTRACT

Prime numbers are fascinating by the way they appear in the set of natural numbers. Despite several results enlightening us about their repartition, the set of prime numbers is often informally qualified as misterious. In the present paper, we introduce a formalism allowing to state a formal conjecture: the set of prime numbers is supernatural. Our conjecture has no analog in the existing literature. We explain that this conjecture is expected to be a hard challenge for any kind of intelligence. However it is really natural and even seems very close to be canonical.

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Prologue

Mathematics establishes theorems from axioms. This was developed by humans in strong correlation with parts of full nature. Two facets of nature are very important for the inspiration behind mathematics: abstracts natural numbers and the physical world as perceived by humans. Some mathematical developments of the XX-th century tend to combine these two inspirations (e.g. the Langlands Program, arithmetic geometries). Many conjectures are produced and proved each year, often as generalizations or combinations of existing previous conjectures. Difficulties sometimes come from technicality and, following Grothendieck's principle of "Rising Sea", one can imagine that combinations of several facts and generalizations will culminate in proofs of some of these generalized conjectures. Recently, mathematics received an other inspiration from nature: the hard concept of human intelligence. A combination of machine artificial intelligence (with formal proof assistants) and human supervision will probably participate in solving some of these generalized conjectures and produce connected ideas. We believe this does not describe all the future of mathematics. Indeed, long times ago, machines, created and directed by hu-

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mans, surpassed humans at carrying heavy objects. Later, machines, created and directed by humans, surpassed humans at computing arithmetical operations on everyday life numbers. Now, machines, created and directed by humans, are surpassing humans at checking mathematical proofs. Possibly, soon, machine will surpass humans at generalizing and combining some proofs and resolve some "generalized" conjectures, trained on existing human ideas. All these beautiful machines were created by humans, involving mathematicians, with the help of previous machines. However, since our understanding of ourselves is still very limited today, it is hard to create a (non-human) machine working exactly as humans and surpassing them for all tasks. We saw that machines will continue to surpass (eventually with assistance) humans for some tasks previously done by them. Today, we believe it is still our job to produce radically new mathematical ideas. Possibly, producing radically new mathematical ideas could then be among the next challenges for some next generations of machines (all of the above seems well known, standard and in any case does not claim originality).

The aim of this paper is to introduce such a new idea in number theory. This idea is, to us, beautiful, unique, extremely deep and completely natural with no technicalities. It is not similar to related ideas in the existing fields of mathematics, more precisely, the only inspiration goes back to an error of Fermat many centuries ago. This idea is about natural and prime numbers but does not involve a particular equation or number, it is about the whole structure of natural arithmetic. Though this conjecture seems obviously true, it seems so hard that we do not find any way of solving this conjecture. It seems very hard to "deform" or "weaken" a bit the conjecture to prove it there and find non-trivial inspiration or hope to prove it in general (a weaker form is given in Corollary 3.2, but it still seems very hard). The only way to prove or refute this conjecture seems to produce radically new ideas.

Introduction

We now introduce our conjecture. Let \mathbb{P} be the set of prime numbers. Fermat [Fermat] conjectured that $\{2^{2^n} + 1 \mid n \text{ positive integers}\} \subset \mathbb{P}$. Euler [Euler1, Euler2] proved that this is wrong, i.e. $\{2^{2^n} + 1 \mid n \text{ positive integers}\} \not\subset \mathbb{P}$. Let \mathbb{I} be the set of positive non-zero integers. We say that a function $f : \mathbb{I} \rightarrow \mathbb{I}$ with variable n is a natural function if it is built using formal (and coherent) combinations of the symbols $n, \{a \in \mathbb{I}\}, +, \times, \wedge$. Here \wedge means elevation to the power. We prove that every natural function is constant or strictly increasing (Proposition 1.6). We then conjecture that for every non-constant natural function f , we have $f(\mathbb{I}) \not\subset \mathbb{P}$. This is our conjecture. "Evidences" for this conjecture are intuition about the "irregularity" of the set of prime numbers, and that it is verified for all "small natural functions" that we have tested, cf. e.g. Proposition 2.2. However, testing it for "higher natural functions" is difficult because the numbers involved are too big after few initial computations.

