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New bounds for the solution and derivatives of the Stein equation for the generalized inverse Gaussian and Kummer distributions

Essomanda KONZOU[*], Angelo Efoévi KOUDOU[†], Kossi E. GNEYOU [‡]

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Abstract

For Lipschitz test functions we propose a new bound of the solution of the Stein equation related to the generalized inverse Gaussian (resp. the Kummer) distribution. This bound is derived using the general approach established in [4] for distributions satisfying a certain differential equation, and thus is optimal for Lipschitz test functions. The main contribution of this paper is to establish an explicit expression of the bound as a function of the parameters of the distribution in terms of the modified Bessel function of the third kind (resp. the confluent hypergeometric function of the second kind). Under a restriction on the parameters we also obtain an optimal bound for the first derivative of the solution. A recurrence formula is established using the iterative technique developed in [4, 5] in order to bound derivatives of any order, for test functions smooth enough.

Keywords: Generalized Inverse Gaussian distribution, Kummer distribution, Stein characterization, Stein equation.

1 Introduction

This paper is a contribution to the literature related to Stein's method and also to that addressing properties of the generalized inverse Gaussian and Kummer distributions.

The generalized inverse Gaussian distribution with parameters $p \in \mathbb{R}, a > 0, b > 0$, which will be denoted $\mathrm{GIG}(p, a, b)$ throughout the paper, has density

$$g_{p,a,b}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0 \tag{1.1}$$

where K_p is the modified Bessel function of the third kind. As readily seen from the expression of the normalizing constant in (1.1), if $W \sim GIG(p, a, b)$ then

$$\mathbb{E}(W) = \sqrt{\frac{b}{a}} \frac{K_{p+1}(\sqrt{ab})}{K_p(\sqrt{ab})}, \qquad \mathbb{E}(W^2) = \frac{b}{a} \frac{K_{p+2}(\sqrt{ab})}{K_p(\sqrt{ab})}. \tag{1.2}$$

For $a > 0$, $b \in \mathbb{R}$, $c > 0$, the Kummer distribution $\mathrm{K}(a, b, c)$ with parameters a, b, c has density

$$k_{a,b,c}(x) = \frac{1}{\Gamma(a)\psi(a, a-b+1; c)} x^{a-1}(1+x)^{-a-b} e^{-cx}, \quad x > 0$$

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where ψ is the confluent hypergeometric function of the second kind. If $Z \sim K(a,b,c)$, then

$$\mathbb{E}(Z) = a\frac{\psi(a+1,2-b;c)}{\psi(a,1-b;c)}; \qquad \mathbb{E}(Z^2) = a(a+1)\frac{\psi(a+2,3-b;c)}{\psi(a,1-b;c)}. \tag{1.3}$$

For more on GIG and Kummer distributions see e.g [8, 10, 13].

Stein's method was introduced in [16], where it was shown that a random variable X has a standard normal distribution if and only if, for all real-valued absolutely continuous function f such that $\mathbb{E}\left|f'(Z)\right| < \infty$ for $Z \sim \mathcal{N}(0,1)$,

$$\mathbb{E}\left[f'(X) - Xf(X)\right] = 0.$$

The related so-called Stein's equation is

$$f'(x) - xf(x) = h(x) - \mathbb{E}h(Z) \tag{1.4}$$

for a bounded function h and a random variable Z following the standard normal distribution. If f_h solves (1.4), then for any random variable X, we have $\left|\mathbb{E}\left[f_h'(X) - Xf_h(X)\right]\right| = \left|\mathbb{E}h(X) - \mathbb{E}h(Z)\right|$. Therefore, in order to control the distance between the distribution of X and the standard normal distribution, bounding $\left|\mathbb{E}h(X) - \mathbb{E}h(Z)\right|$ given h amounts to bounding $\left|\mathbb{E}\left[f_h'(X) - Xf_h(X)\right]\right|$. More details on Stein's method can be found in [2, 12, 14, 6].

Stein's method has been widely applied to other distributions than the standard normal distribution in the literature (see for example [2] for the Poisson distribution, [12] for the gamma distribution, [1] for the exponential distribution), and among other tools one generally needs to bound the solution of the Stein's equation and a few of its derivatives. Works on this subject are generally directed to applications in probability theory and statistics, but they can also be considered for the nice mathematics developed in it. For instance, [4] contributed to this literature by developing a general theory of Stein's method for distributions having positive and locally absolutely continuous density g on an interval and such that

$$\left(s(x)g(x)\right)' = \tau(x)g(x)$$

for some functions s and τ having some prescribed behaviour. See also [15] where such distributions were considered with s a polynomial at most 2 and τ a decreasing linear function. In this paper we are interested in the general bounds provided in [4] for the corresponding Stein equation and for its first derivative for Lipschitz test functions. An advantage of the approach developed in [4] is that, when these bounds are finite, then they are optimal for Lipschitz test functions. But the bound obtained for the derivative by this general approach is not always finite for all distributions and, for concrete examples, much work remains to do, in order either to obtain an explicit expression or at least to prove the finiteness of this bounds. The aim of this paper is to carry out this work in the case of the generalized inverse Gaussian and Kummer distributions, which is not straightforward although the mathematical arguments involved in the proofs are rather simple. To obtain the bound of the solution for these two distributions, we specialize the general bound of [4] to the framework of the functions s and τ being polynomials of second order with some conditions on the coefficients, in which case we obtain an explicit expression for the bound (see Theorems 3.1, 5.1 and 5.3):

for $p \in \mathbb{R}$, $a > 0$, $b > 0$, for any Lipschitz continuous test function h, the solution f_h of the GIG Stein equation satisfies

$$\|f_h\| \leq \frac{2}{\sqrt{ab}}\frac{K_{p+1}(\sqrt{ab})}{K_p(\sqrt{ab})}\|h'\|, \tag{1.5}$$

and for $a > 0$, $b \in \mathbb{R}$, $c > 0$, the solution f_h of the Stein equation related to the Kummer distribution $\mathrm{K}(a,b,c)$ satisfies

$$\|f_h\| \leq \frac{\psi(a+1,2-b;c)}{\psi(a,1-b;c)}\|h'\|,$$

where we denote $||f|| = \sup_{x>0}|f(x)|$ as in the sequel of the paper.

