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# The $\psi$ -Hilfer fractional calculus of variable order and its applications

J. Vanterler da C. Sousa <sup>1</sup>  
ra160908@ime.unicamp.br

J. A. Tenreiro Machado <sup>2</sup>  
jtm@isep.ipp.pt

E. Capelas de Oliveira <sup>3</sup>  
capelas@unicamp.br

<sup>1,3</sup> Department of Applied Mathematics, Imecc-Unicamp, 13083-859, Campinas, SP, Brazil.

<sup>2</sup> Department of Electrical Engineering, Institute of Engineering, Polytechnic of Porto, Porto, Portugal

## Abstract

In this article, we present the  $\psi$ -Hilfer fractional derivatives of variable order (FDVO) of Types I, II and III, versions A and B, as well as their combinations. Moreover, we discuss the  $\psi$ -Caputo FDVO with respect to another function, for the  $\psi$ -Hilfer and the  $\psi$ -Caputo FDVO. In addition, we propose approximations and relations between both derivatives. With regards to the  $\psi$ -Hilfer FDVO Type II, we discuss the stability of the FVO nonlinear systems solutions by means of one-parameter Mittag-Leffler functions of variable order. Examples involving the FDVO Lu and Chen systems, are also presented.

**Key words:** Fractional calculus,  $\psi$ -Hilfer fractional derivative of variable order, approximation methods, stability.

**2010 Mathematics Subject Classification:** 26A33, 33F05, 34A08, 34A12.

## 1 Introduction

The mathematics is driven by looking for patterns and that through rigorous deductions try to answer questions that have capture the attention over the decades and still intrigue scientists [14, 18]. By the year 1695, one of the questions that resulted in consequences of paramount importance for mathematics, emerged in a letter between Leibniz and L'Hôpital, including the following question: "What is the meaning of  $\frac{d^n f(x)}{dx^n}$ ,  $n$  being a fractional number" [10, 47, 48, 49]. Leibniz predicted that such a question would have important consequences in future [45, 46]. Since then, a number definitions of fractional derivatives have been introduced, each with its importance and relevance. We highlight here the Riemann-Liouville and Caputo fractional derivatives, since they are of utmost importance in the theory of fractional calculus [2, 9, 13, 16].

Knowing the broad class of fractional operator definitions where the order is not variable, how do you know which derivative is the best choice for analysing a given set of measurement data [3]? With the increasing complexity of physical phenomena to be modeled, Sousa and Oliveira [26, 28] analysed the application of the  $\psi$ -Hilfer fractional derivative. Indeed, for a given choice of  $\psi$  and for the limits 1 and 0 of the parameter  $\beta$ , we get particular cases

of the classic fractional derivatives. Regarding the  $\psi$ -Hilfer fractional derivative, a number of relevant works were published [17, 27, 29, 30, 31, 32, 36]. For works addressing other types of integrals and derivatives and their applications interested readers can address to [1, 5, 6, 13, 16, 40] and the references therein.

Tavares et al. [38], discussed three new versions of the Caputo fractional derivative of variable order (FDVO) and presented a numerical tool to solve partial differential equations involving such operator. For each version of the Caputo fractional derivative, an approximation formula in terms of the integer order derivative and an estimate for the error of the approximations, were proposed. More recently Almeida [36, 1], discussed three types of Caputo-Hadamard FDVO. Approximation formulas for each operator were presented and estimates for the error were also given. Over the years, researchers started to use fractional integrals of variable order (FIVO) and FDVO to investigate results in the variational context [8, 11, 19, 20, 21, 22, 23, 24, 25, 34, 35, 42]. It is also of particular relevance the work by Almeida, Tavares and Delfim Torres and collaborators, investigating FDVO [4, 5, 7, 12, 15, 37, 39, 41, 42].

In the follow-up of these works and also those by Sousa and Oliveira [26, 28], this paper proposes six types of  $\psi$ -Hilfer FDVO, called Types I, II and III, each with two versions A and B. Some approximations for the type II (versions A and B) of  $\psi$ -Hilfer FDVO are investigated, highlighting their relationship with the  $\psi$ -Caputo and  $\psi$ -Riemann-Liouville fractional derivatives with respect to another function. We verify that when working with FDVO, some common properties of fractional derivatives of fixed order are lost. Nonetheless, the idea is to look at these of FDVO, in the sense of approximations through series, which lead to interesting and important results for the emerging area of fractional calculus of variable order.

We highlight the main results investigated in this article. Therefore, we:

1. Propose 6 types of the  $\psi$ -Hilfer FDVO, called Types I, II and III, versions A and B;
2. Discuss the combined  $\psi$ -Hilfer FDVO;
3. Develop new versions of the  $\psi$ -Caputo FDVO;
4. Present new results and properties regarding the  $\psi$ -Hilfer FDVO;
5. Investigate the stability of solutions for FDVO nonlinear systems.

Besides these aspects, we can raise further questions such as whether it is possible to obtain a version of the Leibniz rule. Such issues are addressed in a critical discussion at the final part the study.

In the rest the article is divided as follows. In section 2, we introduce several results of importance for the development of the article. In section 3, we analyse the six versions of the  $\psi$ -Hilfer FDVO, as well as several formulations for the  $\psi$ -Caputo FDVO with respect to another function. Moreover, the new versions for the combined  $\psi$ -Hilfer FDVO are also discussed. We study approximations for the proposed versions of fractional derivatives and the integration by parts of the  $\psi$ -Hilfer fractional derivative Type II. Some examples include representative graphs and a choice for the function and the values of  $\alpha(x, t)$ ,  $\delta$ ,  $\varepsilon$ ,  $t$ ,  $x$  and  $a$ . In section 4 we consider the  $\psi$ -Hilfer FDVO Type II for illustrating the new concepts.

We investigate the stability of the solutions of fractional nonlinear systems by means of one-parameter Mittag-Leffler functions of variable order. Moreover, we discuss examples involving the FDVO Lu and Chen systems. In section 5, we include the final comments and a critical discussion, summarising the results obtained and pointing towards questions still open.

## 2 Preliminary, concepts and results

In this section, we introduced some fundamental concepts and results that will be of importance throughout the article. For the definitions of FIVO with respect to another function, we use information available in several seminal references [8, 15, 34, 43].

**Definition 2.1** *Let  $0 < \alpha(x, t) < 1$ ,  $I = [a, b]$  ( $-\infty \leq a < (x, t) < b \leq \infty$ ) be a finite or infinite interval,  $f \in L_1([a, b], \mathbb{R})$  and  $\psi \in C^1([a, b], \mathbb{R})$  an increasing function such that  $\psi'(\cdot) \neq 0$ ,  $\forall x, t \in [a, b]$ . The FIVO of a function  $f \in L_1([a, b], \mathbb{R})$  with respect to another function,  $\psi$ , of variable order  $\alpha(x, t)$ , on the left and on the right, are given by*

$$\mathbf{I}_{a+}^{\alpha(x,t);\psi} f(t) = \int_a^t \frac{\psi'(s)}{\Gamma(\alpha(x,s))} (\psi(t) - \psi(s))^{\alpha(x,s)-1} f(s) ds. \quad (2.1)$$

and

$$\mathbf{I}_{b-}^{\alpha(x,t);\psi} f(t) = \int_t^b \frac{\psi'(s)}{\Gamma(\alpha(x,s))} (\psi(s) - \psi(t))^{\alpha(x,s)-1} f(s) ds, \quad (2.2)$$

respectively.

**Remark 2.1** *Some particular cases concerning Eq. (2.1) and Eq. (2.2) should be highlighted:*

a) *Choosing  $\alpha(x, s) = \alpha(x, t)$ , we have*

$$\mathbf{I}_{a+}^{\alpha(x,t);\psi} f(t) = \frac{1}{\Gamma(\alpha(x,t))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} f(s) ds \quad (2.3)$$

and

$$\mathbf{I}_{b-}^{\alpha(x,t);\psi} f(t) = \frac{1}{\Gamma(\alpha(x,t))} \int_t^b \psi'(s) (\psi(s) - \psi(t))^{\alpha(x,t)-1} f(s) ds \quad (2.4)$$

b) *Choosing  $\alpha(x, t) = \alpha(t)$  and  $\alpha(x, s) = \alpha(s)$ , yields*

$$\mathbf{I}_{a+}^{\alpha(t);\psi} f(t) = \int_a^t \frac{\psi'(s)}{\Gamma(\alpha(s))} (\psi(t) - \psi(s))^{\alpha(s)-1} f(s) ds$$

and

$$\mathbf{I}_{b-}^{\alpha(t);\psi} f(t) = \int_t^b \frac{\psi'(s)}{\Gamma(\alpha(s))} (\psi(s) - \psi(t))^{\alpha(s)-1} f(s) ds.$$

c) Taking  $\alpha(x, s) = \alpha(x, t) = \alpha(t)$ , with  $0 < \alpha(t) < 1$ ,  $\forall t \in [a, b]$ , we have

$$\mathbf{I}_{a+}^{\alpha(t); \psi} f(t) = \frac{1}{\Gamma(\alpha(t))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha(t)-1} f(s) ds$$

and

$$\mathbf{I}_{b-}^{\alpha(t); \psi} f(t) = \frac{1}{\Gamma(\alpha(t))} \int_t^b \psi'(s) (\psi(s) - \psi(t))^{\alpha(t)-1} f(s) ds.$$

d) If we consider  $\alpha(x, t) = \alpha$ , a constant, with  $0 < \alpha < 1$ , then we obtain the Riemann-Liouville fractional integral

$$\mathbf{I}_{a+}^{\alpha; \psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s) ds$$

and

$$\mathbf{I}_{b-}^{\alpha; \psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \psi'(s) (\psi(s) - \psi(t))^{\alpha-1} f(s) ds.$$

e) For a particular choice of the order of the FIVO, Eq. (2.1), we can obtain several FIVO that are well known in the literature [1, 8, 21, 33]. Note that the function  $\psi(\cdot)$  has not yet been chosen. Thus, we can conclude that from the choice of  $\psi(\cdot)$ , the variety of possible formulations for the integral is even larger.

f) The law of exponents is not always valid for FIVO, that is, for Eq. (2.1), we have

$$\mathbf{I}_{a+}^{\alpha(x,t); \psi} \mathbf{I}_{a+}^{\beta(x,t); \psi} f(t) \neq \mathbf{I}_{a+}^{\alpha(x,t)+\beta(x,t); \psi} f(t).$$

Also, taking  $\alpha(x, t) = \alpha(t)$  and  $\beta(x, t) = \beta(t)$  we obtain

$$\mathbf{I}_{a+}^{\alpha(t); \psi} \mathbf{I}_{a+}^{\beta(t); \psi} f(t) \neq \mathbf{I}_{a++}^{\alpha(t)+\beta(t); \psi} f(t).$$

For the particular case, when the order is not variable, that is, if  $\alpha(t) = \alpha$  and  $\beta(t) = \beta$ , then it results:

$$\mathbf{I}_{a+}^{\alpha; \psi} \mathbf{I}_{a+}^{\beta; \psi} f(t) = \mathbf{I}_{a+}^{\alpha+\beta; \psi} f(t).$$

In short, Definition 2.1 establishes the  $\psi$ -Riemann-Liouville FIVO. However, we do not have yet formulated a variable order extension for the  $\psi$ -Hilfer fractional derivative, which will be presented in the following definition.

**Definition 2.2** [26, 28] Let  $n - 1 < \alpha < n$ , with  $n \in \mathbb{N}$ ,  $I = [a, b]$  an interval such that  $(-\infty \leq a < b \leq \infty)$  and  $f, \psi \in C^n([a, b], \mathbb{R})$  two functions such that  $\psi(\cdot)$  is increasing and

$\psi(\cdot) \neq 0, \forall t \in I$ . The  $\psi$ -Hilfer fractional derivative, left-side and right-sided,  ${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}(\cdot)$  and  ${}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi}(\cdot)$ , of order  $\alpha$  and type  $\beta$ , with  $0 \leq \beta \leq 1$ , are defined by

$${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(t) = \mathbf{I}_{a+}^{\beta(n-\alpha);\psi} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \mathbf{I}_{a+}^{(1-\beta)(n-\alpha);\psi} f(t) \quad (2.5)$$

and

$${}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(t) = \mathbf{I}_{b-}^{\beta(n-\alpha);\psi} \left( -\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \mathbf{I}_{b-}^{(1-\beta)(n-\alpha);\psi} f(t) \quad (2.6)$$

respectively.

As seen above, we present a variety of FIVO. For a particular choice of  $\alpha(x, t)$  and the function  $\psi(\cdot)$ , we find a number of versions of FIVO scattered in the published literature. Based on the definition of the FIVO in Eq. (2.3) and the  $\psi$ -Hilfer fractional derivative in Eq. (2.5), we will discuss, in the sequel generalized versions of the  $\psi$ -Hilfer FDVO.

### 3 The $\psi$ -Hilfer FDVO

In this section, we introduce a six version for the  $\psi$ -Hilfer FDVO. In addition, we also present versions for the  $\psi$ -Caputo FDVO that are of special importance for obtaining approximations of the  $\psi$ -Hilfer FDVO. In this sense, we present relevant properties and examples for the  $\psi$ -Hilfer FDVO type II. To conclude the section, we discuss several approximations for the  $\psi$ -Caputo and  $\psi$ -Hilfer FDVO and particular cases of these approximations.

**Definition 3.1** Let  $n - 1 < \alpha(x, t) < n$ , with  $n \in \mathbb{N}$ ,  $I = [a, b]$  an interval such that  $(-\infty \leq a < b \leq \infty)$  and  $f, \psi \in C^n([a, b], \mathbb{R})$  two functions such that  $\psi$  is increasing monotonically and  $\psi'(\cdot) \neq 0, \forall x, t \in I$ . The  $\psi$ -Hilfer FDVO, left-sided and right-sided,  ${}^H_i\mathbb{D}_{a+}^{\alpha(x,t),\beta;\psi}(\cdot)$   $\left( {}^H_i\mathbb{D}_{b-}^{\alpha(x,t),\beta;\psi}(\cdot) \right)$  of variable order  $\alpha(x, t)$  and type  $0 \leq \beta \leq 1$  with  $i = 1, 2$ , are given by

a) **Type I (A)**

$${}^H_1\mathbb{D}_{a+}^{\alpha(x,t),\beta;\psi} f(t) = \mathbf{I}_{a+}^{\beta(n-\alpha(x,t));\psi} \left\{ \frac{1}{\Gamma(n - \gamma_n(x, t))} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t \mathcal{A}_{\psi,+}^{\alpha,\beta}(x, s, t) f(s) ds \right\} \quad (3.1)$$

where  $\mathcal{A}_{\psi,+}^{\alpha,\beta}(x, s, t) := \psi'(s) (\psi(t) - \psi(s))^{n-\gamma_n(x,t)-1}$  and  $\gamma_n(x, t) = \alpha(x, t) + \beta(n - \alpha(x, t))$ .

b) **Type I (B)**

$${}^H_1\mathbb{D}_{b-}^{\alpha(x,t),\beta;\psi} f(t) = \mathbf{I}_{b-}^{\beta(n-\alpha(x,t));\psi} \left\{ \frac{1}{\Gamma(n - \gamma_n(x, t))} \left( -\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_t^b \mathcal{A}_{\beta,-}^{\alpha,\psi}(x, s, t) f(s) ds \right\}$$

(3.2)

where  $\mathcal{A}_{\psi,-}^{\alpha,\beta}(x,s,t) := \psi'(s)(\psi(s) - \psi(t))^{n-\gamma_n(x,t)-1}$ .

c) **Type II (A)**

$${}^H\mathbb{D}_{a+}^{\alpha(x,t),\beta;\psi} f(t) = \mathbf{I}_{a+}^{\beta(n-\alpha(x,t));\psi} \left\{ \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \frac{1}{\Gamma(n - \gamma_n(x,t))} \int_a^t \mathcal{A}_{\psi,+}^{\alpha,\beta}(x,s,t) f(s) ds \right\} \quad (3.3)$$

where  $\mathcal{A}_{\psi,+}^{\alpha,\beta}(x,s,t) := \psi'(s)(\psi(t) - \psi(s))^{n-\gamma_n(x,t)-1}$ .

d) **Type II (B)**

$${}^H\mathbb{D}_{b-}^{\alpha(x,t),\beta;\psi} f(t) = \mathbf{I}_{b-}^{\beta(n-\alpha(x,t));\psi} \left\{ \left( -\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \frac{1}{\Gamma(n - \gamma_n(x,t))} \int_t^b \mathcal{A}_{\beta,-}^{\alpha,\psi}(x,s,t) f(s) ds \right\} \quad (3.4)$$

where  $\mathcal{A}_{\beta,-}^{\alpha,\psi}(x,s,t) := \psi'(s)(\psi(s) - \psi(t))^{n-\gamma_n(x,t)-1}$ .