In this text we formally introduce notation and definitions. Then we state our conjecture. Our conjecture was formulated in 2013. We explain that this conjecture can be reformulated as in the title of the article: \mathbb{P} is supernatural. In some very easy cases which seem not inspiring for the general case, we prove this conjecture. Proposition 3.1 shows that if the conjecture is true then $f(\mathbb{I}) \cap (\mathbb{I} \setminus \mathbb{P})$ is an infinite set for all non-constant natural function f . In particular, this "canonical" conjecture implies the existence of infinite composite Fermat numbers (an open question today), cf. Corollary 3.2. Proposition 3.3 explains that if the conjecture is wrong, this would provide a natural function so incredible that we can not believe it.

1. Notation and definition

Let $\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, \dots\}$ be the set of positive integers. Let $\mathbb{I} = \{1, 2, 3, 4, 5, 6, 7, \dots\}$ be the set of non-zero positive integers. Let $\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, \dots\}$ be the set of prime numbers. Let $\mathcal{F}(\mathbb{I}, \mathbb{I})$ be the set of functions from \mathbb{I} to \mathbb{I} .

DEFINITION 1.1. An elevation structure is a 4-uple $(E, +, \times, \wedge)$ where E is a set and

- (i) $+$ is a map $E \times E \rightarrow E$ sending a, b to $a + b$
- (ii) \times is a map $E \times E \rightarrow E$ sending a, b to $a \times b$
- (iii) \wedge is a map $E \times E \rightarrow E$ sending a, b to $a \wedge b$

such that $\forall a, b, c \in E$ the following hold

- (i) $a + b = b + a$
- (ii) $a + (b + c) = (a + b) + c$
- (iii) $b \times a = a \times b$
- (iv) $a \times (b \times c) = (a \times b) \times c$
- (v) $a \times (b + c) = a \times b + a \times c$
- (vi) $(a \wedge b) \times (a \wedge c) = a \wedge (b + c)$
- (vii) $(a \wedge b) \wedge c = a \wedge (b \times c)$.

We write $a_1 \wedge a_2 \wedge \dots \wedge a_n$ instead of $a_1 \wedge (a_2 \wedge (\dots \wedge a_n))$, be careful that $a \wedge (b \wedge c) \neq (a \wedge b) \wedge c$ in general. We write often $a_1^{a_2^{a_n}}$ instead of $a_1 \wedge a_2 \wedge \dots \wedge a_n$. For example a^{b^c} means $a \wedge (b \wedge c)$.

A morphism of elevation structure from $(E, +, \times, \wedge)$ to $(F, +, \times, \wedge)$ is a map $M : E \rightarrow F$ such that $\forall a, b \in E$, we have $M(a + b) = M(a) + M(b)$, $M(a \times b) = M(a) \times M(b)$ and $M(a \wedge b) = M(a) \wedge M(b)$. We obtain a category.

EXAMPLE 1.2.

- (i) $\mathbb{N}, +, \times, \wedge$ is an elevation structure with $a \wedge b := a^b$.
- (ii) $\mathbb{I}, +, \times, \wedge$ is an elevation structure.
- (iii) $\mathcal{F}(\mathbb{I}, \mathbb{I}), +, \times, \wedge$ is an elevation structure with $(f \wedge g)(n) := f(n) \wedge g(n)$.
- (iv) Let $a \in \mathbb{I}$, then the evaluation map $E_a : \mathcal{F}(\mathbb{I}, \mathbb{I}) \rightarrow \mathbb{I}$, $f \mapsto f(a)$, is a morphism of elevation structure from $\mathcal{F}(\mathbb{I}, \mathbb{I}), +, \times, \wedge$ to $\mathbb{I}, +, \times, \wedge$.

Let $P(\mathcal{F}(\mathbb{I}, \mathbb{I}))$ be the set of all parts of $\mathcal{F}(\mathbb{I}, \mathbb{I})$. Let $\mathcal{F}_s \in P(\mathcal{F}(\mathbb{I}, \mathbb{I}))$ be the part of functions constituted of the identity map $n \mapsto n$ and the constant maps $n \mapsto a$ for every $a \in \mathbb{I}$, we also call it the part of symbols.