To bound the first derivative of f_h for both distributions, we prove that the bound derived in [4] is finite for functions s and τ not being necessarily polynomials but having some specific behaviour at 0 and ∞, common to

the two distributions (see Theorem 4.1), without being able to derive an explicit expression. The condition that τ be decreasing, under which the result in [4] applies, yields some restrictions on the parameters. We also use the iterative argument initiated in [4] and further developed in [5], to establish results on the derivatives of higher order (see Theorem 5.2).

In the course of the derivation of our results for the GIG (resp. Kummer) distributions, we retrieve, by using the Stein characterization of the distribution, a well-known recurrence relation and an inequality seemingly new (see Remarks 5.1 and 5.2) for the modified Bessel function of the third kind (resp. the confluent hypergeometric function of the second kind).

Note that bounds for the solution of the Stein equation related to the same distributions, and for its first and second derivatives, were established in [9], but these bounds do not have tractable forms as in the present paper, and are not easily extendable to derivatives of higher order.

The paper is organized as follows. In Section 2 we recall the general bounds of [4], needed in this paper. In section 3 we compute explicitly the bound of the solution in the case where the functions s and τ satisfy some conditions. Bounds of the first derivative of the solution are treated in Section 4. Specific results for GIG and Kummer distributions are given in Section 5.

2 Döbler's general bounds

The general framework of this paper is that of a probability density g on $(0,\infty)$ verifying the following condition:

Assumption A: g is positive, differentiable on $(0,\infty)$ and there exist differentiable functions s and τ on $(0,\infty)$, such that s is positive, $\lim_{x\to 0} s(x)g(x) = \lim_{x\to\infty} s(x)g(x) = 0$ and, for all $x > 0$,

$$(s(x)g(x))' = \tau(x)g(x). \tag{2.1}$$

Distributions satisfying Assumption A can be considered on any interval ([15], [4]) but here we focus on $(0,\infty)$, having in mind the applications of our results to the GIG and Kummer distributions.

As observed in [9] one can obtain, by using the Stein density approach [3, 11, 15], the following Stein characterization of distributions with density g satisfying Assumption A:

A positive random variable X has density g if and only if for any differentiable function f such that $\lim_{x\to 0} s(x)g(x)f(x) = \lim_{x\to\infty} s(x)g(x)f(x) = 0$,

$$\mathbb{E}\left[s(X)f'(X) + \tau(X)f(X)\right] = 0. \tag{2.2}$$

The related Stein equation is

$$s(x)f'(x) + \tau(x)f(x) = h(x) - \mathbb{E}h(W) \tag{2.3}$$

where W is a random variable with density g.

The Stein equation (2.3) (see [15], [4], [7]) has solution

$$\begin{aligned} f_h(x) =& \frac{1}{s(x)g(x)} \int_0^x g(t)\left[h(t) - \mathbb{E}h(W)\right] dt \\ =& \frac{-1}{s(x)g(x)} \int_x^{+\infty} g(t)\left[h(t) - \mathbb{E}h(W)\right] dt. \end{aligned} \tag{2.4}$$

The following result was established in [4]:

Proposition 2.1 *(Proposition 3.9 of [4]). Suppose τ is decreasing. Let W be a random variable with density g. Given a function h, let f_h be the solution of the Stein equation (2.3) given by (2.4). For any bounded function h,*

$$|f_h(x)| \leq \frac{\|h - E(h(W))\|}{2s(q_{0.5})g(q_{0.5})} \tag{2.5}$$

where $q_{0.5}$ is the median of W.

Remark 2.1 *One can wonder how the bound given in Proposition 2.1 compares with the following one established in [9] under the assumptions of Proposition 2.1:*

$$||f_h|| \leq M\,||h(.) - \mathbb{E}h(W)|| \tag{2.6}$$

where

$$M = \max\left(\frac{1}{s(\alpha)g(\alpha)}\int_0^\alpha g(t)dt; \frac{1}{s(\alpha)g(\alpha)}\int_\alpha^{+\infty} g(t)dt\right)$$

and α is the unique zero of τ on $(0,\infty)$.

As observed in Proposition 3.3 of [4], the function $x \mapsto s(x)g(x)$ reaches its maximum at $x = \alpha$. Therefore,

$$\frac{1}{2s(\alpha)g(\alpha)} \leq \frac{1}{2s(q_{0.5})g(q_{0.5})}.$$

We also have, since

$$\frac{1}{2} \leq \max\left(\int_0^\alpha g(t)dt; \int_\alpha^{+\infty} g(t)dt\right) \leq 1,$$

$$\frac{1}{2s(\alpha)g(\alpha)} \leq M \leq \frac{1}{s(\alpha)g(\alpha)}.$$

Thus it is not clear that one of the two bounds in (2.5) and (2.6) is smaller than the other in general. One could perhaps observe that the bound in (2.6) is a little more explicit than the one in (2.5) in the cases, e.g. for GIG and Kummer distributions, where one has an explicit expression of α and not of $q_{0.5}$.

Let us recall the results of [4] needed in this paper.