We also include the **Type III (A)** and **Type III (B)**, with a variable order depending on the integral factor. The version **Type III (A)**, is given by:

$${}^H\mathbb{D}_{a+}^{\alpha(x,t),\beta;\psi} f(t) = \mathbf{I}_{a+}^{\beta(n-\alpha(x,t));\psi} \left\{ \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t \frac{\mathcal{A}_{\psi,+}^{\alpha,\beta}(x,s,t)}{\Gamma(n - \gamma_n(x,s))} f(s) ds \right\} \quad (3.5)$$

where  $\mathcal{A}_{\psi,+}^{\alpha,\beta}(x,s,t) := \psi'(s)(\psi(t) - \psi(s))^{n-\gamma_n(x,s)-1}$ .

The version **Type III (B)**, is given by:

$${}^H\mathbb{D}_{b-}^{\alpha(x,t),\beta;\psi} f(t) = \mathbf{I}_{b-}^{\beta(n-\alpha(x,t));\psi} \left\{ \left( -\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_t^b \frac{\mathcal{A}_{\psi,-}^{\alpha,\beta}(x,s,t)}{\Gamma(n - \gamma_n(x,s))} f(s) ds \right\} \quad (3.6)$$

where  $\mathcal{A}_{\psi,-}^{\alpha,\beta}(x,s,t) := \psi'(s)(\psi(s) - \psi(t))^{n-\gamma_n(x,s)-1}$ .

Choosing  $\alpha(x,t) = \alpha(t)$  and  $\alpha(x,s) = \alpha(s)$  in Eq. (3.5) and Eq. (3.6), we have

$${}^H\mathbb{D}_{a+}^{\alpha(t),\beta;\psi} f(t) = \mathbf{I}_{a+}^{\beta(n-\alpha(t));\psi} \left\{ \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t \frac{\mathcal{A}_{\psi,+}^{\alpha,\beta}(s,t)}{\Gamma(n - \gamma_n(s))} f(s) ds \right\}$$

and

$${}^H\mathbb{D}_{b-}^{\alpha(t),\beta;\psi} f(t) = \mathbf{I}_{b-}^{\beta(n-\alpha(t));\psi} \left\{ \left( -\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_t^b \frac{\mathcal{A}_{\psi,-}^{\alpha,\beta}(s,t)}{\Gamma(n - \gamma_n(s))} f(s) ds \right\}$$

where  $\mathcal{A}_{\psi,+}^{\alpha,\beta}(s,t) := \psi'(s)(\psi(t) - \psi(s))^{n-\gamma_n(s)-1}$  and  $\mathcal{A}_{\psi,-}^{\alpha,\beta}(s,t) := \psi'(s)(\psi(s) - \psi(t))^{n-\gamma_n(t)-1}$ , respectively.

From the choice of  $\psi(\cdot)$  and the limits  $\beta \rightarrow 0$  or  $\beta \rightarrow 1$ , it is possible to obtain a wide class of FDVO as particular cases. Here, we present two particular cases, the  $\psi$ -Caputo and the  $\psi$ -Riemann-Liouville FDVO.

- Taking the limit  $\beta \rightarrow 1$  in both sides of Eq. (3.1), Eq. (3.2), Eq. (3.3) and Eq. (3.4), we have the  $\psi$ -Caputo FDVO:

$${}_1^C \mathbb{D}_{a^+}^{\alpha(x,t);\psi} f(t) = \mathbf{I}_{a^+}^{n-\alpha(x,t);\psi} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n f(t).$$

and

$${}_1^C \mathbb{D}_{b^-}^{\alpha(x,t);\psi} f(t) = \mathbf{I}_{b^-}^{n-\alpha(x,t);\psi} \left( -\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n f(t).$$

- Taking the limit  $\beta \rightarrow 0$  in both sides of Eq. (3.1), Eq. (3.2), Eq. (3.3) and Eq. (3.4), we have the  $\psi$ -Riemann-Liouville FDVO:

$${}_1 \mathbb{D}_{a^+}^{\alpha(x,t);\psi} f(t) = \frac{1}{\Gamma(n-\alpha(x,t))} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t \mathcal{A}_{\psi,+}^{\alpha,1}(x,s,t) f(s) ds;$$

$${}_2 \mathbb{D}_{a^+}^{\alpha(x,t);\psi} f(t) = \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \frac{1}{\Gamma(n-\alpha(x,t))} \int_a^t \mathcal{A}_{\psi,+}^{\alpha,1}(x,s,t) f(s) ds;$$

$${}_1 \mathbb{D}_{b^-}^{\alpha(x,t);\psi} f(t) = \frac{1}{\Gamma(n-\alpha(x,t))} \left( -\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_t^b \mathcal{A}_{\psi,-}^{\alpha,1}(x,s,t) f(s) ds;$$

$${}_2 \mathbb{D}_{b^-}^{\alpha(x,t);\psi} f(t) = \left( -\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \frac{1}{\Gamma(n-\alpha(x,t))} \int_t^b \mathcal{A}_{\psi,-}^{\alpha,1}(x,s,t) f(s) ds,$$

where  $\mathcal{A}_{\psi,+}^{\alpha,1}(x,s,t) := \psi'(s) (\psi(s) - \psi(t))^{n-\alpha(x,t)-1}$  and  $\mathcal{A}_{\psi,-}^{\alpha,1}(x,s,t) := \psi'(s) (\psi(t) - \psi(s))^{n-\alpha(x,t)-1}$ .

**Remark 3.1** Taking  $\alpha(x,t) = \alpha$  and substituting in both sides of Eq. (3.1), Eq. (3.2), Eq. (3.3) and Eq. (3.4), we have the following identities

$${}_1^H \mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f(t) = {}_2^H \mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f(t) = {}^H \mathbb{D}_{a^+}^{\alpha,\beta;\psi} f(t)$$

and

$${}_1^H \mathbb{D}_{b^-}^{\alpha(x,t),\beta;\psi} f(t) = {}_2^H \mathbb{D}_{b^-}^{\alpha(x,t),\beta;\psi} f(t) = {}^H \mathbb{D}_{b^-}^{\alpha,\beta;\psi} f(t).$$

The  $\psi$ -Hilfer FDVO can be written in terms of the  $\psi$ -Caputo and  $\psi$ -Riemann-Liouville FDVO. Therefore, we have:

$${}_2^H \mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f(t) = \mathbf{I}_{a^+}^{\beta(n-\alpha(x,t));\psi} {}_2 \mathbb{D}_{a^+}^{\gamma_n(x,t);\psi} f(t)$$

and

$${}_2^H \mathbb{D}_{b^-}^{\alpha(x,t),\beta;\psi} f(t) = \mathbf{I}_{b^-}^{\beta(n-\alpha(x,t));\psi} (-1)^n {}_2 \mathbb{D}_{b^-}^{\gamma_n(x,t);\psi} f(t).$$

On the other hand, we also have

$${}_2^H \mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f(t) = {}_1^C \mathbb{D}_{a^+}^{\beta(\alpha(x,t)-n)+n;\psi} \mathbf{I}_{a^+}^{n-\gamma_n(x,t);\psi} f(t)$$



and

$${}^H_2\mathbb{D}_{b^-}^{\alpha(x,t),\beta;\psi} f(t) = {}^C_1\mathbb{D}_{b^-}^{\beta(\alpha(x,t)-n)+n;\psi} \mathbf{I}_{b^-}^{n-\gamma n(x,t);\psi} f(t).$$

We can also present other versions for the Caputo FDVO:

$$\begin{aligned} {}^C_1\mathbf{D}_{a+}^{\alpha(t);\psi} x(t) &= {}_1\mathbb{D}_{a+}^{\alpha(t);\psi} (x(t) - x(a)) \\ &= \frac{1}{\Gamma(1-\alpha(t))} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a)) ds, \end{aligned} \quad (3.7)$$

$$\begin{aligned} {}^C_1\mathbf{D}_{b^-}^{\alpha(t);\psi} x(t) &= {}_1\mathbb{D}_{b^-}^{\alpha(t);\psi} (x(t) - x(b)) \\ &= \frac{1}{\Gamma(1-\alpha(t))} \left( -\frac{1}{\psi'(t)} \frac{d}{dt} \right) \int_t^b \psi'(s) (\psi(s) - \psi(t))^{-\alpha(t)} (x(s) - x(b)) ds, \end{aligned} \quad (3.8)$$

$$\begin{aligned} {}^C_2\mathbf{D}_{a+}^{\alpha(t);\psi} x(t) &= {}_2\mathbb{D}_{a+}^{\alpha(t);\psi} (x(t) - x(a)) \\ &= \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) \frac{1}{\Gamma(1-\alpha(t))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a)) ds, \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} {}^C_2\mathbf{D}_{b^-}^{\alpha(t);\psi} x(t) &= {}_2\mathbb{D}_{b^-}^{\alpha(t);\psi} (x(t) - x(b)) \\ &= \left( -\frac{1}{\psi'(t)} \frac{d}{dt} \right) \frac{1}{\Gamma(1-\alpha(t))} \int_t^b \psi'(s) (\psi(s) - \psi(t))^{-\alpha(t)} (x(s) - x(b)) ds. \end{aligned} \quad (3.10)$$

The expressions are important to investigate two results involving the  $\psi$ -Hilfer FDVO.

Based on the definitions of the  $\psi$ -Hilfer FDVO with respect to another function, we now present the combined  $\psi$ -Hilfer FDVO.

**Definition 3.2** Let  $0 < \alpha_1(x,t), \alpha_2(x,t) < 1$ ,  $0 \leq \beta \leq 1$ ,  $\bar{\gamma} = (\gamma_1, \gamma_2)$ ,  $\gamma_1, \gamma_2 \neq 0$  and  $f \in AC([a, b], \mathbb{R})$ . We have:

$${}^H_i\mathbb{D}_{\bar{\gamma}}^{\alpha_1(x,t), \alpha_2(x,t), \beta; \psi} f(t) = \gamma_1 {}^H_i\mathbb{D}_{a+}^{\alpha_1(x,t), \beta; \psi} f(t) + \gamma_2 {}^H_i\mathbb{D}_{b^-}^{\alpha_2(x,t), \beta; \psi} f(t) \quad (3.11)$$

for  $i = 1, 2$ .

Let us now make a brief analysis concerning Eq. (3.11).

1. Taking the limit  $\beta \rightarrow 1$  in both sides of Eq. (3.11), we get the combined  $\psi$ -Caputo FDVO, given by

$$\begin{aligned} {}^H_i\mathbb{D}_{\bar{\gamma}}^{\alpha_1(x,t), \alpha_2(x,t); 1} f(t) &= \gamma_1 {}^C_i\mathbb{D}_{a+}^{\alpha_1(x,t);\psi} f(t) + \gamma_2 {}^C_i\mathbb{D}_{b^-}^{\alpha_2(x,t);\psi} f(t) \\ &= {}^C_i\mathbb{D}_{\bar{\gamma}}^{\alpha_1(x,t), \alpha_2(x,t);\psi} f(t) \end{aligned}$$

for  $i = 1, 2$ .

2. As a particular case, considering  $\psi(t) = t$  and taking the limit  $\beta \rightarrow 1$  in both sides of Eq. (3.11) we obtain the combined Caputo FDVO

$$\begin{aligned} {}_i^H \mathbb{D}_{\bar{\gamma}}^{\alpha_1(x,t), \alpha_2(x,t); 1} f(t) &= \gamma_1 {}_i^C \mathbb{D}_{a^+}^{\alpha_1(x,t)} f(t) + \gamma_2 {}_i^C \mathbb{D}_{b^-}^{\alpha_2(x,t)} f(t) \\ &= {}_i^C \mathbb{D}_{\bar{\gamma}}^{\alpha_1(x,t), \alpha_2(x,t)} f(t) \end{aligned}$$

for  $i = 1, 2$ .

3. Taking the limit  $\beta \rightarrow 0$  in both sides of Eq. (3.11), we get the combined  $\psi$ -Riemann-Liouville FDVO, given by

$$\begin{aligned} {}_i^H \mathbb{D}_{\bar{\gamma}}^{\alpha_1(x,t), \alpha_2(x,t); 0} f(t) &= \gamma_1 {}_i^{\mathbb{D}}_{a^+}^{\alpha_1(x,t); \psi} f(t) + \gamma_2 {}_i^{\mathbb{D}}_{b^-}^{\alpha_2(x,t); \psi} f(t) \\ &= {}_i^{\mathbb{D}}_{\bar{\gamma}}^{\alpha_1(x,t), \alpha_2(x,t); \psi} f(t) \end{aligned}$$

for  $i = 1, 2$ .

4. As a particular case, considering  $\psi(t) = t$  and taking the limit  $\beta \rightarrow 0$  in both sides of Eq. (3.11) we obtain the combined Riemann-Liouville FDVO

$$\begin{aligned} {}_i^H \mathbb{D}_{\bar{\gamma}}^{\alpha_1(x,t), \alpha_2(x,t); 0} f(t) &= \gamma_1 {}_i^{\mathbb{D}}_{a^+}^{\alpha_1(x,t)} f(t) + \gamma_2 {}_i^{\mathbb{D}}_{b^-}^{\alpha_2(x,t)} f(t) \\ &= {}_i^{\mathbb{D}}_{\bar{\gamma}}^{\alpha_1(x,t), \alpha_2(x,t)} f(t) \end{aligned}$$

for  $i = 1, 2$ .

5. From the choice of the function  $\psi(t)$  and the limits  $\beta \rightarrow 0$  and  $\beta \rightarrow 1$  it is possible to obtain other formulations of FDVO.

6.  $\bar{\gamma} = (\gamma_1, \gamma_2) \in [0, 1]^2$  is a vector with  $\gamma_1$  and  $\gamma_2$  both non null.

7. Taking  $\gamma_1 = 0$  and  $\gamma_2 \neq 0$  or  $\gamma_1 \neq 0$  and  $\gamma_2 = 0$  in Eq. (3.11) we have

$${}_i^H \mathbb{D}_{\bar{\gamma}}^{\alpha_1(x,t), \alpha_2(x,t), \beta; \psi} f(t) = \gamma_2 {}_i^H \mathbb{D}_{b^-}^{\alpha_2(x,t), \beta; \psi} f(t)$$

and

$${}_i^H \mathbb{D}_{\bar{\gamma}}^{\alpha_1(x,t), \alpha_2(x,t), \beta; \psi} f(t) = \gamma_1 {}_i^H \mathbb{D}_{a^+}^{\alpha_1(x,t), \beta; \psi} f(t)$$

for  $i = 1, 2$ .

From the versions of the  $\psi$ -Hilfer FDVO presented in Eq. (3.1) - Eq. (3.4), we can analyse important properties as follows.

**Proposition 3.2** *Let  $f$  and  $g$  be two continuous functions and  $\lambda$  and  $\delta$  two arbitrary constants. The  $\psi$ -Hilfer FDVO are linear, that is*

$${}_i^H \mathbb{D}_{a^+}^{\alpha(x,t), \beta; \psi} (\lambda f \pm \delta g)(t) = \lambda {}_i^H \mathbb{D}_{a^+}^{\alpha(x,t), \beta; \psi} f(t) \pm \delta {}_i^H \mathbb{D}_{a^+}^{\alpha(x,t), \beta; \psi} g(t)$$

for  $i = 1, 2$ .

*Proof:* The proof follows directly from the definition.  $\square$

**Theorem 3.3** *Let  $f \in C'([a, b], \mathbb{R})$  and  $x, t \in [a, b]$ . Then, we have*

- a)  ${}_1^H \mathbb{D}_{a^+}^{\alpha(x,t), \beta; \psi} f(t) = {}_2^H \mathbb{D}_{a^+}^{\alpha(x,t), \beta; \psi} f(t) = 0$  at  $t = a$ ;
- b)  ${}_1^H \mathbb{D}_{b^-}^{\alpha(x,t), \beta; \psi} f(t) = {}_2^H \mathbb{D}_{b^-}^{\alpha(x,t), \beta; \psi} f(t) = 0$  at  $t = b$ .