We now define some operators on $P(\mathcal{F}(\mathbb{I}, \mathbb{I}))$.

DEFINITION 1.3.

- (i) Let A_+ be the map sending $P \in P(\mathcal{F}(\mathbb{I}, \mathbb{I}))$ to

$$P \cup \{f \in \mathcal{F}(\mathbb{I}, \mathbb{I}) \mid \exists g \in P, \exists h \in P \text{ and } f = g + h\}.$$

- (ii) Let A_\times be the map sending $P \in P(\mathcal{F}(\mathbb{I}, \mathbb{I}))$ to

$$P \cup \{f \in \mathcal{F}(\mathbb{I}, \mathbb{I}) \mid \exists g \in P, \exists h \in P \text{ and } f = g \times h\}.$$

(iii) Let A_\wedge be the map sending $P \in P(\mathcal{F}(\mathbb{I}, \mathbb{I}))$ to

$$P \cup \{f \in \mathcal{F}(\mathbb{I}, \mathbb{I}) \mid \exists g \in P, \exists h \in P \text{ and } f = g \wedge h\}.$$

We are now able to define the set of natural functions $\mathcal{F}_{Natural}$. Let Σ be the set of all finite words with letters A_+, A_\times, A_\wedge .

DEFINITION 1.4. We put $\mathcal{F}_{Natural} = \bigcup_{\sigma \in \Sigma} \sigma(\mathcal{F}_s) \in P(\mathcal{F}(\mathbb{I}, \mathbb{I}))$. The 4-uple $\mathcal{F}_{Natural}, +, \times, \wedge$ is an elevation structure.

EXAMPLE 1.5.

- (i) $\mathcal{F}_s \subset \mathcal{F}_{Natural}$
- (ii) The set of polynomial functions on \mathbb{I} $\mathcal{F}_{Polynomial}$ is contained in $\mathcal{F}_{Natural}$. More precisely it is contained in $\bigcup_{j, k \geq 0} A_+^j A_\times^k(\mathcal{F}_s)$.
- (iii) Fermat's function $n \mapsto 2^{2^n} + 1$ is contained in $\mathcal{F}_{Natural}$. More precisely it is contained in $A_+ A_\wedge^2(\mathcal{F}_s)$.
- (iv) The functions

$$\begin{aligned} n &\mapsto n^n + n + 1 \\ n &\mapsto 7^n + 6 \\ n &\mapsto n^{4n^n + n^{2^3} + 2^n + 8} + n + 3 \\ n &\mapsto 2^{2^{2^{2^{2^{2^n}}}}} + 1 \end{aligned}$$

are natural functions.

PROPOSITION 1.6. A natural function $f \in \mathcal{F}_{natural}$ is constant or strictly increasing.

Proof. By definition a function f is in $\mathcal{F}_{natural}$ is and only if there exists a finite word σ in letters A_+, A_\times, A_\wedge such that $f \in \sigma(\mathcal{F}_s)$. We need the following definition.

DEFINITION 1.7. The length of a natural function $f \in \mathcal{F}_{Natural}$ is the number

$$\min\{n \in \mathbb{N} \mid f \in \sigma(\mathcal{F}_s) \text{ with } \sigma \text{ a word of length } n\},$$

it is well defined.

Let us prove our result by induction on length. Consider the following assertion.

$$P_n : \text{Proposition 1.6 is true for every } f \in \mathcal{F}_{Natural} \text{ of length } \leq n.$$

Let us prove by induction that P_n is true for every $n \in \mathbb{N}$. We need the following Lemmas.

LEMMA 1.8. Let $h, g \in \mathcal{F}_{Natural}$ be strictly increasing, then $h + g, h \times g, h \wedge g$ are strictly increasing.