Proposition 2.2 *(Proposition 3.13 of [4]). Let W be a random variable with density g satisfying Assumption A, and distribution function F, such that $\mathbb{E}(W) < \infty$. Given a function h, let f_h be the solution of the Stein equation (2.3) given by (2.4). For any Lipschitz continuous test function h,*

1.

$$|f_h(x)| \leq \|h'\| \frac{F(x)\mathbb{E}(W) - \int_0^x yg(y)dy}{s(x)g(x)} = \|h'\| \frac{\int_0^x (\mathbb{E}(W) - yg(y))\,dy}{s(x)g(x)}; \tag{2.7}$$

2. If τ is decreasing, then

$$|f_h'(x)| \leq \|h'\| \frac{\int_0^x F(y)dyG(x) + \int_x^\infty (1 - F(y))\,dyH(x)}{s^2(x)g(x)} \tag{2.8}$$

where, for $x \in (0,\infty)$, the positive functions $H(x)$ and $G(x)$ are defined by

$$H(x) := s(x)g(x) - \tau(x)F(x) \quad \text{and} \quad G(x) := H(x) + \tau(x).$$

Remark 2.2 *1. We slightly reformulated this proposition by putting the assumption of τ being decreasing only in item 2 because we realized that in the proof given in [4], this assumption is not required for item 1.*

2. The right-hand side of (2.7) is always bounded. But for the right-hand side of (2.8), its finiteness must be evaluated for any specific density g. We fill prove that it is the case for GIG and Kummer distributions.

From this general results we elaborate bounds in the case where the functions τ is a polynomial of second order satisfying some conditions, with the purpose to apply this to GIG and Kummer distributions, which happen to fulfill these conditions.

3 On the bound of the solution of the Stein equation

Theorem 3.1 *Consider a density g on $(0,\infty)$ such that Assumption A holds for some functions s and τ, such that τ is polynomial and has the form*

$$\tau(x) = a_2x^2 + a_1x + a_0$$

with $a_2 < 0$ and $a_0 > 0$.

Let f_h be the solution of the Stein equation (2.3) defined by (2.4).

1. *For any Lipschitz continuous test function h,*

$$\|f_h\| \leq \frac{\mathbb{E}(W)}{a_0}\|h'\| \tag{3.1}$$

where W is a random variable with density g.

2. *Suppose τ is decreasing. Then, there exist constants B and C such that, for any Lipschitz continuous test function h,*

$$|f_h'(x)| \leq B\|h'\|, \quad |xf_h'(x)| \leq C\|h'\|.$$

For the proof we first establish the following proposition where we prove that the function on the right-hand-side of (2.7) is bounded and provide an explicit expression of its supremum. This is obtained by showing that the function $U : x \mapsto \dfrac{\int_0^x (\mathbb{E}(W)-t)g(t)dt}{s(x)g(x)}$ is decreasing on $(0,\infty)$ which is not obvious.

Proposition 3.1 *The function $U : x \mapsto \dfrac{\int_0^x (\mathbb{E}(W)-t)g(t)dt}{s(x)g(x)}$ is decreasing on $(0,\infty)$ and*

$$\lim_{x\to 0} U(x) = \frac{\mathbb{E}(W)}{a_0} < \infty, \tag{3.2}$$

$$U(x) \sim_{x\to\infty} \frac{-1}{a_2x}, \tag{3.3}$$

$$\lim_{x\to\infty} U(x) = 0, \tag{3.4}$$

$$\sup_{x>0} U(x) = \frac{\mathbb{E}(W)}{a_0} < \infty, \tag{3.5}$$

$$\sup_{x>0} xU(x) < \infty. \tag{3.6}$$

By applying the Stein characterization given by (2.2) to the function $x \mapsto f(x) = 1$, we have the following lemma:

Lemma 3.1 *Consider a random variable W with density g. Then*

$$\mathbb{E}(\tau(W)) = a_2\mathbb{E}\left(W^2\right) + a_1\mathbb{E}(W) + a_0 = 0. \tag{3.7}$$

Proof of Proposition 3.1

By de l'Hôpital's rule,

$$\begin{aligned}
\lim_{x\to 0} U(x) &= \lim_{x\to 0} \frac{(\mathbb{E}(W)-x)g(x)}{(s(x)g(x))'} \\
&= \lim_{x\to 0} \frac{(\mathbb{E}(W)-x)g(x)}{\tau(x)g(x)} \\
&= \frac{\mathbb{E}(W)}{\tau(0)} \\
&= \frac{\mathbb{E}(W)}{a_0} < \infty,
\end{aligned}$$

which proves (3.2), and

$$\begin{aligned}\lim_{x\to\infty} U(x) &= \lim_{x\to\infty} \frac{(\mathbb{E}(W)-x)g(x)}{(s(x)g(x))'} \\ &= \lim_{x\to\infty} \frac{(\mathbb{E}(W)-x)g(x)}{\tau(x)g(x)} \\ &= \lim_{x\to\infty} \frac{\mathbb{E}(W)-x}{a_2x^2+a_1x+a_0} \\ &= 0,\end{aligned}$$

which proves (3.3) and 3.4.

Let $m := \mathbb{E}(W)$. We have, for all $x > 0$, since $(s(x)g(x))' = \tau(x)g(x)$,

$$\begin{aligned}U'(x) &= \frac{(m-x)s(x)g^2(x) - \tau(x)g(x)\int_0^x (m-t)g(t)\,dt}{(s(x)g(x))^2} \\ &= \frac{(m-x)s(x)g(x) - \tau(x)\int_0^x (m-t)g(t)\,dt}{s^2(x)g(x)}.\end{aligned}$$

Let us prove that $A(x)$ defined by

$$A(x) := (m-x)s(x)g(x) - \tau(x)\int_0^x (m-t)g(t)\,dt \tag{3.8}$$

is negative for all $x > 0$.

