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Optimal epidemic suppression under an ICU constraint: an analytical solution*

Laurent Miclo[†], Daniel Spiro[‡] and Jörgen Weibull[§]

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Abstract

How much and when should we limit economic and social activity to ensure that the health-care system is not overwhelmed during an epidemic? We study a setting where ICU resources are constrained and suppression is costly. Providing a fully analytical solution we show that the common wisdom of “flattening the curve”, where suppression measures are continuously taken to hold down the spread throughout the epidemic, is suboptimal. Instead, the optimal suppression is discontinuous. The epidemic should be left unregulated in a first phase and when the ICU constraint is approaching society should quickly lock down (a discontinuity). After the lockdown, regulation should gradually be lifted, holding the rate of infected constant, thus respecting the ICU resources while not unnecessarily limiting economic activity. In a final phase, regulation is lifted. We call this strategy “filling the box”. The cost under the optimal strategy is obtained in closed form as an explicit function of economic and medical fundamentals. We show that the policy is optimal also when there, in addition, is a small cost associated with the number of infected. The tighter the ICU constraint, the wider is the range of such cost for which the policy is still optimal. This suggests the primary focus of poor countries (with few ICU resources) should be to protect the health-care system, while richer countries (with extensive ICU resources) may strive to reduce the number of infected even more.

Keywords: Epidemic; Optimal control; Health; Suppression; Infection; Corona.

JEL codes: C61; D6; I18.

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[†]Toulouse Institute of Mathematics, Toulouse School of Economics and CNRS. Email: laurent.miclo@math.cnrs.fr. Acknowledges funding from the French National Research Agency (ANR) under the Investments for the Future program (Investissements d’Avenir, grant ANR-17-EURE-0010). The funding source had no impact on the research.

[‡]Corresponding author. Dept. of Economics, Uppsala University, Box 513, 751 20 Uppsala, Sweden. Email: daniel.spiro.ec@gmail.com. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

[§]Stockholm School of Economics. Email: jorgen.weibull@hhs.se. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

1 Introduction

Amid the Covid-19 health and economic crisis one question has been standing at the centre of professional opinion: How much and when should we limit economic and social activity to ensure that the health-care system is not overwhelmed? This question embodies two simultaneous goals when fighting a pandemic. First, to ensure that each infected person gets the best possible care, we need to ensure that the capacity of the health-care system (henceforth the ICU constraint) is never breached. This goal was perhaps best epitomized by the UK slogan “Protect the NHS” and by the Imperial College report (Ferguson et al., 2020). The second goal is to minimize the socioeconomic costs associated with epidemic suppression. Essentially, the more one is suppressing the spread, the costlier it is since, absent a vaccine, suppression boils down to keeping people away from each other, thus limiting economic activity and social life.

This paper extends the standard S.I.R. model (Kermack and McKendrick, 1927) to provide an *analytical* answer to the above question. Our answer departs from common wisdom. During the Covid-19 pandemic, authorities, news reporting and policy makers popularized the ideal policy as “flattening the curve”,¹ i.e., imposing continuous limitations to lower the number of simultaneously infected in all time periods. This would ensure that the peak of the curve never crossed the ICU constraint. We show that this policy is suboptimal. Instead, the optimal policy can be described as “filling the box”, involving a sudden and discontinuous suppression. More precisely, it prescribes (Theorem 1 and Figure 2) leaving the spread unregulated during a first phase. As the number of infected approaches the ICU constraint we enter a second phase where harsh suppression measures are imposed at once (a discontinuity) but afterwards gradually relaxed. The aim of policy in this second phase is to precisely stop the number of infected from exceeding the ICU constraint and keep it constant at that level. The discontinuous tightening followed by gradual relaxation of suppression is optimal since the underlying growth of infections is highest in the beginning of this phase. In a third phase, once the underlying growth of infections subsides and herd immunity under laissez-faire has been obtained, no suppression measures are taken.

We further fully solve for the optimized cost in closed form as a function of economic and medical fundamentals. This is thus a lower bound, within the S.I.R. model, below which no policy whatsoever (in our wide class) can bring down the costs associated with a given ICU capacity constraint. This lower cost bound is higher the lower is the ICU capacity. The lower cost bound is increasing in the basic reproduction number and in the population share of initially infected, for instance, how many infected individuals arrived from other regions or countries at the start. The expression obtained (see equation (20)) makes all these dependencies explicit and allows for quantification.

Our main contribution is methodological; we show how to analytically solve — from first principles without relying on numerical simulations or quantitative

¹See, e.g., the Imperial College report (Ferguson et al., 2020), Branswell (2020), Time (2020), Pueyo (2020), even Donald Trump (The Sun, 2020) and many more.

calculations — a classic epidemic-economic model for the optimal suppression policy. Importantly, the suppression policy is allowed to be fully time varying, including potential discontinuities. This clearly distinguishes our paper both from papers that study a subset of simpler policies analytically (e.g., Loertscher and Muir, 2021), and from papers that study particular policies by way of numerical simulation.²

We focus on analytically solving the mentioned optimization problem, minimization of the socioeconomic costs associated with limitations of economic and social activity, subject to a health-capacity constraint. Clearly, this optimization program neglects many other relevant aspects. In particular, we do not consider non-linear economic costs, vaccine arrival, endogenous expansion of the health-care system and political constraints to suppression. We discuss the possible effects of such extensions on our results in the concluding section. However, since the classical S.I.R. model is ubiquitous within economic-epidemic modeling and taught to so many students around the globe, we think it is important to illuminate the model’s properties and its capability to permit analytical solutions to relevant policy questions. One important aspect is the cost associated with being infected (including potentially harmful long-term health consequences or death), for the afflicted individuals themselves and for society at large. We show that our main result (Theorem 1) is optimal also under sufficiently small such costs (see Theorem 2). Moreover, “filling the box” is optimal under larger infection costs if the ICU constraint is more binding. This has the policy implication that developing countries (with few ICU resources) should primarily focus on ensuring their health-care system is not overwhelmed. Richer countries (with more ICU resources) on the other hand should also focus on reducing the infection numbers *per se*. The intuition for this is that countries with a low ICU capacity need to implement extensive suppression in order to ensure that the capacity is not breached. Therefore, further suppression, to generally reduce infection, becomes redundant.

In the epidemiology literature there exist other papers with analytical solutions for optimal policy.³ For a literature review on the early research see Wickwire (1977). Many papers model vaccinations (Morton and Wickwire, 1974; Ledzewicz and Schättler, 2011; Hu and Zou, 2014, Laguzet and Turinici, 2015; Maurer and de Pinho, 2015), some model screening (Ainseba & Iannelli, 2012). The previous papers focusing on suppression (or quarantine) either restrict the policy (e.g., diLauro et al., 2020, see also Nowzari 2016 for a review) or abstract

²In a rich numerical model, Favero et al. (2020) analyze ways to restart the Italian economy. They do take into account that ICU capacity is limited among many other things and analyze a subset of policy options (containing not only suppression) but not global optimality. Also Gollier (2020a) analyzes the effect on an ICU constraint among many other things in a simulation model. For numerically solved models with other focuses see, e.g., Wearing et al. (2005), Iacoviello and Liuzzi (2008), Lee et al. (2011), Kar and Batabyal (2011), Iacoviello and Stasio (2013), Giamberardino and Iacoviello (2017), Wang (2020), Farboodi et al. (2020), Eichenbaum et al. (2020), and Alvarez et al. (2020).

³There is a much larger literature studying epidemics without controls, of course, see for instance Dickison et al (2012) and Brauer and Castillo-Chavez (2011), Pastor-Satorras et al. (2015) and references therein.

from the fact that increasing the suppression is costly, obviously a key aspect of any economic analysis.⁴

The papers most closely related to ours are Toxvaerd (2020), Kruse and Strack (2020), and Loertscher and Muir (2021). The last paper is the closest. Their model and question are similar to ours. However, their analytical result concerns only constant suppression levels. They solve for optimal time-varying policies only numerically. By contrast, we (a) solve analytically for a fully time-varying policy (Theorem 1), (b) obtain a closed-form solution for the indirect cost function, thus allowing for analytical comparative statics (equation (20)), and (c) obtain a robustness result concerning the cost of the number of infected (Theorem 2).

Toxvaerd (2020), though being fully analytical, has a different research question and, indeed, a different model than us, focusing on individual choice of social distancing (a form of self-suppression). He does not analyze social optimality and abstracts from the ICU capacity which are the focus of our paper. Yet, interestingly, Toxvaerd (2020) finds that individuals may, by their own individual choices, collectively create an infection spread which graphically looks similar to the one that we show is optimal.

Kruse and Strack (2020), finally, also look at optimal suppression with costs which are increasing in suppression. They show existence of an optimizer for a rather general health-cost function but only solve for the optimizer in the special case where the health costs are linear in the number of currently infected.⁵ This is equivalent to assuming that the total number of deaths (over time) is proportional to the total number of infected (the linearity assumption implies bang-bang solutions for suppression) so it does not (directly) matter how many are infected at the same time (in that sense, their paper is similar to Grigorieva et al., 2016, and Grigorieva and Khailov, 2014). Our contribution is thus complementary since in our model the planner wants to avoid too many infected at the same time – respecting the capacity of the health-care system. The extension where we add a cost of the number of infected (Theorem 2) is in essence a convex combination of Kruse and Strack’s (2020) and our model.

⁴Many papers analytically solve for a suppression policy while respecting a budget constraint (Hansen and Day, 2011; Bolzoni et al., 2019) or a time constraint (so that the suppression cannot be too long, Morris et al 2020) but disregarding that more suppression within a time period is costlier than less suppression (Bolzoni et al., 2017; Piunovskiy et al., 2019). This is isomorphic to restricting the suppression policy to be binary since, once there is suppression within a time period, it may as well be at full force. We allow the suppression policy to take any value within a period and change in any way between time periods. Grigorieva et al. (2016) and Grigorieva and Khailov (2014) analyze an objective of minimizing the number of infectious during or at some end period, but the control bears no cost. Abakus (1973) and Behncke (2000) analyze an objective of minimizing the total (over time) number of infected (see Behncke, 2000, Section 3) but the cost of putting a person in quarantine is only taken once so is independent of the length of quarantine. Finally, Gonzales-Eiras and Niepelt (2020) analyze an S.I. model, finding, just like some of the papers above, that the optimal control is binary.

⁵They also show existence of an optimizer when adding that a vaccine may arrive. For dual approaches to the problem solved by Kruse and Strack (2020), see Bliman and Duprez (2021) and Bliman et al. (2021).

2 Model

Our model setup closely follows the canonical Susceptible-Infectious-Removed model (Kermack and McKendrick 1927; see also Brauer and Castillo-Chavez, 2011, for an excellent overview). At any time $t \geq 0$, let $x_1(t)$ be the population share of individuals who at time t are susceptible to the infection, and let $x_2(t)$ be the population share of individuals who are infected at time t . All infected individuals are assumed to be contagious, and population shares are defined with respect to the initial population size, N . Let $m(t)$ be the rate at time t of pairwise meetings between susceptible and infected, and let $q(t)$ be the probability of contagion when an infected person meets a susceptible person at time t . Write $b(t) = m(t)q(t)$. Infected individuals are removed from the population at rate $\alpha > 0$. This may be either because they become immune or because they die. An important assumption is that those who are infected never again become susceptible.

The population dynamic is then defined by the following simple system of ordinary differential equations:

$$\begin{cases} \dot{x}_1(t) = -b(t)x_1(t)x_2(t) \\ \dot{x}_2(t) = b(t)x_1(t)x_2(t) - \alpha x_2(t) \end{cases} \quad (1)$$

The initial condition is $x_1(0) = 1 - \varepsilon$ and $x_2(0) = \varepsilon$, for some $\varepsilon \in (0, 1)$. That is, the infection enters the population at time zero in a population share $\varepsilon > 0$. The state space of this dynamic is $\Delta = \{x = (x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 \leq 1\}$. The only difference from the standard S.I.R. model is that the propagation coefficient $b(t)$, instead of being a constant over time, is here allowed to vary over time.⁶

Indeed, we will view $(b(t))_{t \geq 0}$ as a non-negative function in the hands of a social planner who strives to minimize the economic and social costs of limiting economic activity and/or social life, while never letting the population share of infected individuals, $x_2(t)$, exceed an exogenously given level γ . The latter is interpreted as the capacity of the health-care system to give full treatment to infected patients. We refer to it as the *ICU capacity* or *constraint*.⁷ It is meant to capture a situation, such as under Covid-19, where if the number of simultaneously and seriously ill exceeds the number of respirators some patients may have to be left to die. Not breaching the ICU capacity thus ensures that all get the full care. To allow for risk – e.g., due to intrinsic stochasticity or uncertainty about parameter values – the ICU constraint can of course also include a margin to the actual limit.

We assume that the cost of keeping $b(t)$ below its *natural*, or *unregulated level* β is a linear function of the difference, while there is no cost of moving $b(t)$ above β . The latter assumption is made to “tilt the table” against us in the subsequent analysis, where we will show that it is suboptimal to enhance the

⁶If $x_3(t)$ denotes the population share of removed individuals in a standard S.I.R. model, then its dynamic is $\dot{x}_3(t) = \alpha x_2(t)$, and $x_1(t) + x_2(t) + x_3(t) = 1$ at all times $t \geq 0$.

⁷If, for example, on average 20% of those infected need intensive care and the number of ICUs is C in a population of size N , then $\gamma = 5C/N$.

propagation of the infection even if this can be done at no cost. Formally, the cost functional $C : \mathcal{B} \rightarrow \mathbb{R}_+$ is defined by

$$C(b) = \int_0^\infty [\beta - b(t)]_+ dt. \quad (2)$$

where \mathcal{B} is the class of piecewise continuous functions $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that have finitely many points of discontinuity (allowing for no discontinuity at all). Let $x(0) \equiv (x_1(0), x_2(0))$ be the population state at time $t = 0$ and let

$$c^*(x(0), \gamma) = \inf_{b \in \mathcal{B}(x(0), \gamma)} C(b) \quad (3)$$

Here $\mathcal{B}(x(0), \gamma)$, for any given $x(0)$ and $\gamma > 0$, is the subset of functions in \mathcal{B} for which $x_2(t) \leq \gamma$ at all times $t \geq 0$.⁸ Thus $c^*(x(0), \gamma)$ is the greatest lower bound on the socio-economic cost of keeping the infection from ever surpassing the ICU capacity constraint γ , when the initial population state is $x(0)$. The social planner seeks to find a policy $b \in \mathcal{B}(x(0), \gamma)$ that achieves this theoretically lowest cost. Does such a policy exist? If so, can optimal policies be characterized in terms of the primitives of the S.I.R. model? How does $c^*(x(0), \gamma)$ depend on the health-care capacity γ , the initial population share of infected $x_2(0)$ (say when $x_2(0) + x_1(0) = 1$), the natural propagation rate β , and the recovery rate α ? These are the questions we here raise and answer.