Proof. Let $i > j$ be integers in \mathbb{I} , then $h(i) > h(j)$ and $g(i) > g(j)$. So $h(i) + g(i) > h(j) + g(j)$, consequently $h + g$ is strictly increasing. Since $g(i) > g(j) \geq 1$ and $h(i) > h(j) \geq 1$; we have $h(i) \times g(i) > h(j) \times g(j)$ and consequently $h \times g$ is strictly increasing; we also have $h(i)^{g(i)} > h(j)^{g(j)}$ and consequently $h \wedge g$ is strictly increasing. \square

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LEMMA 1.9. *Let $h \in \mathcal{F}_{Natural}$ be strictly increasing and $g \in \mathcal{F}_{Natural}$ be constant, then $h + g, h \times g, h \wedge g$ are strictly increasing.*

Proof. Let $i > j$ be integers in \mathbb{I} , then $h(i) > h(j)$ and $g(i) = g(j)$. So $h(i) + g(i) > h(j) + g(j)$, consequently $h + g$ is strictly increasing. Since $g(i) = g(j) \geq 1$ and $h(i) > h(j)$, we have $h(i) \times g(i) > h(j) \times g(j)$ and consequently $h \times g$ is strictly increasing. We have $h(i)^{g(i)} > h(j)^{g(j)}$ and consequently $h \wedge g$ is strictly increasing. \square

LEMMA 1.10. *Let $h \in \mathcal{F}_{Natural}$ be strictly increasing and $g \in \mathcal{F}_{Natural}$ be constant, then $g \wedge h$ is strictly increasing except if $g = 1$. If $g = 1$, $g \wedge h$ is constant.*

Proof. Let $i > j$ be integers in \mathbb{I} . If $g = 1$ then $g(i)^{h(i)} = g(j)^{h(j)} = 1$ and $g \wedge h$ is constant. In the other case $g(i) = g(j) > 1$ and $h(i) > h(j)$, consequently $g(i)^{h(i)} > g(j)^{h(j)}$. This finishes the proof. \square

Now let us start our induction. If $n = 0$, then f is constant or the identity map and thus satisfies P_0 . Now assume that P_n is true and let us prove that P_{n+1} is true. Let f of length $n + 1$. By definition, there are h, g of length $\leq n$ such that $f = h + g$ or $f = h \times g$ or $f = g \wedge h$ or $f = h \wedge g$. By induction hypothesis h and g are constant or strictly increasing. Assume first both are constant. Then f is obviously constant. If both are strictly increasing, then f is strictly increasing by Lemma 1.8. If h is strictly increasing and g is constant, then by Lemma 1.9, $h + g, h \times g$ and $h \wedge g$ are strictly increasing, and by Lemma 1.10, $g \wedge h$ is strictly increasing or constant. If h is constant and g is strictly increasing, we deduce, in the same way, that $h + g, h \times g, h \wedge g$ and $g \wedge h$ are strictly increasing or constant. Consequently, f is constant or strictly increasing. This finishes the induction and the proof of Proposition. \square

REMARK 1.11. *The map $\mathbb{N} \rightarrow \mathbb{N}, n \mapsto n^n$ is neither constant nor strictly increasing since $0^0 = 1^1 \neq 2^2$. Its restriction to \mathbb{I} is a strictly increasing natural function.*

2. Statement of the conjecture

2.1 Statement

We now state our conjecture.

CONJECTURE 2.1. *Let $f \in \mathcal{F}_{Natural}$ be a non-constant natural function. Then*

$$f(\mathbb{I}) \not\subseteq \mathbb{P}.$$

PROPOSITION 2.2. *Conjecture 2.1 is true in the following cases.*

- (i) *Let $f \in \mathcal{F}_{Polynomial}$, assume f is non-constant, then $f(\mathbb{I}) \not\subseteq \mathbb{P}$.*
- (ii) *Let $a \in \mathbb{I}_{>1}, b \in \mathbb{N}$ and f be $n \mapsto a^n + b$, then $f(\mathbb{I}) \not\subseteq \mathbb{P}$.*
- (iii) *Let f be $n \mapsto 2^{2^n} + 1$, the Fermat function. Then $f(\mathbb{I}) \not\subseteq \mathbb{P}$.*