We have

$$s(x)g(x) = \int_0^x \tau(t)g(t)\,dt.$$

Thus

$$\begin{aligned}A(x) &= \int_0^x (m-x)\tau(t)g(t)\,dt - \tau(x)\int_0^x (m-t)g(t)\,dt \\ &= \int_0^x \left[(m-x)\tau(t) - (m-t)\tau(x)\right] g(t)\,dt. \end{aligned}\tag{3.9}$$

For any $x > 0$, let

$$B_x(t) := (m-x)\tau(t) - (m-t)\tau(x). \tag{3.10}$$

We have

$$\begin{aligned}B_x(t) &= (m-x)\tau(t) - (m-x+x-t)\tau(x) \\ &= (m-x)\tau(t) - (m-x)\tau(x) + (t-x)\tau(x) \\ &= (m-x)(\tau(t)-\tau(x)) + (t-x)\tau(x) \\ &= (m-x)\left(a_2t^2+a_1t+a_0 - \left(a_2x^2+a_1x+a_0\right)\right) + (t-x)\tau(x) \\ &= (m-x)\left(a_1(t-x) + a_2(t^2-x^2)\right) + (t-x)\tau(x) \\ &= (m-x)(t-x)\left(a_1 + a_2(t+x)\right) + (t-x)\tau(x) \\ &= (t-x)\left[(m-x)\left(a_1+a_2t+a_2x\right) + a_2x^2 + a_1x + a_0\right] \\ &= (t-x)\left[ma_1 + a_2(m-x)t + a_2mx + a_0\right] \\ &= (x-t)\left[a_2(x-m)t - a_2mx \ - a_1m - a_0\right]\end{aligned}$$

Suppose $0 < x \le m$. Define

$$C_x(t) := a_2(x-m)t - a_2mx \ - a_1m - a_0. \tag{3.11}$$

Then, $t \mapsto C_x(t)$ is increasing, so that, for all $t \in (0, x]$,

$$\begin{aligned} C_x(t) &\leq C_x(x) \\ &= a_2(x-m)x - a_2mx \ - a_1m - a_0 \\ &= a_2x^2 - 2a_2mx \ - a_1m - a_0. \end{aligned}$$

We now observe that $a_2x^2 - 2a_2mx \ - a_1m - a_0$, polynomial in x, has discriminant

$$\Delta = 4a_2^2m^2 + 4a_2(a_1m + a_0) = 4a_2\left[a_2m^2 + a_1m + a_0\right] = 4a_2\left[a_2\left(\mathbb{E}(W)\right)^2 + a_1\mathbb{E}(W) + a_0\right]. \tag{3.12}$$

By Lemma 3.1, we have $a_2\mathbb{E}\left(W^2\right) + a_1\mathbb{E}\left(W\right) + a_0 = 0$, which implies

$$a_1\mathbb{E}\left(W\right) + a_0 = -a_2\mathbb{E}(W^2),$$

so that

$$\begin{aligned} \Delta &= 4a_2\left[a_2\left(\mathbb{E}(W)\right)^2 - a_2\mathbb{E}(W^2)\right] \\ &= 4a_2^2[(E(W))^2 - \mathbb{E}(W^2)] \\ &= -4a_2^2Var(W) \\ &< 0. \end{aligned}$$

As a consequence, since $a_2 < 0$, we have $a_2x^2 - 2a_2mx \ - a_1m - a_0 < 0$ for all x. It follows that, if $0 < x \leq m$, then $C_x(t) < 0$ for all $t \in (0, x]$, and therefore

$$B_x(t) = (x-t)C_x(t) \leq 0$$

for all $t \in (0, x]$. Thus, by (3.9), for all $x \in (0, m]$,

$$A(x) = \int_0^x B_x(t)g(t)\,dt \leq 0.$$

Suppose now that $x > m$. The function A defined by (3.8) can be written as

$$\begin{aligned} A(x) &= (m-x)s(x)g(x) + \tau(x)\int_x^\infty (m-t)g(t)\,dt \\ &= \int_x^\infty -\left[(m-x)\tau(t) - (m-t)\tau(x)\right]g(t)\,dt \\ &= \int_x^\infty -B_x(t)g(t)\,dt \\ &= \int_x^\infty (t-x)C_x(t)g(t)\,dt \end{aligned}$$

where $B_x(t)$ is again defined by (3.10) and $C_x(t)$ is defined by (3.11). Since $x > m$, the function $t \mapsto C_x(t)$ is decreasing, so that, for all $t \in [x, \infty)$, $C_x(t) \leq C_x(x)$ and the conclusion follows as previously. Thus U is decreasing and (3.5) follows. (3.6) follows from (3.2), (3.3) and from the continuity of U. □

Remark 3.1 *Let us describe another way to obtain that $C_x(x) < 0$. The discriminant Δ defined by (3.12) can be written as*

$$\Delta = 4a_2\left[a_2\left(\mathbb{E}(W)\right)^2 + a_1\mathbb{E}(W) + a_0\right] = 4a_2\tau(\mathbb{E}(W)). \tag{3.13}$$

Since $\tau(x) = a_2x^2 + a_1x + a_0$ with $a_2 < 0$, then $-\tau$ is strictly convex and, by Jensen's inequality, $\tau(\mathbb{E}(W)) \geq \mathbb{E}(\tau(W))$ and, since W is not constant, $\tau(\mathbb{E}(W)) > \mathbb{E}(\tau(W))$.

By Lemma 3.1, $\mathbb{E}(\tau(W)) = 0$, therefore, $\tau(\mathbb{E}(W)) > 0$ and we conclude by (3.13) that $\Delta < 0$.