*Proof:* a) Consider the  $\psi$ -Hilfer FDVO in terms of the  $\psi$ -Riemann-Liouville FDVO. Taking the norm, we can write

$$\begin{aligned}
{}_2^H \mathbb{D}_{a^+}^{\alpha(x,t), \beta; \psi} f(t) &= \left\| \mathbf{I}_{a^+}^{\beta(n-\alpha(x,t)); \psi} {}_2 \mathbb{D}_{a^+}^{\gamma_n(x,t); \psi} f(t) \right\| \\
&= \left\| \frac{1}{\Gamma(\beta(n-\alpha(x,t)))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} {}_2 \mathbb{D}_{a^+}^{\gamma_n(x,t); \psi} f(s) ds \right\| \\
&\leq \frac{\left\| {}_2 \mathbb{D}_{a^+}^{\gamma_n(x,t); \psi} f(t) \right\|}{\Gamma(\beta(n-\alpha(x,t)))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} ds \\
&= \frac{\left\| {}_2 \mathbb{D}_{a^+}^{\gamma_n(x,t); \psi} f(t) \right\|}{\Gamma(\beta(n-\alpha(x,t)))} \frac{(\psi(t) - \psi(a))^{\alpha(x,t)}}{\alpha(x,t)}
\end{aligned}$$

that is zero at  $t = a$ .

On the other hand, we have

$$\begin{aligned}
\left\| {}_1^H \mathbb{D}_{a^+}^{\alpha(x,t), \beta; \psi} f(t) \right\| &= \left\| \mathbf{I}_{a^+}^{\beta(n-\alpha(x,t)); \psi} \left\{ \frac{1}{\Gamma(\beta, \alpha(x,t))} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \times \int_a^s \mathcal{A}_{\psi,+}^{\alpha, \beta}(x, s, t) f(\tau) d\tau \right\} \right\| \\
&\leq \|f(t)\|_{C_\psi^n} \left\| \mathbf{I}_{a^+}^{\beta(n-\alpha(x,t)); \psi} \left( \frac{1}{\Gamma(n-\gamma_n(x,t))} \frac{(\psi(s) - \psi(a))^{n-\gamma_n(x,t)}}{n-\gamma_n(x,t)} \right) \right\| \\
&\leq \frac{\|f(x, t)\|_{C_\psi^n}}{\Gamma(n-\gamma_n(x,t)+1) \Gamma(\beta(n-\alpha(x,t)))} \times \\
&\quad \times \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\beta(n-\alpha(x,t))-1} (\psi(s) - \psi(a))^{n-\gamma_n(x,t)} ds.
\end{aligned}$$

where  $\mathcal{A}_{\psi,+}^{\alpha, \beta}(x, s, t) := \psi'(s) (\psi(t) - \psi(s))^{n-\gamma_n(x,t)-1}$ .

Integrating by parts and rearranging we obtain

$$\begin{aligned}
&\left\| {}_1^H \mathbb{D}_{a^+}^{\alpha(x,t), \beta; \psi} f(t) \right\| \\
&\leq \frac{\|f(x, t)\|_{C_\psi^n}}{\Gamma(n-\gamma_n(x,t)+1) \Gamma(\beta(n-\alpha(x,t)))} \int_0^{\psi(t)-\psi(a)} \left( 1 - \frac{u}{\psi(t) - \psi(a)} \right)^{\beta(n-\alpha(x,t))-1} u^{n-\gamma_n(x,t)} du.
\end{aligned}$$

Introducing a change of variables  $p = \frac{u}{\psi(t) - \psi(a)}$  and simplifying, yields

$$\left\| {}_1^H \mathbb{D}_{a^+}^{\alpha(x,t), \beta; \psi} f(t) \right\| \leq \frac{\|f(x, t)\|_{C_\psi^n}}{\Gamma(n-\gamma_n(x,t)+1) \Gamma(\beta(n-\alpha(x,t)))} \int_0^1 (1-p)^{\beta(n-\alpha(x,t))-1} p^{n-\gamma_n(x,t)} dp.$$

Using the definition of gamma function, we have

$$\left\| {}^H_1\mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f(t) \right\| \leq \frac{\|f(t)\|_{C_\psi^n}}{\Gamma(n - \gamma_n(x,t) + 1)} \frac{(\psi(t) - \psi(a))^{n - \gamma_n(x,t)}}{\Gamma(\beta(n - \alpha(x,t)))} \frac{\Gamma(\bar{\alpha})\Gamma(n - \gamma_n(x,t) + 1)}{\Gamma(n - \alpha(x,t) + 1)}.$$

From this expression evaluated at  $t = a$  we obtain zero, which conclude the proof. For b) we omit the proof, since the procedure is the one adopted for a).  $\square$

**Lemma 3.4** *Let  $\varepsilon, \delta > 0$ . Consider the function  $f_1(t) = (\psi(t) - \psi(a))^{\delta-1}(\psi(x) - \psi(a))^{\varepsilon-1}$ .*

*Then, we have*

$$\begin{aligned} {}^H_1\mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f_1(t) &= (\psi(x) - \psi(a))^{\varepsilon-1} \mathbf{I}_{a^+}^{\beta(1-\alpha(x,t));\psi} \left\{ \frac{1}{\Gamma(1 - \gamma_1(x,t))} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) \right. \\ &\quad \left. \times B(1 - \gamma_1(x,t), \delta) (\psi(t) - \psi(a))^{\delta - \gamma_1(x,t)} \right\} \end{aligned} \quad (3.12)$$

and

$${}^H_2\mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f_1(t) = \frac{\Gamma(\delta)}{\Gamma(\delta - \alpha(x,t))} (\psi(t) - \psi(a))^{\delta - \alpha(x,t) - 1} (\psi(x) - \psi(a))^{\varepsilon-1} \quad (3.13)$$

where  $\gamma_1(x,t) := \alpha(x,t) - \beta(1 - \alpha(x,t))$ .

*Proof:* First, let us show Eq. (3.12). In fact, by definition of  ${}^H_1\mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi}(\cdot)$  and remembering the relation

$$\begin{aligned} &\mathbf{I}_{a^+}^{1-\gamma_1(x,t);\psi} \left[ (\psi(t) - \psi(a))^{\delta-1} (\psi(x) - \psi(a))^{\varepsilon-1} \right] \\ &= (\psi(x) - \psi(a))^{\varepsilon-1} \mathbf{I}_{a^+}^{1-\gamma_1(x,t);\psi} (\psi(t) - \psi(a))^{\delta-1} \\ &= (\psi(x) - \psi(a))^{\varepsilon-1} \frac{\Gamma(\delta)}{\Gamma(1 - \gamma_1(x,t) + \delta)} (\psi(t) - \psi(a))^{\delta - \gamma_1(x,t)}, \end{aligned}$$

we have

$$\begin{aligned} {}^H_1\mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f(t) &= \mathbf{I}_{a^+}^{\beta(1-\alpha(x,t));\psi} \left\{ \frac{1}{\Gamma(1 - \gamma_1(x,t))} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) \times \right. \\ &\quad \left. \times \Gamma(1 - \gamma_1(x,t)) (\psi(t) - \psi(a))^{\varepsilon-1} \frac{\Gamma(\delta)}{\Gamma(1 - \gamma_1(x,t) + \delta)} (\psi(t) - \psi(a))^{\delta - \gamma_1(x,t)} \right\} \\ &= (\psi(x) - \psi(a))^{\varepsilon-1} \mathbf{I}_{a^+}^{\beta(1-\alpha(x,t));\psi} \left\{ \frac{1}{\Gamma(1 - \gamma_1(x,t))} \times \right. \\ &\quad \left. \times \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) \frac{\Gamma(\delta)\Gamma(1 - \gamma_1(x,t))}{\Gamma(1 - \gamma_1(x,t) + \delta)} (\psi(t) - \psi(a))^{\delta - \gamma_1(x,t)} \right\} \\ &= (\psi(x) - \psi(a))^{\varepsilon-1} \mathbf{I}_{a^+}^{\beta(1-\alpha(x,t));\psi} \left\{ \frac{1}{\Gamma(1 - \gamma_1(x,t))} \times \right. \\ &\quad \left. \times \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) B(1 - \gamma_1(x,t), \delta) (\psi(t) - \psi(a))^{\delta - \gamma_1(x,t)} \right\}. \end{aligned}$$

Consider the relation

$$\begin{aligned} & {}_2\mathbb{D}_{a^+}^{\gamma_1(x,t);\psi} \left( (\psi(t) - \psi(a))^{\delta-1} (\psi(x) - \psi(a))^{\varepsilon-1} \right) \\ &= \frac{\Gamma(\delta)}{\Gamma(\delta - \gamma_1(x,t))} (\psi(t) - \psi(a))^{\delta - \gamma_1(x,t) - 1} (\psi(x) - \psi(a))^{\varepsilon-1}. \end{aligned} \quad (3.14)$$

Applying  $\mathbf{I}_{a^+}^{\gamma_1(x,t) - \alpha(x,t);\psi}(\cdot)$  to both sides of Eq.(3.14) we get

$$\begin{aligned} {}_2^H\mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f_1(t) &= \mathbf{I}_{a^+}^{\gamma_1(x,t) - \alpha(x,t);\psi} {}_2\mathbb{D}_{a^+}^{\gamma_1(x,t);\psi} \left( (\psi(t) - \psi(a))^{\delta-1} (\psi(x) - \psi(a))^{\varepsilon-1} \right) \\ &= \frac{\Gamma(\delta)}{\Gamma(\delta - \gamma_1(x,t))} (\psi(x) - \psi(a))^{\varepsilon-1} \mathbf{I}_{a^+}^{\gamma_1(x,t) - \alpha(x,t);\psi} \left[ (\psi(t) - \psi(a))^{\delta - \gamma_1(x,t) - 1} \right] \\ &= \frac{\Gamma(\delta)}{\Gamma(\delta - \alpha(x,t))} (\psi(t) - \psi(a))^{\delta - \alpha(x,t) - 1} (\psi(x) - \psi(a))^{\varepsilon-1} \end{aligned}$$

which concludes the proof.  $\square$

**Theorem 3.5** *Let  $\varepsilon, \delta > 0$  and consider the function*

$$f_2(t) = \mathbb{E}_{\alpha(x,t)}((\psi(t) - \psi(a))^{\delta-1} (\psi(x) - \psi(a))^{\varepsilon-1})$$

where  $\mathbb{E}_{\alpha(x,t)}(\cdot)$  is the one parameter Mittag-Leffler function of variable order. Then, we have

$${}_2^H\mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f_2(t) = \sum_{k=0}^{\infty} \frac{\Gamma(k(\delta - 1) + 1)}{\Gamma(k(\delta - 1) + 1 - \alpha(x,t))} \frac{(\psi(t) - \psi(a))^{k(\delta-1) - \alpha(x,t)} (\psi(x) - \psi(a))^{k(\varepsilon-1)}}{\Gamma(\alpha(x,t)k + 1)}. \quad (3.15)$$

*Proof:* In fact, by the definition of the one parameter Mittag-Leffler function we can write

$$\begin{aligned} {}_2^H\mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f_2(t) &= {}_2^H\mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} \mathbb{E}_{\alpha(x,t)}((\psi(t) - \psi(a))^{\delta-1} (\psi(x) - \psi(a))^{\varepsilon-1}) \\ &= {}_2^H\mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} \sum_{k=0}^{\infty} \frac{[(\psi(t) - \psi(a))^{\delta-1} (\psi(x) - \psi(a))^{\varepsilon-1}]^k}{\Gamma(\alpha(x,t)k + 1)} \\ &= \sum_{k=0}^{\infty} {}_2^H\mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} \frac{[(\psi(t) - \psi(a))^{\delta-1} (\psi(x) - \psi(a))^{\varepsilon-1}]^k}{\Gamma(\alpha(x,t)k + 1)} \\ &= \sum_{k=0}^{\infty} {}_2^H\mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} \frac{(\psi(t) - \psi(a))^{k(\delta-1)} (\psi(x) - \psi(a))^{k(\varepsilon-1)}}{\Gamma(\alpha(x,t)k + 1)} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(k(\delta - 1) + 1)}{\Gamma(k(\delta - 1) + 1 - \alpha(x,t))} \frac{(\psi(t) - \psi(a))^{k(\delta-1) - \alpha(x,t)} (\psi(x) - \psi(a))^{k(\varepsilon-1)}}{\Gamma(\alpha(x,t)k + 1)} \end{aligned}$$

which concludes the the proof.  $\square$

We illustrate the theoretical discussion held previously by means of some examples. Following Theorem 10, we address the possibility of choosing  $\alpha(x, t)$ , which is directly related to the function  $f$ . Then, let  $\delta, \varepsilon > 0$ ,  $0 < \alpha(x, t) \leq 1$  and consider the function

$$f(x, t) = \sum_{k=0}^{\infty} \frac{\Gamma(k(\delta - 1) + 1)}{\Gamma(k(\delta - 1) + 1 - \alpha(x, t))} \frac{(\psi(t) - \psi(a))^{k(\delta-1)-\alpha(x,t)} (\psi(x) - \psi(a))^{k(\varepsilon-1)}}{\Gamma(\alpha(x, t)k + 1)}.$$

The plots in Fig. 1, Fig. 2 and Fig. 3 consider with  $t$  and  $x$  varying in the interval  $[0, 0.5]$  with space of 0.001. Moreover, three cases are considered, namely  $(\delta, \varepsilon, a) = (2.7, 1.9, 0)$ ,  $(\delta, \varepsilon, a) = (3.7, 0.9, 0)$  and  $(\delta, \varepsilon, a) = (2.9, 2.9, 0)$ .