We focus on situations in which $\varepsilon < \gamma$, that is, when the initial infection level is below the ICU capacity constraint. Moreover, we assume that $\beta > \alpha$. Otherwise the population share of infected individuals does not increase from its initial value, which would imply herd immunity already from the outset, and thus the social planner's optimization program then has a trivial solution; *laissez-faire*, that is, $b(t) \equiv \beta$. The *basic reproduction number*, or *basic reproduction rate*, is $R_0 = \beta/\alpha$.

3 Preliminaries

The class \mathcal{B} of policies allowed for in this analysis is very large (an infinite-dimensional function space). However, it can be shown that for any function $b \in \mathcal{B}$, equations (1) define a unique solution trajectory through any given state $(x_1^0, x_2^0) \in \Delta$ and time $t_0 \geq 0$ (see Appendix for a proof.). Evidently, all constant functions $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with $b(t) = \delta$ for some $\delta > 0$, belong to \mathcal{B} . However, they do not all belong to $\mathcal{B}(x(0), \gamma)$, i.e., they may violate the ICU constraint. It is easy to show that such constant policies belong to $\mathcal{B}(x(0), \gamma)$ if δ is sufficiently low, for any given $\gamma > 0$. Thus, to choose δ as high as possible, while keeping $x_2(t) \leq \gamma$ for all $t \geq 0$, is a feasible policy (belongs to $\mathcal{B}(x(0), \gamma)$), and can

⁸To be more precise, we require that there is a finite set $T \subset \mathbb{R}_+$ such that the function $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous at all other points, and that it is everywhere left-continuous and has a right limit. We also require that b is positive except on at most finitely many connected components.

be called *flattening the curve*.⁹ However, such a policy incurs an infinite cost if $\delta < \beta$, since it lasts forever. An alternative feasible control function, with finite cost, is to only temporarily keep $b(t)$ at a constant level $\delta < \beta$, where δ is such that $x_2(t) \leq \gamma$ for all $t \geq 0$. However, as will be shown below, also such “temporary constant shut down” policies are suboptimal. Before turning to the formal statement of our main result, we analyze some general properties of the dynamic induced by (1).

3.1 Dynamic

Some well-known properties of the solutions to standard S.I.R. models hold also here (see Brauer and Castillo-Chavez, 2011). A key such property is that the population share of susceptible individuals, $x_1(t)$, is non-increasing over time t . Roughly speaking, this follows from the first equation in (1), since $b(t)$ is always non-negative and $x_2(t)$ is positive at all times $t \geq 0$. Being bounded from below by zero, $x_1(t)$ necessarily has a limit value as $t \rightarrow \infty$, which we denote x_1^∞ . According to (1), also the sum $x_2(t) + x_1(t)$ is strictly decreasing over time t , and hence also this sum has a limit value. If $x_2^\infty > 0$, then $x_1(t) + x_2(t) \rightarrow -\infty$ which implies that $x_2^\infty = 0$. In other words, in the very long run, the population share of (currently) infected individuals tends to zero. Denoting by x_3^∞ the total population share of removed individuals during the whole epidemic, we thus have $x_3^\infty = 1 - x_1^\infty$, and Nx_3^∞ is approximately (for large N), the total number of individuals who were infected during the epidemic.

Let us now consider the solution to (1) through any given state $x^o = (x_1^o, x_2^o) \in \Delta$ at any time $t_0 \geq 0$, where $0 < x_1^o < 1$ and $0 < x_2^o < 1$. Dividing both sides of the first equation in (1) by $x_1(t) > 0$ and integrating, we obtain

$$\ln x_1(t) = \ln x_1^o - \int_{t_0}^t b(s) x_2(s) ds \quad \forall t \geq t_0. \quad (4)$$

Moreover, integrating the sum of the two equations in (1), we obtain

$$x_1(t) + x_2(t) = x_1^o + x_2^o - \alpha \int_{t_0}^t x_2(s) ds \quad \forall t \geq t_0. \quad (5)$$

3.2 Constant policy

In particular, if $b(t) = \delta > 0$ for all $t \geq t_0$, for some $\delta > 0$, then for all $t \geq t_0$:

$$\ln \left[\frac{x_1(t)}{x_1^o} \right] = -\delta \int_{t_0}^t x_2(s) ds = \frac{\delta}{\alpha} [x_1(t) - x_1^o + x_2(t) - x_2^o], \quad (6)$$

or

$$x_1(t) + x_2(t) = \frac{\alpha}{\delta} \ln \left[\frac{x_1(t)}{x_1^o} \right] + x_1^o + x_2^o \quad \forall t \geq t_0. \quad (7)$$

⁹In the public debate, “flattening the curve” has not been provided a mathematically exact meaning. We perceive the common usage to suggest a constant suppression level which is also what we will mean formally when using that term in the paper.

This equation is well-known for S.I.R. models. Moreover, (7) implies that $(x_1(t), x_2(t)) \rightarrow (x_1^\infty, 0) \in \Delta$, where x_1^∞ by continuity solves (7) for $x_2(t) = 0$, so

$$x_1^\infty = \frac{\alpha}{\delta} \ln \left[\frac{x_1^\infty}{x_1^o} \right] + x_1^o + x_2^o. \quad (8)$$

Since $x_1(t)$ is strictly decreasing, $x_1^\infty < x_1^o$. It is easily verified that the fixed-point equation (8) has a unique solution $x_1^\infty \in (0, x_1^o)$.¹⁰ Equation (8) will later be used to calculate the population share that were ever infected during the epidemic under the optimal suppression policy.

The value

$$\hat{x}_2 = \sup_{t \geq 0} x_2(t) \quad (9)$$

is the peak infection level. It is obtained when $\dot{x}_2 = 0$, or, equivalently (by (1)), when $x_1 = \alpha/\delta$. From (7), for $x_1^o = 1 - \varepsilon$ and $x_2^o = \varepsilon$, we obtain

$$\hat{x}_2 = 1 + \frac{\alpha}{\delta} \ln \left[\frac{\alpha}{\delta(1 - \varepsilon)} \right] - \frac{\alpha}{\delta}. \quad (10)$$

The right-hand side is a strictly decreasing function of the ratio α/δ . Thus, the infection peak is higher the larger δ is and the smaller α is.

Once the population share $x_1(t)$ of susceptible individuals has fallen below the level α/δ , achieved precisely when $x_2(t) = \hat{x}_2$, herd immunity is obtained; the population share $x_2(t)$ of infected individuals falls, even without intervention. To see this, note that according to (1), $x_1(t) = \beta/\alpha = R_0$ if $\dot{x}_2(t) = 0$ and $b(t) = \beta$. In particular, the limit state as $t \rightarrow \infty$ is Lyapunov stable. That is, there is no risk of a second infection wave, since after any small perturbation of the limit population state $(x_1^\infty, 0) \in \Delta$, obtained by exogenously inserting a small population share of infected individuals, the population share of infected individuals will fall gradually back towards zero, while the population share of susceptible individuals gradually moves towards a somewhat lower, new limit value.

Equation (10) is particularly relevant for the case when $\delta = \beta$, that is, under *laissez-faire*. Because if the peak of the infection wave then does not exceed the ICU capacity constraint, that is, if

$$1 + \frac{\alpha}{\beta} \ln \left(\frac{\alpha}{\beta(1 - \varepsilon)} \right) - \frac{\alpha}{\beta} \leq \gamma, \quad (11)$$

then *laissez-faire* is evidently optimal; $b^*(t) \equiv \beta$ solves (3) at no cost. Since $R_0 = \beta/\alpha$, we note that inequality (11) holds if and only if R_0 is small enough. More precisely, (11) can be rewritten as

$$\gamma \geq 1 - \frac{1}{R_0} [\ln R_0 + \ln(1 - \varepsilon)R + 1]. \quad (12)$$

¹⁰For any $x_1^o > 0$ and $0 < x_2^o \leq 1 - x_1^o$, the right-hand side of (8) is a continuous and strictly increasing function f of $s = x_1^\infty \in (0, x_1^o)$. Moreover, $f(s) \rightarrow -\infty$ as $s \downarrow 0$ and $f(x_1^o) > x_1^o$, $f' > 0$ and $f'' < 0$, so there exists a unique fixed point s^* in $(0, x_1^o)$.

In other words: $c^*(x(0), \gamma) = 0$ for all such γ . Moreover, when the initial infection constitutes an infinitesimal population fraction ($\varepsilon \rightarrow 0$), then (12) boils down to $\gamma \geq 1 - (\ln R_0 + 1)/R_0$. For $\gamma \geq 1$, the condition is trivially met (then the health sector has the capacity to receive the whole population). For $\gamma < 1$, this inequality gives an upper bound on R_0 .¹¹ But if the peak is above the ICU constraint, regulation has to be implemented. This is the topic of the next section.

4 Main result

Our main result, optimal control of an epidemic under a health-care capacity constraint, is fairly easy to state but involved to prove. The formal proof, based on the Hamilton-Jacobi approach, is provided in a separate appendix. Here we state the result and provide intuitions and sketch a heuristic proof.

We summarize below our main result, which treats all cases when laissez-faire does not respect the ICU constraint. If (11) does not hold, which we henceforth assume, then the solution orbit (7) under laissez-faire intersects the capacity constraint $x_2(t) = \gamma$ twice. Let $\tau_1 > 0$ be the first such time and let $x_1(\tau_1)$ be the population share of susceptible individuals at that time. Then

$$\tau_1 = \min \left\{ t \geq 0 : x_1(t) = 1 - \gamma + \frac{\alpha}{\beta} \ln \left(\frac{x_1(t)}{1 - \varepsilon} \right) \right\} \quad (13)$$

where $x_1(t)$ is solved for according to (1) when $b(t) \equiv \beta$, and $x_1(\tau_1)$ is the larger of the two solutions to the associated fixed-point equation in x_1 ,

$$x_1 = \frac{\alpha}{\beta} \ln \left(\frac{x_1}{1 - \varepsilon} \right) + 1 - \gamma. \quad (14)$$

We note that $x_1(\tau_1) > \alpha/\beta$.¹² Let

$$\tau_2 = \tau_1 + \frac{1}{\alpha\gamma} \left[x_1(\tau_1) - \frac{\alpha}{\beta} \right]. \quad (15)$$

Theorem 1 *Suppose that $\varepsilon < \gamma$, $\alpha < \beta$ and (11) does not hold. There exists a solution to program (3), one of which is the policy $b^* \in \mathcal{B}(x(0), \gamma)$ defined by*

$$b^*(t) = \begin{cases} \beta & \text{for } t \leq \tau_1 \\ \beta/[1 + \beta\gamma(\tau_2 - t)] & \text{for } \tau_1 < t \leq \tau_2 \\ \beta & \text{for } t > \tau_2 \end{cases} \quad (16)$$

¹¹This follows from the observation that

$$\frac{\partial}{\partial R_0} \left(\frac{\ln R_0 + 1}{R_0} \right) = -\frac{\ln R_0}{R_0^2},$$

a negative number for all $R_0 > 1$, which holds under our maintained hypothesis that $\alpha < \beta$.

¹²This follows from the observation that the derivative of the right-hand side of (14) is less than unity at $x_1 = x_1(\tau_1)$.

Every optimal policy $b \in \mathcal{B}(x(0), \gamma)$ agrees with b^* on $[0, \tau_2]$ and satisfies $b(t) \geq \beta$ for all $t > \tau_2$.

We note that the optimal policy is laissez-faire both before time τ_1 and after time τ_2 . We also note that the optimal policy has exactly one discontinuity, namely, a sudden shut-down of society at time τ_1 ; then $b^*(t)$ falls from $b^*(\tau_1) = \beta$ to

$$\lim_{t \downarrow \tau_1} b^*(t) = \frac{\beta}{1 + \beta\gamma(\tau_2 - \tau_1)} = \frac{\alpha}{x_1(\tau_1)}. \quad (17)$$

From time τ_1 on, $b^*(t)$ rises continuously until time τ_2 , at which point $b^*(t)$ reaches the level β . In the meantime, between times τ_1 and τ_2 , the population share $x_2(t)$ of infected individuals remains constant at the capacity level γ , while the population share $x_1(t)$ falls linearly over time to the level $\alpha/\beta = 1/R_0$, reached at time τ_2 . The solid curve in the phase-space diagram in Figure 1, below, is the orbit induced by the optimal policy, starting at where $x_1(0)$ is close to 1 and $x_2(0)$ is close to zero. Once the orbit hits the horizontal line where $x_2 = \gamma$, it remains there until it reaches the point where herd immunity is reached, whereupon the orbit without any suppression turns back down. The dotted curve is the orbit taken under laissez-faire.

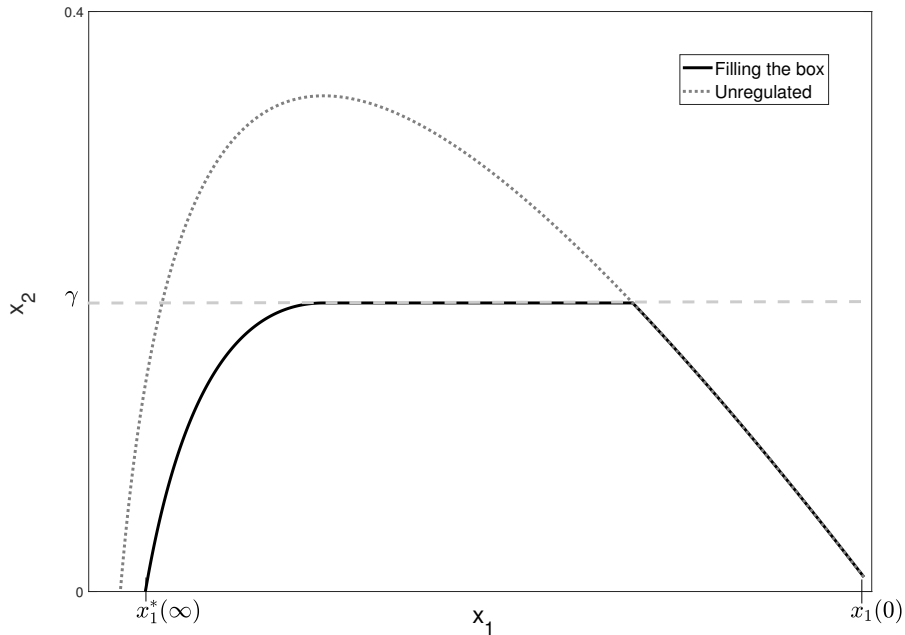


Figure 1: The solution orbit (solid) in the (x_1, x_2) -plane under the optimal policy b^* , and the solution orbit under unregulated spread (dotted). Parameter values used: $\alpha = 0.3$, $\beta = 1$, $\gamma = 0.2$, and $\varepsilon = 0.01$.