4 On the bound of the derivative of the solution of the Stein equation

Theorem 4.1 *Consider a density g on $(0,\infty)$ such that Assumption A holds for some functions s and τ. Suppose there exists a constant $a_2 < 0$ such that*

$$\tau(x) \sim_{x\to\infty} a_2 s(x), \quad \tau(0) > 0, \quad s(x) \sim_{x\to\infty} x^2, \quad s(0) = 0, \quad \tau(0) + s'(0) \neq 0. \tag{4.1}$$

Let

$$G(x) = s(x)g(x) + \tau(x)(1 - F(x)),$$

$$H(x) = s(x)g(x) - \tau(x)F(x),$$

$$R(x) = \frac{G(x)\displaystyle\int_0^x F(t)dt + H(x)\int_x^{+\infty}(1 - F(t))dt}{s(x)^2 g(x)}.$$

Then

$$\lim_{x\to 0} R(x) = \frac{1}{\tau(0) + s'(0)} - \frac{\tau'(0)\mathbb{E}(W)}{\tau(0)(\tau(0) + s'(0))}, \tag{4.2}$$

$$\lim_{x\to\infty} R(x) = 0. \tag{4.3}$$

As a consequence,

$$\sup_{x\geq 0} R(x) < \infty, \tag{4.4}$$

$$\sup_{x\geq 0} xR(x) < \infty. \tag{4.5}$$

and

$$\sup_{x\geq 0} x^2 R(x) < \infty. \tag{4.6}$$

Remark 4.1 *Assertions (4.4), (4.5) and (4.6) follow from the other results of Theorem 4.1, as R is clearly continuous on $(0,\infty)$. We have not obtained an explicit expression of the supremum of R on $(0,\infty)$ as in (3.5).*

The proof of Theorem 4.1 is based on the following proposition.

Proposition 4.1 *Under the notation of Proposition 4.1, consider*

$$R_1(x) = \frac{G(x)\displaystyle\int_0^x F(t)dt}{s(x)^2 g(x)},$$

$$R_2(x) = \frac{H(x)\displaystyle\int_x^{+\infty}(1 - F(t))dt}{s(x)^2 g(x)}.$$

Then

$$\lim_{x\to 0} R_1(x) = \frac{1}{\tau(0) + s'(0)}. \tag{4.7}$$

$$\lim_{x\to 0} R_2(x) = -\frac{\tau'(0)\mathbb{E}(W)}{\tau(0)(\tau(0) + s'(0))}. \tag{4.8}$$

$$R_1(x) \sim_{x\to\infty} -\frac{\tau'(x)}{a_2(x\tau(x) + s(x))}, \quad \lim_{x\to\infty} R_1(x) = 0. \tag{4.9}$$

$$R_2(x) \sim_{x\to\infty} -\frac{1}{\tau(x)}, \quad \lim_{x\to\infty} R_2(x) = 0. \tag{4.10}$$

PROOF :

4.1 Computing $\lim_{x\to 0} R_1(x)$

By the definition of G, we have

$$\lim_{x\to 0} G(x) = \tau(0). \tag{4.11}$$

We also have $\lim_{x\to 0} \int_0^x F(t)dt = 0$ and $\lim_{x\to 0} s(x)^2 g(x) = 0$. By de l'Hôpital's rule, as $x \to 0$,

$$\begin{aligned}
\frac{\int_0^x F(t)dt}{s(x)^2 g(x)} &\sim \frac{F(x)}{(s(x)^2 g(x))'} \\
&= \frac{F(x)}{s(x)g(x)(\tau(x)+s'(x))} \\
&\sim \frac{F(x)}{s(x)g(x)(\tau(0)+s'(0))}.
\end{aligned}$$

Taking once again the derivatives we have

$$\begin{aligned}
\frac{\int_0^x F(t)dt}{s(x)^2 g(x)} &\sim \frac{g(x)}{\tau(x)g(x)(\tau(0)+s'(0))} \\
&= \frac{1}{\tau(x)(\tau(0)+s'(0))}.
\end{aligned} \tag{4.12}$$

Combining (4.12) with (4.11) we get (4.7).

4.2 Computing $\lim_{x\to 0} R_2(x)$

Recall that

$$R_2(x) = \frac{H(x)\int_x^{+\infty}(1-F(t))dt}{s(x)^2 g(x)}$$

We have

$$\lim_{x\to 0} H(x) = 0$$

since $H(x) = G(x) - \tau(x)$. We have

$$\lim_{x\to 0} H(x)\int_x^{+\infty}(1-F(t))dt = 0$$

and

$$\lim_{x\to 0} s(x)^2 g(x) = 0.$$

By partial integration, we have

$$\begin{aligned}
\int_x^{+\infty}(1-F(t))dt &= \int_x^{+\infty} tg(t)dt - x(1-F(x)) \\
&= \mathbb{E}(W) - \int_0^x tg(t)dt - x(1-F(x))
\end{aligned}$$

which implies that

$$\lim_{x\to 0}\int_x^{+\infty}(1-F(t))dt = \mathbb{E}(W).$$

Therefore, by de l'Hôpital's rule, as $x \to 0$,

$$\begin{aligned} R_2(x) &\sim \frac{H(x)}{s(x)^2 g(x)}\mathbb{E}(W) \\ &= \frac{H'(x)}{s(x)g(x)(\tau(x)+s'(x))}\mathbb{E}(W) \\ &= \frac{-\tau'(x)F(x)}{s(x)g(x)(\tau(x)+s'(x))}\mathbb{E}(W) \\ &\sim \mathbb{E}(W)\frac{-\tau'(0)}{\tau(0)+s'(0)} \times \frac{F(x)}{s(x)g(x)} \\ &\sim \mathbb{E}(W)\frac{-\tau'(0)}{\tau(0)+s'(0)} \times \frac{g(x)}{\tau(x)g(x)} \\ &\sim \mathbb{E}(W)\frac{-\tau'(0)}{\tau(0)+s'(0)} \times \frac{1}{\tau(0)} \end{aligned}$$

which proves (4.8).

4.3 Computing $\lim_{x\to\infty} R_1(x)$

By integration by part we have

$$\int_0^x F(t)dt = xF(x) - \int_0^x tg(t)dt,$$

so that, as $x \to \infty$, since $F(x) \to 1$ and $\int_0^x tg(t)dt \to \mathbb{E}(W)$, we have

$$\int_0^x F(t)dt \sim x$$

so that, as $x \to \infty$,

$$R_1(x) \sim \frac{xG(x)}{s(x)^2 g(x)}$$

and, since $s(x) \sim_{x\to\infty} x^2$,

$$R_1(x) \sim \frac{G(x)}{xs(x)g(x)}.$$

By de l'Hôpital's rule,

$$\begin{aligned} R_1(x) &\sim \frac{\tau'(x)(1-F(x))}{x\tau(x)g(x)+s(x)g(x)} \\ &= \frac{\tau'(x)(1-F(x))}{g(x)(x\tau(x)+s(x))}. \end{aligned} \tag{4.13}$$

Furthermore,

$$\begin{aligned} \frac{1-F(x)}{g(x)} &\sim_{x\to\infty} \frac{-g(x)}{g'(x)} \\ &= \frac{-s(x)}{\tau(x)-s'(x)} \\ &\sim_{x\to\infty} \frac{-x^2}{\tau(x)} \\ &= \frac{-1}{a_2}. \end{aligned} \tag{4.14}$$

(4.13) and (4.14) imply (4.9).