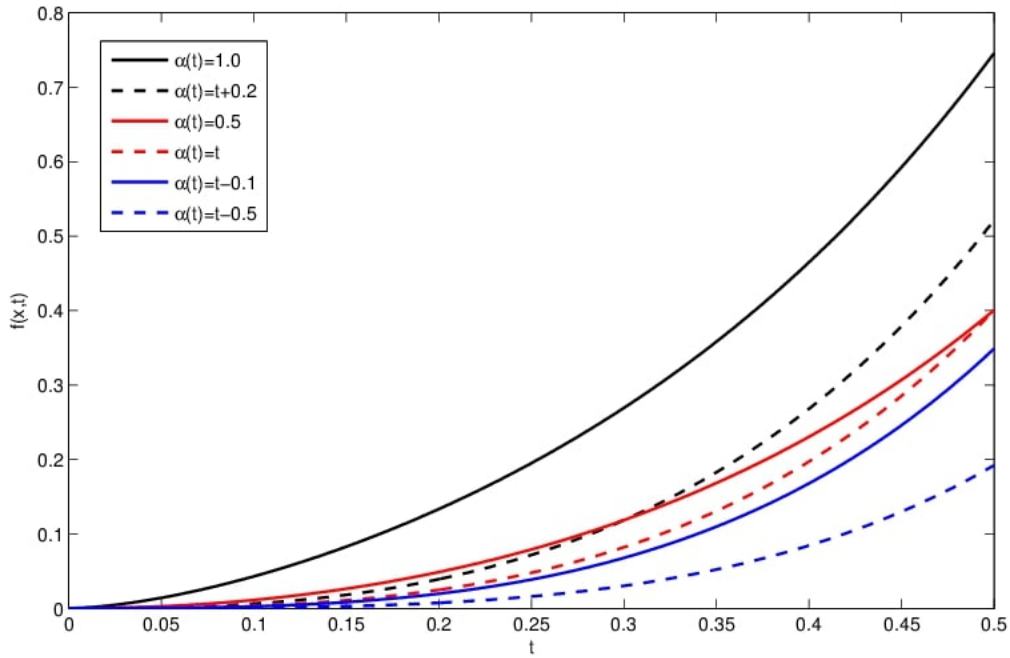


Figure 1:  $(\delta, \varepsilon, a) = (2.7, 1.9, 0)$

**Remark 3.6** In particular, given  $n \leq k \in \mathbb{N}$ , and as  $\delta > n$ , yields

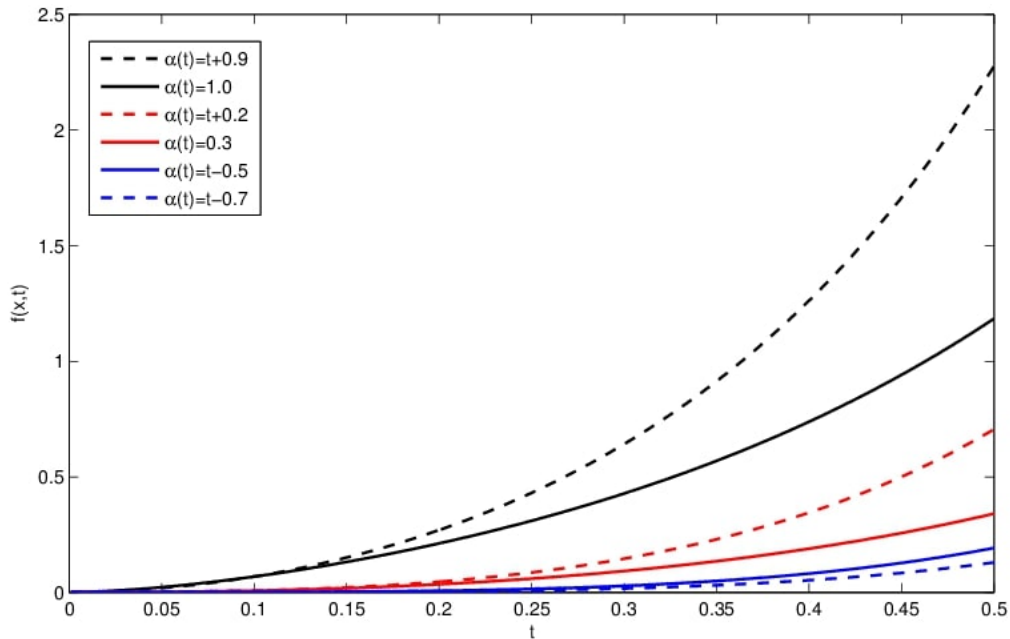
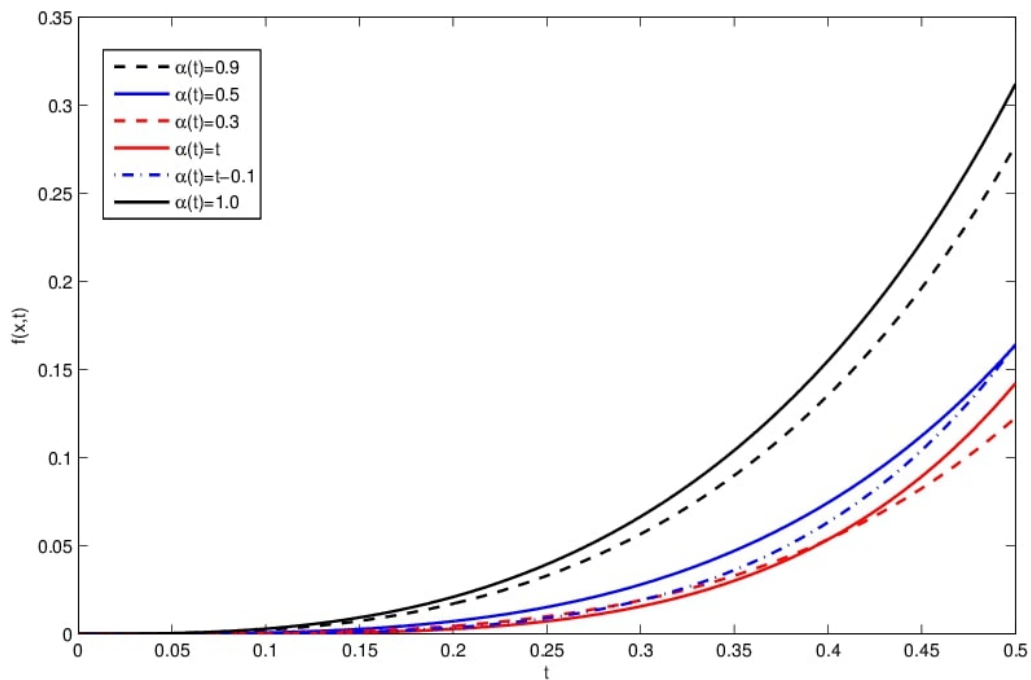
$${}_2^H \mathbb{D}_{a+}^{\alpha(x,t), \beta; \psi} (\psi(t) - \psi(a))^k = \frac{k!}{\Gamma(k + 1 - \alpha(x, t))} (\psi(t) - \psi(a))^{k-\alpha}$$

and

$${}_2^H \mathbb{D}_{b-}^{\alpha(x,t), \beta; \psi} (\psi(b) - \psi(t))^k = \frac{k!}{\Gamma(k + 1 - \alpha(x, t))} (\psi(b) - \psi(t))^{k-\alpha}.$$

On the other hand, for  $n > k \in \mathbb{N}_0$ , we obtain

$${}_2^H \mathbb{D}_{a+}^{\alpha(x,t), \beta; \psi} (\psi(t) - \psi(a))^k = {}_2^H \mathbb{D}_{b-}^{\alpha(x,t), \beta; \psi} (\psi(b) - \psi(t))^k = 0.$$

Figure 2:  $(\delta, \varepsilon, a) = (3.7, 0.9, 0)$ Figure 3:  $(\delta, \varepsilon, a) = (2.9, 2.9, 0)$ 

**Theorem 3.7** Given  $\lambda > 0$ ,  $n - 1 < \alpha(x, t) < n$  and  $0 \leq \beta \leq 1$ . Consider the functions

$f(t) = \mathbb{E}_{\alpha(x,t)}(\lambda(\psi(t) - \psi(a))^{\alpha(x,t)})$  and  $g(t) = \mathbb{E}_{\alpha(x,t)}(\lambda(\psi(b) - \psi(t))^{\alpha(x,t)})$ . Then, we have

$${}_2^H\mathbb{D}_{a+}^{\alpha(x,t),\beta;\psi} f(t) = \lambda f(t) \quad \text{and} \quad {}_2^H\mathbb{D}_{b-}^{\alpha(x,t),\beta;\psi} g(t) = \lambda g(t).$$

*Proof:* Using the definition of the one parameter Mittag-Leffler function and Eq. (3.13) (with  $\varepsilon = 1$ ), we have

$$\begin{aligned} {}_2^H\mathbb{D}_{a+}^{\alpha(x,t),\beta;\psi} f(t) &= {}_2^H\mathbb{D}_{a+}^{\alpha(x,t),\beta;\psi} \mathbb{E}_{\alpha(x,t)}(\lambda(\psi(t) - \psi(a))^{\alpha(x,t)}) \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha(x,t)k + 1)} {}_2^H\mathbb{D}_{a+}^{\alpha(x,t),\beta;\psi} (\psi(t) - \psi(a))^{\alpha(x,t)k} \\ &= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1} (\psi(t) - \psi(a))^{\alpha(x,t)(k-1)}}{\Gamma((k-1)\alpha(x,t) + 1)} \\ &= \lambda f(t). \end{aligned}$$

which concludes the the proof.  $\square$

**Theorem 3.8** Let  $0 < \alpha(x,t) < 1 - \frac{1}{n}$  for all  $(x,t) \in \Delta$  with a number  $n \in \mathbb{N}$  greater than or equal to two, and  $\psi'(t) \neq 0$ . If  $f \in C'([a,b], \mathbb{R})$ ,  $g \in C([a,b], \mathbb{R})$  and  $\mathbf{I}_{b-}^{1-\alpha(x,t);\psi} g \in AC[a,b]$ , then:

a)

$$\begin{aligned} \int_a^b g(t) {}_2^H\mathbb{D}_{a+}^{\alpha(x,t),\beta;\psi} f(t) dt &= \mathbf{I}_{a+}^{1-\gamma(x,t);\psi} f(t) \mathbf{I}_{b-}^{\beta(1-\alpha(x,t));\psi} \left( \frac{g(t)}{\psi'(t)} \right) \Big|_a^b \\ &+ \int_a^b \mathbf{I}_{a+}^{1-\gamma(x,t);\psi} f(t) \psi'(t) {}_2\mathbb{D}_{b-}^{\beta(\alpha(x,t)-1);\psi} \left( \frac{g(t)}{\psi'(t)} \right) dt. \end{aligned} \quad (3.16)$$

b)

$$\begin{aligned} \int_a^b g(t) {}_2^H\mathbb{D}_{b-}^{\alpha(x,t),\beta;\psi} f(t) dt &= -\mathbf{I}_{b-}^{1-\gamma(x,t);\psi} f(t) \mathbf{I}_{a+}^{\beta(1-\alpha(x,t));\psi} \left( \frac{g(t)}{\psi'(t)} \right) \Big|_a^b \\ &+ \int_a^b \mathbf{I}_{b-}^{1-\gamma(x,t);\psi} f(t) \psi'(t) {}_2\mathbb{D}_{a+}^{\beta(\alpha(x,t)-1);\psi} \left( \frac{g(t)}{\psi'(t)} \right) dt. \end{aligned} \quad (3.17)$$

*Proof:* Here we prove Eq. (3.16). The proof of Eq. (3.17) is similar and is omitted. Then,

$$\int_a^b g(t) {}_2^H\mathbb{D}_{a+}^{\alpha(x,t),\beta;\psi} f(t) dt = \int_a^b g(t) {}_2\mathbb{C}\mathbb{D}_{a+}^{\beta(\alpha(x,t)-1)+1;\psi} \mathbf{I}_{a+}^{1-\gamma(x,t);\psi} f(t) dt$$



$$= \int_a^b g(t) {}_2^C \mathbb{D}_{a^+}^{\beta(\alpha(x,t)-1)+1;\psi} h(x,t) dt$$

where we introduced the notation

$$h(x,t) = \mathbf{I}_{a^+}^{1-\gamma(x,t);\psi} f(t).$$

Using the relation

$${}_2^C \mathbb{D}_{a^+}^{\beta(\alpha(x,t)-1)+1;\psi} h(x,t) = \mathbf{I}_{a^+}^{1-\tilde{\beta};\psi} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) f(t)$$

with  $\tilde{\beta} = \beta(\alpha(x,t) - 1) + 1$ , we obtain

$$\begin{aligned} \int_a^b g(t) {}_2^H \mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f(t) dt &= \int_a^b g(t) \mathbf{I}_{a^+}^{\beta(1-\alpha(x,t));\psi} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) h(x,t) dt \\ &= \int_a^b \psi'(t) \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) h(x,t) \mathbf{I}_{b^-}^{\beta(1-\alpha(x,t));\psi} \left( \frac{g(t)}{\psi'(t)} \right) dt \\ &= \int_a^b \frac{d}{dt} h(x,t) \mathbf{I}_{b^-}^{\beta(1-\alpha(x,t));\psi} \left( \frac{g(t)}{\psi'(t)} \right) dt. \end{aligned}$$

Integrating by parts, we can write

$$\begin{aligned} &\int_a^b g(t) {}_2^H \mathbb{D}_{a^+}^{\alpha(x,t),\beta;\psi} f(t) dt \\ &= h(x,t) \mathbf{I}_{b^-}^{\beta(1-\alpha(x,t));\psi} \left( \frac{g(t)}{\psi'(t)} \right) \Big|_a^b - \int_a^b h(x,t) \frac{d}{dt} \mathbf{I}_{b^-}^{\beta(1-\alpha(x,t));\psi} \left( \frac{g(t)}{\psi'(t)} \right) dt \\ &= \mathbf{I}_{a^+}^{1-\gamma(x,t);\psi} f(t) \mathbf{I}_{b^-}^{\beta(1-\alpha(x,t));\psi} \left( \frac{g(t)}{\psi'(t)} \right) \Big|_a^b - \int_a^b \mathbf{I}_{a^+}^{1-\gamma(x,t);\psi} \frac{d}{dt} \mathbf{I}_{b^-}^{\beta(1-\alpha(x,t));\psi} \left( \frac{g(t)}{\psi'(t)} \right) dt \\ &= \mathbf{I}_{a^+}^{1-\gamma(x,t);\psi} f(t) \mathbf{I}_{b^-}^{\beta(1-\alpha(x,t));\psi} \left( \frac{g(t)}{\psi'(t)} \right) \Big|_a^b + \int_a^b \mathbf{I}_{a^+}^{1-\gamma(x,t);\psi} f(t) (\psi'(t)) {}_2^C \mathbb{D}_{b^-}^{\beta(\alpha(x,t)-1)} \left( \frac{g(t)}{\psi'(t)} \right) dt \end{aligned}$$

which completes the proof.  $\square$

**Theorem 3.9** *Assuming the same conditions as to Theorem 5.2 and choosing  $\psi(t) = t$  with limit  $\beta \rightarrow 1$ , we have*

$$\int_a^b g(t) {}_2^C \mathbb{D}_{a^+}^{\alpha(x,t)} f(t) dt = f(t) \mathbf{I}_{b^-}^{1-\alpha(x,t)} g(t) \Big|_a^b + \int_a^b f(t) {}_2^C \mathbb{D}_{b^-}^{\alpha(x,t)-1} g(t) dt.$$

*Proof:* The proof follows directly from the Theorem 3.8.  $\square$

The following theorem highlights the relationship between the two versions of  $\psi$ -Caputo FDVO with respect to another function.

**Theorem 3.10** *The following relations hold between the left fractional operators:*

$$\begin{aligned} {}_1^C \mathbf{D}_{a+}^{\alpha(t); \psi} x(t) &= {}_1^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t) \\ &+ \frac{\alpha'(t)}{\Gamma(2-\alpha(t)) \psi'(t)} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} \left[ \frac{1}{1-\alpha(t)} - \ln(\psi(t) - \psi(s)) \right] ds \end{aligned}$$

and

$$\begin{aligned} {}_1^C \mathbf{D}_{b-}^{\alpha(t); \psi} x(t) &= {}_1^C \mathbb{D}_{b-}^{\alpha(t); \psi} x(t) \\ &+ \frac{\alpha'(t)}{\Gamma(2-\alpha(t)) \psi'(t)} \int_t^b (\psi(s) - \psi(t))^{1-\alpha(t)} \left[ \frac{1}{1-\alpha(t)} - \ln(\psi(s) - \psi(t)) \right] ds. \end{aligned}$$

*Proof:* Using the Definition 3.7 and applying the integration by parts of function  $f(s) = \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a))$ , we obtain

$$\begin{aligned} {}_1^C \mathbf{D}_{a+}^{\alpha(t); \psi} x(t) &= \frac{1}{\Gamma(1-\alpha(t))} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a)) ds \\ &= \frac{1}{\Gamma(1-\alpha(t))} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) \frac{1}{(1-\alpha(t))} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) ds. \end{aligned}$$

Let us recall that

$$\left[ (\psi(t) - \psi(s))^{1-\alpha(t)} \right]' = (\psi(t) - \psi(s))^{1-\alpha(t)} \left[ -\alpha'(t) \ln(\psi(t) - \psi(s)) + \frac{(1-\alpha(t)) \psi'(t)}{\psi(t) - \psi(s)} \right] \quad (3.18)$$

Differentiating the integral Eq. (3.18) and using Eq. (3.18), yields

$$\begin{aligned} {}_1^C \mathbf{D}_{a+}^{\alpha(t); \psi} x(t) &= \frac{\alpha'(t)}{\Gamma(2-\alpha(t)) \psi'(t) (1-\alpha(t))} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) ds \\ &+ \frac{1}{\Gamma(1-\alpha(t)) \psi'(t) (1-\alpha(t))} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} \\ &\times \left[ -\alpha'(t) \ln(\psi(t) - \psi(s)) + \frac{(1-\alpha(t)) \psi'(t)}{\psi(t) - \psi(s)} \right] x'(s) ds \\ &= \frac{\alpha'(t)}{\Gamma(2-\alpha(t)) \psi'(t) (1-\alpha(t))} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) ds \\ &- \frac{\alpha'(t)}{\Gamma(2-\alpha(t)) \psi'(t)} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) \ln(\psi(t) - \psi(s)) ds \\ &+ \frac{1}{\Gamma(1-\alpha(t))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} \frac{x'(s)}{\psi'(s)} ds \\ &= {}_1^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t) + \frac{\alpha'(t)}{\Gamma(2-\alpha(t)) \psi'(t)} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) \\ &\left[ \frac{1}{1-\alpha(t)} - \ln(\psi(t) - \psi(s)) \right] ds. \end{aligned}$$

Thus, we concluded the proof. The proof for the second expression follows the same steps.  $\square$

In particular, for the choice  $\alpha(t) = \alpha$  (constant), we have  ${}_1^C \mathbf{D}_{a+}^{\alpha(t); \psi} x(t) = {}_1^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t)$ . The Eq. (3.18) in Theorem 3.10, can be written as follows

$$\begin{aligned} {}_1^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t) &= {}_1^C \mathbf{D}_{a+}^{\alpha(t); \psi} x(t) \\ &+ \frac{\alpha'(t)}{\Gamma(2 - \alpha(t)) \psi'(t)} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} \left[ \ln(\psi(t) - \psi(s)) - \frac{1}{1 - \alpha(t)} \right] x'(s) ds. \end{aligned}$$

Remembering that

$${}_2^H \mathbb{D}_{a+}^{\alpha(t), \beta; \psi} y(t) = {}_1^C \mathbf{D}_{a+}^{\beta(\alpha(t)-1)+1; \psi} \mathbf{I}_{a+}^{1-\gamma(t); \psi} y(t)$$

with  $\gamma(t) = \alpha(t) + \beta(1 - \alpha(t))$ , choosing  $x(t) = \mathbf{I}_{a+}^{1-\gamma(t); \psi} y(t)$ , and replacing in Eq. (3.18), yields

$$\begin{aligned} {}_2^H \mathbb{D}_{a+}^{\alpha(t), \beta; \psi} y(t) &= {}_1^C \mathbf{D}_{a+}^{\beta(\alpha(t)-1)+1; \psi} x(t) + \frac{\beta \alpha'(t)}{\Gamma(1 - \beta(\alpha(t) - 1)) \psi'(t)} \\ &\times \int_a^t (\psi(t) - \psi(s))^{\beta(1-\alpha(t))} x'(s) \left[ \ln(\psi(t) - \psi(s)) - \frac{1}{\beta(1 - \alpha(t))} \right] ds. \end{aligned} \quad (3.19)$$

Analogously, we have

$$\begin{aligned} {}_2^H \mathbb{D}_{b-}^{\alpha(t), \beta; \psi} y(t) &= {}_1^C \mathbf{D}_{b-}^{\beta(\alpha(t)-1)+1; \psi} x(t) + \frac{\beta \alpha'(t)}{\Gamma(1 - \beta(\alpha(t) - 1)) \psi'(t)} \\ &\times \int_t^b (\psi(s) - \psi(t))^{\beta(1-\alpha(t))} x'(s) \left[ \ln(\psi(s) - \psi(t)) - \frac{1}{\beta(1 - \alpha(t))} \right] ds. \end{aligned}$$

Choosing  $\psi(t) = t$  in Eq. (3.19), leads to

$${}_2^H \mathbb{D}_{a+}^{\alpha(t); \psi} y(t) = {}_1^C \mathbf{D}_{a+}^{\beta(\alpha(t)-1); \psi} x(t) + \frac{\beta \alpha'(t)}{\Gamma(1 - \beta(\alpha(t) - 1))} \int_a^t (t - s)^{\beta(\alpha(t)-1)} x'(s) ds$$

with  $x(t) = \mathbf{I}_{a+}^{1-\gamma(t)} y(t)$ .