Figure 2 depicts the optimal policy as a function of time in comparison to a strategy of flattening the curve, here assumed to take the form: keep $b(t)$ at the level $\delta < \beta$ for which $\hat{x}_2 = \gamma$ (see (10)) until the infection wave has passed its peak, and then return to laissez-faire, $b(t) = \beta$ (outside the time range of the figure). The upper panel shows the dynamics of infections and the lower panel the policy $b(t)$.

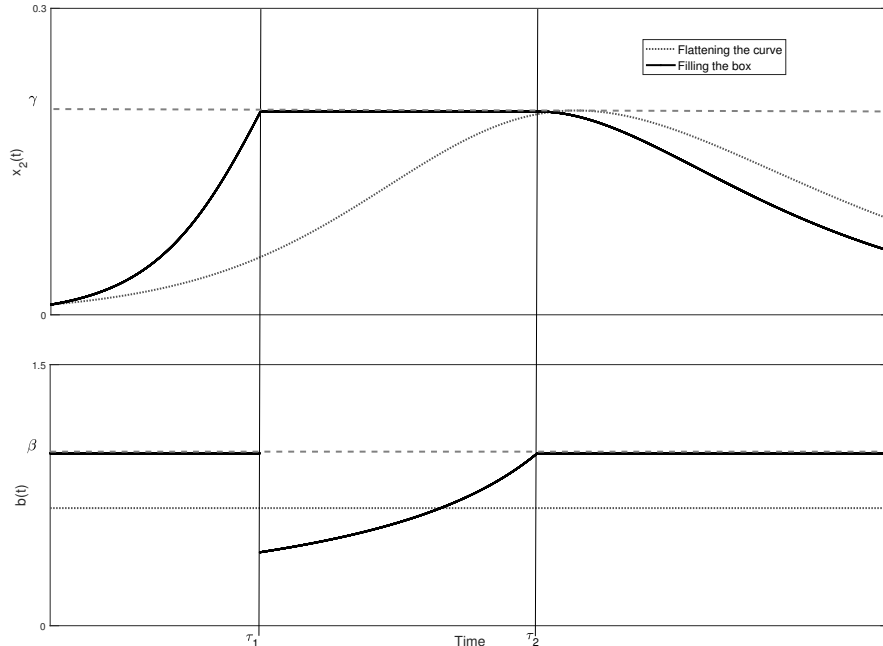


Figure 2: Upper panel: The share of infected over time under the optimal policy (solid) and flattening the curve (dotted). The horizontal dashed line represents the ICU constraint γ . Lower panel: Optimal suppression (solid) and flattening-the-curve suppression (dotted). The horizontal dashed line represents the baseline spread β . Parameter values used: $\alpha = 0.3$, $\beta = 1$, $\gamma = 0.2$, and $\varepsilon = 0.01$.

As can be seen, and as expressed by the theorem, the optimal policy is characterized by leaving the spread unregulated initially, then a sudden shut-down of society (a discontinuity at τ_1), followed by gradual (continuous) opening of society, until time τ_2 , from which onwards the propagation is not regulated. The time axis and the ICU constraint create a square – a box. The economic logic behind the optimal policy is essentially to ensure that we do not close down society while leaving idle ICU resources – “filling the box”. This implies that whenever the natural spread is not threatening the constraint, it should go unregulated. This holds in the early phase when only few have been infected, and in the last phase, when many have already been infected but most of them also have recovered. It is only when the epidemic may breach the ICU constraint – the second phase – that it should be regulated. In order to ensure that the

constraint is not breached, strong suppression has to be imposed when reaching the ICU constraint – a sudden shutdown. The reason for the abruptness of this policy (the discontinuity) is that the natural infection is progressing very quickly at that point, so a sudden break is needed to stop it. This can be seen in the lower panel by the drop at τ_1 . After that, b^* gradually increases. The reason for this is that the suppression only needs to keep the infection just below the ICU constraint. Then since over time the number of susceptible, x_1 , is falling, the number of infected, x_2 , is held endogenously constant and since new infections depends on their product, $b(t) x_2(t) x_1(t)$, it follows that b^* is increasing during the second phase. The policy as a function of the uninfected population share simply is $b^*(t) = \alpha/x_1(t)$, i.e., the recovery rate α determines what share of the susceptible population that can be allowed to be infected.

Three further remarks about optimal policies are now in place. First, it may be noted that the optimal policy never attempts to fully eradicate the spread. In our model, like in all standard S.I.R. models, this is since x_2 only asymptotically goes to zero. Hence, full eradication (a form of extreme corner solution) would imply locking down forever. We discuss this further in the conclusions.

Second, the optimal policy is unique during the first and the second phase but not during the third. The uniqueness during the first phase is not obvious. To see this note that here there is no reason to hold back the spread. Then, given that $b(t) > \beta$ has been assumed to be costless, why would accelerating the spread not be optimal? The answer is that, if one does that, then the ICU capacity is reached at a high speed of infection hence it would require hitting the breaks very hard. This is not optimal. The multiplicity of optimal strategies during the third phase is due to the same assumption – acceleration is free. Hence, not only laissez-faire is optimal, but also acceleration (of which one can think as stimulus for economic interaction). The acceleration cannot be too fast, however, as it may then breach the ICU constraint. Naturally, should we assume that there is a cost of acceleration (even the slightest) this multiplicity disappears and a unique optimal policy emerges also in the third phase – laissez-faire.

Third, compared with the optimal policy, “flattening the curve” implies costs that lead to idle resources. This is visible in the upper panel of Figure 2, where costs are incurred without the spread posing a threat to the health system – both before and after the peak, suppression costs are incurred for no reason. The additional cost of flattening the curve (instead of filling the box) can be seen in the lower panel by comparing the rectangle between β and the dashed-dotted line on the one hand with the area between β and the solid line on the other. It is potentially very large, in particular if the policy maker continues to flatten the curve long after the peak.

4.1 A heuristic

While the interested reader is recommended to consult the Appendix for a rigorous proof of Theorem 1, we here provide a graphical heuristic for some key steps.

Consider the solution *orbits* associated with (1) in the state space Δ , through a given state $x^o = (x_1^o, x_2^o)$, at some arbitrary time, where $0 < x_1^o < 1$ and $0 < x_2^o < \gamma$. In such a population state, some, but not all individuals are susceptible, and the population share of infected is positive but below the health-care capacity. Write $\mathcal{D}_\gamma \subset \Delta$ for this subset of population states, and write (1) more concisely as an autonomous system of ordinary differential equation $\dot{x} = f(x, u(x))$, where $u : \Delta \rightarrow \mathbb{R}_+$ is a *control function* and $f : \Delta \times \mathbb{R}_+ \rightarrow \mathbb{R}^2$ is the vector field that defines the population dynamic — by pointing in the tangential direction of movement. In this (timeless) perspective, (1) becomes

$$\begin{cases} f_1(x, u(x)) = -u(x)x_1x_2 \\ f_2(x, u(x)) = u(x)x_1x_2 - \alpha x_2 \end{cases} .$$

While b is a function of (calendar) time, for a fixed and given initial state at time zero, u is a timeless function of the current state. The *laissez-faire* policy is now represented by the (costless) control function $u^o(x) = \beta$ for all $x \in \Delta$. We will call the solution orbits under this control function the *laissez-faire orbits*. As shown above, the laissez-faire orbit through any given state $x^o \in \Delta$ consists of those population states $x \in \Delta$ that satisfy the equation

$$x_2 = \frac{\alpha}{\beta} \ln \left(\frac{x_1}{x_1^o} \right) + -x_1 + x_1^o + x_2^o. \quad (18)$$

Such laissez-faire orbits are shown in Figure 3.

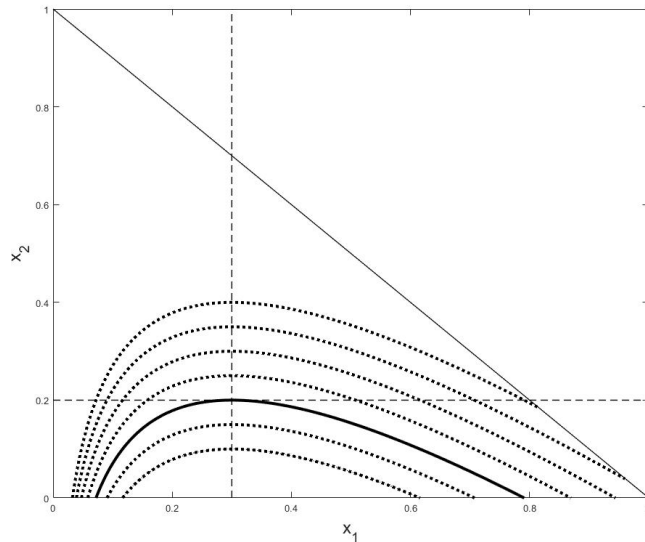


Figure 3: Laissez-faire orbits.

The orbit through any point x^o on any one of these curves will follow its curve counter-clockwise and gradually over time approach a limit point on the

boundary where $x_2 = 0$. The dashed horizontal line represents the capacity constraint $\gamma = 0.2$, and the thick curve is the *maximal laissez-faire orbit* that respects the health-care constraint $x_2 \leq \gamma$. The dashed vertical line indicates states with $x_1 = \alpha/\beta$. The population state at the intersection of these dashed lines, $\tilde{x} = (\alpha/\beta, \gamma)$, is the point at which the maximal laissez-faire orbit touches the health-care capacity constraint. All laissez-faire orbits reach their maximal infection share when $x_1 = \tilde{x}_1 = \alpha/\beta$. At all such states, herd immunity is achieved.

From any population state in the rectangle $A = [0, \alpha/\beta] \times [0, \gamma]$, the solution orbit respects the capacity constraint at all future times. Inside this rectangle, the population state travels counter-clockwise along its orbit, and both x_1 , the number of (currently) susceptible, and x_2 , the number of (currently) infected, decline over time, and x_2 tends towards zero. Consider a population state $x^o \in \mathcal{A}$, the expansion of the set A to also contain all states on or below the maximal laissez-faire orbit. At any such population state (irrespective of time), laissez-faire is clearly an optimal policy; the population state will drift along its laissez-faire orbit, without ever breaching the capacity constraint. Hence, the value function $c^*(\cdot, \gamma)$ is zero on the subset of initial states $x(0) \in \mathcal{A}$.

How should one optimally control the epidemic in population states $x^o \in \mathcal{D}_\gamma$ outside \mathcal{A} ? We here only provide intuitions for why local deviations from the proposed optimal control,

$$u^*(x) = \begin{cases} \alpha/x_1 & \text{if } x_2 = \gamma \\ \beta & \text{otherwise} \end{cases}$$

are suboptimal.

Suppose, first, that the population state $x^o \in \mathcal{D}_\gamma$, outside \mathcal{A} , is exactly at the capacity constraint, that is, $x_1^o > \alpha/\beta$ and $x_2^o = \gamma$. In such a situation, it is necessary to suppress the propagation rate not to violate the constraint. It can be shown that it is optimal to suppress it as little as possible, while not breaching the capacity constraint. The population state will then travel west-ward along the straight line $x_2 = \gamma$. During this travel, which requires costly regulation, gradually less suppression is needed. This is because the population share x_1 of susceptible individuals declines over time while the population share of infected is held constant. When the population share x_1 has reached the critical value α/β , herd immunity has been obtained. From then on, laissez-faire is optimal.

Second, suppose that the population state is above the maximal laissez-faire orbit and with $x_1^o > \alpha/\beta$ and $x_2^o < \gamma$. According to the claimed optimal control u^* , it is best to follow the laissez-faire orbit, the (thin) curve through the point, all the way up to the capacity constraint, and from there behave as described above. Could it be better to deviate upwards or downwards, in comparison with the laissez-faire orbit? It can be shown that it is suboptimal to boost the propagation rate. All that then happens is that the population state will sooner reach the capacity constraint, and will then have a higher share x_1 of susceptible than if laissez-faire had been applied. Since it is costly to travel along the capacity constraint, as noted above, there will be no gain but an additional

suppression cost, to travel an extra stretch along this straight line. Could it be better to deviate downwards? By assumption, this is costly. But by doing so, the capacity constraint can be reached later, and thus be less costly to uphold until $x_1 = \alpha/\beta$. However, one can show that the cost is higher than the benefit.

Suppose that $x_1^o > \alpha/\beta$ and $x_2^o < \gamma$, and suppose that the social planner, by way of a brief temporary suppression induces inward with respect to the laissez-faire orbit through x^o , and thereafter applies laissez-faire. This ends up being suboptimal, as it would be cheapest to wait until the capacity constraint is reached (the population share of susceptible will then be lower). A direct comparison of the cost is not so easy as it may seem (and is related to the linearity of the cost), that is why we will resort to the Hamilton-Jacobi approach in the Appendix to prove Theorem 1.

4.2 The socio-economic cost of respecting the ICU constraint

Having solved the optimization program, one can readily obtain a closed form expression for the minimized socio-economic cost, when starting from a population state $x(0) \in \mathcal{D}_\gamma$ outside \mathcal{A} . With a slight abuse of notation we will write $c^*(\varepsilon, \gamma)$ for $c^*(x(0), \gamma)$ when $x(0) = (1 - \varepsilon, \varepsilon)$ for some $\varepsilon \in (0, 1)$.