Proof. (i) Since f is non-constant, it is strictly increasing by 1.6, so there exists $n \in \mathbb{I}$ such that $f(n) > 1$. Let $p \in \mathbb{P}$ such that $p \mid f(n)$. Then $f(n) = 0 \pmod p$. Since f is polynomial, we have

$$f(n + p) = f(n) \pmod p.$$

So we have

$$p \mid f(n) \quad , \quad p \mid f(n+p) \quad , \quad 1 < f(n) < f(n+p).$$

This implies that $f(n+p)$ is not a prime number. So $f(\mathbb{I}) \not\subseteq \mathbb{P}$.

- (ii) Let $n \in \mathbb{I}_{>1}$. Then $f(n) = a^n + b > 1$. Let $p \in \mathbb{P}$ such that $p \mid f(n)$. Then $a^n + b = 0 \pmod p$. Assume first that $p \nmid a$, then by Fermat's little theorem

$$a^{n+(p-1)} + b = a^n + b = 0 \pmod p.$$

So we have

$$p \mid f(n) \quad , \quad p \mid f(n+(p-1)) \quad , \quad 1 < f(n) < f(n+(p-1)).$$

This implies that $f(n+p-1)$ is not a prime number. Now if $p \mid a$, then $p < f(n)$ and $f(n)$ is not a prime number. So we have proved that $f(\mathbb{I}) \not\subseteq \mathbb{P}$.

- (iii) We have $f(5) = 2^{2^5} + 1 = 4294967297 = 641 \times 6700417$, as Euler [Euler1, Euler2] computed. So $f(\mathbb{I}) \not\subseteq \mathbb{P}$. □

2.2 Reformulation: \mathbb{P} is supernatural

An infinite subset of \mathbb{I} is said to be natural if it is of the form $f(\mathbb{I})$ where $f \in \mathcal{F}_{Natural}$ is non-constant. An infinite subset of \mathbb{I} is said to be supernatural if it does not contain any natural subset of \mathbb{I} . Our conjecture is now equivalent to the following statement: \mathbb{P} is supernatural.

3. Implications and remarks

In this section we discuss implications of our conjecture and comment it.

PROPOSITION 3.1. *Assume Conjecture 2.1 is true. Let $f \in \mathcal{F}_{Natural}$ non-constant, then $\{x \in f(\mathbb{I}) \mid x \text{ is not a prime number}\}$ is infinite.*

Proof. By 1.6, f is strictly increasing. It is enough to prove the following statement. *For all $k \in \mathbb{I}$, there exists $q \in \mathbb{N}_{>k}$, such that $f(q)$ is not a prime number.* So let $k \in \mathbb{I}$. Let us consider the function $g \in \mathcal{F}(\mathbb{I}, \mathbb{I})$ defined by $g(n) = f(n+k)$. The function g is natural. So there exists $d \in \mathbb{I}$ such that $g(d)$ is not a prime number. So $q := d+k$ is such that $q > k$ and $f(q)$ is not a prime number. This finishes the proof. □

COROLLARY 3.2. *Assume Conjecture 2.1 is true. Then there are infinitely many composite (i.e. not prime) Fermat's numbers (this can be seen as a weaker form of our conjecture, but still seems very hard).*

Proof. Apply 3.1 to Fermat's natural function $n \mapsto 2^{2^n} + 1$. □

PROPOSITION 3.3. *Assume Conjecture 2.1 is wrong. Then there exists an "arithmetical" (relying only on $+$, \times , \wedge) formula giving arbitrary big prime numbers.*

Proof. If Conjecture 2.1 is wrong. There exists a non-constant (and thus strictly increasing) natural function giving only prime numbers. It is "arithmetical". □

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REMARK 3.4. In order to define natural functions in Definition 1.4, we have used $+$, \times , \wedge . One can define with the same formalism a larger class of function using moreover Knuth's up arrow $\{\uparrow^n \mid n \geq 1\}$ notation for hyperoperation ($\uparrow^1 = \wedge$). Then one can extend Conjecture 2.1 to this larger class of functions.

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