Choosing  $\psi(t) = t$  and  $\beta \rightarrow 1$  in Eq. (3.19), results in:

$${}_2^H \mathbb{D}_{a+}^{\alpha(t)} y(t) = {}_1^C \mathbf{D}_{a+}^{\beta(\alpha(t)-1)} x(t) + \frac{\alpha'(t)}{\Gamma(2 - \alpha(t))} \int_a^t (t - s)^{1-\alpha(t)} x'(s) ds.$$

**Theorem 3.11** *The following relations hold between the left fractional operators:*

$$\begin{aligned} {}_2^C \mathbf{D}_{a+}^{\alpha(t); \psi} x(t) &= {}_2^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t) - \frac{\alpha'(t)}{\psi'(t) \Gamma(2 - \alpha(t))} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) \\ &\times \left( \tilde{\Psi}(2 - \alpha(t)) + \ln(\psi(t) - \psi(s)) \right) ds \end{aligned} \quad (3.20)$$

with  $\tilde{\Psi}(2 - \alpha(t)) := \frac{\Gamma'(2 - \alpha(t))}{\Gamma(2 - \alpha(t))}$ .

*Proof:* Using the Definition 3.9 and applying the integration by parts of function  $f(s) = \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a))$ , yields

$$\begin{aligned} {}_2^C \mathbf{D}_{a+}^{\alpha(t); \psi} x(t) &= \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) \frac{1}{\Gamma(1 - \alpha(t))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a)) ds \\ &= \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) \frac{1}{\Gamma(2 - \alpha(t))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) ds. \end{aligned} \quad (3.21)$$

Let us remember that,

$$\left( (\psi(t) - \psi(s))^{1-\alpha(t)} \right)' = (\psi(t) - \psi(s))^{1-\alpha(t)} \times \left( -\alpha(t) \ln(\psi(t) - \psi(s)) + \frac{(1 - \alpha(t)) \psi'(t)}{\psi(t) - \psi(s)} \right). \quad (3.22)$$

Differentiating the integral Eq. (3.21) and using Eq. (3.22), we get

$$\begin{aligned} {}_2^C \mathbf{D}_{a+}^{\alpha(t); \psi} x(t) &= \frac{1}{\psi'(t)} \left\{ \left( \frac{1}{\Gamma(2 - \alpha(t))} \right)' \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) ds \right. \\ &\quad \left. + \frac{1}{\Gamma(2 - \alpha(t))} \frac{d}{dt} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) ds \right\} \\ &= \frac{1}{\psi'(t)} \left( \frac{1}{\Gamma(2 - \alpha(t))} \right)' \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) ds \\ &\quad + \frac{1}{\psi'(t)} \frac{1}{\Gamma(2 - \alpha(t))} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) \\ &\quad \left( -\alpha(t) \ln(\psi(t) - \psi(s)) + \frac{(1 - \alpha(t)) \psi'(t)}{\psi(t) - \psi(s)} \right) ds \\ &= -\frac{1}{\psi'(t)} \frac{\alpha'(t) \Gamma'(2 - \alpha(t))}{\Gamma(2 - \alpha(t))^2} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) ds \\ &\quad - \frac{1}{\psi'(t)} \frac{\alpha'(t)}{\Gamma(2 - \alpha(t))} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) \ln(\psi(t) - \psi(s)) ds \\ &\quad + \frac{1}{\Gamma(1 - \alpha(t))} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) ds \\ &= -\frac{1}{\psi'(t)} \frac{\alpha'(t) \tilde{\Psi}(2 - \alpha(t))}{\Gamma(2 - \alpha(t))} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) ds \\ &\quad - \frac{1}{\psi'(t)} \frac{\alpha'(t)}{\Gamma(2 - \alpha(t))} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x'(s) \ln(\psi(t) - \psi(s)) ds \\ &\quad + {}_2^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t), \end{aligned}$$

which concludes the proof.  $\square$

**Theorem 3.12** *The following relations hold between the left fractional operators*

$${}_2^C \mathbf{D}_{a+}^{\alpha(t); \psi} x(t) = {}_1^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t) - \frac{1}{\psi'(t)} \frac{\tilde{\Psi}(1 - \alpha(t))}{\Gamma(1 - \alpha(t))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a)) ds.$$

*Proof:* Using the Definition 3.9 and applying the product property of functions to classical derivative, we have

$$\begin{aligned}
{}_2^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t) &= \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) \frac{1}{\Gamma(1-\alpha(t))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a)) ds \\
&= \frac{1}{\psi'(t)} \left( \frac{1}{\Gamma(1-\alpha(t))} \right)' \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a)) ds \\
&\quad + \frac{1}{\psi'(t)} \frac{d}{dt} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a)) ds \\
&= \frac{1}{\psi'(t)} \frac{\Gamma'(1-\alpha(t))}{\Gamma(1-\alpha(t))^2} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a)) ds \\
&\quad + \frac{1}{\Gamma(1-\alpha(t))} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a)) ds \\
&= {}_1^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t) - \frac{1}{\psi'(t)} \frac{\tilde{\Psi}(1-\alpha(t))}{\Gamma(1-\alpha(t))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\alpha(t)} (x(s) - x(a)) ds
\end{aligned}$$

$$\text{with } \tilde{\Psi}(1-\alpha(t)) = \frac{\Gamma'(1-\alpha(t))}{\Gamma(1-\alpha(t))}. \quad \square$$

The result in the follow-up, is intended to obtain an approximation of the  $\psi$ -Caputo FDVO. Consequently, we highlight some particular cases. In this sense, we will present an approximate version for the  $\psi$ -Hilfer FDVO.

For  $k \in \mathbb{N}$  we consider the two following equalities:

$$\begin{aligned}
\mathcal{A}_k &= \frac{1}{\Gamma(k+1-\alpha(t))} \left[ 1 + \sum_{p=n-k+1}^N \frac{\Gamma(\alpha(t) - n + p)}{\Gamma(\alpha(t) - k) (p - n + k)!} \right], \\
\mathcal{D}_k &= \frac{\Gamma(\alpha(t) - n + k)}{\Gamma(1-\alpha(t)) \Gamma(\alpha(t)) (k-n)!}
\end{aligned}$$

and the function

$$\mathcal{V}_k(t) = \int_a^t (\psi(s) - \psi(a))^k x'(s) ds.$$

**Theorem 3.13** *Let  $x : [a, b] \rightarrow \mathbb{R}$  be a function of class  $C^{n+1}$ , for  $n \in \mathbb{N}$ , and  $f, x \in \mathbb{N}$  with  $N \geq n$ . Then*

$${}_1^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t) = \sum_{k=1}^n \mathcal{A}_k (\psi(t) - \psi(a))^{k-\alpha(t)} x^{(k)}(t) + \sum_{k=n}^N \mathcal{B}_k (\psi(t) - \psi(a))^{n-k-\alpha(t)} \mathcal{V}_{k-n}(t) + \mathcal{E}(t) \quad (3.23)$$

with

$$\mathcal{E}(t) \leq \frac{(t-a) (\psi(t) - \psi(a))^{n-\alpha(t)} \exp \left[ (n-\alpha(t))^2 + n - \alpha(t) \right]}{\Gamma(n+1-\alpha(t)) N^{n-\alpha(t)} (n-\alpha(t))} \max_{s \in [a, t]} |x'_N(s)|.$$

*Proof:* By definition of  ${}^C_1\mathbb{D}_{a+}^{\alpha(t); \psi}(\cdot)$  and performing the integration by parts of function  $f(s) = (\psi(t) - \psi(s))^{-\alpha(t)} x'(s)$ , we have

$$\begin{aligned} {}^C_1\mathbb{D}_{a+}^{\alpha(t); \psi} x(t) &= \frac{1}{\Gamma(1 - \alpha(t))} \int_a^t (\psi(t) - \psi(s))^{-\alpha(t)} x'(s) ds \\ &= \frac{(\psi(t) - \psi(a))^{1-\alpha(t)} x'(a)}{\Gamma(2 - \alpha(t))} + \frac{1}{\Gamma(2 - \alpha(t))} \int_a^t (\psi(t) - \psi(s))^{1-\alpha(t)} x''(s) ds. \end{aligned}$$

Performing integration by parts of function  $f(s) = (\psi(t) - \psi(s))^{1-\alpha(t)} x''(s)$ , yields

$$\begin{aligned} {}^C_1\mathbb{D}_{a+}^{\alpha(t); \psi} x(t) &= \frac{(\psi(t) - \psi(a))^{1-\alpha(t)} x'(a)}{\Gamma(2 - \alpha(t))} + \frac{(\psi(t) - \psi(a))^{2-\alpha(t)} x''(a)}{\Gamma(3 - \alpha(t))} \\ &\quad + \frac{1}{\Gamma(3 - \alpha(t))} \int_a^t (\psi(t) - \psi(s))^{2-\alpha(t)} x'''(s) ds. \end{aligned}$$

Doing again integration by parts of function  $f(s) = (\psi(t) - \psi(s))^{2-\alpha(t)} x'''(s)$ , we get

$$\begin{aligned} &{}^C_1\mathbb{D}_{a+}^{\alpha(t); \psi} x(t) \\ &= \frac{(\psi(t) - \psi(a))^{1-\alpha(t)} x'(a)}{\Gamma(2 - \alpha(t))} + \frac{(\psi(t) - \psi(a))^{2-\alpha(t)} x''(a)}{\Gamma(3 - \alpha(t))} \\ &\quad + \frac{(\psi(t) - \psi(a))^{3-\alpha(t)} x'''(a)}{\Gamma(4 - \alpha(t))} + \frac{1}{\Gamma(4 - \alpha(t))} \int_a^t (\psi(t) - \psi(s))^{3-\alpha(t)} x^{(4)}(s) ds. \end{aligned}$$

Performing this procedure (integration by parts)  $n - 1$  times, yields

$$\begin{aligned} {}^C_1\mathbb{D}_{a+}^{\alpha(t); \psi} x(t) &= \sum_{k=1}^n \frac{(\psi(t) - \psi(a))^{k-\alpha(t)} x^{(k)}(a)}{\Gamma(k + 1 - \alpha(t))} \\ &\quad + \frac{1}{\Gamma(n + 1 - \alpha(t))} \int_a^t (\psi(t) - \psi(s))^{n-\alpha(t)} x^{(n+1)}(s) ds. \end{aligned}$$

By the Taylor's theorem we obtain

$$\begin{aligned} (\psi(t) - \psi(s))^{n-\alpha(t)} &= (\psi(t) - \psi(a))^{n-\alpha(t)} \left( 1 - \frac{\psi(s) - \psi(a)}{\psi(t) - \psi(a)} \right)^{n-\alpha(t)} \\ &= (\psi(t) - \psi(a))^{n-\alpha(t)} \sum_{p=0}^N \binom{n - \alpha(t)}{p} (-1)^p \frac{(\psi(s) - \psi(a))^p}{(\psi(t) - \psi(a))^p} + \mathcal{E}_1(t) \end{aligned}$$

where

$$\mathcal{E}_1(t) = (\psi(t) - \psi(a))^{n-\alpha(t)} \sum_{p=0}^N \binom{n - \alpha(t)}{p} (-1)^p \frac{(\psi(s) - \psi(a))^p}{(\psi(t) - \psi(a))^p}$$

and

$$\binom{n - \alpha(t)}{p} (-1)^p = \frac{\Gamma(\alpha(t) - n + p)}{\Gamma(\alpha(t) - n) p!}. \quad (3.24)$$

Using the Eq. (3.24), we obtain the formula

$$\begin{aligned} {}_1^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t) &= \sum_{k=1}^n \frac{(\psi(t) - \psi(a))^{k-\alpha(t)} x^{(k)}(a)}{\Gamma(k+1-\alpha(t))} + \frac{(\psi(t) - \psi(a))^{n-\alpha(t)}}{\Gamma(n+1-\alpha(t))} \sum_{k=1}^{\infty} \frac{\Gamma(\alpha(t) - n + p)}{\Gamma(\alpha(t) - n) p!} \times \\ &\times \frac{1}{(\psi(t) - \psi(a))^p} \int_a^t (\psi(s) - \psi(a)) x^{(n+1)}(s) ds + \mathcal{E}(t) \end{aligned}$$

where

$$\mathcal{E}(t) = \frac{1}{\Gamma(k+1-\alpha(t))} \int_a^t \mathcal{E}_1(t) x^{(n+1)}(s) ds.$$

If we split the sum into the first term  $p = 0$  and the remaining terms, that is  $p = 1, \dots, N$ , and if use integration by parts, taking  $u(s) = (\psi(s) - \psi(a))^p$  and  $v'(s) = x^{(n+1)}(s)$ , then we get

$$\begin{aligned} {}_1^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t) &= \sum_{k=1}^{n-1} \frac{(\psi(t) - \psi(a))^{k-\alpha(t)} x^{(k)}(a)}{\Gamma(k+1-\alpha(t))} + \frac{1}{\Gamma(n+1-\alpha(t))} \\ &\left( 1 + \sum_{p=n-k+1}^N \frac{\Gamma(\alpha(t) - n + p)}{\Gamma(\alpha(t) - k)(p - n + k)} \right) (\psi(t) - \psi(a))^{n-\alpha(t)} x^{(n)}(a) \\ &+ \frac{(\psi(t) - \psi(a))^{n-1-\alpha(t)}}{\Gamma(n-\alpha(t))} \sum_{p=1}^N \frac{\Gamma(\alpha(t) - n + p)}{\Gamma(\alpha(t) + 1 - n)(p-1)! (\psi(t) - \psi(a))^{p-1}} \\ &\times \int_a^t (\psi(s) - \psi(a))^{p-1} x^{(n)}(s) ds + \mathcal{E}(t). \end{aligned}$$

Repeating the process, that is, splitting the second sum (first term  $p = k$  plus the remaining ones  $p = k+1, \dots, N$ ) and integration by parts the integral that appears in the sum  $p = k+1, \dots, N$ , we obtain the desired formula.