Corollary 1 *Suppose $\varepsilon < \gamma$, $\alpha < \beta$, and that (11) does not hold. Then*

$$c^*(\varepsilon, \gamma) = \frac{1}{\gamma} \left[\frac{\beta - \alpha}{\alpha} - \ln \frac{\beta(1 - \varepsilon)}{\alpha} \right] - \frac{\beta}{\alpha}. \quad (19)$$

Proof. In force of Theorem 1:

$$\begin{aligned} c^*(\varepsilon, \gamma) &= C(b^*, x(0)) = \beta \int_{\tau_1}^{\tau_2} \left(1 - \frac{1}{1 + \beta\gamma(\tau_2 - t)} \right) dt \\ &= \beta(\tau_2 - \tau_1) - \beta \int_{\tau_1}^{\tau_2} \frac{dt}{1 + \beta\gamma(\tau_2 - t)} \\ &= \beta(\tau_2 - \tau_1) - \frac{1}{\gamma} \int_1^{1 + \beta\gamma(\tau_2 - \tau_1)} \frac{ds}{s} \\ &= \beta(\tau_2 - \tau_1) - \frac{1}{\gamma} \ln [1 + \beta\gamma(\tau_2 - \tau_1)] \end{aligned}$$

where we have used the substitution $s = 1 + \beta\gamma(\tau_2 - t)$. In force of (15):

$$\tau_2 - \tau_1 = \frac{1}{\alpha\gamma} \left(x_1 - \frac{\alpha}{\beta} \right)$$

where $x_1 > \alpha/\beta$ is the largest solution to the fixed-point equation (14). Hence,

$$\begin{aligned}
c^*(\varepsilon, \gamma) &= \frac{1}{\gamma} \left[\frac{\beta}{\alpha} \left(x_1 - \frac{\alpha}{\beta} \right) - \ln \frac{\beta x_1}{\alpha} \right] \\
&= \frac{1}{\gamma} \left(\frac{\beta x_1}{\alpha} - 1 - \ln \frac{\beta}{\alpha} - \ln x_1 \right) \\
&= \frac{1}{\gamma} \left(\frac{\beta}{\alpha} (1 - \gamma) - \ln(1 - \varepsilon) - 1 - \ln \frac{\beta}{\alpha} \right) \\
&= \frac{1}{\gamma} \left(\frac{\beta}{\alpha} - \ln(1 - \varepsilon) - 1 - \ln \frac{\beta}{\alpha} \right) - \frac{\beta}{\alpha}
\end{aligned}$$

where we used the fixed-point equation (14). ■

From equation (19) it is clear that the lower the ICU capacity (γ) is the more costly implementing the policy will be. Likewise, the cost is increasing in the initial share of infected. It may seem surprising that the size ε of the initial infection plays a role, since under the optimal policy no actions are anyhow taken during the first phase of the epidemic. However, if the initial infection is large, contagion is faster already from the outset, which necessitates a harder lock-down once the ICU capacity is reached. This result can be interpreted as a motivation for closing borders when a new virus hits another country. Limiting interaction with that country then may decrease the initial infection ε which will lower the total costs. It can also be shown that the cost is increasing in the basic level of infection spread, β . Moreover, since in this model the basic reproduction rate R_0 equals β/α , the expression for the minimal cost can be rewritten entirely in terms of three fundamental parameters, the ICU capacity constraint, γ , the initial infection size, ε , and R_0 :

$$c^*(\varepsilon, \gamma) = \frac{R_0 - 1 - \ln R_0 - \ln(1 - \varepsilon)}{\gamma} - R_0. \quad (20)$$

For infinitesimally small initial infection levels ($\varepsilon \downarrow 0$), the expression of the cost boils down to a simple formula:

$$\hat{c}(R_0, \gamma) = \lim_{\varepsilon \downarrow 0} c^*(\varepsilon, \gamma) = \frac{R_0 - 1 - \ln R_0}{\gamma} - R_0 \quad (21)$$

This lower bound is continuous in R_0 , taking the value zero when R_0 is such that the infection peak, \hat{x}_2 , in the absence of regulation, equals γ . For R_0 above this critical level (see equation (11)), $\hat{c}(\cdot, \gamma)$ is strictly increasing in R_0 . Moreover, the dependence is non-linear. An increase in the ICU capacity γ has a moderate effect on the minimal suppression cost if R_0 is low (near its critical unit value), but a relatively strong cost effect if R_0 is high.

5 Infection costs

The policy of filling the box may need to be mitigated in practice when including other considerations than just the existence of an ICU constraint. However, fill-

ing the box does remain optimal under some perturbations of our simple model. We are now in a position to present such an instance, where the cost suffered by infected people is added. This cost may include both personal sufferings as well as treatment costs or a cost to delays in other medical activities when treating a large number of infected during an epidemic. Indeed, there is a range of small such costs for which the same filling-the-box policy as found above remains optimal. When the costs are large enough to fall outside this range, the optimal policy may take another mathematical form, a form that can be found by numerical simulation, but for which we do not see any possibility to pin it down analytically in closed form.

Let $a \geq 0$ and generalize the cost functional C defined in (2) to C_a , defined by

$$C_a(x(0), b) = \int_0^\infty [\beta - b(t)]_+ dt + a \int_0^\infty x_2(t) dt. \quad (22)$$

(now this quantity depends on $x(0)$ through the second integral). Since the last integral equals the total number of infected during the epidemic, a is a cost per infected person. For any $a \geq 0$, the corresponding optimized cost is defined like in (3) by

$$c_a^*(x(0), \gamma) = \inf_{b \in \mathcal{B}(x(0), \gamma)} C_a(b, x(0)). \quad (23)$$

Our next result establishes that exactly the same filling-the-box policy (b^* or u^*) that was found to be optimal when $a = 0$ remains optimal also when a is positive but small enough, and we also establish a closed-form expression for $c_a^*(x(0), \gamma)$ for values of a in this range.

To rigorously pin down this result in our formal model, we restrict the analysis to initial states $x(0) \in \Delta$ where the initial population share of infected is strictly below the health-care capacity, but where this constraint will be hit in finite time under *laissez-faire*. Formally, $x(0)$ belongs to the subset \mathcal{D}_γ but not to \mathcal{A} .

In order to state the result, some more notation is needed. First, let $a_0 \geq 0$ denote the maximal a -value for which the original filling-the-box policy (for the case $a = 0$) is still optimal. Second, let λ be the unique solution $x_1^\infty \in (0, \alpha/\beta)$ of equation (8) for $\delta = \beta$ and $x^o = \tilde{x}$ (recalling that $\tilde{x}_1 = \alpha/\beta$ and $\tilde{x}_2 = \gamma$, see Section 4.1, see also Figure 4 in the Appendix):

$$\lambda = \frac{\alpha}{\beta} \left[\ln \lambda - \ln \left(\frac{\alpha}{\beta} \right) + 1 \right] + \gamma \quad (24)$$

In other words, λ , a number in the interval $(0, \alpha/\beta)$, is the population share that remains uninfected during the whole epidemic, granted the population state at some point in time passes through the population state \tilde{x} . And this is always true under the optimal filling-the-box policy when $a = 0$. Third, and finally, let ρ be the unique solution in $(0, 1)$ of the equation $\rho^2 + \rho + \ln(1 - \rho) = 0$ ($\rho \approx 0.684$). Moreover, let

$$a_0 = \begin{cases} \frac{\lambda}{\alpha - \beta \lambda} \cdot \gamma & \text{if } \gamma \leq \gamma_0 \\ \frac{1 - \rho}{\beta \rho} \cdot \gamma_0 & \text{if } \gamma > \gamma_0 \end{cases}. \quad (25)$$

where $\gamma_0 = \rho^2 \alpha / \beta$.

Theorem 2 *Suppose that $x(0) \in \mathcal{D}_\gamma$ and $x(0) \notin \mathcal{A}$. The filling-the-box policy b^* defined in (16) is optimal if and only if $a \in [0, a_0]$. Moreover, the optimized cost for any $a \in [0, a_0]$ is*

$$c_a^*(x(0), \gamma) = c^*(x(0), \gamma) + a \cdot \frac{x_1(0) + x_2(0) - \lambda}{\alpha}. \quad (26)$$

The formal proof of the theorem can be found in the appendix. Part of the intuition for the—perhaps surprising—result that the same policy is optimal even when a is positive but sufficiently small, is latent in the heuristic sketched in Section 4.1. Because, as suggested there, in the beginning of an epidemic, an “outward” deviation (boosting its spread) is always suboptimal (even more so when $a > 0$). Also an “inward” deviation (suppressing the epidemic spread) in the beginning of an epidemic is suboptimal, because, as we argued, it is cheaper to postpone such a deviation, and the maximal postponement is until the capacity constraint is reached. The cost saved by waiting is positive. Hence, by continuity, it is still positive for when $a > 0$ is sufficiently small. That laissez-faire is also optimal in the late stage of an epidemic is a more subtle issue, and we have to refer the interested reader to the appendix.

An economic intuition for the optimality of the filling-the-box policy also in the presence of a small cost associated with the number of infected is that a side effect of filling the box is that the total number of infected is lowered, compared to complete laissez-faire, despite not having it as an explicit aim. However, the filling-the-box policy may not reduce the infection in the cheapest way, compared to if *only* the number of infected mattered and there were no ICU constraint (a solution of which can be found in Kruse and Strack, 2020). But given that the ICU constraint is respected, any further reductions of infections are not worth the cost when the per unit cost a of infection is low. Another way of interpreting the result is that when a policy maker has two considerations – respecting an ICU constraint and reducing the number of infected – then the former takes precedent and the latter can be ignored if the infection cost is low.

As can be verified, λ is a continuous function of γ , for any given $0 < \alpha < \beta$. Hence also the upper bound a_0 is a continuous function of γ , given $0 < \alpha < \beta$. Our final result is that this upper bound on the optimality of the filling-the-box policy b^* is strictly decreasing in γ for all $\gamma < \gamma_0$:

Corollary 2 *For given α and β , with $0 < \alpha < \beta$, equation (25) defines a_0 as a strictly decreasing function of $\gamma \in (0, \gamma_0)$.*

By the same heuristic as above, a_0 is greater the lower is γ , *ceteris paribus*. Because when γ is lower, the initial part of the laissez-faire orbit hits the constraint sooner, when the population share of still susceptible is larger, and hence the suppression cost is higher, which means that there is a wider range of a -values for initial laissez-faire to still be optimal.

The policy implication of the corollary is that countries with less developed health care (few ICU resources) should focus on ensuring that the spread is not

overwhelming the health-care system. Meanwhile, more developed countries (with more ICU resources) should also focus on reducing the infection numbers *per se*. The intuition for this is that countries with a tight ICU constraint need to implement extensive suppression just to respect the constraint (i.e., the side effect on reduced infection is large). Further suppression, to reduce infection more generally, is then redundant. When the ICU constraint is not very tight, on the other hand, less suppression is needed to respect it hence further suppression is needed if one also cares about the number of infected.

We finally note that under the filling-the-box policy, the resulting additional cost term in (26) is amenable to comparative statics in terms of the primitives of the model. To see this, first note that the second integral in the generalized cost function (22) equals the total population share ever infected during the epidemic, multiplied by $1/\alpha$, the average duration of the infection in an individual. Hence, if we write y for $\int_0^\infty x_2(t) dt$, then the fixed-point equation for λ can be used to establish that y is the unique solution, in the open interval between $1/\alpha - 1/\beta$ and $1/\alpha$, of the fixed-point equation

$$y = \frac{1-\gamma}{\alpha} - \frac{1}{\beta} \ln\left(\frac{1}{\alpha} - y\right) + \frac{1}{\beta} \left(\ln \frac{1}{\beta} - 1\right). \quad (27)$$

We thus have $c_a^*(x(0), \gamma) = c^*(x(0), \gamma) + ay$, where (19) and (27) permit comparative statics analyses. A topic we, for the sake of brevity, leave for future studies.

6 Discussion

This paper provides an *analytical* answer to the question: What is the optimal time-varying suppression policy to avoid a collapsed health-care system when suppression is costly? We have shown that the general recommendation of “flattening the curve” is suboptimal. Instead the optimal policy essentially prescribes “filling the box”. In a first phase, the spread is unregulated until the number of infected reaches the ICU constraint. A second phase begins by discontinuous suppression followed by gradual relaxation until, in a third phase, the spread is left unregulated again.

The main contribution of the paper is methodological, to analytically solve a S.I.R. model with economic costs that are linearly increasing in suppression, and respecting an I.C.U. constraint. We obtain closed-form expressions both for the optimal policy and for the theoretically lowest possible cost of respecting a given I.C.U. constraint. We also analyze under which infection costs “filling the box” remains optimal. With these results we hope to contribute to the policy debate on how to fight an epidemic, and, in particular, the current pandemic.

Our model rests upon a number of simplifying assumptions. We here briefly discuss the robustness of our result on optimal policy with respect to some of the modelling assumptions.

First, in our model (like in the standard S.I.R. model) a complete wipe out of the spread is infeasible. The number of infected only asymptotically

approaches zero, even if full suppression ($b = 0$) were implemented. The reason is the technical assumption that the duration of an individual’s infection has no upper bound (it is treated as if it has an exponential distribution). Naturally, if there instead would exist a maximal duration within which all recover with probability one, then it could be optimal to go for a full wipe out right away.

Second, if account was taken of a potential vaccine arriving within a reasonable time frame, then suppressing the spread more than what our policy prescribes could be optimal. See, e.g., Auld (2003) and McAdams (2020) for modeling and discussion of vaccines.

Third, another factor that could suggest early suppression is if the ICU constraint can be expanded over time (for Covid-19 equivalent to an increased number of respirators or development of a cure or improved treatment). Another possibility is that one learns about the parameters of the disease. However, for that to motivate early suppression, one has to assume that the suppression itself does not distort the signal, see Gollier (2020b).

Fourth, if the cost of suppression is strictly convex (rather than as here linear), in particular if mild suppression is very cheap, then that would again motivate some suppression early on. However, most likely, it would still be optimal to discontinuously increase suppression.

Fifth, and finally, the only constraint considered by our model is the health-care capacity. In practice there may exist, for instance, political constraints one how harsh policies that can be implemented or how fast a policy can be changed. Our results remain the same if these other constraints do not bind under our optimal policy. But if, for instance, a fast increase in suppression is not politically feasible then suppression may need to start before infections reach the ICU constraint.

Despite these limitations of our model, we do hope that the present analysis, focusing on a restricted but clear question and providing a clear answer, provides a useful stepping stone for future analyses of more complex and richer models.

7 Appendix

This appendix provides proofs of Theorem 1, Theorem 2 and Corollary 2. The proofs are based on the Hamilton-Jacobi approach to optimal control. This approach is quite useful here since we do have a conjectured candidate solution to the optimization program. The Euler-Lagrange approach is more appropriate for situations in which the analyst has no candidate solution to verify. This is, for example, the case when $a > a_0$ in Theorem 2, and the Euler-Lagrange approach may be useful for analyses of generalizations of the present model, a task we leave for future studies.