4.4 Computing $\lim_{x\to\infty} R_2(x)$

$$\begin{aligned} H(x) &= s(x)g(x) - \tau(x)F(x) \\ &= -\tau(x)F(x)\left(1 - \frac{s(x)g(x)}{\tau(x)F(x)}\right) \\ &\sim_{x\to\infty} -\tau(x). \end{aligned}$$

Therefore,

$$\begin{aligned} R_2(x) &= \frac{\int_x^{+\infty}(1-F(t))dt}{s(x)g(x)} \times \frac{H(x)}{s(x)} \\ &\sim_{x\to\infty} \frac{F(x)-1}{\tau(x)g(x)} \times \frac{-\tau(x)}{s(x)} \\ &= \frac{1-F(x)}{s(x)g(x)} \\ &\sim_{x\to\infty} \frac{-g(x)}{\tau(x)g(x)} = \frac{-1}{\tau(x)} \to 0, \end{aligned}$$

which proves (4.10)

□

In the following session we apply the results to the GIG and Kummer distributions.

5 Application to the Stein equation for GIG and Kummer distributions

5.1 Bounds in the GIG case

Recall that the density of the GIG distribution with parameters $p \in \mathbb{R}$, $a > 0$, $b > 0$ is

$$g_{p,a,b}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0, \tag{5.1}$$

where K_p is the modified Bessel function of the third kind.

Let

$$s(x) = x^2 \quad \text{and} \quad \tau_{p,a,b}(x) = \frac{b}{2} + (p+1)x - \frac{a}{2}x^2. \tag{5.2}$$

Then, as observed in [7], the GIG density $g_{p,a,b}$ satisfies

$$(s(x)g_{p,a,b}(x))' = \tau_{p,a,b}(x)g_{p,a,b}(x). \tag{5.3}$$

Following (2.2) we have the following Stein characterization of the GIG distribution:

A random variable X follows the GIG distribution with density $g_{p,a,b}$ if and only if, for all real-valued and differentiable function f such that $\lim_{x\to\infty} g_{p,a,b}(x)f(x) = \lim_{x\to 0} g_{p,a,b}(x)f(x) = 0$, and such that the following expectation exists, we have:

$$\mathbb{E}\left[X^2 f'(X) + \left(\frac{b}{2} + (p+1)X - \frac{a}{2}X^2\right) f(X)\right] = 0.$$

The corresponding Stein equation is

$$x^2 f'(x) + \left(\frac{b}{2} + (p+1)x - \frac{a}{2}x^2\right) f(x) = h(x) - \mathbb{E}h(W) \tag{5.4}$$

where h is a bounded function and W a random variable following the GIG distribution with parameters p, a, b. As in the general case (2.4), the solution of Stein's equation (5.4) is given by

$$\begin{aligned} f_h(x) =& \frac{1}{s(x)g_{p,a,b}(x)} \int_0^x g_{p,a,b}(t)\left[h(t) - \mathbb{E}h(W)\right] dt \\ =& \frac{-1}{s(x)g_{p,a,b}(x)} \int_x^{+\infty} g_{p,a,b}(t)\left[h(t) - \mathbb{E}h(W)\right] dt. \end{aligned} \tag{5.5}$$

5.1.1 Bound of the solution

Equations (5.1), (5.2) and (5.3) show that $\tau_{p,a,b}$ and $g_{p,a,b}$ satisfy the assumptions of Theorem 3.1 with

$$s(x) = x^2, \quad \tau(x) = \tau_{p,a,b}(x) = \frac{b}{2} + (p+1)x - \frac{a}{2}x^2, \quad g = g_{p,a,b}.$$

If W is a random variable following the GIG distribution with parameters p, a, b then $\mathbb{E}(W)$ is given by (1.2). Apply Theorem 3.1, we have the following new bound of the solution of the Stein equation for the GIG distribution.

Theorem 5.1 *Let $p \in \mathbb{R}$, $a > 0$, $b > 0$. Let f_h be the solution of the GIG Stein equation defined by (5.5). For any Lipschitz continuous test function h,*

$$\|f_h\| \leq \frac{2}{\sqrt{ab}} \frac{K_{p+1}(\sqrt{ab})}{K_p(\sqrt{ab})} \|h'\|. \tag{5.6}$$

.

Remark 5.1 *1. The fact that $\tau_{p,a,b}(\mathbb{E}(W)) \geq 0$ (see Remark 3.1) implies that $\mathbb{E}(W) \leq \alpha$, where*

$$\alpha = \frac{p+1+\sqrt{(p+1)^2+ab}}{a}$$

is the unique zero of $\tau_{p,a,b}$ on $(0, \infty)$. This yields the following property of the modified Bessel function of the third kind: for all, $p \in \mathbb{R}$, $a > 0$, $b > 0$,

$$\sqrt{ab}\frac{K_{p+1}(\sqrt{ab})}{K_p(\sqrt{ab})} \leq p+1+\sqrt{(p+1)^2+ab}. \tag{5.7}$$