We now seek the upper bound formula for  $\mathcal{E}(t)$ . Using the two relations

$$\left| \frac{(\psi(s) - \psi(a))^p}{(\psi(t) - \psi(a))^p} \right| \leq 1, \text{ if } s \in [a, t],$$

$$\left| \binom{n-\alpha(t)}{p} \right| \leq \frac{\exp \left[ (n-\alpha(t))^2 + n-\alpha(t) \right]}{p^{n+1-\alpha(t)}}$$

we can write

$$\begin{aligned} |\mathcal{E}_1(t)| &\leq (\psi(t) - \psi(a))^{n-\alpha(t)} \sum_{p=N+1}^{\infty} \left| \binom{n-\alpha(t)}{p} \right| (-1)^p \left| \frac{(\psi(s) - \psi(a))^p}{(\psi(t) - \psi(a))^p} \right| \\ &\leq (\psi(t) - \psi(a))^{n-\alpha(t)} \sum_{p=N+1}^{\infty} \frac{\exp \left[ (n-\alpha(t))^2 + n-\alpha(t) \right]}{p^{n+1-\alpha(t)}} \\ &\leq (\psi(t) - \psi(a))^{n-\alpha(t)} \frac{\exp \left[ (n-\alpha(t))^2 + n-\alpha(t) \right]}{N^{n-\alpha(t)} (n-\alpha(t))}. \end{aligned}$$

Then, we have

$$\begin{aligned} \mathcal{E}(t) &\leq \frac{1}{\Gamma(n+1-\alpha(t))} \int_a^t (\psi(t) - \psi(a))^{n-\alpha(t)} \frac{\exp\left[(n-\alpha(t))^2 + n - \alpha(t)\right]}{N^{n-\alpha(t)}(n-\alpha(t))} x^{(n+1)}(p) dp \\ &\leq \frac{(t-a)(\psi(t) - \psi(a))^{n-\alpha(t)} \exp\left[(n-\alpha(t))^2 + n - \alpha(t)\right]}{\Gamma(n+1-\alpha(t)) N^{n-\alpha(t)}(n-\alpha(t))} \max_{s \in [a,t]} |x'_N(s)|. \end{aligned}$$

This concludes the proof.  $\square$

In the Theorem 3.13 we have  $\lim_{N \rightarrow \infty} \mathcal{E}(t) = 0$ . Consequently, it results

$$\begin{aligned} {}_1^C \mathbb{D}_{a+}^{\alpha(t); \psi} x(t) &\approx \sum_{k=1}^n \mathcal{A}_k (\psi(t) - \psi(a))^{k-\alpha(t)} x^{(k)}(t) \\ &\quad + \sum_{k=n}^N \mathcal{B}_k (\psi(t) - \psi(a))^{n-k-\alpha(t)} \mathcal{V}_{k-n}(t). \end{aligned}$$

**Remark 3.14** Remembering that

$${}_2^H \mathbb{D}_{a+}^{\alpha(t), \beta; \psi} y(t) = {}_1^C \mathbb{D}_{a+}^{\beta(\alpha(t)-1)+1; \psi} \mathbf{I}_{a+}^{1-\gamma(t); \psi} y(t)$$

with  $\gamma(t) = \alpha(t) + \beta(1 - \alpha(t))$ , choosing  $x(t) = \mathbf{I}_{a+}^{1-\gamma(t); \psi} y(t)$ , and replacing in Eq. (3.23), yields

$$\begin{aligned} {}_2^H \mathbb{D}_{a+}^{\alpha(t), \beta; \psi} y(t) &= \sum_{k=1}^n \mathcal{A}_k (\psi(t) - \psi(a))^{k-\beta(\alpha(t)-1)-1} x^{(k)}(t) \\ &\quad + \sum_{k=n}^N \mathcal{B}_k (\psi(t) - \psi(a))^{n-k-\beta(\alpha(t)-1)-1} \mathcal{V}_{k-n}(t) + \mathcal{E}(t) \end{aligned} \tag{3.25}$$

with

$$\begin{aligned} \mathcal{E}(t) &\leq \frac{(t-a)(\psi(t) - \psi(a))^{n-\beta(\alpha(t)-1)-1}}{\Gamma(n-\beta(\alpha(t)-1))} \\ &\quad \times \frac{\exp\left[(n-\beta(\alpha(t)-1)-1)^2 + n - \beta(\alpha(t)-1) - 1\right]}{N^{n-\beta(\alpha(t)-1)-1}(n-\beta(\alpha(t)-1)-1)} \max_{s \in [a,t]} |x'_N(s)|; \\ \mathcal{A}_k &= \frac{1}{\Gamma(k-\beta(\alpha(t)-1))} \left[ 1 + \sum_{p=n-k+1}^N \frac{\Gamma(\beta(\alpha(t)-1) + 1 - n + p)}{\Gamma(\beta(\alpha(t)-1) + 1 - k)(p-n+k)!} \right]; \\ \mathcal{B}_k &= \frac{\Gamma(\beta(\alpha(t)-1) + 1 - n + k)}{\Gamma(\beta(1-\alpha(t))) \Gamma(\beta(\alpha(t)-1) + 1)(k-n)!} \end{aligned}$$

and the function

$$\mathcal{V}_k(t) = \int_a^t (\psi(s) - \psi(a))^k x'(s) ds.$$



1. Taking  $\psi(t) = t$  in Eq. (3.25), we get

$$\begin{aligned} {}_2^H \mathbb{D}_{a+}^{\alpha(t), \beta} y(t) &= \sum_{k=1}^n \mathcal{A}_k (t-a)^{k-\beta(\alpha(t)-1)-1} x^{(k)}(t) \\ &+ \sum_{k=n}^N \mathcal{B}_k (t-a)^{n-k-\beta(\alpha(t)-1)-1} \mathcal{V}_{k-n}(t) + \mathcal{E}(t) \end{aligned}$$

with

$$\mathcal{E}(t) \leq \frac{(t-a)^{n-\beta(\alpha(t)-1)}}{\Gamma(n-\beta(\alpha(t)-1))} \frac{\exp\left[(n-\beta(\alpha(t)-1)-1)^2 + n-\beta(\alpha(t)-1)-1\right]}{N^{n-\beta(\alpha(t)-1)}(n-\beta(\alpha(t)-1)-1)} \max_{s \in [a, t]} |x'_N(s)|,$$

$$\mathcal{A}_k = \frac{1}{\Gamma(k-\beta(\alpha(t)-1))} \left[ 1 + \sum_{p=n-k+1} \frac{\Gamma(\beta(\alpha(t)-1)+1-n+p)}{\Gamma(\beta(\alpha(t)-1)+1-k)(p-n+k)!} \right],$$

$$\mathcal{B}_k = \frac{\Gamma(\beta(\alpha(t)-1)+1-n+k)}{\Gamma(\beta(1-\alpha(t)))\Gamma(\beta(\alpha(t)-1))(k-n)!},$$

and

$$\mathcal{V}_k(t) = \int_a^t (s-a)^k x'(s) ds.$$

Note that  $\lim_{N \rightarrow \infty} \mathcal{E}(t) = 0$

$${}_2^H \mathbb{D}_{a+}^{\alpha(t), \beta} y(t) \approx \sum_{k=1}^n \mathcal{A}_k (t-a)^{k-\beta(\alpha(t)-1)-1} x^{(k)}(t) + \sum_{k=n}^N \mathcal{B}_k (t-a)^{n-k-\beta(\alpha(t)-1)-1} \mathcal{V}_{k-n}(t)$$

2. Taking  $\psi(t) = t$  and  $\beta \rightarrow 1$ , results

$${}_2^C \mathbb{D}_{a+}^{\alpha(t)} y(t) = \sum_{k=1}^n \mathcal{A}_k (t-a)^{k-\alpha(t)} x^{(k)}(t) + \sum_{k=n}^N \mathcal{B}_k (t-a)^{n-k-\alpha(t)} \mathcal{V}_{k-n}(t) + \mathcal{E}(t)$$

with

$$\mathcal{E}(t) \leq \frac{(t-a)^{n-\alpha(t)+1}}{\Gamma(n-\alpha(t)+1)} \frac{\exp\left[(n-\alpha(t))^2 + n-\alpha(t)\right]}{N^{n-\alpha(t)}(n-\alpha(t))} \max_{s \in [a, t]} |x'_N(s)|,$$

$$\mathcal{A}_k = \frac{1}{\Gamma(k-\alpha(t)+1)} \left[ 1 + \sum_{p=n-k+1} \frac{\Gamma(\alpha(t)-n+p)}{\Gamma(\alpha(t)-k)(p-n+k)!} \right],$$

$$\mathcal{B}_k = \frac{\Gamma(\alpha(t)-n+k)}{\Gamma(1-\alpha(t))\Gamma(\alpha(t))(k-n)!},$$

and

$$\mathcal{V}_k(t) = \int_a^t (s-a)^k x'(s) ds.$$

Note that  $\lim_{N \rightarrow \infty} \mathcal{E}(t) = 0$

$${}_2^C \mathbb{D}_{a+}^{\alpha(t)} y(t) \approx \sum_{k=1}^n \mathcal{A}_k (t-a)^{k-\alpha(t)} x^{(k)}(t) + \sum_{k=n}^N \mathcal{B}_k (t-a)^{n-k-\alpha(t)} \mathcal{V}_{k-n}(t).$$

We presented some particular choices of  $\psi(\cdot)$  and the limits  $\beta \rightarrow 0$  and  $\beta \rightarrow 1$ . However, the results are not restricted to those discussed here, it is possible to consider other cases that are not included for the sake of parsimony.

## 4 Fractional variable order nonlinear system

In this section, we investigate the stability of solutions for a dynamic system via the  $\psi$ -Hilfer FDVO type II and the one-parameter Mittag-Leffler function of variable order by means of two theorems. In order to discuss the investigated results, some examples are presented involving FDVO Lu and Chen systems.

For the investigation of the main result of this section, we chose  $\alpha(\xi, t) = \alpha(t)$  in the definition  ${}_2^H \mathbb{D}_{a+}^{\alpha(\xi, t), \beta; \psi}(\cdot)$ , so that we have  ${}_2^H \mathbb{D}_{a+}^{\alpha(t), \beta; \psi}(\cdot)$ .

The one-parameter Mittag-Leffler function is given by

$$\mathbb{E}_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (4.1)$$

$z \in \mathbb{R}$  where  $\alpha \in \mathbb{C}$ ,  $Re(\alpha) > 0$  and  $\Gamma(z)$  is a Gamma function, given by

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad Re(z) > 0.$$

It is easy to see that in the limit  $\alpha \rightarrow 1$ , we have  $\mathbb{E}_1(z) = \exp(z)$ .

On the other hand, motivate by Eq. (4.1), we have one-parameter Mittag-Leffler function of variable order  $\alpha(t)$  given by

$$\mathbb{E}_{\alpha(t)}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha(t)k + 1)} \quad (4.2)$$

where  $\alpha(t) \in \mathbb{C}$ ,  $z \in \mathbb{R}$ .

Consider the following fractional differential equations, given by

$${}_2^H \mathbb{D}_{0+}^{\alpha(t), \beta; \psi} \xi(t) = A \xi(t). \quad (4.3)$$

By Theorem 3.7, the general solution of Eq. (4.3), is given by

$$\xi(t) = \xi_0 \mathbb{E}_{\alpha(t)} \left( A (\psi(t) - \psi(0))^{\alpha(t)} \right) \quad (4.4)$$

where  $\mathbb{E}_{\alpha(t)}(\cdot)$  is Eq. (4.2).

Choosing,  $\psi(t) = t$  in Eq. (4.4), yields

$$\xi(t) = \xi_0 \mathbb{E}_{\alpha(t)} \left( At^{\alpha(t)} \right). \quad (4.5)$$

On the other hand, choosing  $\psi(t) = t$  and taking the limit  $\alpha(t) \rightarrow \alpha$  and  $\alpha(t) \rightarrow 1$ , yields

$$\xi(t) = \xi_0 \mathbb{E}_{\alpha} (At^{\alpha}) \quad (4.6)$$

and

$$\xi(t) = \xi_0 \mathbb{E}_1 (At) = \xi_0 \exp (At), \quad (4.7)$$

respectively

Consider the following  $\psi$ -Hilfer fractional non-autonomous systems given by

$${}^H_2\mathbb{D}_{0+}^{\alpha(t),\beta;\psi} \xi(t) = g(\xi(t)) \xi(t) \quad (4.8)$$

with initial condition  $x(t_0)$ , where  ${}^H_2\mathbb{D}_{0+}^{\alpha(t),\beta;\psi}(\cdot)$  is  $\psi$ -Hilfer fractional derivative of variable order  $0 < \alpha(t) < 1$ ,  $\xi \in \mathbb{R}$ ,  $g(\cdot)$  is nonlinear continuous function and locally Lipschitz about  $x$ .

**Theorem 4.1** *Let  $\xi(t) \in \mathbb{R}$ , if  $g(\xi(t)) \leq \xi$ , then  $|x(t)| \leq |\xi_0| \mathbb{E}_{\alpha(t)} \left( \xi(\psi(t) - \psi(0))^{\alpha(t)} \right)$ .*

*Proof:* Let  $x(t) \in \mathbb{R}$ ,  $g(\xi(t)) \leq \xi$  and  ${}^H_2\mathbb{D}_{0+}^{\alpha(t),\beta;\psi} \xi(t) = g(\xi(t)) \xi(t)$ . Define  $\xi(t) = -y(t)$ , and we get

$${}^H_2\mathbb{D}_{0+}^{\alpha(t),\beta;\psi} (-y(t)) = g(-y(t)) (-y(t)).$$

Then

$${}^H_2\mathbb{D}_{0+}^{\alpha(t),\beta;\psi} |\xi(t)| = g(|\xi(t)|) |\xi(t)| g(|\xi(t)|) \leq \xi.$$

It can be gained

$$\begin{aligned} |\xi(t)| &= \frac{1}{\Gamma(\alpha(t))} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha(t)-1} g(|x(s)|) |x(s)| ds \\ &\leq \frac{1}{\Gamma(\alpha(t))} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha(t)-1} \xi |\xi(s)| ds \\ &= |\xi_0| \mathbb{E}_{\alpha(t)} \left( \xi(\psi(t) - \psi(0))^{\alpha(t)} \right). \end{aligned}$$

Then, yields

$$|\xi(t)| \leq |\xi_0| \mathbb{E}_{\alpha(t)} \left( \xi(\psi(t) - \psi(0))^{\alpha(t)} \right).$$

□

The following results are a direct result of Theorem 4.1.

**Corollary 4.2** *Let us choose  $\alpha(t) = \alpha$  in Theorem 4.1. For  $\xi(t) \in \mathbb{R}$ , if  $g(\xi(t)) \leq \xi$ , then  $|\xi(t)| \leq |\xi_0| \mathbb{E}_{\alpha} (\xi(\psi(t) - \psi(0))^{\alpha})$ .*

**Corollary 4.3** *Let us choose  $\alpha(t) = \alpha$  and  $\psi(t) = t$  in Theorem 4.1. For  $\xi(t) \in \mathbb{R}$ , if  $g(\xi(t)) \leq \xi$ , then  $|\xi(t)| \leq |\xi_0| \mathbb{E}_\alpha(\xi t^\alpha)$ .*

**Corollary 4.4** *Let us choose  $\alpha(t) = 1$  in Theorem 4.1. For  $\xi(t) \in \mathbb{R}$ , if  $g(\xi(t)) \leq \xi$ , then  $|\xi(t)| \leq |\xi_0| \exp(\xi t)$ .*

**Theorem 4.5** *Consider the  $\psi$ -Hilfer fractional system of variable order*

$${}^H_2\mathbb{D}_{0+}^{\alpha(t),\beta;\psi} \xi(t) = f(\xi(t)) \quad (4.9)$$

when  $\xi(t) = [\xi_1(t), \xi_2(t), \dots, \xi_n(t)]$ ,  $f(\xi(t)) = [f_1(\xi(t)), f_2(\xi(t)), \dots, f_n(\xi(t))]$ ,  $0 < \alpha(t) < 1$  and  $0 \leq \beta \leq 1$ .

If

$$\frac{[\xi(t)]^T {}^H_2\mathbb{D}_{0+}^{\alpha(t),\beta;\psi} \xi(t)}{[\xi(t)]^T \xi(t)} = \frac{[\xi(t)]^T f(\xi(t))}{[\xi(t)]^T \xi(t)} \leq \theta, \quad (4.10)$$

then  $|\xi(t)| \leq |\xi_0| \mathbb{E}_{\alpha(t)}(\theta(\psi(t) - \psi(0))^{\alpha(t)})$ .

*Proof:* From Eq. (4.9), yields

$$\begin{aligned} \frac{d}{dt} [{}^H_2\mathbb{D}_{0+}^{\alpha(t),\beta;\psi} \xi(t)] &= \frac{d}{dt} [f(\xi(t))] \\ &= J(\xi) Df(\xi) \end{aligned}$$

where  $J(\xi) := \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$  is the Jacobian matrix and  $Df(\xi) := \left[ \frac{d}{dt} \xi_1(t) \frac{d}{dt} \xi_1(t) \cdots \frac{d}{dt} \xi_1(t) \right]^T$ .