The discussion in the main text, including the heuristics in Section 4.1, is focused on a “new” epidemic, that is, a population in which initially nobody is immune. Technically, this amounts to the assumption $x_1(0) + x_2(0) = 1$. By contrast, in this appendix we also allow for “old” epidemics, that is, initial population states where $x_1(0) + x_2(0) < 1$. The heuristic in Section 4.1 also

presumes that the control problem is time homogenous, that is, that it is sufficient to know the current population state $x(t)$, and not the time t since the epidemic started. By contrast, the appendix sets out without this presumption, by allowing the control to depend in part also directly on time. At the end of Section 7.2, we show how the Hamilton-Jacobi approach justifies the “timeless” approach in Section 4.1. While the discussion in that section only considers small local variations of the candidate solution, we here consider also global variations, a much more challenging task.

The plan of this appendix is as follows: in the next section we recall the general methodology we will follow, fixing the notation and introducing the Hamilton-Jacobi equation. In Section 7.2, we will check that this equation is satisfied when the control is given by the *filling-the-box* policy, thus proving Theorem 1. The last section will extend these computations to the case of subcritical infection costs $a \leq a_0$, ensuring the validity of Theorem 2 and Corollary 2.

7.1 The Hamilton-Jacobi approach

We begin by recalling some general features of the Hamilton-Jacobi approach to optimal control. For more details, specially concerning regularity questions, we refer to Section 1.4 of Soner [57] (the following lemmas correspond to Theorems 2 and 3 there) or to Section 5.1.4 of Liberzon [39].

Consider a controlled dynamical system

$$\dot{x}(t) = f(x(t), b(t)) \quad \forall t \geq 0 \quad (28)$$

where the trajectory $X := (x(t))_{t \geq 0}$ is taking values in a closed domain $\mathcal{D} \subset \mathbb{R}^d$, the control policy $B := (b(t))_{t \geq 0}$ is taking values in a closed set $\mathcal{K} \subset \mathbb{R}$. We assume that B is right-continuous and admits left limits (at any positive time). It follows the control policy B is measurable and locally bounded. The function f goes from $\mathcal{D} \times \mathcal{K}$ to \mathbb{R}^d . In particular, X cannot cross the boundary $\partial\mathcal{D}$, which is seen as the set of constraints on the system (28). This property leads to restrictions on the admissible controls. Namely, at any time $t \geq 0$ when X hits the boundary, say $x(t) = x \in \partial\mathcal{D}$, the value of the control $b(t)$ must be such that there exists a continuous function \tilde{b} on $[t, t + \epsilon]$, with $\epsilon > 0$ and $\tilde{b}(t) = b(t)$, such that the solution \tilde{x} of (28) with $\tilde{x}(t) = x$ and b replaced by \tilde{b} , remains in \mathcal{D} for any time belonging to $[t, t + \epsilon]$. Denote $\mathcal{K}(x)$ the set of such $b(t)$. When $x \in \mathcal{D} \setminus \partial\mathcal{D}$, it is convenient to define $\mathcal{K}(x) := \mathcal{K}$. When \mathcal{D} is smooth, the requirement $b(t) \in \mathcal{K}(x)$ amounts to

$$\langle f(x, b(t)), \nu(x) \rangle \leq 0$$

where $\nu(x)$ is the outward normal of $\partial\mathcal{D}$ at x .

When \mathcal{D} is the intersection of a finite number of smooth domains $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_N$, the previous requirement amounts to

$$\langle f(x, b(t)), \nu_n(x) \rangle \leq 0$$

for all $1 \leq n \leq N$ such that $x \in \partial\mathcal{D}_n$. In our application, we will be in this situation with $N = 4$ and $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and \mathcal{D}_4 half-planes.

For any initial condition $x(0) \in \mathcal{D}$, we would like to minimize the cost associated to a policy B via

$$C(x(0), B) := \int_0^{+\infty} L(x(t), b(t)) dt$$

where L is a measurable function going from $\mathcal{D} \times \mathcal{K}$ to \mathbb{R}_+ . The non-negativeness and the measurability of L , as well as the measurability of $t \mapsto (x(t), b(t))$, ensure that the above integral is well-defined, taking values in $\mathbb{R}_+ \sqcup \{+\infty\}$. More precisely, we are interested in the following quantity as a function of $x := x(0) \in \mathcal{D}$, the initial value of X ,

$$V(x) := \inf_B C(x, B)$$

where the infimum is taken over all admissible control policies.

Introduce the Hamiltonian H on $\mathcal{D} \times \mathbb{R}^d$ via

$$H(x, p) := \inf\{\langle p, f(x, b) \rangle + L(x, b) : b \in \mathcal{K}(x)\} \quad \forall (x, p) \in \mathcal{D} \times \mathbb{R}^d. \quad (29)$$

The principle behind the Hamilton-Jacobi method consists in the characterization of the function V described in the two following results. More precisely, Lemma 1 and Lemma 2 respectively give conditions ensuring that a candidate function $U : \mathcal{D} \rightarrow \mathbb{R}_+$ satisfies $U \leq V$ and $U \geq V$.

Lemma 1 *Let $U : \mathcal{D} \rightarrow \mathbb{R}_+$ be a differentiable function satisfying the Hamilton-Jacobi equation*

$$H(x, \nabla U(x)) = 0 \quad \forall x \in \mathcal{D} \quad (30)$$

and such that for any control policy B , the transversality condition is satisfied:

$$\lim_{t \rightarrow +\infty} U(x(t)) = 0. \quad (31)$$

Then we have $U \leq V$.

To get a reverse inequality, let be given a control policy B and consider X the associated controlled trajectory, solution of (28) and starting from $x \in \mathcal{D}$.

Lemma 2 *Let $U : \mathcal{D} \rightarrow \mathbb{R}_+$ be a function such that for a.e. $t \geq 0$,*

$$\langle \nabla U(x(t)), f(x(t), b(t)) \rangle + L(x(t), b(t)) \leq 0$$

then $U(x) \geq V(x)$.

To conclude, for any $(x, p) \in \mathcal{D} \times \mathbb{R}^d$, consider the set of minimizers in (29)

$$\mathcal{M}(x, p) := \{b \in \mathcal{K}(x) : H(x, p) = \langle p, f(x, b) \rangle + L(x, b)\}$$

Corollary 3 *Let be given a function $U : \mathcal{D} \rightarrow \mathbb{R}_+$ satisfying the conditions of Lemma 1. Consider a policy B and the associated trajectory X , starting from $x \in \mathcal{D}$. Assume that for a.e. $t \geq 0$, $b(t) \in \mathcal{M}(x(t), \nabla U(x(t)))$, then $U(x) = V(x)$.*

As consequence a strategy to compute V is as follows. First find a solution U of the Hamilton-Jacobi equation. Second for a.e. time $t \geq 0$, pick up a control $b(t) \in \mathcal{M}(x(t), \nabla U(x(t)))$, to construct the corresponding controlled trajectory X . Finally check the validity of (31). In particular it follows that the optimal solution we will get depends on time t only through $x(t)$, enabling us to draw corresponding orbits, as in Figures 1 and 3, thus establishing a link with Section 4.1.

7.2 Proof of Theorem 1

Let us come back to our S.I.R. minimization problem, with parameters $\alpha, \beta, \gamma > 0$ and $a = 0$. Recall (1): we are interested in the controlled evolution described by

$$\begin{cases} \dot{x}_1(t) = -b(t)x_1(t)x_2(t) \\ \dot{x}_2(t) = b(t)x_1(t)x_2(t) - \alpha x_2(t) \end{cases} \quad (32)$$

where, with the notations of the previous section, we have $X := (x_1(t), x_2(t))_{t \geq 0}$ (so the dimension is $d = 2$) and $B := (b(t))_{t \geq 0}$. The constraint is represented by the domain

$$\mathcal{D} := \{x \in \Delta : x_2 \leq \gamma\}, \quad \mathcal{K} := \mathbb{R}_+$$

(this set is \mathcal{D}_γ in Section 4.1). The interior $\overset{\circ}{\mathcal{D}}$ of \mathcal{D} is

$$\overset{\circ}{\mathcal{D}} = \{x \in \Delta : x_1 > 0, x_2 > 0, x_1 + x_2 < 1 \text{ and } x_2 < \gamma\}.$$

A priori $\partial\mathcal{D}$ consists in the four segments around the trapezoid \mathcal{D} (see Figure 3), but only the segment $\mathcal{S} \subset \partial\mathcal{D}$ linking the point $(\alpha/\beta, \gamma)$ to $(1 - \gamma, \gamma)$ is relevant for our purposes (the other parts are not attainable from $\overset{\circ}{\mathcal{D}}$). In this section and in the next ones, we will check the Hamilton-Jacobi on $\overset{\circ}{\mathcal{D}}$. Here it will essentially be sufficient to check on $\mathcal{D} \setminus \mathcal{A}$ (where \mathcal{A} was introduced in Section 4.1 and will be given an alternative characterization below).

With the notations of the previous section, we have

$$\begin{cases} f(x, b) & := & (-bx_1x_2, bx_1x_2 - \alpha x_2) \\ L(x, b) & := & (\beta - b)_+ \end{cases} \quad \forall (x, b) \in \mathcal{D} \times \mathcal{K}. \quad (33)$$

For $x \in \mathcal{S}$, let us compute $\mathcal{K}(x)$. The outward normal on \mathcal{S} is the vector $(0, 1)$ (except at the corner $(1 - \gamma, \gamma)$ where it is not defined), so

$$\begin{aligned} \mathcal{K}(x) &= \{b \geq 0 : \langle f(x, b), (0, 1) \rangle \leq 0\} \\ &= \{b \geq 0 : bx_1\gamma - \alpha\gamma \leq 0\} = [0, \alpha/x_1]. \end{aligned}$$

The important computation corresponds to the determination of the Hamiltonian H :

Lemma 3 We have for all $(x, p) \in \mathring{\mathcal{D}} \times \mathbb{R}^2$,

$$H(x, p) = \begin{cases} \min\{1, (p_2 - p_1)x_1x_2\}\beta - \alpha p_2x_2 & \text{if } p_1 \leq p_2 \\ -\infty & \text{if } p_1 > p_2. \end{cases}$$

where p_1 and p_2 are the components of the vector p .

Furthermore, the corresponding sets of minimizers are, still for all $(x, p) \in \mathring{\mathcal{D}} \times \mathbb{R}^2$,

$$\mathcal{M}(x, p) = \begin{cases} \{0\} & \text{if } p_1 \leq p_2 \text{ and } (p_2 - p_1)x_1x_2 - 1 \geq 0 \\ \{\beta\} & \text{if } p_1 \leq p_2 \text{ and } (p_2 - p_1)x_1x_2 - 1 < 0 \\ \emptyset & \text{if } p_1 > p_2. \end{cases}$$

Proof. Let $(x, p) \in \mathring{\mathcal{D}} \times \mathbb{R}^2$ be fixed. He have

$$\begin{aligned} H(x, p) &= \inf\{-p_1bx_1x_2 + p_2(bx_1x_2 - \alpha x_2) + (\beta - b)_+ : b \in \mathbb{R}_+\} \\ &= \min\{I_1, I_2\} - \alpha p_2x_2 \end{aligned}$$

with

$$\begin{aligned} I_1 &:= \beta + \min\{[(p_2 - p_1)x_1x_2 - 1]b : b \in [0, \beta]\} \\ I_2 &:= \inf\{(p_2 - p_1)x_1x_2b : b \geq \beta\}. \end{aligned}$$

Let us first deal with I_2 . We consider two cases:

(i) If $p_1 \leq p_2$, then the mapping $[\beta, +\infty) \ni b \mapsto (p_2 - p_1)x_1x_2b$ is non-decreasing, so its minimal value is attained at $b = \beta$ (as well as on the whole interval $[0, \beta]$, if $p_1 = p_2$) and $I_2 = (p_2 - p_1)x_1x_2\beta$.

(ii) If $p_1 > p_2$, we have $x_1x_2 > 0$ since $x \in \mathring{\mathcal{D}}$, thus the mapping $[\beta, +\infty) \ni b \mapsto (p_2 - p_1)x_1x_2b$ is decreasing, its minimal value is ‘‘attained’’ at $+\infty$ and $I_2 = -\infty$.

Let us now come to I_1 . Again we consider two cases:

(i) If $(p_2 - p_1)x_1x_2 - 1 \geq 0$, then the mapping $[0, \beta] \ni b \mapsto [(p_2 - p_1)x_1x_2 - 1]b$ is non-decreasing, so its minimal value is attained at $b = 0$ (as well as on the whole interval $[0, \beta]$, if $(p_2 - p_1)x_1x_2 = 1$) and $I_1 = \beta$.

(ii) If $(p_2 - p_1)x_1x_2 - 1 < 0$, then the mapping $[0, \beta] \ni b \mapsto [(p_2 - p_1)x_1x_2 - 1]b$ is decreasing, so its minimal value is attained at $b = \beta$ and $I_1 = (p_2 - p_1)x_1x_2\beta$.

These observations lead us to divide $\mathring{\mathcal{D}} \times \mathbb{R}^2$ into the three regions:

- On $\{(x, p) \in \mathring{\mathcal{D}} \times \mathbb{R}^2 : p_2 - p_1 \geq 0 \text{ and } (p_2 - p_1)x_1x_2 - 1 \geq 0\}$, we have $I_2 = (p_2 - p_1)x_1x_2\beta \geq 0$ and $I_1 = \beta$, so that $H(x, p) = -\alpha p_2x_2 + \min\{1, (p_2 - p_1)x_1x_2\}$.
- On $\{(x, p) \in \mathring{\mathcal{D}} \times \mathbb{R}^2 : p_2 - p_1 \geq 0 \text{ and } (p_2 - p_1)x_1x_2 - 1 < 0\}$, we have $I_2 = (p_2 - p_1)x_1x_2\beta$ and $I_1 = (p_2 - p_1)x_1x_2\beta = I_2$, so that $H(x, p) = (p_2 - p_1)x_1x_2\beta - \alpha p_2x_2$.
- On $\{(x, p) \in \mathring{\mathcal{D}} \times \mathbb{R}^2 : p_2 - p_1 < 0\}$, since $I_2 = -\infty$, we necessarily have $H(x, p) = -\infty$.

It follows that for all $(x, p) \in \hat{\mathcal{D}} \times \mathbb{R}^2$,

$$H(x, p) = \begin{cases} \min\{1, (p_2 - p_1)x_1x_2\}\beta - \alpha p_2x_2 & \text{if } p_1 \leq p_2 \\ -\infty & \text{if } p_1 > p_2 \end{cases}$$

as announced in the lemma.