2. Using Lemma 3.1 and plugging in (3.7) the expressions of $\mathbb{E}(W)$ and $\mathbb{E}(W^2)$ given in (1.2) we obtain

$$\frac{K_{p+2}(\sqrt{ab})}{K_p(\sqrt{ab})} = 1 + \frac{2(p+1)}{\sqrt{ab}} \frac{K_{p+1}(\sqrt{ab})}{K_p(\sqrt{ab})}$$

which is equivalent to the following well-known recurrence relation for the modified Bessel function of the third kind:

$$K_{p+2}(\sqrt{ab}) = K_p(\sqrt{ab}) + \frac{2(p+1)}{\sqrt{ab}} K_{p+1}(\sqrt{ab}). \tag{5.8}$$

5.1.2 Bounds of the derivatives of the solution

Let $F_{p,a,b}$ be the distibution function of $W \sim GIG(p, a, b)$. Define

$$G(_{p,a,b}(x) = x^2 g_{p,a,b}(x) + \tau_{p,a,b}(x)(1 - F_{p,a,b}(x)), \tag{5.9}$$

$$H_{p,a,b}((x) = x^2 g_{p,a,b}(x) - \tau_{p,a,b}(x) F_{p,a,b}(x). \tag{5.10}$$

Define

$$A_{p,a,b} = \frac{2}{\sqrt{ab}} \frac{K_{p+1}(\sqrt{ab})}{K_p(\sqrt{ab})}. \tag{5.11}$$

By (4.4) and (4.5) in Theorem 4.1, if $p \leq -1$, then

$$B_{p,a,b} := \sup_{x>0} \frac{G_{p,a,b}((x)\int_0^x F_{p,a,b}((t)dt + H_{p,a,b}(x)\int_x^{+\infty}(1-F_{p,a,b}(t))dt}{x^4 g_{p,a,b}((x)} < \infty \tag{5.12}$$

and

$$C_{p,a,b} := \sup_{x>0} \frac{G_{p,a,b}((x)\int_0^x F_{p,a,b}(t)dt + H_{p,a,b}((x)\int_x^{+\infty}(1-F_{p,a,b}(t))dt}{x^3 g_{p,a,b}(x)} < \infty. \tag{5.13}$$

An interesting feature of Theorem 5.1 is that we have an explicit expression of the constant $A_{p,a,b}$ in terms of the modified Bessel function of the third kind. We did not succed in deriving such an explicit expression for the constants $B_{p,a,b}$ and $C_{p,a,b}$, but still a useful contribution of this paper is that they are finite. They can be computed numerically, given p, a and b. The following theorem gives the bounds of the successive derivatives of the solution of the GIG Stein equation.

Theorem 5.2 *Let $p \in \mathbb{R}$, $a > 0$, $b > 0$. Let f_h be the solution of the GIG Stein equation defined by (5.5). Let $x > 0$.*

1. *Suppose $p \leq -1$. If h is differentiable with bounded derivative, then*

$$|f_h'(x)| \leq B_{p,a,b} \|h'\|, \quad |xf_h'(x)| \leq C_{p,a,b} \|h'\|.$$

2. *Suppose $p \leq -3$. If h is twice differentiable with bounded first and second derivatives, then*

$$|f_h''(x)| \leq B_{p+2,a,b} \left[\|h''\| + (A_{p,a,b} + (p+1)B_{p,a,b} + aC_{p,a,b}) \|h'\|\right],$$

$$|xf_h''(x)| \leq C_{p+2,a,b} \left[\|h''\| + (A_{p,a,b} + (p+1)B_{p,a,b} + aC_{p,a,b}) \|h'\|\right].$$

3. *Let $k \geq 1$. Suppose $p \leq -2k+1$. If h is at least k times differentiable with $h^{(j)}$ bounded for $j = 1, \ldots, k$, then*

$$\left|f_h^{(k)}(x)\right| \leq B_{p+2k-2,a,b} \left\|h_{k-1}'\right\| \quad \text{and} \quad \left|xf_h^{(k)}(x)\right| \leq C_{p+2k-2,a,b} \left\|h_{k-1}'\right\|$$

where

$$h_{k-1}(x) = h^{(k-1)}(x) - (k-1)(p+k-1-ax)f^{(k-2)}(x) + \left(3 + \frac{(k-4)(k+1)}{2}\right) af^{(k-3)}(x).$$

Proof of items 2 and 3

f_h' satisfies the differential equation

$$x^2 f''(x) + \left(\frac{b}{2} + (p+3)x - \frac{a}{2}x^2\right) f'(x) = h'(x) - (p+1-ax)f(x). \tag{5.14}$$

Let $h_1(x) = h'(x) - (p+1-ax)f_h(x)$. Then, by equation (5.14), f_h' solves the differential equation

$$x^2 f'(x) + \left(\frac{b}{2} + (p+3)x - \frac{a}{2}x^2\right) f(x) = h_1(x) \tag{5.15}$$

which is the $GIG(p+2, a, b)$ Stein equation. Since $p \leq 3$, we have $p+2 \leq -1$ and by item 1 of Theorem 3.1,

$$|f_h''(x)| \leq B_{p+2,a,b} \|h_1'\| \quad \text{and} \quad |xf_h''(x)| \leq C_{p+2,a,b} \|h_1'\|.$$

We have

$$\begin{aligned}|h_1'(x)| &= |h''(x) - (p+1-ax)f_h'(x) + af_h(x)| \\ &\leq \|h''\| - (p+1)\|f_h'\| + a\,|xf_h'(x)| + a\,\|f_h\| \\ &\leq \|h''\| - (p+1)B_{p,a,b}\,\|h'\| + aC_{p,a,b}\,\|h'\| + a\frac{2}{\sqrt{ab}}\frac{K_{p+1}(\sqrt{ab})}{K_p(\sqrt{ab})}\,\|h'\| \\ &= \|h''\| + \left(\frac{2\sqrt{a}K_{p+1}(\sqrt{ab})}{\sqrt{b}K_p(\sqrt{ab})} - (p+1)B_{p,a,b} + aC_{p,a,b}\right)\|h'\|,\end{aligned}$$

which proves item 2.