For any  $x(t)$ , if

$$\frac{[x(t)]^T {}^H_2\mathbb{D}_{0+}^{\alpha(t),\beta;\psi} x(t)}{[x(t)]^T x(t)} = \frac{[\xi(t)]^T f(\xi(t))}{[\xi(t)]^T x(t)} \leq \theta$$

yields

$$\frac{\left[ \frac{d}{dt} \xi(t) \right]^T \frac{d}{dt} ({}^H_2\mathbb{D}_{0+}^{\alpha(t),\beta;\psi} x(t))}{\left[ \frac{d}{dt} \xi(t) \right]^T \frac{d}{dt} \xi(t)} = \frac{\left[ \frac{d}{dt} \xi(t) \right]^T \frac{d}{dt} f(t)}{\left[ \frac{d}{dt} \xi(t) \right]^T \frac{d}{dt} \xi(t)}$$

$$= \frac{\left[ \frac{d}{dt} \xi(t) \right]^T J \frac{d}{dt} f(t)}{\left[ \frac{d}{dt} \xi(t) \right]^T \frac{d}{dt} \xi(t)} \leq \theta. \quad (4.11)$$

Suppose  $\lambda$  as one of eigenvalues of the Jacobian matrix  $J(\xi)$  and there should exist a zone zero vector  $\mu$  corresponding to the eigenvalue  $\lambda$ . We obtain

$$J(\xi) \mu = \lambda \mu \quad (4.12)$$

and

$$\mu^H [J(\xi)]^H = \bar{\lambda} \mu^H. \quad (4.13)$$

Multiplying  $\mu$  on both side of Eq. (4.13) on the right side,  $\mu^H$  on both sides of Eq. (4.12), on the left side, and adding their respective sides, yields

$$\mu^H \left( [J(\xi)]^H + J(\xi) \right) \mu = (\lambda + \bar{\lambda}) \mu^H \mu. \quad (4.14)$$

The Eq. (4.14), can be written as follows

$$\frac{\mu^H \left( [J(x)]^H + J(\xi) \right) \mu}{\mu^H \mu} = (\lambda + \bar{\lambda}). \quad (4.15)$$

Multiplying both sides of Eq. (4.15) and using Eq. (4.11), yields

$$\begin{aligned} Re(\lambda) &= \frac{(\lambda + \bar{\lambda})}{2} \\ &= \frac{\mu^H \left( [J(x)]^H + J(\xi) \right) \mu}{2\mu^H \mu} \\ &= \frac{\mu^H [J(\xi)]^H \mu}{\mu^H \mu} \\ &= \frac{\mu^H J(\xi) \mu}{\mu^H \mu} \leq \theta. \end{aligned} \quad (4.16)$$

The Eq. (4.16) shows a direct way to estimative the maximum of the real part of the eigenvalue of Jacobian matrix  $J(\xi)$ .

For any  $t$ , when  $\delta t \rightarrow 0$ , yields

$$\lim_{\delta t \rightarrow 0} \xi(t + \delta t) = \xi(t) \mathbb{E}_{\alpha(t)} \left( J(\xi) (\psi(t\delta) - \psi(0))^{\alpha(t)} \right).$$

Then

$$\begin{aligned} \lim_{\delta t \rightarrow 0} |\xi(t + \delta t)| &= \left| \xi(t) \mathbb{E}_{\alpha(t)} \left( J(\xi) (\psi(t\delta) - \psi(0))^{\alpha(t)} \right) \right| \\ &\leq |\xi(t)| \mathbb{E}_{\alpha(t)} \left[ Re(\lambda_{\max} J(\xi)) (\psi(t\delta) - \psi(0))^{\alpha(t)} \right] \end{aligned}$$

$$\leq |\xi(t)| \mathbb{E}_{\alpha(t)} \left[ \theta (\psi(t\delta) - \psi(0))^{\alpha(t)} \right].$$

For any  $t$ ,

$$\lim_{\delta t \rightarrow 0} |\xi(t + \delta t)| \leq |\xi(t)| \mathbb{E}_{\alpha(t)} \left[ \theta (\psi(t\delta) - \psi(0))^{\alpha(t)} \right]$$

and so it can be deduced

$$|\xi(t)| \leq |\xi(0)| \mathbb{E}_{\alpha(t)} \left[ \theta (\psi(t) - \psi(0))^{\alpha(t)} \right].$$

□

Some important cases, resulting from Theorem 4.5 are presented below.

**Corollary 4.6** *Taking limit  $\beta \rightarrow 1$  in Eq. (4.9), we have the Caputo fractional system*

$${}^c\mathbf{D}_{0+}^{\alpha(t); \psi} \xi(t) = f(\xi(t))$$

when  $\xi(t) = [\xi_1(t), \xi_2(t), \dots, \xi_n(t)]$ ,  $f(\xi(t)) = [f_1(\xi(t)), f_2(\xi(t)), \dots, f_n(\xi(t))]$ ,  $0 < \alpha(t) < 1$ .

If

$$\frac{[\xi(t)]^T {}^c\mathbf{D}_{0+}^{\alpha(t); \psi} \xi(t)}{[\xi(t)]^T \xi(t)} = \frac{[\xi(t)]^T f(\xi(t))}{[\xi(t)]^T \xi(t)} \leq \theta,$$

then  $|\xi(t)| \leq |\xi_0| \mathbb{E}_{\alpha(t)} \left( \theta (\psi(t) - \psi(0))^{\alpha(t)} \right)$ .

**Corollary 4.7** *Taking limit  $\beta \rightarrow 0$  and  $\psi(t) = t$  in Eq. (4.9), we have the Riemann-Liouville fractional system*

$${}^{RL}\mathbf{D}_{0+}^{\alpha(t)} x(t) = f(x(t))$$

when  $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]$ ,  $f(x(t)) = [f_1(x(t)), f_2(x(t)), \dots, f_n(x(t))]$ ,  $0 < \alpha(t) < 1$ .

If

$$\frac{[x(t)]^T {}^{RL}\mathbf{D}_{0+}^{\alpha(t)} x(t)}{[x(t)]^T x(t)} = \frac{[x(t)]^T f(x(t))}{[x(t)]^T x(t)} \leq \theta$$

then  $|x(t)| \leq |x_0| \mathbb{E}_{\alpha(t)} \left( \theta t^{\alpha(t)} \right)$ .

So, we can conclude that, if

$$\frac{[x(t)]^T {}^H\mathbf{D}_{0+}^{\alpha(t); \beta; \psi} x(t)}{[x(t)]^T x(t)} = \frac{[x(t)]^T f(x(t))}{[x(t)]^T x(t)} \leq 0,$$

the  $\psi$ -Hilfer fractional system Eq. (4.9) is stable.

Next, we discuss the stability of some fractional systems involving the  $\psi$ -Hilfer fractional derivative, especially the fractional Lu and Chen systems [50, 51]. The fractional Lu and Chen systems towards the Riemann-Liouville fractional derivatives can be found in the following references.

**Example 1.** Consider the  $\psi$ -Hilfer fractional system

$$\begin{cases} \frac{H}{2} \mathbb{D}_{0+}^{\alpha(t), \beta; \psi} x_1(t) &= x_2(t) \\ \frac{H}{2} \mathbb{D}_{0+}^{\alpha(t), \beta; \psi} x_2(t) &= -x_1(t) - 1 \end{cases}. \quad (4.17)$$

Note that  $[x(t)]^T x(t) = x_1^2(t) + x_2^2(t)$  and  $[x(t)]^T f(x(t)) = -x_2$ . So, we have

$$\frac{[x(t)]^T \frac{H}{2} \mathbb{D}_{0+}^{\alpha(t), \beta; \psi} x(t)}{[x(t)]^T x(t)} = \frac{-x_2(t)}{x_1^2(t) + x_2^2(t)} \leq 0.$$

By Theorem 4.5, results

$$|x(t)| \leq |x(t_0)|$$

and the system Eq. (4.17) is stable.

**Example 2.** Consider the  $\psi$ -Hilfer fractional variable order Lu system

$$\begin{cases} \frac{H}{2} \mathbb{D}_{0+}^{\alpha(t), \beta; \psi} x &= a(y - x) \\ \frac{H}{2} \mathbb{D}_{0+}^{\alpha(t), \beta; \psi} y &= -xz + cy \\ \frac{H}{2} \mathbb{D}_{0+}^{\alpha(t), \beta; \psi} z &= xy - bz \end{cases} \quad (4.18)$$

where  $a = 6$ ,  $b = 9$  and  $c = 10$ .

Using the Theorem 4.5, yields

$$\begin{aligned} \frac{(x(t))^T \frac{H}{2} \mathbb{D}_{0+}^{\alpha(t), \beta; \psi} x(t)}{(x(t))^T x(t)} &= \frac{(x(t))^T f(x(t))}{(x(t))^T x(t)} \\ &= \frac{xa(y - x) + y(-xz + cy) + z(xy - bz)}{x^2 + y^2 + z^2} \\ &= \frac{axy - ax^2 + cy^2 - bz^2}{x^2 + y^2 + z^2} \\ &\leq \frac{\frac{a}{2}x^2 + (\frac{a}{2} + c)y^2 - bz^2}{x^2 + y^2 + z^2} \\ &\leq \max\left(\frac{a}{2}, c + \frac{a}{2}, -b\right) \\ &= \max(3, 13, -8) = 13. \end{aligned}$$

We have

$$|x(t)| \leq |x(t_0)| \mathbb{E}_{\alpha(t)} \left( 13 (\psi(t) - \psi(t_0))^{\alpha(t)} \right)$$

for any  $t > t_0$  and

$$|y(t)| = |x(t_0)| \mathbb{E}_{\alpha(t)} \left( 13 (\psi(t) - \psi(t_0))^{\alpha(t)} \right). \quad (4.19)$$

**Example 3** Consider the  $\psi$ -Hilfer fractional variable order Chen system

$$\begin{cases} \frac{H}{2} \mathbb{D}_{0+}^{\alpha(t), \beta; \psi} x &= a(y - x) \\ \frac{H}{2} \mathbb{D}_{0+}^{\alpha(t), \beta; \psi} y &= (c - a)x + -xz + cy \\ \frac{H}{2} \mathbb{D}_{0+}^{\alpha(t), \beta; \psi} z &= xy - bz \end{cases} \quad (4.20)$$

where  $a = 13$ ,  $b = 5$  and  $c = -8$ .

Using the Theorem 4.5, yields

$$\begin{aligned} \frac{(x(t))^T \frac{H}{2} \mathbb{D}_{0+}^{\alpha(t), \beta; \psi} x(t)}{(x(t))^T x(t)} &= \frac{(x(t))^T f(x(t))}{(x(t))^T x(t)} \\ &= \frac{xa(y - x) + y(-xz + cy) + z(xy - bz) + y(c - a)x}{x^2 + y^2 + z^2} \\ &= \frac{-ax^2 - cy^2 - bz^2 - cxy}{x^2 + y^2 + z^2} \\ &\leq \frac{ax^2 - cy^2 - bz^2 - c\left(\frac{x^2 + y^2}{2}\right)}{x^2 + y^2 + z^2} \\ &\leq \max\left(-a - \frac{c - 3c}{2}, -b\right) \\ &= \max(-9, 12, -5) = 12. \end{aligned}$$

We have

$$|x(t)| \leq |x(t_0)| \mathbb{E}_{\alpha(t)}\left(12(\psi(t) - \psi(t_0))^{\alpha(t)}\right)$$

for any  $t > t_0$  and

$$|y(t)| = |x(t_0)| \mathbb{E}_{\alpha(t)}\left(12(\psi(t) - \psi(t_0))^{\alpha(t)}\right). \quad (4.21)$$

## 5 Conclusion and open questions

This article investigated the  $\psi$ -Hilfer FDVO. New approximations for the  $\psi$ -Caputo fractional derivative were discussed and from them, approximations for the  $\psi$ -Hilfer FDVO. The integration by parts of the  $\psi$ -Hilfer FDVO and particular cases, as well as two examples involving the Mittag-Leffler function with respect to another function were highlighted. Three examples illustrated the proposed concepts and the results of choosing a function and values of  $\alpha(x, t)$ ,  $\delta$ ,  $\varepsilon$ ,  $t$ ,  $x$  and  $a$ . From the particular choice of the  $\psi$ -Hilfer FDVO, the stability of solutions of the fractional nonlinear systems were studied by means of the one-parameter Mittag-Leffler function of variable order. Examples involving the fractional variable order Lu and Chen systems in the sense of  $\psi$ -Hilfer fractional derivative illustrated the proposed formulations.

The results, allow further research work in other problems involving fractional differential equations of variable order. We can point, for example, questions about population growth, since the solution of the problem is directly linked to the Mittag-Leffler function (see Theorem 10). Of course, other approaches involving practical applications, certainly justify and validate this new approach for the  $\psi$ -Hilfer to the FDVO.



We can raise some questions, namely:

1. Is a new formulation of the generalized Leibniz rule possible?
2. What is Laplace transform of the  $\psi$ -Hilfer FDVO?
3. Is it possible to investigate the existence and uniqueness of solutions of fractional differential equations in the direction of  $\psi$ -Hilfer FDVO in the context of sectorial and quasi-sectorial operators?
4. Is it possible to obtain mild solutions for abstract differential equations in the sense of the  $\psi$ -Caputo FDVO?
5. Can we obtain a new approximation for the  $\psi$ -Hilfer FDVO with Bernstein polynomials and perform numerical experiments?

We note that in recent literature, we find several works on FDVO. However, many questions arise as pointed out in the critical discussion held above and need further research.

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## References

- [1] Almeida, R.. "Caputo–Hadamard fractional derivatives of variable order." *Numer. Func. Anal. Opt.* 38.1 (2017): 1-19.
- [2] Almeida, R.. "A Caputo fractional derivative of a function with respect to another function." *Commun. Nonlinear Sci. Numer. Simul.* 44 (2017): 460-481.
- [3] Almeida, R.. "Caputo–Hadamard fractional derivatives of variable order." *Numer. Func. Anal. Opt.* 38.1 (2017): 1-19.
- [4] Almeida, R., Nuno R. O. Bastos, and M. Teresa T. Monteiro. "A fractional Malthusian growth model with variable order using an optimization approach." *Statistics, Opt. & Infor. Computing* 6.1 (2018): 4-11.
- [5] Almeida, R., Nuno R. O. Bastos, and M. Teresa T. Monteiro. "Modeling some real phenomena by fractional differential equations." *Math. Meth. Appl. Sci.* 39.16 (2016): 4846-4855.
- [6] Almeida, R., and Delfim F. M. Torres. "A discrete method to solve fractional optimal control problems." *Nonlinear Dyn.* 80.4 (2015): 1811-1816.

- [7] Almeida, R., Nuno R. O. Bastos, and M. Teresa T. Monteiro. "A fractional Malthusian growth model with variable order using an optimization approach." *Statistics, Opt. & Infor. Comput.* 6.1 (2018): 4-11.
- [8] Atanackovic, Teodor, M. Janev, S. Pilipovic and D. Zorica. "An expansion formula for fractional derivatives of variable order." *Open Phy.* 11.10 (2013): 1350-1360.
- [9] Karniadakis, G. E., *Handbook of Fractional Calculus with Applications. Numerical Methods*, vol. 3 2019, De Gruyter: Berlin, Germany.
- [10] Debnath, L. "A brief historical introduction to fractional calculus." *Inter. J. Math. Education Sci. Tech.* 35.4 (2004): 487-501.
- [11] Hajipour, M., A. Jajarmi, D. Baleanu and H. Sun. "On an accurate discretization of a variable-order fractional reaction-diffusion equation." *Commun. Nonlinear Sci. Numer. Simul.* 69 (2019): 119-133.
- [12] Herzallah, M. A. E., and D. Baleanu. "Fractional-order Euler–Lagrange equations and formulation of Hamiltonian equations." *Nonlinear Dyn.* 58.1-2 (2009): 385.
- [13] Samko, S. G., A. A. Kilbas, and O. I. Marichev. *Fractional integrals and derivatives. Vol. 1.* Yverdon-les-Bains, Switzerland: Gordon and Breach Science Publishers, Yverdon, 1993.
- [14] Nash, J. F. and M. T. Rassias, *Open problems in mathematics*, 2016, New York: Springer.
- [15] Odziejewicz, T., A. Malinowska, and D. Torres. "Noether's theorem for fractional variational problems of variable order." *Open Phys.* 11.6 (2013): 691-701.
- [16] Oliveira, D. S., and E. Capelas de Oliveira. "Hilfer–Katugampola fractional derivatives." *Comput. Appl. Math.* 37.3 (2018): 3672-3690.
- [17] Oliveira, E. Capelas de, and J. Vanterler da C. Sousa. "Ulam–Hyers–Rassias stability for a class of fractional integro-differential equations." *Results Math.* 73.3 (2018): 111.
- [18] Polya, G., and J. Kilpatrick. *The Stanford mathematics problem book: With hints and solutions.* Courier Corporation, 2013.
- [19] Pooseh, S., R. Almeida, and D. F. M. Torres. "Expansion formulas in terms of integer-order derivatives for the Hadamard fractional integral and derivative." *Numer. Funct. Anal. Opt.* 33.3 (2012): 301-319.
- [20] Pooseh, S., R. Almeida, and D. F. M. Torres. "Numerical approximations of fractional derivatives with applications." *Asian J. Control* 15.3 (2013): 698-712.
- [21] Samko, Stefan G., and B. Ross. "Integration and differentiation to a variable fractional order." *Int. Transf. Special Funct.* 1.4 (1993): 277-300.
- [22] Samko, S.. "Fractional integration and differentiation of variable order: an overview." *Nonlinear Dyn.* 71.4 (2013): 653-662.