The corresponding sets of minimizers follow from the above analysis. ■

Similar considerations lead to:

Lemma 4 *We have for any $(x, p) \in \mathcal{S} \times \mathbb{R}^2$,*

$$H(x, p) = \begin{cases} \beta - \alpha\gamma p_2 & \text{if } \gamma(p_2 - p_1)x_1 - 1 \geq 0 \\ \beta - \alpha\gamma p_1 - \alpha/x_1 & \text{if } \gamma(p_2 - p_1)x_1 - 1 < 0 \end{cases}$$

and

$$\mathcal{M}(x, p) = \begin{cases} \{0\} & , \text{ if } \gamma(p_2 - p_1)x_1 - 1 > 0 \\ [0, \alpha/x_1] & \text{ if } \gamma(p_2 - p_1)x_1 - 1 = 0 \\ \{\alpha/x_1\} & \text{ if } \gamma(p_2 - p_1)x_1 - 1 < 0. \end{cases}$$

Proof. Let $(x, p) \in \mathcal{S} \times \mathbb{R}^2$ be fixed.

$$\begin{aligned} H(x, p) &= \min\{-p_1bx_1x_2 + p_2(bx_1x_2 - \alpha x_2) + (\beta - b)_+ : b \in [0, \alpha/x_1]\} \\ &= \min\{-p_1bx_1x_2 + p_2(bx_1x_2 - \alpha x_2) + \beta - b : b \in [0, \alpha/x_1]\} \\ &= \min\{-p_1bx_1\gamma + p_2(bx_1\gamma - \alpha\gamma) + \beta - b : b \in [0, \alpha/x_1]\} \\ &= \min\{[\gamma(p_2 - p_1)x_1 - 1]b : b \in [0, \alpha/x_1]\} + \beta - \alpha\gamma p_2 \end{aligned}$$

where for the second equality we used that for $x \in \mathcal{S}$, we have $x_1 \geq \alpha/\beta$, so $\alpha/x_1 \leq \beta$, and for the third equality we used that $x_2 = \gamma$ for $x \in \mathcal{S}$.

We consider two cases:

(i) If $\gamma(p_2 - p_1)x_1 - 1 \geq 0$, then the mapping that takes b from $[0, \alpha/x_1]$ to $[\gamma(p_2 - p_1)x_1 - 1]b$ is non-decreasing, so its minimal value is attained at $b = 0$, $H(x, p) = \beta - \alpha\gamma p_2$.

(ii) If $\gamma(p_2 - p_1)x_1 - 1 < 0$, then the mapping that takes b from $[0, \alpha/x_1]$ to $[\gamma(p_2 - p_1)x_1 - 1]b$ is decreasing, so its minimal value is attained at $b = \alpha/x_1$ and

$$\begin{aligned} H(x, p) &= [\gamma(p_2 - p_1)x_1 - 1] \frac{\alpha}{x_1} + \beta - \alpha\gamma p_2 \\ &= \alpha\gamma(p_2 - p_1) - \frac{\alpha}{x_1} + \beta - \alpha\gamma p_2 \\ &= -\alpha\gamma p_1 - \frac{\alpha}{x_1} + \beta. \end{aligned}$$

The desired results follow. Note that in the first case, we have to separate the sub-cases $\gamma(p_2 - p_1)x_1 - 1 > 0$ and $\gamma(p_2 - p_1)x_1 - 1 = 0$, where the sets of minimizers are respectively $\{0\}$ and $[0, \alpha/x_1]$.

■

Let us now construct a function U , inspired by the results obtained in our previous paper [45] (see also Corollary 1), that will satisfy the assumptions of Corollary 3.

First, let us introduce some more notations. For any $x \in \Delta$, let

$$\varphi(x, \cdot) := (\varphi_1(x, t), \varphi_2(x, t))_{t \in [0, +\infty)}$$

be the trajectory starting from x and “controlled” by the *laissez-faire* policy. It amounts to taking $b \equiv \beta$ in (28), where the mappings f and L considered in (33) have been extended to $\Delta \times \mathcal{K}$ (then no boundary condition has to be taken into account). As in Section 4.1, the trajectory $\varphi(x, \cdot)$ will also be called *laissez-faire* in the sequel. We need to extend the domain to all of Δ because for some x , $\varphi(x, \cdot)$ does take values in $\Delta \setminus \mathcal{D}$, i.e. the *laissez-faire* policy is not always respecting the capacity constraint $x_2(t) \leq \gamma$. We have seen in (7) that $\varphi(x, \cdot)$ satisfies

$$\varphi_2(x, t) = x_2 + \frac{\alpha}{\beta} \ln \left(\frac{\varphi_1(x, t)}{x_1} \right) - \varphi_1(x, t) + x_1 \quad \forall t \geq 0. \quad (34)$$

Denote $\tau_1(x)$ the hitting time of \mathcal{S} when the state at time zero is x in \mathcal{D} :

$$\tau_1(x) := \inf\{t \geq 0 : X_2(t) = \gamma\}.$$

By the usual conventions, this hitting time is infinite, when \mathcal{S} is not hit by $\varphi(x, \cdot)$.

Let \mathcal{A} be the closure of initial points $x \in \mathcal{D}$ with $\tau_1(x) = +\infty$. Note it corresponds to the same set \mathcal{A} introduced in Section 4.1. Its boundary in \mathcal{D} consists of the points $x \in \mathcal{D}$ such that $\varphi(x, \cdot)$ passes through the point $(\alpha/\beta, \gamma)$. For $x \in \mathcal{A}$, we take $U(x) := 0$.

Let us define U on \mathcal{S} . Consider $x = (x_1, \gamma) \in \mathcal{S}$, we take

$$U(x) := \frac{1}{\gamma} \left(\frac{\beta}{\alpha} x_1 - 1 - \ln \left(\frac{\beta x_1}{\alpha} \right) \right).$$

For $x \in \mathcal{D} \setminus \mathcal{A}$, the trajectory $\varphi(x, \cdot)$ hits \mathcal{S} , say at a point $y(x) \in \mathcal{S}$. We take $U(x) := U(y(x))$.

Here is a more explicit expression:

Lemma 5 *For any $x := (x_1, x_2) \in \mathcal{D} \setminus \mathcal{A}$, we have*

$$U(x) = \frac{1}{\gamma} \left(\frac{\beta}{\alpha} x_2 - \ln \left(\frac{\beta x_1}{\alpha} \right) + \frac{\beta}{\alpha} x_1 - 1 - \frac{\beta}{\alpha} \gamma \right)$$

and

$$\nabla U(x) = \frac{\beta}{\gamma \alpha} \left(1 - \frac{\alpha}{\beta x_1}, 1 \right)$$

(for $x \in \mathcal{S}$, the interpretation of $\nabla U(x)$ is that for any v pointing toward \mathcal{D} , the differential of U at x in the direction of v is given by $dU(x)[v] = \langle \nabla U(x), v \rangle$).

Proof. For fixed $x := (x_1, x_2) \in \mathcal{D} \setminus \mathcal{A}$, denote $y(x) := (\xi(x), \gamma)$. According to (34), the coordinate $\xi(x)$ is a solution ξ to the equation

$$\gamma = x_2 + \frac{\alpha}{\beta} \ln \left(\frac{\xi}{x_1} \right) - \xi + x_1$$

(there are two solutions to this equation and $\xi(x)$ is the largest, corresponding to the first time $\tau_1(x)$ when $\varphi_2^*(x, \cdot)$ goes through γ , it will pass at this level a second time, in its way to herd immunity).

As a consequence, we get

$$\begin{aligned} U(x) &= \frac{1}{\gamma} \left(\frac{\beta}{\alpha} \xi(x) - 1 - \ln \left(\frac{\beta \xi(x)}{\alpha} \right) \right) \\ &= \frac{1}{\gamma} \left(\frac{\beta}{\alpha} \xi(x) - \ln(\xi(x)) - \ln \left(\frac{\beta}{\alpha} \right) - 1 \right) \\ &= \frac{1}{\gamma} \left(\frac{\beta}{\alpha} x_2 - \ln(x_1) + \frac{\beta}{\alpha} x_1 - \frac{\beta}{\alpha} \gamma - \ln \left(\frac{\beta}{\alpha} \right) - 1 \right) \\ &= \frac{1}{\gamma} \left(\frac{\beta}{\alpha} x_2 - \ln \left(\frac{\beta x_1}{\alpha} \right) + \frac{\beta}{\alpha} x_1 - 1 - \frac{\beta}{\alpha} \gamma \right). \end{aligned}$$

The gradient of U is immediately obtained by differentiation of this expression (extended on a neighborhood of points belonging to \mathcal{S}). ■

We deduce:

Lemma 6 *The function U satisfies the Hamilton-Jacobi equation on $\mathcal{D} \setminus \mathcal{A}$.*

Proof. First let us consider $x \in \mathring{\mathcal{D}} \setminus \mathcal{A}$. Let us write ∇_1 and ∇_2 the partial derivatives with respect to x_1 and x_2 . Note that

$$\nabla_1 U(x) = \frac{\beta}{\gamma \alpha} \left(1 - \frac{\alpha}{\beta x_1} \right) < \frac{\beta}{\gamma \alpha} = \nabla_2 U(x)$$

so according to Lemma 3

$$\begin{aligned} H(x, \nabla U(x)) &= \min\{1, (\nabla_2 U(x) - \nabla_1 U(x))x_1 x_2\} \beta - \alpha \nabla_2 U(x) x_2 \\ &= \min \left\{ 1, \frac{1}{\gamma x_1} x_1 x_2 \right\} \beta - \frac{\beta}{\gamma} x_2 \\ &= \min \left\{ 1, \frac{1}{\gamma} x_2 \right\} \beta - \frac{\beta}{\gamma} x_2. \end{aligned}$$

Since $x_2 < \gamma$, the latter expression vanishes.

Second, for $x = (x_1, \gamma) \in \mathcal{S}$, we have

$$\gamma(\nabla_2 U(x) - \nabla_1 U(x))x_1 - 1 = \gamma \frac{1}{\gamma x_1} x_1 - 1 = 0.$$

According to Lemma 4, we infer

$$\begin{aligned} H(x, \nabla U(x)) &= \beta - \alpha\gamma\nabla_1 U(x) - \frac{\alpha}{x_1} \\ &= \beta - \beta\gamma\frac{1}{\gamma}\left(1 - \frac{\alpha}{\beta x_1}\right) - \frac{\alpha}{x_1} = 0. \end{aligned}$$

■

Following the roadmap presented after the proof of Corollary 3, we are led to choose at any time $t \geq 0$, a policy $b(t) \in \mathcal{M}(x(t), \nabla U(x(t)))$. In the following cases, write $x = (x_1, x_2)$ for the position $x(t)$.

(i) When $x \in \mathcal{D} \setminus \mathcal{A}$, the above proof shows that $\nabla_1 U(x) \leq \nabla_2 U(x)$ and similarly we see that

$$(\nabla_2 U(x) - \nabla_1 U(x))x_1 x_2 = \frac{1}{\gamma x_1} x_1 x_2 = \frac{x_2}{\gamma} < 1.$$

Thus according to Lemma 3, we must take $b(t) = \beta$.

(ii) When $x \in \mathcal{S}$, the proof of Lemma 6 shows that $(\nabla_2 U(x) - \nabla_1 U(x))x_1 \gamma - 1 = 0$. Thus according to Lemma 4, we must take $b(t) \in [0, \alpha/x_1]$. Taking exactly $b(t) = \alpha/x_1$, namely $b(t) = \alpha/x_1(t)$ where $x_1(t)$ is the first coordinate of $x(t)$, we recover the *filling-the-box* policy, at least until X hits the point $(\alpha/\beta, \gamma)$.

To end the proof of Theorem 1, it remains to check (31), where X is the trajectory controlled as above and where the initial point $x(0)$ belongs to $\mathcal{D} \setminus \mathcal{A}$. Remark: X ends up hitting the point $(\alpha/\beta, \gamma)$ after moving along \mathcal{S} . Since $U(\alpha/\beta, \gamma) = 0$, afterward the trajectory X stays in the set where U vanishes, following the *laissez-faire* trajectory passing through $(\alpha/\beta, \gamma)$, as explained in Section 4.1.

7.3 Proof of Theorem 2 and Corollary 2

Let us add to the instantaneous cost a term ax_2 , with $a > 0$, corresponding to the costs suffered by infectious people. Namely in our S.I.R. minimization problem, we replace L by

$$L_a(x, b) := L(x, b) + ax_2 \quad \forall x \in \mathcal{D}, \forall b \in \mathcal{K}$$

(here and in the sequel, a quantity having no index a corresponds to $a = 0$, i.e. the quantity considered in the previous section).

Since the additional term does not contain b , we get for the corresponding Hamiltonian and minimizer set:

$$\begin{cases} H_a(x, p) &= H(x, p) + ax_2 \\ \mathcal{M}_a(x, p) &= \mathcal{M}(x, p) \end{cases} \quad \forall (x, p) \in \mathcal{D} \times \mathbb{R}^2.$$

Our goal here is to investigate the modifications induced by this change of the cost and to show that for $a > 0$ sufficiently small, that will be precisely quantified, the *filling-the-box* policy remains optimal.

Our first task is to compute the induced surplus of cost for the *filling-the-box* policy, whose definition is recalled below.

For $x \in \mathcal{D}$, let $\varphi^*(x, \cdot) := (\varphi_1^*(x, t), \varphi_2^*(x, t))_{t \in [0, +\infty)}$ be the trajectory controlled by the *filling-the-box* policy $B := (b(t))_{t \in [0, +\infty)}$. Denote by τ_2^* the first time $\varphi_2^*(x, \cdot)$ hits α/β and τ_1^* the first time $\varphi_1^*(x, \cdot)$ hits γ , except that if $\varphi_2^*(x, \cdot)$ never hits γ , we take $\tau_1^* = \tau_2^*$. By convention we take $\tau_1^* = \tau_2^* = 0$ when $x_1 \leq \alpha/\beta$. The control B is given by

$$b(t) := \begin{cases} \beta & \text{if } t < \tau_1^* \\ \alpha/\varphi_1^*(x, t) & \text{if } t \in [\tau_1^*, \tau_2^*] \\ \beta & \text{if } t > \tau_2^* \end{cases} \quad \forall t \geq 0.$$

Note that $\varphi^*(x, \cdot)$ coincides with the *laissez-faire* trajectory $\varphi(x, \cdot)$ (see the paragraph before (34)) up to the time $\tau_1^* = \tau_1(x)$, with the notations of Section 7.2.

We begin by computing the surplus of cost on the first period $[0, \tau_1^*]$.