Let us prove item 3. By induction, the function $f_h^{(k-1)}$ satisfies the differential equation

$$x^2 f^{(k)}(x) + \left(\frac{b}{2} + (p-1+2k)x - \frac{a}{2}x^2\right) f^{(k-1)}(x) = h_{k-1}(x) \tag{5.16}$$

where

$$h_{k-1}(x) = h^{(k-1)}(x) - (k-1)(p+k-1-ax)f^{(k-2)}(x) + \left(3 + \frac{(k-4)(k+1)}{2}\right) af^{(k-3)}(x). \tag{5.17}$$

Equation (5.16) shows that $f_h^{(k)}$ solves the Stein equation of the distribution $\mathrm{GIG}(p+2(k-1), a, b)$ with right-hand-side h_k given by (5.17). Since $p \leq -2k+1$, we have $(p+2(k-1) \leq -1$ and all the assumptions of Theorem 3.1 and Theorem 4.1 are satisfied. Thus,

$$\left|f_h^{(k)}(x)\right| \leq B_{p+2(k-1),a,b}\left\|h_{k-1}'\right\| \quad \text{and} \quad \left|xf_h^{(k)}(x)\right| \leq C_{p+2(k-1),a,b}\left\|h_{k-1}'\right\|.$$

We have

$$\begin{aligned}\left|h_{k-1}'(x)\right| &= \left|h^{(k)}(x) - (k-1)(p+k-1-ax)f^{(k-1)}(x) + (k-1)af^{(k-2)}(x) + \left(3 + \frac{(k-4)(k+1)}{2}\right) af^{(k-2)}(x)\right| \\ &\leq \left\|h^{(k)}\right\| - (k-1)(p+k-1)\left\|f^{(k-1)}\right\| + (k-1)a\left|xf_h^{(k-1)}(x)\right| + \left(3 + \frac{(k-4)(k+1)}{2}\right) a\left\|f^{(k-2)}\right\|.\end{aligned}$$

By induction on k, we can bound h_{k-1}'.

□

5.2 Bounds for the Kummer distribution case

Let

$$s(x) = x(1+x) \quad \text{and} \quad \tau(x) = (1-b)x - cx(1+x) + a. \tag{5.18}$$

Then the density $k_{a,b,c}$ of the Kummer distribution given by

$$k_{a,b,c}(x) = \frac{1}{\Gamma(a)\psi(a, a-b+1; c)} x^{a-1}(1+x)^{-a-b}e^{-cx}, \quad x > 0$$

for $a > 0$, $b \in \mathbb{R}$, $c > 0$, satisfies

$$(s(x)k_{a,b,c}(x))' = \tau(x)k_{a,b,c}(x).$$

As earlier, we have the following Stein characterization of the Kummer distribution:

A random variable X follows the Kummer distribution with density $k_{a,b,c}$ if and only if, for all differentiable function f such that the expectation exists,

$$\mathbb{E}\left[X(1+X)f'(X) + [(1-b)X - cX(1+X) + a]\,f(X)\right] = 0.$$

The corresponding Stein equation is

$$x(x+1)f'(x) + [(1-b)x - cx(1+x) + a]\, f(x) = h(x) - \mathbb{E}h(W) \tag{5.19}$$

where h is a bounded function and W has K(a,b,c) distribution.
The bounded solution of (5.19) is

$$\begin{aligned} f_h(x) =& \frac{1}{x(1+x)k_{a,b,c}(x)} \int_0^x k_{a,b,c}(t)\,[h(t) - \mathbb{E}h(W)]\,dt \\ =& \frac{-1}{x(1+x)k_{a,b,c}(x)} \int_x^{+\infty} k_{a,b,c}(t)\,[h(t) - \mathbb{E}h(W)]\,dt. \end{aligned} \tag{5.20}$$

As in the case of the GIG distribution, the assumptions of Theorem 3.1 are fulfilled with

$$s(x) = x(x+1), \quad \tau(x) = (1-b)x - cx(1+x) + a, \quad g = k_{a,b,c}.$$

Therefore, we have the following new bound of the solution f_h of the Stein equation related to the Kummer distribution, using the expectation given by (1.3):

Theorem 5.3 *Let f_h be the solution of the Kummer Stein equation defined by (5.20). For any Lipschitz continuous test function h,*

$$\|f_h\| \le \frac{\psi(a+1, 2-b; c)}{\psi(a, 1-b; c)} \|h'\|. \tag{5.21}$$

For the bounds of the derivatives, Theorem 5.2 holds, with all the constants defined by replacing the GIG density with the Kummer one.

Remark 5.2 *1. As in Remark 5.1, the fact that $\tau(\mathbb{E}(W)) \ge 0$ implies that $\mathbb{E}(W) \le \beta$, where*

$$\beta = \frac{1 - b - c + \sqrt{(1-b-c)^2 + 4ac}}{2c}$$

is the unique zero of τ on $(0, \infty)$. And we have the following property of the confluent hypergeometric function of the second kind: for all, $a > 0$, $b \in \mathbb{R}$, $c > 0$,

$$2ac\frac{\psi(a+1, 2-b; c)}{\psi(a, 1-b; c)} \le 1 - b - c + \sqrt{(1-b-c)^2 + 4ac}. \tag{5.22}$$

2. Using Lemma 3.1 and plugging in (3.7) the expressions of $\mathbb{E}(W)$ and $\mathbb{E}(W^2)$ given in (1.3) we obtain

$$-ac(a+1)\frac{\psi(a+2, 3-b; c)}{\psi(a, 1-b; c)} + (1-b-c)a\frac{\psi(a+1, 2-b; c)}{\psi(a, 1-b; c)} + a = 0$$

from which one gets the reccurence relation

$$\psi(a+2, b+2; c) = \frac{1}{c(a+1)}\left((b-c)\psi(a+1, b+1; c) + \psi(a, b; c)\right). \tag{5.23}$$

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