- [23] Sierociuk, D., W. Malesza, and M. Macias. "On a new definition of fractional variable-order derivative." Proceedings of the 14th Inter. Carpathian Control Conference (ICCC). IEEE, 2013.
- [24] Sierociuk, D., W. Malesza, and M. Macias. "On a new symmetric fractional variable order derivative." Theoretical developments and applications of non-integer order systems. Springer, Cham, 2016. 29-39.
- [25] Sierociuk, D., W. Malesza, and M. Macias. "Derivation, interpretation, and analog modelling of fractional variable order derivative definition." Appl. Math. Model. 39.13 (2015): 3876-3888.
- [26] Sousa, J. Vanterler da C., and E. Capelas de Oliveira. "On the  $\psi$ -Hilfer fractional derivative." Commun. Nonlinear Sci. Numer. Simul. 60 (2018): 72-91.
- [27] Sousa, J. Vanterler da C., Kishor D. Kucche, and E. Capelas De Oliveira. "Stability of  $\psi$ -Hilfer impulsive fractional differential equations." Appl. Math. Lett. 88 (2019): 73-80.
- [28] Sousa, J. Vanterler da C., and E. Capelas de Oliveira. "Leibniz type rule:  $\psi$ -Hilfer fractional operator." Commun. Nonlinear Sci. Numer. Simul. 77 (2019): 305-311.
- [29] Sousa, J. Vanterler da C., and E. Capelas De Oliveira. "Ulam–Hyers stability of a nonlinear fractional Volterra integro-differential equation." Appl. Math. Lett. 81 (2018): 50-56.
- [30] Sousa, J. Vanterler da C., E. Capelas de Oliveira and L. A. Magna: Fractional calculus and the ESR test. AIMS Math. 2.4 (2017): 692-705.
- [31] Sousa, J. Vanterler da C., M. N. N. Santos, L. A. Magna and E. Capelas de Oliveira. "Validation of a fractional model for erythrocyte sedimentation rate." Comput. Appl. Math. 37.5 (2018): 6903-6919.
- [32] Sousa, J. Vanterler da C., and E. Capelas de Oliveira. "On the  $\Psi$ -fractional integral and applications." Comput. Appl. Math. 38.1 (2019): 4.
- [33] Sousa, J. Vanterler da C., and E. Capelas de Oliveira. "Two new fractional derivatives of variable order with non-singular kernel and fractional differential equation." Comput. Appl. Math. 37.4 (2018): 5375-5394.
- [34] Sun, H., A. Chang, Y. Zhang and W. Chen. "A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications." Fract. Cal. Appl. Anal. 22.1 (2019): 27-59.
- [35] Sun, H., W. Chen, and Y. Chen. "Variable-order fractional differential operators in anomalous diffusion modeling." Physica A: Stat. Mech. Appl. 388.21 (2009): 4586-4592.
- [36] Tavares, D., R. Almeida, and D. F. M. Torres. "Combined fractional variational problems of variable order and some computational aspects." J. Comput Appl. Math. 339 (2018): 374-388.

- [37] Tavares, D., R. Almeida, and D. F. M. Torres. "Constrained fractional variational problems of variable order." *IEEE/CAA J. Automatica Sinica* 4.1 (2017): 80-88.
- [38] Tavares, D., R. Almeida, and D. F. M. Torres. "Caputo derivatives of fractional variable order: numerical approximations." *Commun. Nonlinear Sci. Numer. Simul.* 35 (2016): 69-87.
- [39] Tavares, D., R. Almeida, and D. F. M. Torres. "Optimality conditions for fractional variational problems with dependence on a combined Caputo derivative of variable order." *Optimization* 64.6 (2015): 1381-1391.
- [40] Teodoro, G. Sales, J. A. Tenreiro Machado, and E. Capelas De Oliveira. "A review of definitions of fractional derivatives and other operators." *J. Comput. Phys.* 388 (2019): 195-208.
- [41] Valério, D., and J S. Da Costa. "Variable-order fractional derivatives and their numerical approximations." *Signal Processing* 91.3 (2011): 470-483.
- [42] Valério, D., and J. S. da Costa. "Variable order fractional controllers." *Asian J. Control* 15.3 (2013): 648-657.
- [43] Zhang, S.. "Existence result of solutions to differential equations of variable-order with nonlinear boundary value conditions." *Commun. Nonlinear Sci. Numer. Simul.* 18.12 (2013): 3289-3297.
- [44] Yang, X.-J., and J. A. Tenreiro Machado. "A new fractional operator of variable order: application in the description of anomalous diffusion." *Physica A: Stat. Mech. Appl.* 481 (2017): 276-283.
- [45] Machado, J. A. Tenreiro, V. Kiryakova, and F. Mainardi. "Recent history of fractional calculus." *Commun. Nonlinear Sci. Numer. Simul.* 16.3 (2011): 1140-1153.
- [46] Machado, J. A. Tenreiro, and V. Kiryakova. "Recent history of the fractional calculus: data and statistics." *Handbook of Fractional Calculus with Applications* 1 (2019): 1-21.
- [47] Leibniz, G. W. "Letter from Hanover, Germany, to GFA L'Hopital, September 30; 1695." *Mathematische Schriften* 2 (1849): 301-302.
- [48] Leibniz, G. W. "Letter from Hanover, Germany to Johann Bernoulli, December 28, 1695." *Leibniz Mathematische Schriften*. Olms-Verlag, Hildesheim, Germany (1962): 226.
- [49] Leibniz, G. W. "Letter from Hanover, Germany to John Wallis, May 28, 1697." *Leibniz Mathematische Schriften*. Olms-Verlag, Hildesheim, Germany (1962): 25.
- [50] Li, C., and G. Chen. "Chaos in the fractional order Chen system and its control." *Chaos, Solitons & Fractals* 22.3 (2004): 549-554.
- [51] Lu, J. G. "Chaotic dynamics of the fractional-order Lü system and its synchronization." *Phys. Lett. A* 354.4 (2006): 305-311.

## Appendix

From the Theorems 5.1 and 5.2, we now discuss the operator limitation  $\mathbf{I}_{a^+}^{\alpha(x,t);\psi}(\cdot)$  and a version of the integration by parts.

**Theorem 5.1** *Let  $\frac{1}{n} < \alpha(x,t) < 1, \forall x, t \in [a, b]$  with a number  $n \in \mathbb{N}$  greater or equal than two and  $\psi \in C'([a, b], \mathbb{R})$  and function such that  $\psi'(\cdot) \neq 0, \forall x, t \in [a, b]$ . The  $\psi$ -Riemann-Liouville FIVO  $\alpha(x, t)$ ,*

$$\mathbf{I}_{a^+}^{\alpha(x,t);\psi} : L_1([a, b], \mathbb{R}) \longmapsto L_1([a, b], \mathbb{R})$$

*is a linear and bounded operator.*

*Proof:* First, note that the operator  $\mathbf{I}_{a^+}^{\alpha(x,t);\psi}(\cdot)$  is linear. Let  $\frac{1}{n} < \alpha(x,t) < 1$  and  $f \in L_1([a, b], \mathbb{R})$ . We define the following function

$$\mathbf{H}(x, t, s) := \begin{cases} \left| \psi'(s) (\psi(s) - \psi(t))^{\alpha(x,t)-1} \right| |f(s)|, & \text{if } s < t, \\ 0, & \text{if } t \leq s, \end{cases}$$

$$\forall (x, t, s) \in \Omega := [a, b] \times [a, b] \times [a, b].$$

By hypothesis,  $\frac{1}{n} < \alpha(x,t) < 1$ , and in this sense, for  $s+1 \leq t$  we have  $(\psi(t) - \psi(s))^{\alpha(x,t)-1} < 1$ . On the other hand, for  $s < t < s+1$  we have  $(\psi(t) - \psi(s))^{\alpha(x,t)-1} < (\psi(t) - \psi(s))^{\frac{1}{n}-1}$ .

Therefore, we can write

$$\begin{aligned} \int_a^b \left( \int_a^s \mathbf{H}(x, s, t) dt \right) ds &= \int_a^b \left( \int_s^b \left| \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} \right| |f(t)| \right) ds \\ &\leq \int_a^b |f(t)| \left( \int_s^b \left| \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} \right| dt \right) ds \\ &= \int_a^b |f(t)| \left( \int_s^{s+1} \left| \psi'(t) \right| \left| (\psi(t) - \psi(s))^{\alpha(x,t)-1} \right| dt + \int_{s+1}^b \left| \psi'(t) \right| \left| (\psi(t) - \psi(s))^{\alpha(x,t)-1} \right| dt \right) ds \\ &\leq \int_a^b |f(t)| \left( \int_s^{s+1} \left| \psi'(t) \right| \left| (\psi(t) - \psi(s))^{\frac{1}{n}-1} \right| dt + \int_{s+1}^b \left| \psi'(t) \right| dt \right) ds \\ &= \int_a^b |f(t)| \left( \int_s^{s+1} \left| \psi'(t) \right| \left| (\psi(t) - \psi(s))^{\frac{1}{n}-1} \right| dt + \psi(b) - \psi(s+1) \right) ds \\ &\leq \int_a^b |f(t)| \left\{ n(\psi(s+1) - \psi(s))^{\frac{1}{n}} + \psi(b) - \psi(s+1) \right\} ds \\ &\leq \psi(b) \|f\| \int_a^b ds + n \|f\| \int_a^b (\psi(s+1) - \psi(s))^{\frac{1}{n}} ds - \|f\| \int_a^b \psi(s+1) ds \\ &\leq \psi(b) \|f\| (b-a) + n \|f\| (b-a) - \|f\| \psi(b+1)(b-a) \end{aligned}$$

$$= \|f\| (b-a)[\psi(b) - \psi(b+1) + n] < \infty.$$

Remembering the inequality

$$\frac{x^2 + 1}{x + 1} \leq \Gamma(x + 1) \leq \frac{x^2 + 2}{x + 2}$$

and using the Fubini's theorem, we have that  $h$  is integrable on  $\Omega$  and

$$\begin{aligned} \left\| \mathbf{I}_{a^+}^{\alpha(x,t);\psi}(\cdot) \right\| &= \int_a^b \left| \frac{1}{\Gamma(\alpha(x,t))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} f(s) ds \right| dt \\ &\leq \int_a^b \frac{1}{\Gamma(\alpha(x,t))} \int_a^t \left| \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} f(s) \right| ds dt \\ &\leq \int_a^b \left( \frac{\alpha^2(x,t) + \alpha(x,t)}{\alpha^2(x,t) + 1} \int_a^t \left| \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} f(s) \right| ds \right) dt \\ &\leq \int_a^b \left( \int_a^t \left| \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} f(s) \right| ds \right) dt \\ &= \int_a^b \left( \int_a^t \mathbf{H}(x,s,t) ds \right) dt \leq (b-a)(\psi(b) - \psi(b+1) + n) \|f\| < \infty. \end{aligned}$$

Therefore, results

$$\left\| \mathbf{I}_{a^+}^{\alpha(x,t);\psi}(\cdot) \right\| \leq (b-a)(\psi(b) - \psi(b+1) + n) \|f\|$$

that completes the proof.  $\square$

**Theorem 5.2** Let  $\frac{1}{n} < \alpha(x,t) < 1, \forall t, x \in [a, b]$  with a number  $n \in \mathbb{N}$  greater or equal than two,  $f, g \in C([a, b], \mathbb{R})$  and  $\psi \in C'([a, b], \mathbb{R})$  and function such that  $\psi'(\cdot) \neq 0, \forall x, t \in [a, b]$ .

Then,

$$\int_a^b g(t) \mathbf{I}_{a^+}^{\alpha(x,t);\psi} f(t) dt = \int_a^b \psi'(t) f(t) \mathbf{I}_{b^-}^{\alpha(x,t);\psi} \left( \frac{g(t)}{\psi'(t)} \right) dt.$$

*Proof:* First, note that the operator  $\mathbf{I}_{a^+}^{\alpha(x,t);\psi}(\cdot)$  is linear. We define the following function

$$\mathbf{H}(x,t,s) := \begin{cases} \left| \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} g(t) f(s) \right|, & \text{if } s < t, \\ 0, & \text{if } t \leq s, \end{cases}$$

$$\forall (x,t,s) \in \Omega := [a, b] \times [a, b] \times [a, b].$$

As  $f, g \in C([a, b], \mathbb{R})$  and using the Bolzano's theorem,  $f, g$  have maximum and minimum. Therefore, there are constants  $c_1, c_2 > 0$ , such that  $|g(t)| \leq c_1$  and  $|f(t)| \leq c_2$  with  $t \in [a, b]$ .

By hypothesis,  $\frac{1}{n} < \alpha(x,t) < 1$ , and, in this sense, for  $1 \leq t - s$  we have  $(\psi(t) - \psi(s))^{\alpha(x,t)-1} < 1$ . On the other hand, for  $1 > t - s$  we have  $(\psi(t) - \psi(s))^{\alpha(x,t)-1} < (\psi(t) - \psi(s))^{\frac{1}{n}-1}$ .

We can write

$$\begin{aligned}
\int_a^b \left( \int_a^t \mathbf{H}(x, s, t) ds \right) dt &= \int_a^b \left( \int_a^t \left| \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} f(s)g(t) \right| ds \right) dt \\
&\leq \int_a^b \int_a^t \left| \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} \right| |g(t)| |f(s)| ds dt \\
&\leq c_1 c_2 \int_a^b \int_a^t \left| \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} \right| ds dt \\
&= c_1 c_2 \int_a^b \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} ds dt \\
&= c_1 c_2 \int_a^b \left( \int_a^{t-1} \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} ds + \int_{t-1}^t \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} ds \right) dt \\
&< c_1 c_2 \int_a^b \left( \int_a^{t-1} \psi'(t) ds + \int_{t-1}^t \psi'(s) (\psi(t) - \psi(s))^{\frac{1}{n}-1} ds \right) dt \\
&= c_1 c_2 \int_a^b \left( \psi(t-1) - \psi(a) + n (\psi(t-1) - \psi(t))^{\frac{1}{n}} \right) dt \\
&< c_1 c_2 (\psi(b-1) - \psi(a) + n) \int_a^b dt \\
&= c_1 c_2 (\psi(b-1) - \psi(a) + n) (b-a) < \infty.
\end{aligned}$$

On the other hand, knowing the inequality

$$\frac{x^2 + 1}{x + 1} \leq \Gamma(x + 1) \leq \frac{x^2 + 2}{x + 2}$$

we can write

$$\int_a^b g(t) \mathbf{I}_{a+}^{\alpha(x,t); \psi} f(t) dt = \int_a^b \left( \frac{1}{\Gamma(\alpha(x,t))} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} g(t) f(s) ds \right) dt$$

and using the Fubini's theorem, we can obtain

$$\begin{aligned}
\int_a^b g(t) \mathbf{I}_{a+}^{\alpha(x,t); \psi} f(t) dt &= \int_a^b \left( \frac{1}{\Gamma(\alpha(x,t))} \int_s^b \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} g(t) f(s) dt \right) ds \\
&= \int_a^b \psi'(s) f(s) \left( \frac{1}{\Gamma(\alpha(x,t))} \int_s^b \psi'(s) (\psi(t) - \psi(s))^{\alpha(x,t)-1} \frac{g(t)}{\psi'(t)} dt \right) ds \\
&= \int_a^b \psi'(s) f(s) \mathbf{I}_{b-}^{\alpha(x,t); \psi} \left( \frac{g(s)}{\psi'(s)} \right) ds
\end{aligned}$$

which is the desired result.  $\square$

**Theorem 5.3** [36, 38, 20, 15, 37] *Assuming the same conditions as Theorem 5.2 and choosing  $\psi(t) = t$ , we have*

$$\int_a^b g(t) \mathbf{I}_{a+}^{\alpha(x,t)} f(t) dt = \int_a^b f(t) \mathbf{I}_{b-}^{\alpha(x,t)} g(t) dt.$$

*Proof:* The proof follows directly from the Theorem (5.2).

□