Lemma 7 *When $\tau_1^* < \tau_2^*$, we have*

$$\int_0^{\tau_1^*} L_a(\varphi^*(x, t), b(t)) dt = \frac{a}{\alpha}(x_1 + x_2 - \xi(x) - \gamma)$$

where $\xi(x) := \varphi_1^*(x, \tau_1^*)$, was introduced in the proof of Lemma 5.

When $\tau_1^* = \tau_2^*$, we have

$$\int_0^{\tau_1^*} L_a(\varphi^*(x, t), b(t)) dt = \frac{a}{\alpha}(x_1 + x_2 - \chi(x) - \alpha/\beta)$$

where $\chi(x) := \varphi_2^*(x, \tau_1^*)$.

Proof. In all cases, since $b(t) = \beta$ for $t \in [0, \tau_1^*)$, we have

$$\begin{aligned} \int_0^{\tau_1^*} L_a(\varphi^*(x, t), b(t)) dt &= \int_0^{\tau_1^*} L_a(\varphi^*(x, t), \beta) dt \\ &= a \int_0^{\tau_1^*} \varphi_2^*(x, t) dt \\ &= -\frac{a}{\alpha} \int_0^{\tau_1^*} (-\alpha)\varphi_2^*(x, t) dt \\ &= -\frac{a}{\alpha} \int_0^{\tau_1^*} \frac{d}{dt}(\varphi_1^*(x, t) + \varphi_2^*(x, t)) dt \\ &= -\frac{a}{\alpha} [\varphi_1^*(x, t) + \varphi_2^*(x, t)]_0^{\tau_1^*} \\ &= -\frac{a}{\alpha} (\varphi_1^*(x, \tau_1^*) + \varphi_2^*(x, \tau_1^*) - (x_1 + x_2)). \end{aligned}$$

According to the situation $\tau_1^* < \tau_2^*$ or $\tau_1^* = \tau_2^*$, we deduce the desired result.

Next we compute the cost on the second period $[\tau_1^*, \tau_2^*]$, it is only needed when $\tau_1^* < \tau_2^*$. ■

Lemma 8 For any τ_1^* less than τ_2^* :

$$\int_{\tau_1^*}^{\tau_2^*} L_a(\varphi^*(x, t), b(t)) dt = \frac{\beta + a\gamma}{\alpha\gamma} \left(\xi(x) - \frac{\alpha}{\beta} \right) - \frac{1}{\gamma} \ln \left(\frac{\beta\xi(x)}{\alpha} \right).$$

Proof. We have

$$\begin{aligned} \int_{\tau_1^*}^{\tau_2^*} L_a(\varphi^*(x, t), b(t)) dt &= \int_{\tau_1^*}^{\tau_2^*} \left(L(\varphi^*(x, t), b(t)) + aX_2^{(x)}(t) \right) dt \\ &= \int_{\tau_1^*}^{\tau_2^*} L(\varphi^*(x, t), b(t)) dt + a\gamma(\tau_2^* - \tau_1^*). \end{aligned}$$

The integral in the last r.h.s. was computed in Corollary 1, we have

$$\int_{\tau_1^*}^{\tau_2^*} L(\varphi^*(x, t), b(t)) dt = \frac{1}{\gamma} \left(\frac{\beta}{\alpha} \xi(x) - \ln(\xi(x)) - \ln \left(\frac{\beta}{\alpha} \right) - 1 \right).$$

The time difference $\tau_2^* - \tau_1^*$ was computed in (15):

$$\tau_2^* - \tau_1^* = \frac{\xi(x) - \alpha/\beta}{\alpha\gamma}$$

as claimed. ■

Finally follows the computation of the contribution to the cost of the last period $[\tau_2^*, +\infty)$. Note that contrary to the situation $a = 0$ considered in Section 7.2, we can no longer overlook this period.

Lemma 9

$$\int_{\tau_2^*}^{+\infty} L_a(\varphi^*(x, t), b(t)) dt = \begin{cases} \frac{a}{\alpha}(\alpha/\beta + \gamma - \lambda) & \text{if } \tau_1^* < \tau_2^* \\ \frac{a}{\alpha}(\alpha/\beta + \chi(x) - l(x)) & \text{if } \tau_1^* = \tau_2^* \end{cases}$$

where λ is the smallest solution to the equation (24) (see also Figure 4 below), and where $l(x)$, for $x \in \mathcal{A}$, is the smallest solution to the following equation in l ,

$$l = \frac{\alpha}{\beta} \ln \left(\frac{\beta l}{\alpha} \right) + \frac{\alpha}{\beta} + \chi(x). \quad (35)$$

Proof. For $t \geq \tau_2^*$, we have $b(t) = \beta$, so as in the proof of Lemma 7,

$$\begin{aligned} \int_{\tau_2^*}^{+\infty} L_a(\varphi^*(x, t), b(t)) dt &= a \int_{\tau_2^*}^{+\infty} \varphi_2^*(x, t) dt \\ &= -\frac{a}{\alpha}(\lambda(x) - \varphi_1^*(x, \tau_2^*) - \varphi_2^*(x, \tau_2^*)) \end{aligned}$$

where $\lambda(x) := \lim_{t \rightarrow +\infty} \varphi_1^*(x, t)$ and where we took into account that $\lim_{t \rightarrow +\infty} \varphi_2^*(x, t) = 0$.

From (34), $\lambda(x)$ is the smallest solution to the equation in l

$$l = \frac{\alpha}{\beta} \ln \left(\frac{l}{\varphi_1^*(x, \tau_2^*)} \right) + \varphi_1^*(x, \tau_2^*) + \varphi_2^*(x, \tau_2^*).$$

Let us now separate the two possible cases.

(i) When $\tau_1^* < \tau_2^*$, we have $(\varphi_1^*(x, \tau_2^*), \varphi_2^*(x, \tau_2^*)) = (\alpha/\beta, \gamma)$, so $l(x)$ does not depend on x (except that $x \in \mathcal{D} \setminus \mathcal{A}$ is required) and is the smallest solution λ to (24).

(ii) When $\tau_1^* = \tau_2^*$, we have $(\varphi_1^*(x, \tau_2^*), \varphi_2^*(x, \tau_2^*)) = (\alpha/\beta, \chi(x))$, so $l(x)$ does depend on x and is the smallest solution to (35). ■

As a consequence of the three previous lemmas, we get:

(i) When $\tau_1^* < \tau_2^*$, i.e. $x \in \mathcal{D} \setminus \mathcal{A}$, we have

$$\begin{aligned} &\int_0^{+\infty} L_a(\varphi^*(x, t), b(t)) dt \\ &= \int_0^{\tau_1^*} L_a(\varphi^*(x, t), b(t)) dt + \int_{\tau_1^*}^{\tau_2^*} L_a(\varphi^*(x, t), b(t)) dt \\ &\quad + \int_{\tau_2^*}^{+\infty} L_a(\varphi^*(x, t), b(t)) dt \\ &= \frac{a}{\alpha}(x_1 + x_2 - \xi(x) - \gamma) + \frac{\beta + a\gamma}{\alpha\gamma} \left(\xi(x) - \frac{\alpha}{\beta} \right) - \frac{1}{\gamma} \ln \left(\frac{\beta\xi(x)}{\alpha} \right) \\ &\quad + \frac{a}{\alpha}(\alpha/\beta + \gamma - \lambda) \\ &= \int_0^{+\infty} L(\varphi^*(x, t), b(t)) dt + \frac{a}{\alpha}(x_1 + x_2 - \xi(x) - \gamma) + \frac{a}{\alpha} \left(\xi(x) - \frac{\alpha}{\beta} \right) \\ &\quad + \frac{a}{\alpha}(\alpha/\beta + \gamma - \lambda) \\ &= \int_0^{+\infty} L(\varphi^*(x, t), b(t)) dt + \frac{a}{\alpha}(x_1 + x_2 - \lambda) \\ &= U(x) + \frac{a}{\alpha}(x_1 + x_2 - \lambda) \end{aligned}$$

where we took into account that

$$U(x) = \int_0^{+\infty} L(\varphi^*(x, t), b(t)) dt$$

when $a = 0$, as shown in the previous section.

(ii) When $\tau_1^* = \tau_2^*$, i.e. $x \in \mathcal{A}$ (recall that \mathcal{A} is the set of points below the *laissez-faire* trajectory passing through $(\alpha/\beta, \gamma)$), we have similarly, only using Lemmas 7 and 9,

$$\begin{aligned} \int_0^{+\infty} L_a(\varphi^*(x, t), b(t)) dt &= \frac{a}{\alpha}(x_1 + x_2 - \chi(x) - \alpha/\beta) + \frac{a}{\alpha}(\alpha/\beta + \chi(x) - l(x)) \\ &= U(x) + \frac{a}{\alpha}(x_1 + x_2 - l(x)) \end{aligned}$$

where we used that $U(x) = 0$ for $x \in \mathcal{A}$ and where $l(x)$ is defined in Lemma 9.

Since we want to test if the *filling-the-box* policy is optimal, let us define the function U_a via

$$\begin{aligned} \forall x \in \mathcal{D}, \quad U_a(x) &:= \int_0^{+\infty} L_a(\varphi^*(x, t), b(t)) dt \\ &= U(x) + \begin{cases} \frac{a}{\alpha}(x_1 + x_2 - \lambda) & \text{if } x \in \mathcal{D} \setminus \mathcal{A} \\ \frac{a}{\alpha}(x_1 + x_2 - l(x)) & \text{if } x \in \mathcal{A}. \end{cases} \end{aligned} \quad (36)$$

In particular, we have (where T stands for transposition)

$$\nabla U_a(x) = \nabla U(x) + \frac{a}{\alpha} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \quad \forall x \in \mathcal{D} \setminus \mathcal{A}$$

and

$$\nabla U_a(x) = \nabla U(x) - \frac{a}{\alpha} \nabla l(x) + \frac{a}{\alpha} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \quad \forall x \in \mathcal{A} \quad (37)$$

We deduce:

Lemma 10 *The function U_a satisfies the Hamilton-Jacobi equation on $\mathcal{D} \setminus \mathcal{A}$.*

Proof. Note that on $\mathcal{D} \setminus \mathcal{A}$, we have

$$\nabla_2 U_a - \nabla_1 U_a = \nabla_2 U - \nabla_1 U.$$

In Section 7.2, we checked that on $\mathring{\mathcal{D}} \setminus \mathcal{A}$, $\nabla_2 U - \nabla_1 U < 0$, and that for $x \in \mathcal{S}$, $\gamma(\nabla_2 U(x) - \nabla_1 U(x))x_1 - 1 = 0$, so we get:

(i) On $\mathring{\mathcal{D}} \setminus \mathcal{A}$,

$$\begin{aligned} H_a(x, \nabla U_a(x)) &= H(x, \nabla U_a(x)) + ax_2 \\ &= \min\{1, (\nabla_2 U_a(x) - \nabla_1 U_a(x))x_1 x_2\} \beta - \alpha \nabla_2 U_a(x) x_2 + ax_2 \\ &= \min\{1, (\nabla_2 U(x) - \nabla_1 U(x))x_1 x_2\} \beta - \alpha \left(\nabla_2 U(x) + \frac{a}{\alpha} \right) x_2 + ax_2 \\ &= H(x, \nabla U(x)) - ax_2 + ax_2 \\ &= H(x, \nabla U(x)) \\ &= 0 \end{aligned}$$

according to Section 7.2.

(ii) On \mathcal{S} ,

$$\begin{aligned}
H_a(x, \nabla U_a(x)) &= H(x, \nabla U_a(x)) + ax_2 \\
&= \beta - \alpha\gamma \nabla_2 U_a(x) + a\gamma \\
&= \beta - \alpha\gamma \left(\nabla_2 U(x) + \frac{a}{\alpha} \right) + a\gamma \\
&= \beta - \alpha\gamma \nabla_2 U(x) \\
&= H(x, \nabla U(x)) \\
&= 0
\end{aligned}$$

according to Section 7.2. ■

To check if the Hamilton-Jacobi equation is satisfied on \mathcal{A} , we need to investigate $\nabla l(x)$. Taking into account (34) and that for $x \in \mathcal{A}$, the control is the laissez-faire policy all the time, $l(x)$ is the smallest solution to the following equation in l ,

$$0 = x_2 + \frac{\alpha}{\beta} \ln \left(\frac{l}{x_1} \right) - l + x_1$$

or equivalently,

$$l - \frac{\alpha}{\beta} \ln(l) = x_2 - \frac{\alpha}{\beta} \ln(x_1) + x_1.$$

Except in the non-relevant case where $x_2 = 0$ and $x_1 \geq \alpha/\beta$, we have $l(x) < \alpha/\beta$. It follows that

$$\nabla l(x) = \frac{1}{1 - \frac{\alpha}{\beta l(x)}} \begin{pmatrix} 1 - \alpha/(\beta x_1) \\ 1 \end{pmatrix}^T.$$

Since $U = 0$ on \mathcal{A} , we deduce from (37) that

$$\begin{aligned}
\forall x \in \mathcal{A}, \quad \nabla U_a(x) &= \frac{a}{\alpha} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}^T - \frac{1}{1 - \frac{\alpha}{\beta l(x)}} \begin{pmatrix} 1 - \alpha/(\beta x_1) \\ 1 \end{pmatrix}^T \right) \\
&= \frac{a}{\alpha \left(1 - \frac{\alpha}{\beta l(x)} \right)} \begin{pmatrix} 1 - \frac{\alpha}{\beta l(x)} - (1 - \alpha/(\beta x_1)) \\ 1 - \frac{\alpha}{\beta l(x)} - 1 \end{pmatrix}^T \\
&= \frac{a}{\beta \left(1 - \frac{\alpha}{\beta l(x)} \right)} \begin{pmatrix} \frac{1}{x_1} - \frac{1}{l(x)} \\ -\frac{1}{l(x)} \end{pmatrix}^T.
\end{aligned}$$

Recall that $l(x) < \alpha/\beta$, so that $\nabla_1 U_a(x) < \nabla_2 U_a(x)$. We deduce that for $x \in \mathcal{A}$:

$$\begin{aligned}
H(x, \nabla U_a(x)) &= H(x, \nabla U_a(x)) + ax_2 \\
&= \min\{1, (\nabla_2 U_a(x) - \nabla_1 U_a(x))x_1 x_2\} \beta - \alpha \nabla_2 U_a(x) x_2 + ax_2 \\
&= \min\left\{1, \left(\frac{-a/x_1}{\beta\left(1 - \frac{\alpha}{\beta l(x)}\right)}\right) x_1 x_2\right\} \beta - \alpha \left(-\frac{a/l(x)}{\beta\left(1 - \frac{\alpha}{\beta l(x)}\right)}\right) x_2 + ax_2 \\
&= \min\left\{1, \frac{-ax_2 l(x)}{\beta l(x) - \alpha}\right\} \beta + \frac{a\alpha x_2}{\beta l(x) - \alpha} + ax_2 \\
&= \min\left\{1, \frac{ax_2 l(x)}{\alpha - \beta l(x)}\right\} \beta - \frac{a\beta x_2 l(x)}{\alpha - \beta l(x)}.
\end{aligned}$$

We are led to consider two cases.

(i) If $\frac{ax_2 l(x)}{\alpha - \beta l(x)} \leq 1$ (for a given $x \in \mathcal{A}$ with $x_2 > 0$, this will be true for $a > 0$ small enough), then

$$H(x, \nabla U_a(x)) = \frac{ax_2 l(x) \beta}{\alpha - \beta l(x)} - \frac{ax_2 l(x) \beta}{\alpha - \beta l(x)} = 0$$

so the Hamilton-Jacobi equation is satisfied.

(ii) If $\frac{ax_2 l(x)}{\alpha - \beta l(x)} > 1$ (for a given $x \in \mathcal{A}$ with $x_2 > 0$, this will be true for $a > 0$ large enough), then

$$H(x, \nabla U_a(x)) = \beta - \frac{a\beta x_2 l(x)}{\alpha - \beta l(x)} < \beta - \beta = 0$$

and the Hamilton-Jacobi equation is not satisfied.

The following result shows that the former case can be satisfied uniformly in $x \in \mathcal{A}$. Recall (see the paragraph preceding Theorem 2) that ρ is the unique solution in $(0, 1)$ of the equation $\rho^2 + \rho + \ln(1 - \rho) = 0$ and that $\gamma_0 = \rho^2 \alpha / \beta$. In addition, define $\lambda_0 := (1 - \rho) \alpha / \beta$.

Proposition 3

$$\sup_{x \in \mathcal{A}} \frac{x_2 l(x)}{\alpha - \beta l(x)} = \begin{cases} \frac{\gamma \lambda}{\alpha - \beta \lambda} & \text{if } \gamma \leq \gamma_0 \\ \frac{\gamma_0 \lambda_0}{\alpha - \beta \lambda_0} & \text{if } \gamma > \gamma_0. \end{cases}$$

Proof. Let

$$F(x) := \frac{x_2 l(x)}{\alpha - \beta l(x)} \quad \forall x \in \mathcal{A}.$$

Consider $x \in \mathcal{A}$. If $x_2 = 0$, then $F(x) = 0$, so we are only interested in the case $x_2 > 0$ and then $l(x) < \alpha/\beta$. For given $l \in (0, \alpha/\beta)$, let

$$A(l) := \{x \in \mathcal{A} : l(x) = l\}$$

There are three alternatives for this set:

(i) For $l \in (0, \lambda)$, $A(l)$ is a curve going from $(l, 0)$ to a point $(x_1(l), \gamma)$, where $x_1(l) \in (0, \alpha/\beta)$, see Figure 4 (recall that λ was defined as the smallest solution to the equation (24), the largest one is $\tilde{\lambda}$ in the following figures).

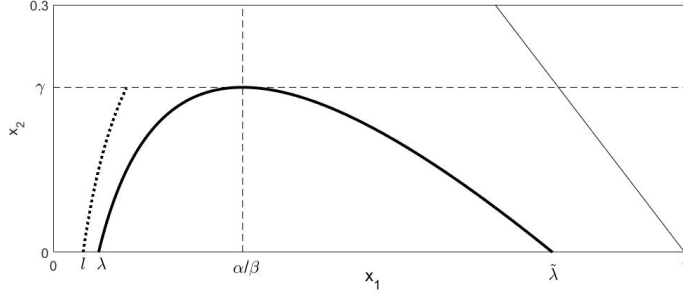


Figure 4: When $l \in (0, \lambda)$, the set $A(l)$ is the dotted line.

(ii) For $l = \lambda$, $A(l)$ is the union of the whole set $\mathcal{D} \setminus \mathcal{A}$ with the curve going from the point $(\lambda, 0)$ to $(\alpha/\beta, \gamma)$, see Figure 5.

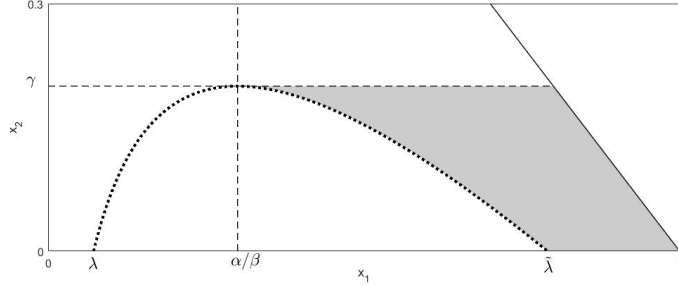


Figure 5: When $l = \lambda$ the set $A(l)$ is the bold line and the shaded area.

(iii) For $l \in (\lambda, \alpha/\beta)$, $A(l)$ is the curve going from the point $(l, 0)$ to $(\tilde{l}, 0)$ passing through $(\alpha/\beta, \chi(l))$, see Figure 6.

For $l \leq \lambda$, the highest second coordinate x_2 for the elements of $A(l)$ is γ . It follows that

$$\begin{aligned} \sup \left\{ F(x) : x \in \bigcup_{0 < l \leq \lambda} A(l) \right\} &= \sup \left\{ \frac{\gamma l}{\alpha - \beta l} : l \in (0, \lambda] \right\} \\ &= \frac{\gamma \lambda}{\alpha - \beta \lambda}. \end{aligned} \quad (38)$$

For $l \in (\lambda, \alpha/\beta)$, the highest second coordinate x_2 for the elements of $A(l)$ is

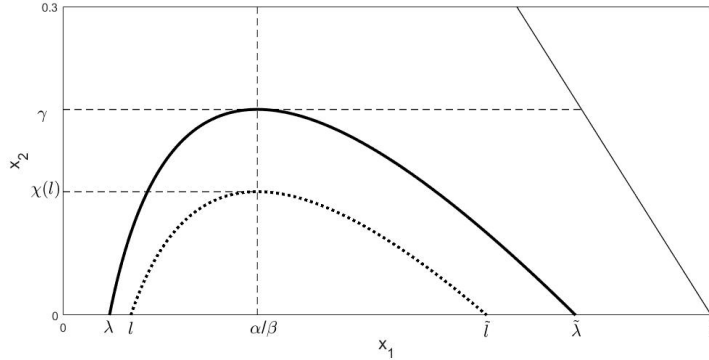


Figure 6: When $l \in (\lambda, \alpha/\beta)$, the set $A(l)$ is the dotted line.

$\chi(l)$. Thus we have

$$\sup \left\{ F(x) : x \in \bigcup_{l \in (\lambda, \alpha/\beta)} A(l) \right\} = \sup \left\{ \frac{\chi(l)l}{\alpha - \beta l} : l \in [\lambda, \alpha/\beta] \right\}.$$

Note that l and $\chi(l)$ are related via (35) (with $\chi(x)$ replaced by $\chi(l)$), so that

$$\chi(l) = l - \frac{\alpha}{\beta} - \frac{\alpha}{\beta} \ln \left(\frac{\beta l}{\alpha} \right).$$

Using this expression and introducing the variable $r := 1 - \frac{\beta}{\alpha}l$, which belongs to $(0, 1 - \frac{\beta}{\alpha}\lambda]$, we get

$$\frac{\beta^2}{\alpha} \frac{\chi(l)l}{\alpha - \beta l} = f(r) := (r - 1) \left(1 + \frac{1}{r} \ln(1 - r) \right)$$

and thus

$$\sup \left\{ F(x) : x \in \bigcup_{l \in (\lambda, \alpha/\beta]} A(l) \right\} = \frac{\alpha}{\beta^2} \sup \{ f(r) : r \in (0, 1 - \beta\lambda/\alpha] \}.$$

We compute that for any $r \in (0, 1)$, $f'(r) = \frac{g(r)}{r^2}$, with $g(r) := r^2 + r + \ln(1 - r)$. Since

$$g'(r) = \frac{r(1 - 2r)}{1 - r} \quad \forall r \in (0, 1)$$

g is increasing on $(0, 1/2)$ and decreasing on $(1/2, 1)$. Since $g(0) = 0$ and $\lim_{r \rightarrow 1^-} g(r) = -\infty$, there exists a unique $\rho \in (0, 1)$ such that $g(\rho) = 0$, g being positive and negative respectively on $(0, \rho)$ and $(\rho, 1)$.

It follows that

$$\sup\{f(r) : r \in [0, 1 - \beta\lambda/\alpha]\} = f(\min\{\rho, 1 - \beta\lambda/\alpha\}).$$

Taking into account that $\chi(\lambda) = \gamma$, we get that

$$\frac{\alpha}{\beta^2} f(1 - \beta\lambda/\alpha) = \frac{\gamma\lambda}{\alpha - \beta\lambda}$$

and consequently

$$\sup \left\{ F(x) : x \in \bigcup_{l \in (\lambda, \alpha/\beta)} A(l) \right\} = \begin{cases} \frac{\gamma\lambda}{\alpha - \beta\lambda} & \text{if } 1 - \beta\lambda/\alpha \leq \rho \\ \frac{\alpha}{\beta^2} f(\rho) & \text{if } 1 - \beta\lambda/\alpha > \rho. \end{cases} \quad (39)$$

Considering the constants λ_0 and γ_0 introduced in the proposition, one may write

$$\frac{\alpha}{\beta^2} f(\rho) = \frac{\gamma_0\lambda_0}{\alpha - \beta\lambda_0}$$

and the condition $1 - \beta\lambda/\alpha \leq \rho$ is equivalent to $\lambda \geq \lambda_0$ or $\gamma \leq \gamma_0$. It follows that (39) can be written

$$\sup \left\{ F(x) : x \in \bigcup_{l \in (\lambda, \alpha/\beta)} A(l) \right\} = \begin{cases} \frac{\gamma\lambda}{\alpha - \beta\lambda} & \text{if } \gamma \leq \gamma_0 \\ \frac{\gamma_0\lambda_0}{\alpha - \beta\lambda_0} & \text{if } \gamma > \gamma_0 \end{cases}$$

and comparing with (38), we conclude the desired result. ■

Remark 4 *In Proposition 3, the alternative $\gamma > \gamma_0$ is only possible if the point $(\alpha/\beta, \gamma_0)$ belongs to the interior of the simplex, i.e. if $\alpha/\beta + \gamma_0 < 1$. This condition is equivalent to*

$$\frac{\alpha}{\beta} - \frac{\alpha}{\beta}(\rho + \ln(1 - \rho)) < 1$$

or, recalling the definition of ρ ,

$$\frac{\alpha}{\beta}(1 + \rho^2) < 1$$

which is satisfied for $\alpha/\beta < 1/(1 + \rho^2)$.

Define $a_0 > 0$ as in (25). It follows from the previous considerations that for $a \in [0, a_0]$, U_a satisfies the Hamilton-Jacobi on \mathcal{D} . More precisely, it appears that *filling the box* is an optimal policy if and only if $a \in [0, a_0]$. This ends the proof of Theorem 2.

Proof of Corollary 2

Recall that λ is solution of

$$\lambda - \frac{\alpha}{\beta} \ln(\lambda) = \frac{\alpha}{\beta} \ln\left(\frac{\beta}{\alpha}\right) + \frac{\alpha}{\beta} + \gamma.$$

Differentiating with respect to γ , we get

$$\left(1 - \frac{\alpha}{\beta\lambda}\right) \frac{d\lambda}{d\gamma} = 1$$

from which it follows that λ is decreasing with respect to γ (this monotonicity is obvious on a picture of the phase space), since $\lambda < \alpha/\beta$. Moreover,

$$\begin{aligned} \frac{da_0}{d\gamma} &= \frac{\lambda}{\alpha - \beta\lambda} + \gamma \frac{d}{d\gamma} \frac{\lambda}{\alpha - \beta\lambda} \\ &= \frac{\lambda}{\alpha - \beta\lambda} + \gamma \frac{\alpha}{(\alpha - \beta\lambda)^2} \frac{d\lambda}{d\gamma} \\ &= \frac{\lambda}{\alpha - \beta\lambda} + \gamma \frac{\alpha}{(\alpha - \beta\lambda)^2} \frac{1}{1 - \frac{\alpha}{\beta\lambda}} \\ &= \frac{\lambda}{\alpha - \beta\lambda} - \gamma \frac{\alpha\beta\lambda}{(\alpha - \beta\lambda)^3} \\ &= \frac{\lambda}{(\alpha - \beta\lambda)^3} ((\alpha - \beta\lambda)^2 - \gamma\alpha\beta). \end{aligned}$$

Since $\frac{\lambda}{(\alpha - \beta\lambda)^3}$ is positive for $\lambda \in (0, \alpha/\beta)$, the sign of $\frac{da_0}{d\gamma}$ is that of

$$\begin{aligned} f(\lambda) &:= (\alpha - \beta\lambda)^2 - \gamma\alpha\beta \\ &= (\alpha - \beta\lambda)^2 - \alpha\beta \left(\lambda - \frac{\alpha}{\beta} \ln(\lambda) - \frac{\alpha}{\beta} \ln\left(\frac{\beta}{\alpha}\right) - \frac{\beta}{\alpha} \right) \\ &= 2\alpha^2 - 3\alpha\beta\lambda + \beta^2\lambda^2 + \alpha^2 \ln\left(\frac{\beta\lambda}{\alpha}\right). \end{aligned}$$

Thus for $\lambda \in (0, \alpha/\beta)$,

$$\begin{aligned} f'(\lambda) &= -3\alpha\beta + 2\beta^2\lambda + \frac{\alpha^2}{\lambda} \\ &= \frac{2\beta^2\lambda^2 - 3\alpha\beta\lambda + \alpha^2}{\lambda}. \end{aligned}$$

The two roots of the numerator are $\alpha/(2\beta)$ and α/β , so that f is increasing on $(0, \alpha/(2\beta))$ and decreasing on $(\alpha/(2\beta), \alpha/\beta)$. We compute that $\lim_{\lambda \rightarrow 0^+} f(\lambda) = -\infty$ and $f(\alpha/\beta) = 0$, so there exists a unique $\tilde{l} \in (0, \alpha/\beta)$ such that $f(\tilde{l}) = 0$. Comparing with the equation satisfied by ρ , we get $\tilde{l} = \lambda_0$. It follows that f is negative on $(0, \lambda_0)$, that is, a_0 is decreasing on $(0, \lambda_0)$. ■

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